

REFLECTION GROUPS AND COMPOSITE MAPPINGS IN CLASS \mathcal{C}^r

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ABSTRACT. Let W be a finite reflection group acting orthogonally on \mathbb{R}^n and $P = \{p_1, \dots, p_n\}$ be a set of basic polynomial invariants. We show that the algebra of composite mappings $P^*(\mathcal{E}^r(P(\mathbb{R}^n)))$ is a Fréchet subspace of the space of r -regular jets of order hr , where h is the highest degree of the polynomials p_i (the largest Coxeter number of the irreducible components of W). The algebra homomorphism P^* is a Fréchet isomorphism that identifies this space with the space of functions of class \mathcal{C}^r of the polynomial invariants. This study needs the Whitney 1-regularity property of $P(\mathbb{R}^n)$ and by lack of a reference we had to complete a proof of this property given in [7] with a lemma that was not proved for all Coxeter groups.

1. INTRODUCTION

Let W be a finite reflection group acting orthogonally on \mathbb{R}^n . The algebra of W -invariant polynomials is generated by n algebraically independent W -invariant homogeneous polynomials and the degrees of these basic invariants are uniquely determined [12], [5]. Let p_1, \dots, p_n be a set of basic invariants, we define the ‘Chevalley’ mapping

$$P : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad P(x) = (p_1(x), \dots, p_n(x)).$$

Glaeser’s theorem [9] shows that W -invariant functions of class \mathcal{C}^∞ , may be expressed as \mathcal{C}^∞ functions of the basic invariants, and actually that the subalgebra $P^*(\mathcal{C}^\infty(\mathbb{R}^n))$ of composite mappings of the form $F \circ P$ with F of class \mathcal{C}^∞ , is closed in $\mathcal{C}^\infty(\mathbb{R}^n)$. In finite class of differentiability, the situation is not this simple. Let h be the highest degree of the coordinate polynomials in P , equal to the greatest Coxeter number of the irreducible components of W . In [4] it is shown that if a function $f \in \mathcal{C}^r(\mathbb{R}^n)$ is invariant by W , there exists an F of class $[r/h]$, the integer part of r/h , such that $f = F \circ P$. A general counterexample shows that this result is the best possible. However if F is of class \mathcal{C}^r , in general $f = F \circ P$ is of class \mathcal{C}^r and not of class \mathcal{C}^{hr} . Here we study the algebra of composite mappings $P^*(\mathcal{C}^r(\mathbb{R}^n))$. The functions in this algebra induce jets in a Fréchet space which may be identified with the space of functions of class \mathcal{C}^r of the invariants. For this study, we need to show first that the image set $P(\mathbb{R}^n)$ is Whitney 1-regular.

Definition 1.1. [15][13] A compact set $K \subset \mathbb{R}^n$, connected by rectifiable arcs, is *Whitney 1-regular* if the geodesic distance in K is equivalent to the Euclidean distance: there exists $k_K > 0$ such that for all $(x, x') \in K^2$, there exists a rectifiable arc from x to x' in K with length $\ell(x, x') \leq k_K |x - x'|$.

Key words and phrases. Whitney regularity property, Finite Coxeter groups, Chevalley theorem, Morse Theory.

A closed set is Whitney 1-regular if it is a union of Whitney 1-regular compact sets.

The 1-regularity of $P(\mathbb{R}^n)$ which is of interest by itself, was conjectured in [4] and already in [7]. By lack of a better reference we state and prove this property in section 3, using the results in [7] and the results given in [10] for the symmetric group that apply for any Coxeter group, *mutatis mutandis*

2. WHITNEY FUNCTIONS AND r -REGULAR, m -CONTINUOUS JETS.

One may find a study of Whitney functions in [13], the notations of which will be used freely.

A jet of order $m \in \mathbb{N}$, on a locally closed set $E \subset \mathbf{R}^n$ is a collection $A = (a_k)_{\substack{k \in \mathbb{N}^n \\ |k| \leq m}}$ of real valued functions a_k continuous on E . At each point $x \in E$ the jet A determines a polynomial $A_x(X)$, and we sometimes speak of continuous polynomial fields instead of jets. As a function, A_x acts upon vectors $x' - x$ tangent to \mathbf{R}^n at x and we write somewhat inconsistently:

$$A_x : x' \mapsto A_x(x') = \sum_k \frac{1}{k!} a_k(x) (x' - x)^k.$$

The space $J^m(E)$ of jets of order m on E is naturally provided with the Fréchet topology induced by the family of semi-norms: $|A|_m^{K_n} = \sup_{\substack{x \in K_n \\ |k| \leq m}} |\frac{1}{k!} a_k(x)|$, where K_n runs through a countable exhaustive collection of compact sets of E . By formal derivation of A of order $q \in \mathbb{N}^n$, $|q| \leq m$, we get jets of the form $(a_{q+k})_{|k| \leq m - |q|}$ inducing polynomials

$$(D^q A)_x(x') = \left(\frac{\partial^{|q|} A}{\partial x^q} \right)_x(x') = a_q(x) + \sum_{\substack{k > q \\ |k| \leq m}} \frac{1}{(k - q)!} a_k(x) (x' - x)^{k - q}.$$

For $0 \leq |q| \leq r \leq m$, we put:

$$(R_x A)^q(x') = (D^q A)_{x'}(x') - (D^q A)_x(x').$$

Definition 2.1. Let A be a jet of order m on E . For $r \leq m$, A is r -regular on E , if and only if for all compact set K in E , for $(x, x') \in K^2$, and for all $q \in \mathbb{N}^n$ with $|q| \leq r$, it satisfies the Whitney conditions.

$$(\mathcal{W}_q^r) \quad (R_x A)^q(x') = o(|x' - x|^{r - |q|}), \text{ when } |x - x'| \rightarrow 0.$$

Remark 2.2. If $m > r$ there is no need to consider the truncated field A^r in stead of A in the conditions (\mathcal{W}_q^r) . Actually $(R_x A^r)^q(x')$ and $(R_x A)^q(x')$ differ by a sum of terms $[a_k(x)/(k - q)!] (x' - x)^{k - q}$, with a_k uniformly continuous on K and $|k| - |q| > r - |q|$.

The space of r -regular jets of order m on E , is naturally provided with the Fréchet topology defined by the family of semi-norms:

$$\|A\|_{r,m}^{K_n} = |A|_m^{K_n} + \sup_{\substack{(x, x') \in K_n^2 \\ x \neq x', |k| \leq r}} \left(\frac{|(R_x A)^k(x')|}{|x - x'|^{r - |k|}} \right).$$

This Fréchet space is denoted by $\mathcal{E}^{r,m}(E)$.

If $r = m$, $\mathcal{E}^r(E)$ is the space of Whitney fields of order r or Whitney functions of class \mathcal{C}^r on E . If E is open, $\mathcal{E}^r(E)$ is the space of Taylor polynomial fields of functions in $\mathcal{C}^r(E)$ and these two spaces may be identified.

In general the norms $\|\cdot\|_r^K$ and $|\cdot|_r^K$ are not equivalent on $\mathcal{E}^r(K)$. Nevertheless, Whitney showed:

Proposition 2.3. [15], [13] *If K is 1-regular, the norms $\|\cdot\|_r^K$ and $|\cdot|_r^K$ are equivalent on $\mathcal{E}^r(K)$.*

Conversely, assuming that the compact K is connected by rectifiable arcs (or is a finite union of sets connected by rectifiable arcs), Glaeser has proved :

Proposition 2.4. [8], [13] *If the norms $\|\cdot\|_1^K$ and $|\cdot|_1^K$ are equivalent on $\mathcal{E}^1(K)$, then K is 1-regular.*

For an example of functions in $\mathcal{E}^{r,m}(E)$, let us consider the Chevalley polynomial mapping P and the induced mapping

$$P^* : \mathcal{E}^r(P(\mathbb{R}^n)) \longrightarrow \mathcal{E}^r(\mathbb{R}^n), \quad P^*(F) = F \circ P.$$

For any $(a, x) \in \mathbb{R}^n \times \mathbb{R}^n$, by the Taylor formula for F between $P(a)$ and $P(x)$, we have:

$$(2.1) \quad F[P(x)] = F[P(a)] + \sum_{1 \leq |\beta| \leq r} \frac{1}{\beta!} D^\beta F[P(a)] (P(x) - P(a))^\beta + o(|P(x) - P(a)|^r),$$

either by considering an extension of F to $\mathcal{E}^r(\mathbb{R}^n)$ by Whitney's extension theorem or by considering an integral Taylor's remainder, along a integrable path satisfying the inequality given by the 1-regularity property.

Expanding $P(x) - P(a)$ by the polynomial Taylor formula we get a polynomial in $x - a$ of degree h . Hence for $f = F \circ P$:

$$(2.2) \quad f(x) = f(a) + \sum_{1 \leq |\alpha| \leq hr} \frac{1}{\alpha!} f_\alpha(x) (x - a)^\alpha + o(|P(x) - P(a)|^r).$$

On a compact $K \subset \mathbb{R}^n$ containing $[a, x]$, there exists a constant C_K such that $|P(x) - P(a)|^r \leq C_K |x - a|^r$, and $f \in \mathcal{E}^{r,hr}(\mathbb{R}^n)$.

If $f \in P^*(\mathcal{E}^r(P(\mathbb{R}^n)))$, it belongs to $\mathcal{E}^{r,hr}(\mathbb{R}^n)$, but the only partial derivatives that will appear are those of order $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| \leq hr$, that may be obtained by the composition process with $|\beta| \leq r$ and degree of $p_i = k_i$.

About the countable exhaustive sequence of compact sets used to define the topologies we may choose any such sequence of invariant compact sets K_n in \mathbb{R}^n and their images $P(K_n)$ in $P(\mathbb{R}^n)$ (or any such sequence of compact sets k_n in $P(\mathbb{R}^n)$ and their reciprocal image $P^{-1}(k_n)$ that are W -invariant and compact since P is proper).

Remark 2.5. If for some closed set $E \subset \mathbb{R}^n$ we consider the field of Taylor's polynomials of order r of f on E , say $T_E^r f$, its extension given by Whitney's Theorem

[14] will not be in $P^*(\mathcal{E}^r(\mathbb{R}^n))$. It is the field f with $f_x = \sum_{|k| \leq hr} \frac{1}{k!} a_k(x) (x - a)^k$

that carries the information about the fact that $f = P^*(F)$ and not the truncated field f^r .

3. WHITNEY 1-REGULARITY OF $P(\mathbb{R}^n)$.

In this section, we give a proof of Whitney 1-regularity of the image $P(\mathbb{R}^n)$ of the Chevalley mapping associated with any finite reflection group. In [7], theorem 3.1 below was stated but lemma 3.2 was not proved for all Coxeter groups.

Since the 1-regularity is not altered by diffeomorphism, it does not depend on the choice of coordinates. It does not depend on the choice of the set of basic invariants either, since a change of basic invariants is an invertible polynomial map on the target.

When W is reducible we may choose coordinates such that the Chevalley map P is the product of the Chevalley maps P^i s associated with the irreducible components W^i s of W acting on the subspaces \mathbb{R}^{n_i} of \mathbb{R}^n . If for each i , $P^i(\mathbb{R}^{n_i})$ is 1-regular, so is $P(\mathbb{R}^n)$. As a consequence it is sufficient to prove the regularity when W is irreducible and from now on in this section, we will assume W to be a finite Coxeter group.

We will assume as we may that the degrees of the coordinate polynomials p_1, \dots, p_n are in increasing order: $2 = k_1 \leq \dots \leq k_n = h$.

Let \mathcal{R} be the set of reflections different from identity in W . For each $\tau \in \mathcal{R}$, let λ_τ be a linear form the kernel of which is the hyperplane $H_\tau = \{x \in \mathbb{R}^n | \tau(x) = x\}$. The jacobian of P is $J_P = c \prod_{\tau \in \mathcal{R}} \lambda_\tau$ for some constant $c \neq 0$. The critical set is the union of the H_τ when τ runs through \mathcal{R} .

3.1. Strata of $P(\mathbb{R}^n)$ and minors of the jacobian. Let C be a Weyl Chamber, a connected component of the regular set of P . There is a stratification of \mathbb{R}^n by the regular set $\bigcup_{w \in W} w(C)$, the reflecting hyperplanes H_τ and their intersections. The mapping P induces an analytic diffeomorphism of C onto the interior of $P(\mathbb{R}^n)$ and an homeomorphism that carries the stratification from the fundamental domain \bar{C} onto $P(\mathbb{R}^n)$. The walls of \bar{C} are contained in n hyperplanes $(H_\omega)_{\omega \in \Omega}$, where Ω is a subset of cardinal n in \mathcal{R} .

A stratum S of dimension k in \bar{C} is the intersection of $(n - k)$ of the H_ω . The λ_τ that are linear combinations of the λ_ω vanishing on S will also vanish there, so that $p \geq n - k$ hyperplanes $H_\tau, \tau \in \mathcal{R}$ will intersect along S .

The points in S have the same isotropy subgroup W_S , generated by the reflections in the p hyperplanes H_τ containing S . In a neighborhood of S , since P is W_S invariant we can write $P = Q \circ V$, where Q is invertible and V is a Chevalley mapping for W_S .

We write $W_S = W^0 \times W^1 \times \dots \times W^\ell$ where W^0 is the identity on S and the W^i s are the irreducible components of W_S . If we choose coordinates fitted to the orthogonal direct sum $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n_1} \oplus \dots \oplus \mathbb{R}^{n_\ell}$, we have $V = \text{Id}_k \times V_1 \times \dots \times V_\ell$.

The equation of an H_τ containing S depends of the $x_{k+1}, x_{k+2}, \dots, x_n$ but perhaps not all of them. There is a partition among the x_i s, $k+1 \leq i \leq n$ and a corresponding one among the λ_τ . Let $r < s$, x_{k+r} and x_{k+s} are equivalent if there is a sequence $(x_{k+\ell}), k+r < k+\ell < k+s$ and forms λ_τ such that any two consecutive $x_{k+\ell}$ appear in the equation of a form λ_τ . Two forms λ_τ are equivalent if two equivalent x_{k+r} appear in their equations. In each class, the H_τ are the reflecting hyperplanes of an irreducible component of W_S .

The jacobian matrix J_V is block diagonal. Let $I_k, J_{V_1}, \dots, J_{V_\ell}$ be the diagonal blocks, the determinants $|J_{V_1}|, \dots, |J_{V_\ell}|$, all vanish on S . The $k \times k$ minor in the upper left corner is 1 but all the bordering $(k+1) \times (k+1)$ minors vanish on S .

When restricted to a stratum S , $P|_S = Q \circ V|_S$ is an analytic isomorphism on its image and onto each of its projections on the spaces \mathbf{R}^ℓ , $k \leq \ell \leq n-1$. The closure \bar{S} of S , is a fundamental domain for a subgroup of W generated by reflections in hyperplanes that do not contain S , and Q is a polynomial mapping, invariant by this subgroup. On \bar{S} , $P = Q$ is an homeomorphism onto $P(\bar{S})$ and each of its projections on the spaces \mathbf{R}^ℓ .

3.2. The varieties $P_k^{-1}[P_k(x)]$. We set $P_k = (p_1, p_2, \dots, p_k) : \mathbb{R}^n \rightarrow \mathbb{R}^k$, and analogously $Q_k = (q_1, q_2, \dots, q_k)$. We denote with Π_k the image $P_k(\mathbb{R}^n)$.

Let $m^k \in \Pi_k$, then

$$P_k^{-1}(m^k) = \{x \in \mathbb{R}^n; p_1(x) = m_1, \dots, p_k(x) = m_k\}$$

is a compact algebraic variety of co-dimension k .

Theorem 3.1. [7] *For almost all m^k , the intersection of a fundamental domain \bar{C} and $P_k^{-1}(m^k)$ is either empty or contractible.*

By Sard's lemma, for almost all m^k , the variety $P_k^{-1}(m^k)$ is a non singular manifold. Of course if m^k is on the border of Π_k , the variety is singular, but even if m^k belongs to the interior of Π_k , it may be the image of both regular and singular points.

Lemma 3.2. *Let $P_k^{-1}(m^k)$ be a non singular manifold. On this manifold, p_{k+1} is a Morse function with no critical point on the strata of positive dimension in $P_k^{-1}(m^k) \cap \bar{C}$. Its critical points are the zero dimensional strata of the intersection.*

Proof. On the non critical manifold $P_k^{-1}(m^k)$, there is a $k \times k$ minor of the matrix $\partial(p_1, \dots, p_k)/\partial(x_1, \dots, x_n)$ that does not vanish. We may assume that it is the principal minor $\partial(p_1, \dots, p_k)/\partial(x_1, \dots, x_k)$ which is not 0.

The critical points of p_{k+1} on $P_k^{-1}(m^k)$ are the points where:

$$\frac{\partial p_{k+1}}{\partial x_i} - \lambda_k \frac{\partial p_k}{\partial x_i} - \dots - \lambda_1 \frac{\partial p_1}{\partial x_i} = 0, \quad \text{for } i = 1, \dots, n.$$

For this system to have solutions in λ_i , $1 \leq i \leq k$, the $(k+1) \times (k+1)$ bordering minors $\partial(p_1, \dots, p_k, p_{k+1})/\partial(x_1, \dots, x_k, x_{k+s})$, $s = 1, \dots, n-k$ must all be zero. Therefore the jacobian and all minors of order $\geq (k+1) \times (k+1)$ are 0 and the rank of P is k at such points. These points belong to strata S of dimension k in \bar{C} . The critical points are the points of intersection of $P_k^{-1}(m^k)$ and strata S ; they are points of 0-dimensional strata in $P_k^{-1}(m^k) \cap \bar{C}$.

If $P_k^{-1}(m^k)$ is not singular, strata of lower dimension do not intersect it since on such a stratum all $k \times k$ minors vanish. On strata of dimension $> k$, one of the $(k+1) \times (k+1)$ minors does not vanish and there may not be critical points.

Let us show that the critical points of p_{k+1} on $P^{-1}(m^k)$ are non degenerate. At a critical point

$$\frac{\partial p_{k+1}}{\partial x_i} - \lambda_k \frac{\partial p_k}{\partial x_i} - \dots - \lambda_1 \frac{\partial p_1}{\partial x_i} = \sum_j \left(\frac{\partial q_{k+1}}{\partial v_j} - \lambda_k \frac{\partial q_k}{\partial v_j} - \dots - \lambda_1 \frac{\partial q_1}{\partial v_j} \right) \left(\frac{\partial v_j}{\partial x_i} \right) = 0.$$

When $1 \leq i \leq k$, $\partial v_j / \partial x_i = 1$ if $i = j$, and $= 0$ otherwise. The only terms remaining in the sum are the $(\partial q_{k+1} / \partial v_i - \lambda_k \partial q_k / \partial v_i - \dots - \lambda_1 \partial q_1 / \partial v_i)$ and these are 0.

When $k+1 \leq i \leq n$, $\partial v_j / \partial x_i = 0$ either because v_j does not depend on x_i or is an homogeneous polynomial of degree ≥ 2 that vanishes with its derivatives on S , but the $(\partial q_{k+1} / \partial v_i - \lambda_k \partial q_k / \partial v_i - \dots - \lambda_1 \partial q_1 / \partial v_i)$ are not all 0, since Q is invertible.

Actually the vectors $(\partial P / \partial x_1, \dots, \partial P / \partial x_k) = (\partial Q / \partial x_1, \dots, \partial Q / \partial x_k)$ define the tangent plane to $P(S)$ of dimension k . The $\partial P / \partial x_s, k+1 \leq s \leq n$ are linear combinations of them, hence the vanishing of the bordering minors. The $\partial Q / \partial x_s, k+1 \leq s \leq n$, however, span the complement of the tangent space, and they are linearly independent. As a consequence the minors of order $\geq k+1$ in the jacobian of Q do not vanish, and for $i \geq k+1$, $\partial q_{k+1} / \partial v_i - \lambda_k \partial q_k / \partial v_i - \dots - \lambda_1 \partial q_1 / \partial v_i \neq 0$.

In restriction to the kernel of the first differential, which is the orthogonal of S , in the quadratic differential

$$\begin{aligned} & \frac{\partial^2 p_{k+1}}{\partial x_i \partial x_j} - \lambda_k \frac{\partial^2 p_k}{\partial x_i \partial x_j} - \dots - \lambda_1 \frac{\partial^2 p_1}{\partial x_i \partial x_j} \\ = & \sum_{r, s \geq k+1} \left(\frac{\partial^2 q_{k+1}}{\partial v_r \partial v_s} - \lambda_k \frac{\partial^2 q_k}{\partial v_r \partial v_s} - \dots - \lambda_1 \frac{\partial^2 q_1}{\partial v_r \partial v_s} \right) \left(\frac{\partial v_r}{\partial x_i} \right) \left(\frac{\partial v_s}{\partial x_j} \right) \\ & + \sum_{r \geq k+1} \left(\frac{\partial q_{k+1}}{\partial v_r} - \lambda_k \frac{\partial q_k}{\partial v_r} - \dots - \lambda_1 \frac{\partial q_1}{\partial v_r} \right) \left(\frac{\partial^2 v_r}{\partial x_i \partial x_j} \right) \end{aligned}$$

the mixed derivatives are all 0 either because v_j does not depend on x_i or x_j or is an homogeneous polynomial of degree > 2 or a sum of squares of x_i s. For the same reasons many terms in the pure derivatives also vanish, but some do not. In each $V_s : \mathbb{R}^{n_s} \rightarrow \mathbb{R}^{n_s}$, let v_s^1 be the first W^s -invariant which is the sum of the squares of the $x_i \in \mathbb{R}^{n_s}$. By the above remark, $\partial q_{k+1} / \partial v_s^1 - \lambda_k \partial q_k / \partial v_s^1 - \dots - \lambda_1 \partial q_1 / \partial v_s^1$ does not vanish, and for each $x_i \in \mathbb{R}^{n_s}$ of which v_s^1 actually depends, we have $\partial^2 v_s^1 / \partial x_i^2 = 2$, so that:

$$\frac{\partial^2 p_{k+1}}{\partial x_i^2} - \lambda_k \frac{\partial^2 p_k}{\partial x_i^2} - \dots - \lambda_1 \frac{\partial^2 p_1}{\partial x_i^2} = 2 \left(\frac{\partial q_{k+1}}{\partial v_s^1} - \lambda_k \frac{\partial q_k}{\partial v_s^1} - \dots - \lambda_1 \frac{\partial q_1}{\partial v_s^1} \right) \neq 0$$

Accordingly $p_{k+1} = q_{k+1} \circ v$ is a Morse function on $P_k^{-1}(m^k)$.

Observe that for each irreducible component, the quadratic differential is definite with the sign of $\partial q_{k+1} / \partial v_s^1 - \lambda_k \partial q_k / \partial v_s^1 - \dots - \lambda_1 \partial q_1 / \partial v_s^1$, but the sign may be different for different irreducible components.

By the equivariant Morse lemma [2], in the neighborhood of a critical point at the intersection of S and $P_k^{-1}(m^k)$, p_{k+1} is W_S -locally equivalent to a W_S -invariant quadratic form which is the direct sum of definite quadratic forms in each \mathbb{R}^{n_i} . \square

Lemma 3.3. [7] *The reconstruction of the topology of a level set of a function on $\mathbf{R}_+^a \oplus \mathbf{R}_+^b$ in the neighborhood of the critical point $0 \oplus 0$ with the quadratic differential $Q_+ \oplus Q_-$ is trivial if $a, b > 0$ and consists of the birth (death) of a simplex otherwise.*

Theorem 3.1 may now be proved by induction on k [7].

In particular, for almost all $m^k \in \Pi_k$, $P_k^{-1}(m^k) \cap \bar{C}$ is connected.

Corollary 3.4. *Every variety $P_k^{-1}(m^k) \cap \bar{C}$, $k = 1, \dots, n$, is connected or empty.*

The corollary may be derived from 3.1, exactly as in [10]. Basically if m^k is not in the regular image, we have to consider two cases. First if for some j , $1 < j < k$, $m^j = (m_1, m_2, \dots, m_j)$ belongs to the projection of a stratum of dimension less than j but m^k itself does not belong to the projection of any stratum of dimension

less than k , then $P_k^{-1}(m^k)$ is connected since almost all the $P_k^{-1}(n^k)$ are, for n^k close enough to m^k . Then there is the case when m^k is on the projection of a stratum of dimension less than k . For a given m^{k-1} this may happen but for a finite number of m^k . Since $P_1^{-1}(m^1)$ is connected, this case is taken care of by an induction, using the following:

Lemma 3.5. [10] *Let C be a connected compact set in \mathbf{R}^n , and f a real valued function continuous on C . If all but a finite number of the level sets of f in C are connected, then they are all connected.*

As a consequence, $P(P_k^{-1}(m^k))$ is connected or empty and in particular, we have:

Corollary 3.6. *The fibres of the projections: $\Pi_{k+1} \rightarrow \Pi_k$ are connected: they are points or intervals.*

3.3. End of the proof. Let us consider the lift-up $(p_1, \dots, p_k) \mapsto p_{k+1}$ of the projection $P_k(S) \subset \Pi_k$ of a stratum of dimension k . The derivatives are obtained by solving the system:

$$\frac{\partial p_{k+1}}{\partial x_i} = \sum_{j=1}^k \frac{\partial p_{k+1}}{\partial p_j} \frac{\partial p_j}{\partial x_i}, \quad i = 1, \dots, k$$

by Cramer's method. Considering the subgroup of W generated by the reflections in the hyperplanes that do not contain S , for $j = 1, \dots, k$ the $\partial p_{k+1}/\partial p_j$, are quotients of two polynomials anti-invariant by this subgroup. So, they are rational fractions the numerator and the denominator of which are invariant homogeneous polynomials that do not vanish but at the origin. Since the degree of the numerator is greater than the degree of the denominator the rational fractions have continuous extensions on \bar{S} .

If we restrict ourselves to some compact subset K of $P(\mathbf{R}^n)$, determined for instance by $p_1 \leq a$, $a > 0$, the x_i s and as a consequence the $\partial p_{k+1}/\partial p_j$, $j = 1, \dots, k$ are bounded on $\bar{S} \cap K$.

P is an homeomorphism of \bar{S} onto its image $P(\bar{S})$, and so is P_k (on any compact it is continuous and one to one). Hence p_{k+1} which is continuous with respect to the variables (x_1, \dots, x_k) on \bar{S} , is also continuous in the variables (p_1, \dots, p_k) on $P_k(\bar{S})$ and moreover by the previous paragraph it is Lipschitz.

The border of Π_{k+1} is contained in the images $P_{k+1}(\bar{S})$ of closed strata of dimension k . These images are graphs of functions p_{k+1} on $P_k(\bar{S})$. By 3.6, the graph of one of the p_{k+1} , say p_{k+1}^{\max} , is above the others and another, say the graph of p_{k+1}^{\min} , is below the others. In Π_k , the images of the closure of strata of dimension k will intersect along the images of strata of lesser dimension. Above such points of intersection the mapping p_{k+1}^{\max} (resp. p_{k+1}^{\min}) may and will change but globally p_{k+1}^{\max} (resp. p_{k+1}^{\min}) will still be continuous and Lipschitz since the functions above the two strata are glued by their common value above the stratum of lesser dimension along which they intersect.

Now, it would be easy to give for any two points A and B in a compact $K \subset P(\mathbf{R}^n)$, the construction of a continuous arc $AB \subset K$ of length $\ell(AB) \leq k_K |AB|$, following the method in [10] and already in [3]. This kind of construction justifies the statement in [1]: *the prism between graphs of Lipschitz functions over a Whitney 1-regular domain is Whitney 1-regular.*

Π_1 is Whitney 1-regular. By induction, assuming Π_k to be Whitney 1-regular, the prism Π_{k+1} over it and between the graphs of the Lipschitz functions p_{k+1}^{\min} and p_{k+1}^{\max} will also be Whitney 1-regular. Hence all the Π_k and in particular $\Pi_n = P(\mathbf{R}^n)$ are Whitney 1-regular and we can state:

Theorem 3.7. *The image of the Chevalley mapping $P(\mathbf{R}^n)$ is Whitney 1-regular.*

Corollary 3.8. *On $\mathcal{E}^r(P(\mathbf{R}^n))$ the semi-norms $\|\cdot\|_r^K$ and $|\cdot|_r^K$ are equivalent.*

4. $P^*(\mathcal{E}^r(P(\mathbf{R}^n)))$ IS A CLOSED SUBALGEBRA OF $\mathcal{E}^{r,hr}(\mathbf{R}^n)$

Besides a geometric property of $P(\mathbf{R}^n)$ which is of interest by itself, Whitney 1-regularity plays a part when studying the algebras of composite mappings invariant by reflection groups in finite class of differentiability.

In [4] we had already noticed that even though the proof would be basically the same, the 1-regularity would make it simpler because the continuity on $P(\mathbf{R}^n)$ of the field F of class \mathcal{C}^r on the interior of $P(\mathbf{R}^n)$ would bring its r -regularity without any need of an extension to $P^{-1}(\mathbf{R}^n)$.

Here we are interested in $P^*(\mathcal{E}^r(P(\mathbf{R}^n)))$. Let us begin with a lemma.

Definition 4.1. We say that $f \in J^{hr}(\mathbf{R}^n)$ pointwise belongs to $P^*(J^r(P(\mathbf{R}^n)))$, if for each x there exists an $F \in J^r(P(\mathbf{R}^n))$ such that $f_x = (F \circ P)_x$.

This means that for each $x \in \mathbf{R}^n$, the polynomial $f_x(X)$ is W -invariant. Of course it is necessary that f be W -invariant but it is not sufficient.

Lemma 4.2. *If $f \in J^{hr}(\mathbf{R}^n)$ pointwise belongs to $P^*(J^r(P(\mathbf{R}^n)))$, and we write $f = F \circ P$, then for $0 \leq |\beta| \leq r$, $F_\beta \circ P(x)$ is a linear combination of some $f_\alpha(x)$, $0 \leq |\alpha| \leq hr$. In particular the F_β are continuous, $F \in J^r(P(\mathbf{R}^n))$ and $f \in P^*(J^r(P(\mathbf{R}^n)))$.*

Clearly $f_0(x) = F_0(P(x))$.

By induction, let us first assume $r = 1$. We identify

$$f_x(x') = f_0(x) + \sum_{1 \leq |\alpha| \leq h} \frac{1}{\alpha!} f_\alpha(x) (x'_1 - x_1)^{\alpha_1} \dots (x'_\ell - x_\ell)^{\alpha_\ell}, \text{ with}$$

$$f_x(x') = F_0(P(x)) + \sum_1^l F_i \circ P(x) \left(\sum_{|\alpha|=1}^{k_i} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} p_i}{\partial x^\alpha(x)} (x' - x)^\alpha \right)$$

where F_i stands for F_β , $\beta_i = 1, \beta_j = 0$ if $i \neq j$.

For $|\alpha| = k_i$, $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$, we get:

$$(4.1) \quad f_\alpha = F_i \circ P \frac{\partial^{k_i} p_i}{\partial x^\alpha} + \sum_{s>i} F_s \circ P \frac{\partial^{k_i} p_s}{\partial x^\alpha}.$$

In particular for α with $|\alpha| = h$, $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$, we have:

$$f_\alpha = F_n \circ P \frac{\partial^h p_n}{\partial x^\alpha},$$

and since p_n is of degree h , there is a $\partial^h p_n / \partial x^\alpha$ which is not 0. Hence the result for F_n .

Solving the equations (4.1) in succession gives the result for the $F_i \circ P$, $i = 1, \dots, n$.

For more explicit computations, observe that if W is reducible, it would be sufficient to study each irreducible component in each subspace \mathbb{R}^{n_i} and gather the results at the end. For an irreducible component, we may use the polynomial invariants given in [11]. Disregarding D_n for a while, for all the other groups the k_i are distinct and there is an invariant set of real linear forms $\{L_1, \dots, L_v\}$ such that their symmetric functions $\sum_{i=1}^v L_i^k$ are W -invariant, and we may take $p_i(X) = \sum_{j=1}^v [L_j(X)]^{k_i}$ with k_i as determined in [6]. At least one of the $L_j(X)$ contains a monomial in X_1 , bringing in $p_i(X)$ a monomial in $X_1^{k_i}$ that cannot be canceled since the k_i are all even, with 2 exceptions: A_n and $I_2(p)$. For $I_2(p)$ we may choose $p_1(X) = X_1^2 + X_2^2$ and $p_2(X) = \sum_{i=1}^p (X_1 \cos 2i\theta + X_2 \sin 2i\theta)^p$ in which the coefficient of X_1^p is $\sum_{i=1}^p (\cos 2i\theta)^p \neq 0$. For A_n we may take $L_i(X) = X_i$, $i = 1, \dots, n+1$ and there is no possible cancelation either.

Finally, for D_n we can choose as basic invariant polynomials $p_j(x) = \sum_{i=1}^n x_i^{2j}$, $j = 1, \dots, n-1$ and $p_n(x) = x_1 x_2 \dots x_n$. We may use the above method when $1 \leq j \leq n-1$, and consider $\partial^n p_n / \partial x_1 \dots \partial x_n = 1$ to get the continuity of $\tilde{F}_n \circ P$.

Anyway $\partial^{k_n} p_n / \partial x_1^{k_n} = k_n! c_n$ for some coefficient $c_n \neq 0$, while for $j < n$, $\partial^{k_n} p_j / \partial x_1^{k_n} = 0$, since the greatest exponent of x_1 in $p_j(x)$ is $k_j < k_n$.

The identification shows that $c_n F_n \circ P(x) = (1/k_n!) f_{k_n, 0, \dots, 0}(x)$ with $c_n \neq 0$.

Assuming that when $s > i$, the $F_s \circ P$ are linear combinations of f_α , $|\alpha| \leq h$, since $p_i(x)$ contains a monomial in $x_1^{k_i}$, we have $\partial^{k_i} p_i / \partial x_1^{k_i} = k_i! c_i$ for some coefficient $c_i \neq 0$, while as above for $j < i$, $\partial^{k_i} p_j / \partial x_1^{k_i} = 0$. The identification now gives: $(1/k_i!) f_{k_i, 0, \dots, 0} = c_i F_i \circ P + \sum_{s>i} F_s \circ P (1/k_i!) \partial^{k_i} p_s / \partial x_1^{k_i}$.

By using the induction assumption we get the result for $F_i \circ P$, and by decreasing induction for all the $F_j \circ P$, $j = 1, \dots, n$.

Let us assume that the lemma is true when $r \leq k$. When $|\beta| = k$, $F_\beta \circ P$ is a linear combination of some f_α s with $|\alpha| \leq hk$. By using the basis step of the induction for the function $g(x) = F_\beta \circ P(x) = G \circ P(x)$, we get that the $G_i \circ P$ are linear combinations of g_α , $|\alpha| \leq h$ and using the induction assumption for g we get that for $|\gamma| = k+1$, $F_\gamma \circ P$ is a linear combination of f_α s with $|\alpha| \leq h(k+1)$. This achieves the induction and the proof of first part of the lemma.

Since $f \in J^{hr}(\mathbb{R}^n)$, the f_α , $|\alpha| \leq hr$ are continuous. Hence the $F_\beta \circ P$ are also continuous for $|\beta| \leq r$ and since P is proper, the F_β themselves are continuous, so that $F \in J^r(P(\mathbb{R}^n))$. \square

From the lemma, we get at once that there exists a numerical constant C_K that depends only on K and W , such that $|F|_r^{P(K)} \leq C_K |f|_{hr}^K$. Then, by using the 1-regularity there is a constant C_K^1 such that $\|F\|_r^{P(K)} \leq C_K^1 |f|_{hr}^K$.

Let us consider a function of class \mathcal{C}^r of the polynomial invariants. In terms of the variables x it is a function in $P^*(\mathcal{E}^r(P(\mathbb{R}^n)))$.

The algebra homomorphism:

$$P^* : \mathcal{E}^r(P(\mathbb{R}^n)) \longrightarrow \mathcal{E}^{r, rh}(\mathbb{R}^n), \quad P^*(F) = F \circ P,$$

is injective and surjective onto its image. From (2.1) and (2.2) we see that a ‘truncated’ Faà di Bruno formula applies and $|f|_{rh}^K \leq C_K^2 |F|_r^{P(K)}$ for some constant C_K^2 that depends on the compact K , *a fortiori* $|f|_{rh}^K \leq C_K^2 \|F\|_r^{P(K)}$, and the linear mapping P^* is continuous. (Observe that we didn’t use the Whitney 1-regularity).

Theorem 4.3. $P^*(\mathcal{E}^r(P(\mathbb{R}^n)))$ is closed in $\mathcal{E}^{r, rh}(\mathbb{R}^n)$. P^* is an isomorphism of Fréchet space from $\mathcal{E}^r(P(\mathbb{R}^n))$ onto its image.

Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $P^*(\mathcal{E}^r(P(\mathbb{R}^n)))$. This sequence converges in $\mathcal{E}^{r, rh}(\mathbb{R}^n)$. For each n there exists an $F_n \in \mathcal{E}^r(P(\mathbb{R}^n))$ such that $f_n = F_n \circ P$, and since $\|F\|_{P(K)}^r \leq C_K^1 |f|_K^{hr}$, the sequence $(F_n)_{n \in \mathbb{N}}$ is a Cauchy sequence for the topology defined by the semi-norms $\|F\|_{P(K)}^r$. Hence $(F_n)_{n \in \mathbb{N}}$ converges to an $F \in \mathcal{E}^r(P(\mathbb{R}^n))$. If we take the limit in $f_n = F_n \circ P$ we see that $f = F \circ P$ is also in $P^*(\mathcal{E}^r(P(\mathbb{R}^n)))$. So $P^*(\mathcal{E}^r(P(\mathbb{R}^n)))$ is closed in the Fréchet space $\mathcal{E}^{r, rh}(\mathbb{R}^n)$ and then a Fréchet space itself.

We have noticed that P^* is injective, surjective onto its image and continuous. Since its image is a Fréchet space, by Banach theorem, P^* is an isomorphism between its source and its image.

This means that P^* identifies the space of functions of class \mathcal{C}^r of the invariant polynomials (by [14], a function in $\mathcal{E}^r(P(\mathbb{R}^n))$ has an extension to $\mathcal{C}^r(\mathbb{R}^n)$) and a Fréchet subspace of W -invariant functions in $\mathcal{C}^r(\mathbb{R}^n)^W$.

Proposition 4.4. *$P^*(\mathcal{E}^r(P(\mathbb{R}^n)))$ is closed in $\mathcal{E}^{r, rh}(\mathbb{R}^n)$ if and only if $P(\mathbb{R}^n)$ is Whitney 1-regular.*

Let us assume that $P^*(\mathcal{E}^r(P(\mathbb{R}^n)))$ is closed and consider a sequence $(F_n)_{n \in \mathbb{N}}$ in $\mathcal{E}^1(P(\mathbb{R}^n))$, which is Cauchy for the topology induced by the semi-norms $|\cdot|_1^{P(K)}$. The sequence $(f_n = F_n \circ P)_{n \in \mathbb{N}}$ in $P^*(\mathcal{E}^1(P(\mathbb{R}^n))) \subset \mathcal{E}^{1, h}(\mathbb{R}^n)$ which is also Cauchy by Faa di Bruno's formula, converges to an f in $P^*(\mathcal{E}^1(P(\mathbb{R}^n)))$ which is closed by assumption. As a consequence, the limit F of $(F_n)_{n \in \mathbb{N}}$ with $f = F \circ P$ is in $\mathcal{E}^1(P(\mathbb{R}^n))$. Thus $\mathcal{E}^1(P(\mathbb{R}^n))$ is complete for the topology induced by the semi-norms $|\cdot|_1^{P(K)}$ and the Banach isomorphism theorem shows that this topology is equivalent to the topology induced by the semi-norms $\|\cdot\|_1^{P(K)}$. Glaeser's proposition 2.4 then shows that $P(\mathbb{R}^n)$ is Whitney 1-regular.

Remark 4.5. Finally, one might wish to prove the 1-regularity of $P(\mathbb{R}^n)$ by using Glaeser's proposition 2.4. By Banach theorem, we just have to prove that $\mathcal{E}^1(P(\mathbb{R}^n))$ is complete for the topology induced by the semi-norms $|\cdot|_1^{P(K)}$. So we consider a sequence $(F_n)_{n \in \mathbb{N}} \subset \mathcal{E}^1(P(\mathbb{R}^n))$ which is Cauchy for the topology induced by the semi-norms $|\cdot|_1^{P(K)}$. The sequence $(f_n = F_n \circ P)_{n \in \mathbb{N}}$ is Cauchy in $\mathcal{E}^{1, h}(\mathbb{R}^n)$ since as we already noticed $|f|_h^K \leq C_K^2 |F|_1^{P(K)}$, hence (f_n) converges to an $f \in \mathcal{E}^{1, h}(\mathbb{R}^n)$, which is of the form $F \circ P$ with $F = \lim_n F_n$.

f induces a formally holomorphic jet still denoted by $f \in \mathcal{H}^{1, h}(\mathbb{R}^n)$ (see [13] and [4]). If there was an extension \tilde{f} to $\mathcal{H}^{1, h}(P^{-1}(\mathbb{R}^n))$ of the form $\tilde{f} = \tilde{F} \circ P$, \tilde{F} might be identified with F on $P(\mathbb{R}^n)$, would be of class \mathcal{C}^1 on the regular image of P in \mathbb{R}^n , and 1-continuous everywhere by 4.2. Conclusion, since the critical image is contained in the null set of the discriminant polynomial, \tilde{F} which is continuous everywhere and of class \mathcal{C}^1 when the discriminant does not vanish, would be of class \mathcal{C}^1 everywhere in \mathbb{R}^n , and in particular $F \in \mathcal{E}^1(P(\mathbb{R}^n))$. Unfortunately, no such extension is available. As mentioned above, Whitney's extension does not take into account the part of jet f beyond degree r . It is possible to provide a linear and continuous version of Łojasiewicz extension operator (see [13]) that would give an $\tilde{f} \in \mathcal{H}^{1, h}(P^{-1}(\mathbb{R}^n))$, but it is not an algebra isomorphism and \tilde{f} would not be of the form $\tilde{F} \circ P$ with a \tilde{F} of degree 1 and lemma 4.2 would not apply.

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