Incentives for Demand-Response Programs with Nonlinear, Piece-Wise Continuous Electricity Cost Functions

Carlos Barreto and Alvaro A. Cárdenas

***Abstract*—In this paper we identify some of the limitations with previous electricity cost functions used for demand-response (DR) programs in the retail market of the smart grid. In particular, we prove that while previous models lower the demand, they do not guarantee that they will try to flatten the demand. We then introduce a new nonlinear piece-wise continuous electricity cost function to model the way several consumers (from small utilities to retail consumers) are billed for electricity consumption. We show that the new cost function lowers the peaks in DR programs and design new incentive mechanisms to ensure that distributed agents converge to the Pareto-efficient solution of the system. However, we prove that any incentives mechanism either imposes additional taxes or requires external subsidies to operate. The paper finalizes with two examples of such mechanisms.**

***Index Terms*—Electricity market, dynamic pricing, game theory, mechanism design.**

I. Introduction

One of the goals of the smart grid is to make consumers active participants and decision makers in the retail electricity market through *demand response* (DR) programs [1]. DR programs attempt to achieve better energy efficiency and reduce new capital investments by controlling consumer loads, which might be responsive to conditions in the electricity market. For example, by providing incentives consumers might redistribute their load more evenly—e.g., consume more energy when there is high wind or solar energy in the grid, and reduce consumption during peak demand times. Moreover, an efficient use of resources might defer the need for grid expansion and reduce the investments on fast generators, which are only used to supply peak demands.

In DR programs consumers will have a choice between cost and convenience. A good way to capture the behavior of strategic decision makers are game-theoretic models, where all participants attempt to optimize their own utility functions [2]. Previous work that used game-theoretic models for DR focus on improving social welfare and in optimizing the use of resources in the system [3]–[9]. Nevertheless, the efficiency metrics used, such as Pareto efficiency, are not designed to reduce peaks in demand. Furthermore, in the literature price functions are assumed to be lineal, since the cost of generation is assumed to be quadratic [10]. Therefore, these previous models do not capture non-linearities associated with peak-reduction incentives. These peak-reduction incentives are particularly important for small utilities and co-ops, who have to pay not only for the amount of electricity consumed, but also for the largest electricity peak during the billing period. This happens because the upstream provider has to build enough capacity to provide energy during these peaks. Particularly, the costs charged to customers due to peaks are significant with respect to the cost of the overall energy consumed in a billing period (see some evidence in [11]).

In this paper we address these concerns by the following steps. First, in Section II we introduce the basic notation and two DR models, which use linear price functions: (i) a centralized optimization problem that models direct-load control DR; and (ii) an equivalent decentralized optimization problem modeling selfish agents. This second DR formulation models dynamic pricing. Section III is dedicated to show that although previous work reduces the peak consumption of electricity, they do not target a Peak-to-Average Ratio (PAR) reduction of electricity consumption. Thus, these DR schemes might not be appropriate when peaks have a significant contribution to the cost-function. Next, we introduce in Section IV a new nonlinear price functions to the formulation, and show how it can effectively reduce electricity peaks in a population (for the centralized optimization DR model).

The main contribution of the paper is presented in Section V, where we deal with the design of incentives to reduce electricity peak prices in the decentralized case, using nonlinear price functions. Specifically, we find that it is not possible to design incentives in which rewards are equal to penalties. In other words, a decentralized DR scheme either imposes additional taxes or requires external subsidies to operate. Then, we deduce two mechanisms with the aforementioned properties. Particularly, it is interesting to note that the incentive scheme that requires external incentives was obtained before for the linear price case, by imposing some fairness properties on the incentives (see [5]). This suggest some generality of that mechanism for problems of resource allocation.

II. Background

A. Retail Electricity Market Model

We consider a population of N consumers $\mathcal{V} = 1, \ldots N$. We divide a period of 24 hours in a set of T time intervals denoted $\tau = \{\tau_1, \ldots, \tau_T\}$. Formally, we define the set

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τ as a partition of $[0, 24)$, where $\cup_{t\in\{1,\ldots,T\}}\tau_t = \tau$ and $\cap_{t\in\{1,\ldots,T\}}\tau_t = \varnothing$. Let q_i^t be the electricity consumption of the i^{th} user in the t^{th} time interval. The daily electricity consumption of the i^{th} user is represented by the vector $\boldsymbol{q}_i = [q_i^1, \ldots, q_i^T]^\top \in \Re^T_{\geq 0}$. The population consumption at a given time t is defined by the vector $\boldsymbol{q}^t = [q_1^t, , q_2^t \ldots, q_N^t]^\top \in \Re^N_{\geq 0}$. On the other hand, the joint electricity consumption of the whole population is denoted by $\boldsymbol{q} = [\boldsymbol{q}_1^\top, \ldots, \boldsymbol{q}_N^\top]^\top$ and the vector $\boldsymbol{q}_{-i} = [\boldsymbol{q}_1^\top, \ldots, \boldsymbol{q}_{i-1}^\top, \boldsymbol{q}_{i+1}^\top, \ldots, \boldsymbol{q}_N^\top]^\top \in \mathbb{R}^{T\cdot(N-1)}_{\geq 0}$ represents the consumption of the population, except for the i^{th} agent. Without loss of generality, we assume that the electricity consumption of the i^{th} user satisfies $q_i^t \geq 0$, in each time instant t. A *valuation function* $v_i^t(q_i^t)$ models the *valuation* that the i^{th} user gives to an electricity consumption of q_i^t units in the t^{th} time interval. Finally, let $p(\cdot) : \Re \to \Re$ be the price of electricity charged to consumers. The aggregated consumption at a given time t is defined as $\|\boldsymbol{q}^t\|_1 = \sum_{j=1}^N q_j^t$.

B. Centralized Optimization DR Model

Assuming that the electricity generation cost is the same for all t, we can express the profit function of each individual as their valuation of electricity minus their electricity bill, i.e., $U_i(\boldsymbol{q}) = \sum_{t=1}^T \left(v_i^t(q_i^t) - q_i^t p\left(\|\boldsymbol{q}^t\|_1\right)\right),$ where $p : \Re_+ \to \Re_+$ is the price per unit of consumption. The consumers welfare function is maximized by solving the following optimization problem [12]:

$$\begin{aligned} \underset{\boldsymbol{q}}{\text{maximize}} \quad & \sum_{i=1}^N U_i(\boldsymbol{q}) = \sum_{i=1}^N \left(\sum_{t=1}^T \left(v_i^t(q_i^t) - q_i^t p\left(\left\|\boldsymbol{q}^t\right\|_1 \right) \right) \right) \\ \text{subject to} \quad & q_i^t \geq 0, i = \{1, \ldots, N\}, t = \{1, \ldots, T\}. \end{aligned} \tag{1}$$

Here we make some assumptions on the problem characteristics in order to guarantee that the problem has a maximum and it is unique.

Assumption 1.

i. The valuation function $v_i^t(\cdot)$ is differentiable, concave, and non-decreasing.
ii. The price $p(\cdot)$ is differentiable, convex, and non-decreasing.

Assumption 2. The maximum of a concave function is inside the feasible set, i.e., the following inequality is satisfied for all i: $\frac{\partial}{\partial q_i^t} U_i(\boldsymbol{0}) > 0$.

Notice that this optimization problem assumes a central planner coordinating the consumption of each user, and thus, it is a model of DR program like direct-load control, where a central agent directly controls electricity consumption of different agents. A drawback is that the central planner requires full information to find the optimal solution.

C. Decentralized Optimization DR Model

A decentralized DR scheme models the case when all agents keep their valuation of electricity to themselves, and have autonomous control of their consumption. Particularly, we need to consider strategic agents that will try to selfishly maximize their own profit. The analysis of strategic interactions among rational agents can be done by using game theory [2]. The outcome of a distributed system is the equilibrium of a game between users, which is the solution of the following optimization problem:

$$\begin{aligned} \underset{\boldsymbol{q}_i}{\text{maximize}} \quad & U_i(\boldsymbol{q}_i, \boldsymbol{q}_{-i}) = \sum_{t=1}^T \left(v_i^t(q_i^t) - q_i^t p(\left\|\boldsymbol{q}^t\right\|_1) \right) \\ \text{subject to} \quad & q_i^t \geq 0, i = \{1, \ldots, N\}, t = \{1, \ldots, T\}. \end{aligned} \tag{2}$$

The solution of the decentralized optimization problem is sub-optimal with respect to the solution of the centralized scheme (Eq. (2) and (1), respectively) [12]. Furthermore, delegation of actions among users also involves a larger consumption of resources, with respect to the centralized scheme. This situation is known as the tragedy of the commons [5].

An alternative to mitigate the inefficiency of the decentralized scheme is to design incentives that modify the behavior of users for the good of the population. For instance, we can modify the profit function of each user (by adding some price signal) to make the new game efficient in the sense of Pareto. Consider a new profit function for the i^{th} agent [5]:

$$W_i(\boldsymbol{q}_i^t, \boldsymbol{q}_{-i}^t) = \sum_{t=1}^T v_i^t(q_i^t) - q_i^t p\left(\left\|\boldsymbol{q}^t\right\|_1\right) + I_i(\boldsymbol{q}^t) \tag{3}$$

where incentives are of the form

$$I_i(\boldsymbol{q}^t) = \left(\left\|\boldsymbol{q}_{-i}^t\right\|_1\right)\left(h_i(\left\|\boldsymbol{q}_{-i}^t\right\|) - p\left(\left\|\boldsymbol{q}^t\right\|_1\right)\right). \tag{4}$$

The incentives assign rewards or penalties according to the contribution made by an agent to the society. In particular, the function $h_i : \Re \to \Re$ is a design parameter that estimates the externalities introduced by each individual. It can be shown that these incentives can lead to an optimal equilibrium in a strategic environment. Note that the incentives modify the price paid by each user according to their relative consumption. However, two different users receive different incentives if their consumption is different.

III. Limitations of Previous DR Models

One of the objectives of DR is to shave electricity consumption peaks. Nevertheless, Pareto efficiency does not guarantee a flat demand profile. To see why, let us impose some constraints on the valuation functions. We assume that the valuation functions belong to a family of functions $v : \Re_+ \times \Re_+ \to \Re$. Here, the valuation function of the i^{th} agent at the t^{th} time interval is defined as $v_i^t(q) := v(q, \alpha_i^t)$, where α_i^t is a parameter that characterizes the form of the valuation. The family of valuation functions satisfy the following properties:

Assumption 3. The family of valuation functions $v : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfies the following conditions:

- If $\alpha_i^{k_1} > \alpha_j^{k_2}$, then $v(q, \alpha_i^{k_1}) > v(q, \alpha_j^{k_2})$, for all $i, j \in \mathcal{P}$ and $\alpha_i^{k_1}, \alpha_j^{k_2}, q \in \mathbb{R}_{\geq 0}$.
- $\lim_{q\to 0} v(q, \alpha) = 0$, for $\alpha \in \mathbb{R}_{\geq 0}$.

Now we are ready to prove that an efficient outcome might not be flat.

Proposition 1. *Consider a centralized maximization problem described by Eq. (1) that satisfies Assumptions 1, 2, and 3. If* *agents have different valuations at each time period, then the aggregated demand is greater in the time period with greater valuation. That is, if there are t_1 and t_2, such that for all i $\alpha_i^{t_1} > \alpha_i^{t_2}$, then $\|\boldsymbol{q}^{t_1}\|_1 > \|\boldsymbol{q}^{t_2}\|_1$.*

Proof. The proof is made by contradiction. Let us assume that the aggregated consumption at time t_1 is less than or equal to the aggregated consumption at time t_2, i.e., $\|\boldsymbol{q}^{t_1}\|_1 \le \|\boldsymbol{q}^{t_2}\|_1$. The first-order conditions (FOC) for two different time periods t_1 and t_2 for all individual $i \in \mathcal{V}$ are:

$$\dot{v}(q_i^{t_1}, \alpha_i^{t_1}) - p(\|q^{t_1}\|_1) - \|q^{t_1}\|_1 \, \dot{p}(\|q^{t_1}\|_1) = 0, \quad (5)$$

$$\dot{v}(q_i^{t_2}, \alpha_i^{t_2}) - p(\|q^{t_2}\|_1) - \|q^{t_2}\|_1 \, \dot{p}(\|q^{t_2}\|_1) = 0. \quad (6)$$

Note that $p(\cdot)$ and $\dot{p}(\cdot)$ are increasing functions. We can use our contradiction assumption in Eq. (5) and (6) to obtain the following inequality:

$$\dot{v}(q_i^{t_2}, \alpha_i^{t_2}) > \dot{v}(q_i^{t_1}, \alpha_i^{t_1}),$$

for all $i \in \mathcal{V}$. Now, recall from Assumption 3 that the derivative of two valuation functions with different parameters ($\alpha_i^{t_1}$ and $\alpha_i^{t_2}$) evaluated at the same point ($q_i^{t_2}$) satisfies the following inequality:

$$\dot{v}(q_i^{t_2}, \alpha_i^{t_1}) > \dot{v}(q_i^{t_2}, \alpha_i^{t_2}) > \dot{v}(q_i^{t_1}, \alpha_i^{t_1}), \quad (7)$$

for all $i \in \mathcal{V}$, since $\alpha_i^{t_1} > \alpha_i^{t_2}$ for all users in the society. Now, in order to satisfy Eq. (7), $q_i^{t_1}$ has to be greater that $q_i^{t_2}$, for all $i \in \mathcal{V}$ (recall that $\dot{v}(\cdot, \alpha)$ is decreasing). However, if $q_i^{t_1}$ is greater than $q_i^{t_2}$ for every individual, then the aggregated demand at time t_1 has to be greater than the aggregated demand at time t_2, i.e., $\|\boldsymbol{q}^{t_1}\| > \|\boldsymbol{q}^{t_2}\|$. This is in contradiction with our initial assumptions. Hence, we conclude that $\|\boldsymbol{q}^{t_1}\|_1 > \|\boldsymbol{q}^{t_2}\|_1$. □

From this result we conclude that the total demand might be time changing, however, it is not clear if the PAR improves in the optimal outcome. We illustrate the differences between optimal and sub-optimal outcomes by means of the following numerical example. In particular, in this example PAR reduction is not a consequence of an efficient outcome.

Example 1. We select some typical functions previously used in the literature [4], [13]

$$v_i^t(q_i^t) = \alpha_i^t \log(1 + q_i^t), \; \alpha_i^t > 0,$$

$$p(\|\boldsymbol{q}\|_1) = \beta \|\boldsymbol{q}\|_1, \; \beta > 0.$$

These functions satisfy Assumptions 3, 1, and 2. In order to model time varying valuations along a day, we assign to α_i^k a value proportional to the actual consumption in an electrical system. In this case, we consider $T = 24$ time periods and define the valuations of each individual using consumption measurements provided by the Colombian electricity system administrator [14] (a detailed implementation of the simulations can be found in [15]). In Fig. 1 we can see how the inefficient outcome (Nash equilibrium) has less total utility for all parties and produces more power consumption than the efficient outcome (Pareto equilibrium). While in this example

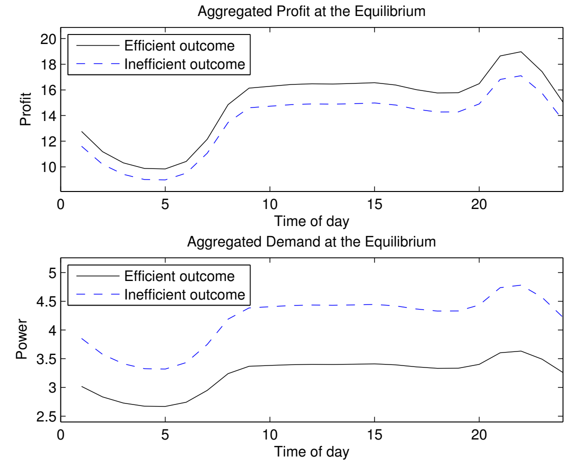

Fig. 1: Profit and demand at both efficient and inefficient outcomes.

we can see how if the electric utility designs incentives that guarantee the optimal outcome of the system, then the whole system is more efficient (including a reduction in electricity consumption) than in the Nash equilibrium (without incentives), there can still be peaks of electricity that can potentially be reduced under a different setting.

In this example we find that the inefficient outcome (Nash equilibrium) has a Peak-to-Average Ratio (PAR) of 1.1139, while the efficient outcome (Pareto solution) has a PAR of 1.1228. Note that the efficient outcome has a slight major PAR.

IV. Nonlinear Piece-Wise Continuous Price Functions for the Centralized DR Model

We now incorporate in the DR models the cost of peaks. Let us introduce the following price per unit of electricity consumption:

$$\hat{p}(\boldsymbol{q}) = p(\boldsymbol{q}) + p_k(\boldsymbol{q})$$

where the term $p_k(\boldsymbol{q})$ can be interpreted as an additional price (a peak tax) or incentive designed by an upstream electricity distributor or a regulatory entity to reduce demand peaks. The function $p_k : \Re \to \Re$ is defined as follows:

$$p_k(z) = \begin{cases} 0, & \text{if } z \le k, \\ \gamma(z-k)^2, & \text{if } z > k, \end{cases}$$

where γ is a positive real number. Note that $p(\cdot)$ is a continuous differentiable function. Note also that this new price function is nonlinear piece-wise continuous, and serves as a justification for further analysis of more complex price functions in DR programs.

Example 2. We now include the additional cost function $p_k(\cdot)$ in our numerical example. In this case, the threshold k corresponds to the average consumption at the efficient outcome, i.e., $k = 3.3$, and our goal is to penalize significantly peaks above it $\gamma = 15$. With this new set up we obtain the Pareto efficient solution for all agents and observe that the PAR of the demand is 1.0446. This value is lower than the PAR

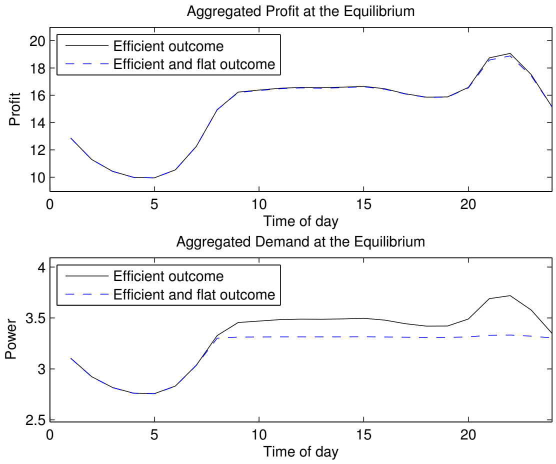

Fig. 2: Profit and demand at both efficient and flat outcomes.

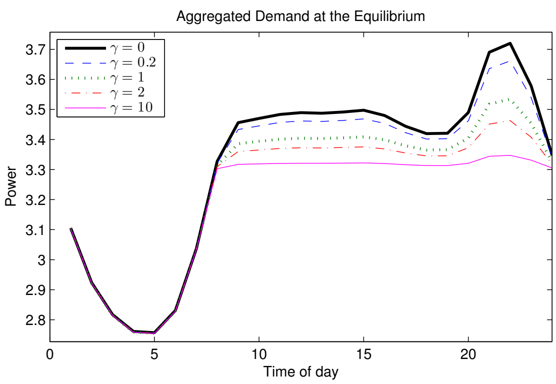

Fig. 3: Aggregated power demand for different values of γ.

of the efficient demand in the Example 1 (case with $\gamma = 0$), which is 1.1228 (see Fig. 2).

Note that the maximum consumption is close to the threshold specified, causing a reduction in prices. Furthermore, aggregated profit of the population is not affected notably. Inspection the profit of each individual, however, reveals that some individuals improve their utility with the peak tax, while others do worse. This happens because some agents do not have high electricity valuations and can improve their utilities with lower prices associated with peak reduction.

Fig. 3 shows the demand profile of the population for different values of γ. We can see that the impact of the tax is more drastic with higher tax increase rates.

V. Designing Incentives for the Decentralized DR Model with Nonlinear Price Functions

We have shown how nonlinear price functions can be used to flatten the demand in the centralized DR model. Nevertheless, we now need to show how to design incentives for the decentralized DR model such that the selfish agents converge to the same Pareto optimal solution described in the previous section. To simplify the notation we drop the superscript t from the consumption profiles.

First, we address the problem of whether or not there is an incentive mechanism that satisfies the property of budget balance, i.e., we want to see if there is an incentive mechanism of the form in Eq. (4) that satisfies

$$\sum_{i\in\mathcal{V}} I_i(\boldsymbol{q}) = \sum_{i\in\mathcal{V}} \left\|\boldsymbol{q}_{-i}\right\|_1 \left(h_i(\left\|\boldsymbol{q}_{-i}\right\|_1) - p\left(\|\boldsymbol{q}\|_1\right)\right) = 0.$$

This is an ideal property because rewards are equal to penalties. In other words, a budget balanced systems neither require additional taxes nor requires external subsidies to operate.

The budget balance property can be rewritten as

$$p\left(\|\boldsymbol{q}\|_1\right) = \frac{\sum_{i\in\mathcal{V}} \left(\left\|\boldsymbol{q}_{-i}\right\|_1 h_i(\left\|\boldsymbol{q}_{-i}\right\|_1)\right)}{\sum_{i\in\mathcal{V}} \left\|\boldsymbol{q}_{-i}\right\|_1} \tag{8}$$

To prove the budget balance impossibility of our case, we need to find some properties that must be satisfied by the function $h_i(\cdot)$. Specifically, we prove that if the price function $p(\cdot)$ is concave, then $h_i(\cdot)$ must be concave as well. First, we analyze Eq. (8) for the particular case of a uniform demand profile.

Proposition 2. *An incentive that is budget balanced must satisfy*

$$p\left(\|\hat{\boldsymbol{q}}\|_1\right) = h_i(\left\|\hat{\boldsymbol{q}}_{-i}\right\|_1), \tag{9}$$

given a uniform demand profile $\hat{\boldsymbol{q}}$ for all $i \in \mathcal{V}$.

Proof. Let us consider a uniform consumption profile $\hat{\boldsymbol{q}}$ where all individuals have the same consumption, i.e., $\hat{\boldsymbol{q}} = \varepsilon\, \boldsymbol{e}$, for $\varepsilon > 0$ and $\boldsymbol{e}$ a vector of ones in $\Re^N$. According to this, Eq. (8) can be expressed as:

$$p\left(\|\hat{\boldsymbol{q}}\|_1\right) = \sum_{i\in\mathcal{V}} h_i(\left\|\hat{\boldsymbol{q}}_{-i}\right\|_1) \Big/ N\,. \tag{10}$$

Recall that $h_i(\left\|\boldsymbol{q}_{-i}\right\|_1) = h_j(\left\|\boldsymbol{q}_{-j}\right\|_1)$ if and only if $\boldsymbol{q}_{-i} = \boldsymbol{q}_{-j}$, for all $i, j \in \mathcal{V}$. Since the demand profile is uniform, we have $\hat{\boldsymbol{q}}_{-i} = \hat{\boldsymbol{q}}_{-j}$, and consequently Eq. (10) is equivalent to $p\left(\|\hat{\boldsymbol{q}}\|_1\right) = h_i(\left\|\hat{\boldsymbol{q}}_{-i}\right\|_1) = h(\left\|\hat{\boldsymbol{q}}_{-i}\right\|_1)$. □

Remark 1. *Note that Eq. (9) must be satisfied for every demand profile $\hat{\boldsymbol{q}}$. Thus, if $p(\cdot)$ is convex, then $h(\cdot)$ must be convex on $\hat{\boldsymbol{q}}_{-i}$.*

Now we are ready to prove that there is no function $h_i(\cdot)$ that satisfies Eq. (8) for all possible demand profiles of a society.

Theorem 1. *Consider a system described by Eq. (3) that satisfies Assumptions 3, 1, and 2 and an incentive scheme of the form in Eq. (4). Then, there is no function $h(\cdot)$ that satisfies the budget balance property.*

Proof. We prove this theorem by contradiction. We restrict our analysis to convex functions $h_i(\cdot)$, since this is a necessary condition to achieve budget balance (see Remark 1). First, let us consider a population of two agents, denoted as customers 1 and 2. Consequently, the demand profile is expressed as $\|\boldsymbol{q}\|_1 = |q_1| + |q_2|$, while $\left\|\boldsymbol{q}_{-i}\right\|_1 = |q_j|$, for $i \neq j$. In this case, Eq. (8) can be rewritten as

$$p(|q_1| + |q_2|) = \frac{|q_2| h_1(|q_2|) + |q_1| h_2(|q_1|)}{|q_1| + |q_2|}. \tag{11}$$

Now, let us consider the case in which the aggregated consumption is constant, i.e., $\|\boldsymbol{q}\|_1 = Q$. Let us introduce the following variable change:

$$\rho = q_1/(q_1+q_2)\,, \qquad 1-\rho = q_2/(q_1+q_2)\,.$$

where $0 \le \rho \le 1$. We can use this variable change and the fact that $h_1(z) = h_2(z) = h(z)$ (see Eq. (9)) to transform Eq. (11) into:

$$p(Q) = (1-\rho)h\big((1-\rho)Q\big) + \rho\, h\big(\rho\, Q\big) \tag{12}$$

Note that the right hand side of Eq. (12) must be constant for every vector $\boldsymbol{q}$ that satisfies $\|\boldsymbol{q}\|_1 = Q$, i.e., for any ρ belonging to the subspace $[0,1]$. However, the right hand side of Eq. (12) is not constant on ρ. For example, note that if ρ is equal to one or zero, then the right hand side of Eq. (12) is equal to $h(Q)$. On the other hand, if ρ is equal to $1/2$, then the right hand side of Eq. (11) is equal to $h(1/2Q)$. Note that $h(Q) \neq h(1/2Q)$. This statement can be extended for larger populations. Hence, it is not possible to find a convex function that satisfies the property of budget balance. □

The previous result is negative in the sense that it establishes a limitation for the design of DR incentives. Now, we look into the problem of analyzing the conditions in which the mechanism either imposes taxes over customers or requires external subsidies. First, let us consider the case when require external subsidies.

A. Incentives that Require External Subsidies

The following theorem shows the existence of an incentive scheme in which incentives granted to the population are greater than penalties. Hence, it is necessary to obtain external subsidies to support the incentive scheme.

Theorem 2. *Consider a system described by Eq. (3) that satisfies Assumptions 3, 1, 2 and an incentive scheme satisfying the form in Eq. (4). If we use the following convex function* $h_i : \Re \to \Re$*, defined as*

$$h_i(\|\boldsymbol{q}_{-i}\|_1) = p\left(N/(N-1)\,\|\boldsymbol{q}_{-i}\|_1\right), \tag{13}$$

the incentives scheme satisfies the weakly budget balance property, i.e., $\sum_{i\in\mathcal{V}} I_i(\boldsymbol{q}) \ge 0$.

Proof. Recall that external subsidies are required when the following condition is satisfied:

$$\sum\nolimits_{i\in\mathcal{V}} I_i(\boldsymbol{q}) \ge 0. \tag{14}$$

This expression can be used with Eq. (4) to obtain

$$p\left(\|\boldsymbol{q}\|_1\right) \le \frac{\sum_{i\in\mathcal{V}}\left(\|\boldsymbol{q}_{-i}\|_1\, h_i(\|\boldsymbol{q}_{-i}\|_1)\right)}{\sum_{i\in\mathcal{V}}\|\boldsymbol{q}_{-i}\|_1}. \tag{15}$$

Now, let us consider

$$\theta = \sum\nolimits_{i\in\mathcal{V}} \|\boldsymbol{q}_{-i}\|_1 = (N-1)\,\|\boldsymbol{q}\|_1\,. \tag{16}$$

Also, let us introduce the following variable change (similar to the one used in Theorem 1):

$$\rho_i = \|\boldsymbol{q}_{-i}\|_1 \Big/ \sum\nolimits_{i\in\mathcal{V}} \|\boldsymbol{q}_{-i}\|_1\,. \tag{17}$$

Thus, $\sum_{i\in\mathcal{V}}\rho_i = 1$ and $\|\boldsymbol{q}_{-i}\|_1 = \theta\rho_i$. Now we can rewrite Eq. (15) as

$$p\left(\|\boldsymbol{q}\|_1\right) \le \sum\nolimits_{i\in\mathcal{V}} \rho_i\, h_i\left(\theta\rho_i\right). \tag{18}$$

Recall that $h_i(\cdot)$ is convex, thus, the following condition is satisfied:

$$h_i\left(\sum\nolimits_{i\in\mathcal{V}}\lambda_i z_i\right) \le \sum\nolimits_{i\in\mathcal{V}}\lambda_i h_i\left(z_i\right), \tag{19}$$

where $z_i \in \Re_+$, and

$$\sum\nolimits_{i\in\mathcal{V}}\lambda_i = 1. \tag{20}$$

Note that the strict equality is satisfied when $\lambda_i = \lambda_j$ and $z_i = z_j$ for all $i,j\in\mathcal{V}$. Now, choosing $z_i = \theta\rho_i$ and $\lambda_i = \rho_i$ (which satisfies Eq. (20)), Eq. (19) can be rewritten as:

$$h_i\left(\theta\sum\nolimits_{i\in\mathcal{V}}\rho_i^2\right) \le \sum\nolimits_{i\in\mathcal{V}}\rho_i h_i\left(\theta\rho_i\right). \tag{21}$$

At this point we have developed the tools to prove that Eq. (13) satisfies the (weakly) budget balance property in Eq. (14). First, let us consider a trivial case, when all individuals consume the same amount of power, i.e., the demand profile can be defined as $\hat{\boldsymbol{q}} = k\boldsymbol{e}$, with $k > 0$. In this case, we want to verify that the incentive is budget balanced, i.e., satisfies Eq. (9).

Observe that with a uniform demand $\rho_i = \rho_j = 1/N$, Eq. (21) holds with strict equality. Thus, we can replace Eq. (21) in Eq. (18) to obtain

$$p\left(\|\hat{\boldsymbol{q}}\|_1\right) \le h_i\left(\theta\sum\nolimits_{i\in\mathcal{V}}\rho_i^2\right) = h_i\left((N-1)/N\,\|\hat{\boldsymbol{q}}\|_1\right).$$

Moreover, with a uniform distribution $N-1/N\,\|\hat{\boldsymbol{q}}\|_1 = \|\hat{\boldsymbol{q}}_{-i}\|_1$, and therefore, Eq. (9) is satisfied.

Now, we need to prove that Eq. (18) is satisfied for all demand profiles $\boldsymbol{q}$. Note that if replace Eq. (13) in Eq. (21) we obtain

$$p\left(N/(N-1)\theta\sum\nolimits_{i\in\mathcal{V}}\rho_i^2\right) \le \sum\nolimits_{i\in\mathcal{V}}\rho_i p\left(N/(N-1)\theta\rho_i\right). \tag{22}$$

Note also that $\min_{\rho_1,\ldots,\rho_N}\sum_{i\in\mathcal{V}}\rho_i^2 = 1/N$ (the minimum takes place when $\rho_i = 1/N$, for all $i\in\mathcal{V}$). Hence, the left part of Eq. (22) has a lower bound given by

$$p\left(\theta/(N-1)\right) \le p\left(N/(N-1)\theta\sum\nolimits_{i\in\mathcal{V}}\rho_i^2\right). \tag{23}$$

We can use Eq. (16), (23), and (22) to show that the following is satisfied for any demand profile $\boldsymbol{q}$

$$p\left(\|\boldsymbol{q}\|_1\right) \le \sum\nolimits_{i\in\mathcal{V}}\rho_i\, p\left(N/(N-1)\theta\rho_i\right),$$

which is equivalent to Eq. (18). It is interesting that the same rule is obtained for the linear price case, imposing some fairness properties on the incentives [5]. □

B. Incentives that Impose Taxes

Now we show the existence of an incentive scheme in which punishments are grater than rewards, i.e., this scheme is supported through taxes imposed on the population.

Theorem 3. *Consider a system described by Eq. (3) that satisfies Assumptions 3, 1, and 2 and a incentives scheme of the form in Eq. (4). Then, the convex function $h_i : \Re \to \Re$ defined as*

$$h_i(\|\boldsymbol{q}_{-i}\|_1) = p\left(\|\boldsymbol{q}_{-i}\|/(N-1)\right), \tag{24}$$

can be used to satisfy the weakly budget balance property, i.e., $\sum_{i\in\mathcal{V}} I_i(\boldsymbol{q}) \leq 0$.

Proof. First, note that the mechanism does not require external subsidies when $\sum_{i\in\mathcal{V}} I_i(\boldsymbol{q}) \leq 0$. Hence, from Eq. (4) we can extract

$$p\left(\|\boldsymbol{q}\|_1\right) \geq \frac{\sum_{i\in\mathcal{V}}\left(\|\boldsymbol{q}_{-i}\|_1 h_i(\|\boldsymbol{q}_{-i}\|_1)\right)}{\sum_{i\in\mathcal{V}}\|\boldsymbol{q}_{-i}\|_1}.$$

Making the variable change of Eq. (16) and (17) results in

$$p\left(\|\boldsymbol{q}\|_1\right) \geq \sum_{i\in\mathcal{V}} \rho_i\, h_i\left(\theta\rho_i\right). \tag{25}$$

In general, this inequality can be satisfied if

$$p\left(\|\boldsymbol{q}\|_1\right) \geq \max_{\rho_1,\ldots,\rho_N} \sum_{i\in\mathcal{V}} \rho_i\, h_i\left(\theta\rho_i\right). \tag{26}$$

It can be shown that the right hand part of Eq. (26) is maximum for some $\rho_i = 1$ and $\rho_j = 0$, for $i \neq j$. Thus, we can define the following lower bound of Eq. (25):

$$p\left(\|\boldsymbol{q}\|_1\right) = p\left(\theta/(N-1)\right) \geq h\left(\theta\right).$$

Now, let us introduce the candidate function

$$h_i\left(\|\boldsymbol{q}_{-i}\|_1\right) = p\left(\|\boldsymbol{q}_{-i}\|_1/(N-1)\right). \tag{27}$$

If we replace Eq. (27) into Eq. (25) see that

$$p\left(\theta/(N-1)\right) \geq p\left(\theta/(N-1)\right).$$

Since this inequality is true, we verify that Eq. (27) satisfies the budget condition in Eq. (26). □

C. Importance of the Incentives Scheme

Let us denote by $I_i^{\mathcal{S}}(\cdot)$ and $I_i^{\mathcal{T}}(\cdot)$ two incentives schemes that use the results from Sections V-A and V-B, respectively. Note that $I_i^{\mathcal{S}}(\cdot)$ assigns more incentives to each customer because the estimation of $h_i(\cdot)$ in Eq. (13) is larger than the estimation made in Eq. (24). This has a positive effect in the welfare of the society. However, the implementation of $I_i^{\mathcal{S}}(\cdot)$ might not be feasible because it might require external subsidies.

On the other hand, $I_i^{\mathcal{T}}(\cdot)$ provides lower incentives, and the welfare of the whole population might be lower, with respect to $I_i^{\mathcal{S}}(\cdot)$. However, its implementation might be more convenient since it does not require external subsidies. Part of the future work is to analyze if customers have incentives to join a system that implements $I_i^{\mathcal{T}}(\cdot)$.

VI. Conclusions and Future Work

In this work we showed that popular electricity cost functions in the literature do not achieve the peak minimization goal of practical DR programs. We then introduced a new nonlinear pricing function that achieves this objective (using a centralized scheme). Later, we showed how there is no incentive mechanism that, in a distributed way, can achieve Pareto efficiency with a balanced budget. We finalized the paper by introducing two incentive rules, which either require external subsidies or impose taxes on the population to operate.

A drawback of the analysis is that the consumption is assumed to be independent in each time interval. That is, agents do not keep a memory of their previous consumption or future consumption needs. For example, if an agent does not use energy for an interval of time, it is very likely that its valuation of energy will increase in time. Incorporating the previous and future demand into their current energy valuation is an open formulation problem that we plan to address in the future.

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