Journal of Mathematics Research; Vol. 8, No. 1; 2016
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

On Some Properties of *-annihilators and *-maximal Ideals in Rings with Involution

Maya A. Shatila

Correspondence: Maya A. Shatila, Department of Mathematics and Computer Science, Faculty of Science, Beirut Arab University, Beirut, Lebanon. E-mail: mayachatila@hotmail.com

Received: November 3, 2015 Accepted: November 19, 2015 Online Published: January 7, 2016

doi:10.5539/jmr.v8n1p1 URL: http://dx.doi.org/10.5539/jmr.v8n1p1

Abstract

We describe the $*-$ *right annihilator* ($*-$ *left anihilator*) of a subset of a ring and we investigate the relationships between the *right annihilator* and $*-$ *right annihilator*. These connections permit the transfer of various properties from *annihilators* to $*-$ *annihilators* . It is known that the quotient ring constructed from a ring and a maximal ideal is a field, whereas we prove that the quotient ring constructed from a ring and a *-maximal ideal is not a *-field. Equivalent definitions to $*-$regular ring are given.

Keywords: involution, *-annihilator, *-maximal ideal, *-regular ring

1. Introduction

A ring A is said to be a ring with involution or simply *-ring if there is a unary operation $*$: A→ A such that for all a, b ∈ A we have:

$$a^{**} = a, (ab)^* = b^*a^*, (a+b)^* = a^* + b^*$$

In this paper, only associative rings are considered. For more details concerning the ring with involution see (Rowen, 1988).

An ideal I of an involution ring A $(I \lhd A)$ is called **-ideal* $(I \lhd^* A)$, if it is closed under involution; that is $I^* = I$. An involution * of a *-ring R is said to be proper (semiproper) if $x^*x = 0$ $(x^*Rx = 0)$ implies $x = 0$ for every x ∈ R. In (Rowen, 1988), the right annihilator of a ∈ A, denoted by r(a), is defined as r(a) = {b ∈ A| ab = 0}. Similarly, the left annihilator of a is l(a) = {b ∈ A|ba = 0}.

A ring (resp. *-ring) A is semiprime (resp. *-semiprime) if $I^2 = 0$ for every nonzero ideal (resp. *-ideal) I of A. A ring A is called reduced if it has no nonzero nilpotent elements ($a^n = 0$ for any $a \in A$ and positive integer n). (*see* (*Berberianetal.*, 1988), (*Rowen*, 1988)) . A ring A is called regular if for every a ∈ A, a ∈ aAa. Equivalently, every principal one-sided ideal of A is generated by an idempotent (see (von Neuman, 1960)).

An element e of A is called idempotent (projection) if $e^2 = e$ (and $e^* = e$. Equivalently, $e = ee^*$).

2. Properties of *-annihilators

Let A be a ring with involution which does not necessary have identity. Recall that the *right annihilator* of a subset S of A is defined as $S^r = \{x \in A/Sx = 0\}$. Now, let S be a non empty subset of the *-ring A, define the $*-$ of S to be the self adjoint subset $S^r_* = \{x \in A/Sx = 0 \text{ and } Sx^* = 0\}$. Similarly, the $*-$ *left annihilator* can be defined. It is clear that $S^r_* \subseteq S^r$. However the converse is not true as shown in the following example.

Example 1. *Consider the ring A of all* 2×2 *matrices rings over the real field* $\mathbb{R}$, $M_2(\mathbb{R})$*, with transpose of matrices as invotution. Let* $S = \left\{\begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} / a \in \mathbb{R}\right\}$, *then* $S^r = \left\{\begin{pmatrix} b & c \\ -b & -c \end{pmatrix} / b, c \in \mathbb{R}\right\}$ *and* $S^r_* = \left\{\begin{pmatrix} -t & t \\ t & -t \end{pmatrix} / t \in \mathbb{R}\right\}$*.It is clear that in this example the right annihilator of S is not a* $*-$ *right annihilator of S.*

In (Anderson et al., 1992), it is proved that the *right annihilator* of S is a two sided ideal, a similar proof is given in the following proposition to show that the $*-$ *right annihilator* of a right ideal S of A is a $*-$ *ideal* of A.

Proposition 2. *If S is a right (resp. left) ideal of a *- ring A, then the* $*-$ *right annihilator* S^r_* *(resp. left) is a* $*-$ *ideal of A.*

Proof. Let x,y be two elements of the $*-$ *right annihilator* S^r_*, a ∈A. Then $S(x-y) \subseteq Sx - Sy = 0$ and $S(x-y)^* \subseteq Sx^* - Sy^* = 0$. Also, $S(ax) = (Sa)x \subseteq Sx = 0, S(ax)^* = (Sx^*)a^* = 0$ and similarly $S(xa) = 0 = S(xa)^*$. □

The $*-$ *annihilator* of a non empty subset S is defined by $S_* = S^r_* \cap S^l_*$.If S is self adjoint, then it is clear that $S^r_* = S^l_* = S_*$. The following is an immediate corollary of the previous proposition.

Corollary 3. *If S is a *-ideal of A, then $S^r_* = S^l_* = S_*$ is also a *-ideal of A.*

Our main goal is to give some properties of *-annihilators.

Theorem 4. *Let S, T be subsets of a ring A, then:*

1. $S^r_* = (S^*)^l_*$
2. $S^r_* = \cap_{(a\in s)} (a)^r_*$
3. $(S \cup T)^r_* = S^r_* \cap T^r_*$, $(S \cup T)^l_* = S^l_* \cap T^l_*$

Proof. 1. let $x \in S^r_*, Sx = 0$ and $Sx^* = 0, x^*S^* = 0$ and $xS^* = 0$, so $x \in (S^*)^l_*$ and $S^r_* \subseteq (S^*)^l_*$. Let $x \in (S^*)^l_*$, $xS^* = 0$ and $x^*S^* = 0, Sx^* = 0$ and $Sx = 0$, So, $x \in S^r_*$ and $(S^*)^l_* \subseteq S^r_*$ therefore $S^r_* = (S^*)^l_*$.

2. Let $x \in S^r_*, Sx = 0$ and $Sx^* = 0$, $ax = 0$ and $ax^* = 0$ for every $a \in S$ then $x \in (a)^r_*$ for every $a \in S$ hence $x \in \cap (a)^r_*$.

Let $x \in \cap_{(a\in s)} (a)^r_*$, $ax = 0$ and $ax^* = 0$ for every $a \in S$, So$x \in S^r_*$. Hence, $S^r_* = \cap_{(a\in s)} (a)^r_*$.

3. Let $x \in (S \cup T)^l_*, x(S \cup T) = 0$ and $x^*(S \cup T) = 0, (xS = 0$ and $xT = 0)$ and $(x^*S = 0$ and $x^*T = 0)$. Then $x \in S^l_*$ and $x \in T^l_*$ and $x \in S^l_* \cap T^l_*$. Let $x \in S^l_* \cap T^l_*$, $x \in S^l_*$ and $x \in T^l_*, (xS = 0$ and $x^*S = 0)$ and $(xT = 0$ and $x^*T = 0)$ so, $(xS = 0$ and $xT = 0)$ and $(x^*S = 0$ and $x^*T = 0)$, finally, $x(S \cup T) = 0$ and $x^*(S \cup T) = 0$, Hence $x \in (S \cup T)^l_*$. □

Proposition 5. *If A is reduced then $S^r_* = S^l_*$.*

Proof. Let $x \in S^r_*$ then $Sx = 0$ and $Sx^* = 0$, $yx = 0$ and $yx^* = 0$ for every $y \in S$, we also have $(xy)^2 = xyxy = 0$ and $(x^*y)^2 = x^*yx^* y = 0$. But A is reduced then it has no non zero nilpotent element. Thus, $xy = 0$ and$x^*y = 0$ for every $y \in S$. So, $x \in S^l_*$. Similarly, we get $S^l_* \subseteq S^r_*$. Hence, $S^r_* = S^l_*$. □

Proposition 6. *If * -is a proper (semi proper) involution then $S \cap S^l_* = 0$*

Proof. Let $x \in S \cap S^l_*, x \in S$ and $x \in S^l_*$ which implies that $xS = 0$ and$x^*S = 0$, but $x \in S$ then $x^2 = 0$ and $x^*x = 0$. But $^*-$ is a proper involution then $x^*x = 0$ gives $x = 0$ (due to (Berberian, 1988)). Hence $S \cap S^l_* = 0$ □

By a similar reasoning we obtain that $S \cap S^l_* = 0$ if $^*-$ is a semi proper involution or if A is a reduced ring.

In general, for any subset S of A, $S \subsetneq (S^r_*)^l_*$.

Example 7. $S = \left\{ \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} / a \in \mathbb{R} \right\}, T = S^r_* = \left\{ \begin{pmatrix} -t & t \\ t & -t \end{pmatrix} / t \in \mathbb{R} \right\},\ T^l_* = \left\{ \begin{pmatrix} b & b \\ b & b \end{pmatrix} / b \in \mathbb{R} \right\}, S \subsetneq (S^r_*)^l_*$.

If S is self adjoint, then $S \subseteq (S^r_*)^l_*$.

Proposition 8. *If $S = S^*$ then $S \subseteq (S^r_*)^l_*$; moreover $S \subseteq (S^r_*)^r_*$ and $S \subseteq \left(S^l_*\right)^l_*$*

Proof. Let $T = S^r_*$. To show that $S \subseteq T^l_*$ we need to show that $ST = 0$ and$S^*T = 0$ but $S = S^*$ then it is enough to show $ST = 0$.

But $T = S^r_*$ gives $ST = 0$ and $ST^* = 0$, hence $ST = 0$ and $S \subseteq (S^r_*)^l_*$ .Notice that if $S = S^*$ then $S^r_* = S^l_*$ and $S \subseteq \left(S^l_*\right)^l_*$ and $S \subseteq (S^r_*)^r_*$ □

Corollary 9. *If A is semiprime ring and $S \lhd A$ then $S^r_* = S^l_*$.(same reasoning as (Herstein), corollary 1, p.6)*

Corollary 10. *Every element of S^r_* is a *-zero divisor. (definition of *-zero divisor is given in (Anderson, et al., 2010))*

Proof. Let $x \in S^r_*$ then $Sx = 0$ and $Sx^* = 0$ then there exist $y \in S$ such that $yx = 0$ and $yx^* = 0$. Hence x is *a $*-$zero divisor.* □

The converse is not true; not every $*-$ *zero divisor* of a ring belongs to S^r_*.

Example 11. *Let $R = A \oplus A^{op}$ with exchange involution $(a, b)^* = (b, a)$, $A = Z_6$, $(2, 0)$ is a $*-$zero divisor, $(2, 0)(3, 0) = (0, 0)$ and $(2, 0)(0, 3) = (0, 0)$, but $(2, 0) \notin S^r_*S = Z_3 \oplus Z_3$ since there exist $(1, 3) \in S$ such that $(2, 0)(1, 3) \neq (0, 0)$.*

3. *-maximal Ideal

Motivated by a theorem in ring theory which said that an ideal I of a ring A is maximal if and only if the quotient ring A/I is a field, the involutive version will be shown in this section. birkenmeier has defined $*-$prime ideal and $*-$maximal ideal in a ring with involution in (Birkenmeier et al., 1997), he showed that every prime (maximal) ideal is $*-$prime ($*-$maximal) ideal.

The ring A considered in this section is commutative.

Every maximal ideal of A is a $*-$maximal ideal of A but the converse is not true. Indeed, consider the ring $R = Z_4 \oplus Z_4$ with exchange involution $(a,b)^* = (b,a)$. $I = \{0,2\}$ is a maximal ideal of Z_4, then $J = I \oplus I$ is a $* -$ *maximal ideal* of $Z_4 \oplus Z_4$ under the exchange involution. But J is not maximal since it is contained in $Z_4 \oplus I$.

Proposition 12. *Let A be a $*-$ ring, every $*-$ maximal ideal of A is a $*-$prime ideal of A.*

Proof. Let M be a $*-$ maximal ideal of A. if M is a maximal ideal of A then M is a prime ideal and therefore M is a $*-$prime ideal of A. if M is not a maximal ideal K of A then there exists a maximal K of A such that: $K + K^* = A$ and $K \cap K^* = M$ (see (Birkenmeier et al., 1997)). K is a maximal ideal of A then K is prime, So K is $*-$prime and $K \cap K^*$ is $*-$prime (see (Birkenmeier et al., 1997)), Then M is $*-$prime ideal of A. □

Proposition 13. *Let A be a commutative $*-$ ring with identity and $M \lhd^* A$. If the factor ring A/M is a $*-$ field then M is a $*-$ maximal ideal of A.*

Proof. Let A/M is a $*-$ field then A/M is a field then M is a maximal ideal of A and M is a $*-$ maximal ideal of A. □

The converse is not always true; the following example shows that is if M is a $*-$ maximal ideal of A then A/M is not a $*-$field.

Example 14. *Let $A = Z_4 \oplus Z_4$, $M = I \oplus I$ with $I = \{0,2\}$ is a $*-$ maximal ideal of A under the exchange involution $(a,b)^* = (b,a)$, but $\bar{O} \neq \overline{(2,1)} \in A/M$ is not invertible for the reason that $\overline{(2,1)}$ is a zero divisor $\overline{(2,1)(2,0)} = \left(\overline{0},\overline{0}\right)$ henceA/M is not a field and not a $*-$field.*

Proposition 15. *Every $*-$ field is a $*-$ integral domain.*

Proof. Let A be a $*-$field with a, b and c are non zero elements in A such that $ab = ac$ and $a^*b = a^*c$, a admits an inverse element $a^{-1}, a^{-1}ab = a^{-1}ac$ and $a^{-1}a^*b = a^{-1}a^*c$ then $b = c$ and A is a $*-$ integral domain since the cancellation property holds true. □

4. *-regular Ring

Definition 16. *Refer to* $(vonNeuman, 1960)$, *A $*-$ring A is called *-regular, if every principal one-sided ideal of A is generated by a projection.*

Theorem 17. *For every $*-$ring A, the following statements are equivalent:*

1. A is $* -$ *regular.*
2. $a \in Aa^*a$ for every $a \in A$
3. $a \in aa^*A$ for every $a \in A$
4. $a \in Aa^*a \cap aa^*A$ for every $a \in A$

Proof. (1) $\Rightarrow$ (2) Let A be $* -$ *regular*, then for every $a \in A$, $aA = eA$ for some projection e of A. Hence $a = ea$ and $e = ar$ for some $r \in A$.Thus $a = e^*a = r^*a^*a \in Aa^*a$.

(2) $\Rightarrow$ (3) Let the condition be satisfied. Then for every $a \in A$, we have $a^* \in A\,(a^*)^*\,(a^*) = Aaa^*$. Take the involution, then $a \in aa^*A$.

(3) $\Rightarrow$ (4) obvious

(4) $\Rightarrow$ (1) we have $a = xa^*a$ for some $x \in A$. But $(xa^*)\,(xa^*)^* = xa^*ax^* = ax^*$ implies $(xa^*)\,(xa^*)^* = (xa^*)$ which means that xa^* is a projection. Then $a = ea$ for some projection e of A implies $aA = eA$ and hence A is $* -$ *regular.* □

References

Aburawash, U. A., & Sola, K.B. (2010). *-Zero divisors and *-prime ideals. *East-West J.of mathematics, 12*(1), 27-31.

Anderson, F. W., & Fuller, K. R. (1992). Rings and categories of modules, Graduate Texts in Mathematics 13 (2 ed.), New York: Springer-Verlag, pp. x+376, ISBN 0-387-97845-3, MR 1245487. http://dx.doi.org/10.1007/978-1-4612-4418-9.

Berberian, S. K. (1988). Baer rings and baer *-rings, University of Texas at Austin.

Birkenmeier, G. F, Groenewald, N. J., & Heatherly, H. E. (1997). Minimal and maximal ideal in rings with involution. *Beitrage Algebra Geom., 38*(2), 217-225.

Herstein, I. N. Rings with involution. Chicago lectures in mathematics, ISSN 0069-3286. http://dx.doi.org/10.1016/0001-8708(78)90072-5.

Rowen, L. H. (1988). Ring theory I, Academic Press, San Diego. http://dx.doi.org/10.1016/S0079-8169(08)60985-X

von Neumann, J. (1960), Continuous geometry, Edited by I. Halperin, Princeton Univerxity Press, Princeton, N. J. http://dx.doi.org/10.2307/3612654

Copyrights