AN IMPROVED BOUND FOR SULLIVAN'S CONVEX HULL THEOREM

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ABSTRACT. Sullivan showed that there exists K_0 such that if $\Omega \subset \hat{\mathbb{C}}$ is a simply connected hyperbolic domain, then there exists a conformally natural K_0-quasiconformal map from Ω to the boundary $\mathrm{Dome}(\Omega)$ of the convex hull of its complement which extends to the identity on $\partial\Omega$. Explicit upper and lower bounds on K_0 were obtained by Epstein, Marden, Markovic and Bishop. We improve on these bounds, by showing that one may choose $K_0 \leq 7.1695$.

1. INTRODUCTION

In this paper we consider the relationship between the Poincaré metric on a hyperbolic simply connected domain Ω in $\hat{\mathbb{C}} = \partial\mathbb{H}^3$ and the geometry of the boundary $\mathrm{Dome}(\Omega)$ of the convex core of its complement in $\mathbb{H}^3$. Sullivan [15] (see also Epstein-Marden [10]) showed that there exists $K_0 > 0$ such if Ω is simply connected, then there is a conformally natural K_0-quasiconformal map $f : \Omega \to \mathrm{Dome}(\Omega)$ which extends to the identity on $\partial\Omega$. Epstein, Marden and Markovic provided upper and lower bounds for the value of K_0.

Theorem 1.1. (Epstein-Marden-Markovic [11, 12]) *There exists $K_0 \leq 13.88$ such that if $\Omega \subset \hat{\mathbb{C}}$ is a simply connected hyperbolic domain, then there is a conformally natural K_0-quasiconformal map $f : \Omega \to \mathrm{Dome}(\Omega)$ which extends continuously to the identity on $\partial\Omega \subset \hat{\mathbb{C}}$. Moreover, one may not choose $K_0 \leq 2.1$.*

We recall that f is said to be *conformally natural* if whenever A is a conformal automorphism of $\hat{\mathbb{C}}$ which preserves Ω, then $\bar{A} \circ f = f \circ \Omega$ where $\bar{A}$ is the extension of A to an isometry of $\mathbb{H}^3$. If one does not require that the quasiconformal map $f : \Omega \to \mathrm{Dome}(\Omega)$ be conformally natural, Bishop obtained a better uniform bound on the quasiconformality constant. Epstein and Markovic [13] showed that even in this setting one cannot uniformly bound the quasiconformality constant above by 2.

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Theorem 1.2. (Bishop [3]) *There exists $K_1 \leq 7.88$ such that if $\Omega \subset \hat{\mathbb{C}}$ is a simply connected hyperbolic domain, then there is a K_1-quasiconformal map $f : \Omega \to \mathrm{Dome}(\Omega)$ which extends continuously to the identity on $\partial\Omega \subset \hat{\mathbb{C}}$.*

In this paper, we obtain a bound in the conformally natural setting, which improves on both of these bounds.

Theorem 1.3. *There exists $K_0 \leq 7.1695$ such that if $\Omega \subset \hat{\mathbb{C}}$ is a simply connected hyperbolic domain, then there is a conformally natural K_0-quasiconformal map $f : \Omega \to \mathrm{Dome}(\Omega)$ which extends continuously to the identity on $\partial\Omega \subset \hat{\mathbb{C}}$.*

Outline of argument: One may realize $\mathrm{Dome}(\Omega)$ as the image of a pleated plane $P_\mu : \mathbb{H}^2 \to \mathbb{H}^3$ whose bending is encoded by a measured lamination μ. Given $L > 0$, we define the *L-roundness* $||\mu||_L$ of μ to be the best upper bound on the total bending of $P_\mu(\alpha)$ where α is an open geodesic segment in $\mathbb{H}^2$ of length L. (This generalizes the notion of roundness introduced by Epstein-Marden-Markovic [11].) Our first bound improves on an earlier bounds of Bridgeman [4, 5] on roundness.

Theorem 3.1. *If $L \in (0, 2\sinh^{-1}(1))$, μ is a measured lamination on $\mathbb{H}^2$ and P_μ is an embedding, then*

$$||\mu||_L \leq 2\cos^{-1}\left(-\sinh\left(\frac{L}{2}\right)\right).$$

We then generalize work of Epstein-Marden-Markovic [11, Theorem 4.2, part 2] and an unpublished result of Epstein and Jerrard [9].which give criteria for P_μ to be an embedding.

Theorem 4.1. *There exists an increasing function $G : (0, \infty) \to (0, \pi)$ with $G(1) \approx 0.948$, such that if μ is a measured lamination on $\mathbb{H}^2$ such that*

$$||\mu||_L < G(L),$$

then P_μ is a bilipschitz embedding which extends continuously to a map $\hat{P}_\mu : \mathbb{H}^2 \cup \mathbb{S}^1 \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$ so that $\hat{P}_\mu(\mathbb{S}^1)$ is a quasi-circle.

With these bounds in place, we may adapt the techniques of Epstein, Marden and Markovic [11, 12] to complete the proof of our main result.

2. Pleated planes and L-roundness

In this section, we recall the definition of the pleated plane associated to a measured lamination, and introduce the notion of L-roundness.

Let $G(\mathbb{H}^2)$ be the set of unoriented geodesics on the hyperbolic plane $\mathbb{H}^2$. One may identify $G(\mathbb{H}^2)$ with $(\mathbb{S}^1 \times \mathbb{S}^1 - \Delta)/\mathbb{Z}_2$. A *geodesic lamination* on $\mathbb{H}^2$ is a closed subset $\lambda \subset G(\mathbb{H}^2)$ which does not contain any intersecting geodesics. A *measured lamination* μ on $\mathbb{H}^2$ is a non-negative measure μ on $G(\mathbb{H}^2)$ supported on a geodesic lamination $\lambda = \mathrm{supp}(\mu)$. A geodesic arc α

in $\mathbb{H}^2$ is said to be transverse to μ, if it is transverse to every geodesic in the support of μ. If α is transverse to μ, we define

$$i(\mu, \alpha) = \mu\left(\{\gamma \in G(\mathbb{H}^2) \mid \gamma \cap \alpha \neq \emptyset\}\right).$$

If α is not transverse to μ, then it is contained in a geodesic in $\mathrm{supp}(\mu)$ and we let $i(\mu, \alpha) = 0$.

Given a measured lamination μ on $\mathbb{H}^2$, we may define a pleated plane $P_\mu : \mathbb{H}^2 \to \mathbb{H}^3$, well-defined up to post-composition by an isometry of $\mathbb{H}^3$. P_μ is an isometry on the components of $\mathbb{H}^2 - \mathrm{supp}(\mu)$, which are called flats. If μ is a finite-leaved lamination, then P_μ is simply obtained by bending, consistently rightward, by the angle $\mu(l)$ along each leaf l of μ. Since any measured lamination is a limit of finite-leaved laminations, one may define P_μ in general by taking limits (see [10, Theorem 3.11.9]).

If $\Omega \subset \hat{\mathbb{C}}$ is a simply connected hyperbolic domain, let $\mathrm{Dome}(\Omega)$ denote the boundary of the convex hull of its complement $\hat{\mathbb{C}} - \Omega$. Thurston [16] showed that there exists a lamination μ on $\mathbb{H}^2$ such that $\mathrm{Dome}(\Omega) = P_\mu(\mathbb{H}^2)$ and $P_\mu : \mathbb{H}^2 \to \mathrm{Dome}(\Omega)$ is an isometry. (See Epstein-Marden [10, Chapter 1], especially sections 1.11 and 1.12, for a detailed exposition.)

Lemma 2.1. *If Ω is a hyperbolic domain, there is a lamination μ on $\mathbb{H}^2$ such that P_μ is a locally isometric covering map with image $\mathrm{Dome}(\Omega)$.*

For any point $p \in \mathrm{Dome}(\Omega)$, a *support plane* at p is a totally geodesic plane through p which is disjoint from the interior of the convex hull of $\hat{\mathbb{C}} - \Omega$. The exterior angle, denoted $\angle(P, Q)$, between two intersecting support planes P and Q is the angle between their normal vectors at a point of intersection.

Let $\alpha : [a, b] \to \mathbb{H}^2$ be a unit-speed geodesic where J is a bounded interval in $\mathbb{R}$. If $\alpha(t)$ lies on a leaf l of μ with $\mu(l) > 0$, then there is a maximal family $\{Q_l^\theta\}_{\theta \in [0, \mu(l)]}$ of support planes to $\mathrm{Dome}(\Omega)$ through $P_\mu(\alpha(t))$, all of which contain $P_\mu(l)$ In all other cases, $\mathrm{Dome}(\Omega)$ has has a unique support plane at $P_\mu(\alpha(t))$. One may concatenate all the support planes to points in $P_\mu(\alpha([a, b]))$ to obtain a continuous family $\{P_t\}_{t \in [0, k]}$ of support planes along α, so that P_0 is the leftmost support plane to $\mathrm{Dome}(\Omega)$ at $P_\mu(\alpha(a))$ and P_k is the rightmost support plane to $\mathrm{Dome}(\Omega)$ at $P_\mu(\alpha(b))$. Moreover, there exists a continuous non-decreasing function $q : [0, k] \to [a, b]$ so that P_t is a support plane to $\mathrm{Dome}(\Omega)$ at $P_\mu(\alpha(q(t)))$ for all t. If $0 = t_0 < t_1 < \cdots < t_n = k$ and P_t intersects both $P_{t_{i-1}}$ and P_{t_i} for all $t \in [t_{i-1}, t_i]$, then

$$i(\mu, \alpha) \leq \sum_{i=1}^n \angle(P_{t_{i-1}}, P_{t_i}).$$

See Section 4 of [6], especially Lemma 4.1, for a more careful discussion.

For a measured lamination μ on $\mathbb{H}^2$, Epstein, Marden and Markovic [11] defined the *roundness* of μ to be

$$||\mu|| = \sup i(\mu, \alpha)$$

where the supremum is taken over all open unit length geodesic arcs in $\mathbb{H}^2$. The roundness bounds the total bending of P_μ on any segment of length 1 and is closely related to average bending, which was introduced earlier by the first author in [4]. In this paper, it will be useful to consider the *L-roundness* of a measured lamination for any $L > 0$

$$||\mu||_L = \sup i(\alpha, \mu)$$

where now the supremum is taken over all open geodesic arcs of length L in $\mathbb{H}^2$. We note that the supremum over open geodesic arcs of length L, is the same as that over half open geodesic arcs of length L.

In [5], the first author obtained an upper bound on the L-roundness of an embedded pleated plane.

Theorem 2.2. (Bridgeman [5]) *There exists a strictly increasing homeomorphism $F : [0, 2\sinh^{-1}(1)] \to [\pi, 2\pi]$ such that if μ is a measured lamination on $\mathbb{H}^2$ and P_μ is an embedding, then*

$$||\mu||_L \leq F(L).$$

for all $L \leq 2\sinh^{-1}(1)$. In particular,

$$||\mu|| \leq F(1) = 2\pi - 2\sin^{-1}\left(\frac{1}{\cosh(1)}\right) \approx 4.8731.$$

Epstein, Marden and Markovic [11] provided a criterion guaranteeing that a pleated plane is a bilipschitz embedding.

Theorem 2.3. (Epstein-Marden-Markovic [11, Theorem 4.2, part 2]) *If μ is a measured lamination on $\mathbb{H}^2$ such that $||\mu|| \leq c_2 = 73$, then P_μ is a bilipschitz embedding which extends to an embedding $\hat{P}_\mu : \mathbb{H}^2 \cup \mathbb{S}^1 \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$ such that $\hat{P}_\mu(\mathbb{S}^1)$ is a quasi-circle.*

In [12], Epstein, Marden and Markovic comment "Unpublished work by David Epstein and Dick Jerrard should prove that $c_2 > .948$, though detailed proofs have not yet been written". The authors contacted David Epstein who kindly provided their notes outlining the proof. In section 4 we prove a generalization of their result using the approach outlined in their notes.

3. An upper bounds on L-roundness for embedded pleated planes

In this section, we adapt the techniques of [5] to obtain an improved bound on the L-roundness for an embedded pleated plane.

Theorem 3.1. *If $L \in (0, 2\sinh^{-1}(1))$, μ is a measured lamination on $\mathbb{H}^2$ and P_μ is an embedding, then*

$$||\mu||_L \leq c_1(L) = 2\cos^{-1}\left(-\sinh\left(\frac{L}{2}\right)\right).$$

Proof. Since $F(2\sinh^{-1}(1)) = 2\pi$, Theorem 3.1 follows from Theorem 2.2 when $L = 2\sinh^{-1}(1)$. Therefore, we may assume that $L < 2\sinh^{-1}(1)$.

Let $\alpha : [0, L] \to \mathbb{H}^2$ be a geodesic arc of length $L < 2\sinh^{-1}(1)$. Let $\{P_t \mid t \in [0, k]\}$ be the continuous one-parameter family of support planes to α and let $q : [0, k] \to [0, L]$ be the continuous non-decreasing map such that P_t is a support plane to $\mathrm{Dome}(\Omega)$ at $\alpha(q(t))$ for all t.

We now recall the proof of Lemma 4.3 in [6]. If P_0 intersects P_t for all $t \in [0, k)$, then $i(\alpha, \mu) \le \pi$ and we are done. If not, there exists $a \in (0, k)$ such that P_a has an ideal intersection point with P_0 and P_t intersects P_0 for all $t \in (0, a)$. If there exists $t \in (a, k]$ so that P_t is disjoint from P_a, then Lemma 3.2 in [6] implies that $\alpha([0, q(t)])$ has length at least $2\sinh^{-1}(1)$, which would be a contradiction. Therefore, if $t \in (a, k]$, then P_a intersects P_t. One of the key arguments in the proof of [6, Lemma 4.3] gives that P_0 must be disjoint from P_k (since otherwise one could extend $\alpha([0, 1])$ to a closed curve by appending arcs in $P_0 \cup P_k$ and then project onto $\mathrm{Dome}(\Omega)$ to find a homotopically non-trivial curve on $\mathrm{Dome}(\Omega)$.)

Let ϕ be the interior angle of intersection between P_a and P_k. The interior angle of intersection between P_t and P_0 varies continuously from π to 0 as t varies between 0 and a and achieves the value 0 only at a. There exists $c \in (0, a)$ such that P_c has an ideal intersection with P_k and P_t intersects P_k for all $t \in (c, a)$ (since otherwise we could again argue that $i(\mu, \alpha) \le \pi$). The interior angle of intersection of P_t with P_k varies from 0 to ϕ as t varies from c to a. Thus, there exists some $b \in (c, a)$ such that P_b intersects P_0 and P_k in the same interior angle $\theta > 0$. Therefore, by [6, Lemma 4.1], we have

$$i(\mu, \alpha) \le 2\pi - 2\theta.$$

Consider the plane R perpendicular to P_0, P_b and P_k. Consider the three geodesics $g_s = P_s \cap R$, where $s = 0$, b or k. Notice that g_b intersects both g_0 and g_k with interior angle θ. Let $\bar{\alpha}$ be the orthogonal projection of α to R. Then $\bar{\alpha}$ is a curve in R with $\bar{\alpha}(q(s)) \in g_s$ for $s = 0, b, k$. Let β be the shortest curve joining a point of g_0 to a point on g_k which intersects g_k. One may easily check that β consists of two geodesic arcs β_0 and β_1 such that β_0 intersects g_0 perpendicularly, β_1 intersects g_k perpendicularly and β_0 and β make the same angle with g_b at their common point of intersection.

Since g_0 and g_k do not intersect, β is shortest when the geodesics g_0 and g_k have a common ideal point. In this case, the geodesics g_0, g_b and g_k form an isosceles triangle T with an ideal vertex (see Figure 1). One may apply hyperbolic trigonometry formulae [2, Theorem 7.9.1] and [2, Theorem 7.11.2] to check that in this case

$$\cos(\theta) = \sinh(\ell(\beta)/2).$$

So, in general

$$\ell(\beta) \ge 2\sinh^{-1}(\cos(\theta)).$$
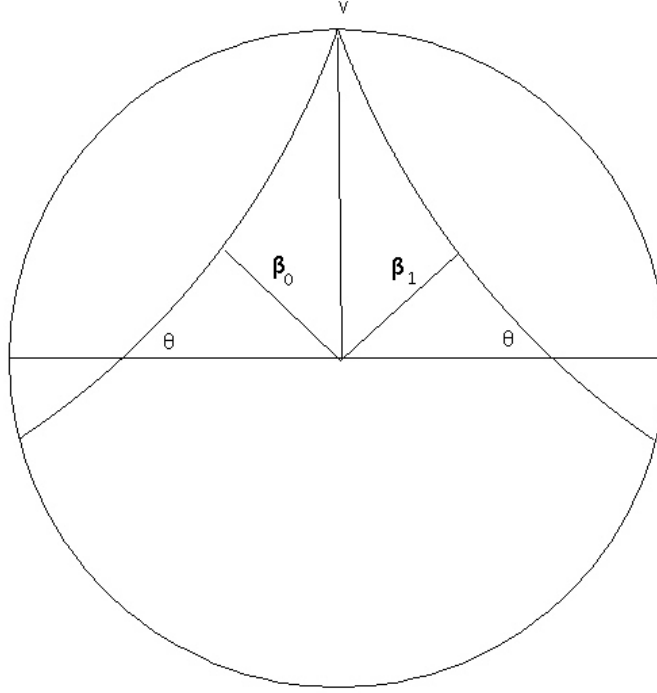
FIGURE 1. The triangle T and its decomposition

Since, by construction, $\ell(\beta) \leq \ell(\alpha) = L$, we see that

$$L \geq 2\sinh^{-1}(\cos(\theta))$$

which implies that

$$\theta \geq \cos^{-1}(\sinh(L/2)).$$

Therefore,

$$i(\mu, \alpha) \leq 2\pi - 2\cos^{-1}(\sinh(L/2)) = 2\cos^{-1}(-\sinh(L/2))$$

for any closed geodesic α of length L. Therefore, the same bound holds for all open geodesics of length L and the result follows. □

4. A NEW CRITERION FOR EMBEDDEDNES OF PLEATED PLANES

In this section, we provide a new criterion which guarantees the embeddedness of a pleated plane which generalizes earlier work of Epstein-Marden-Markovic [11] (see Theorem 2.3) and an unpublished result of Epstein-Jerrard [9]

Theorem 4.1. *There exists an increasing function* $G : (0, \infty) \to (0, \pi)$, *such that if* μ *is a measured lamination on* $\mathbb{H}^2$ *and*

$$||\mu||_L < G(L),$$

then P_μ is a bilipschitz embedding which extends continuously to a map $\hat{P}_\mu : \mathbb{H}^2 \cup \mathbb{S}^1 \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$ such that $\hat{P}_\mu(\mathbb{S}^1)$ is a quasi-circle.

Since $G(1) \approx 0.948$, we recover the result claimed by Epstein and Jerrard as a special case.

Corollary 4.2. (Epstein-Jerrard [9]) *If μ is a measured lamination on $\mathbb{H}^2$ such that*

$$||\mu|| < .948$$

then P_μ is a bilipschitz embedding which extends continuously to a map $\hat{P}_\mu : \mathbb{H}^2 \cup \mathbb{S}^1 \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$ such that the image of $\mathbb{S}^1$ is a quasi-circle.

We begin by finding an embedding criterion for piecewise geodesics. This portion of the proof follows Epstein and Jerrard's outline quite closely. Such a criterion is easily translated into a criterion for the embeddedness of pleated planes associated to finite-leaved laminations. We then further show that, in the finite-leaved lamination case, the pleated planes are in fact quasi-isometric embeddings with uniform bounds on the quasi-isometry constants. The general case is handled by approximating a general pleated plane by pleated planes associated to finite-leaved laminations.

Remark 4.3. As in [11, Theorem 4.2] we can consider a horocycle C in $\mathbb{H}^2$ and a sequence of points on C with hyperbolic distance between consecutive points being L. Connecting consecutive points, one obtains an embedded piecewise geodesic γ in $\mathbb{H}^3$. Let $P_\mu(\mathbb{H}^2)$ be the pleated plane in $\mathbb{H}^3$ obtained by extending each flat in γ to a flat in $\mathbb{H}^3$. One may check that

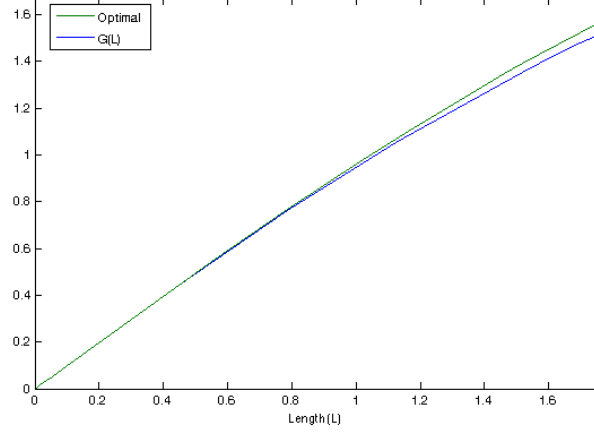$$||\mu||_L = 2\sin^{-1}\left(\tanh\left(\frac{L}{2}\right)\right).$$

Since $2\sin^{-1}(\tanh(1/2)) \approx .96076$, Theorem 4.1 is nearly optimal. Comparing the bounds for all $L \in [0, 2\sinh^{-1}(1)]$, we see they are also close to optimal (see Figure 2).

4.1. **Piecewise geodesics.** Let J be an interval in $\mathbb{R}$. A map $\gamma : J \to \mathbb{H}^3$ will be called a "piecewise geodesic" if it is unit speed and there exists a discrete subset $\{t_i\}$, parameterized by an interval in $\mathbb{Z}$, in J such that $t_i < t_{i+1}$ for all j and $\gamma([t_j, t_{j+1}])$ is a geodesic arc. We will call t_i (or $\gamma(t_i)$) the *bending points* of γ. The *bending angle* ϕ_i at t_i is the angle between $\gamma([t_{i-1}, t_i])$ and $\gamma([t_i, t_{i+1}])$. (If there is a last bending point t_r, we define $t_{r+1} = \sup J$.) Let

$$s(t) = d_{\mathbb{H}^3}(\gamma(0), \gamma(t)).$$

If $L > 0$, by analogy with the definition of L-roundness, we may define $||\gamma||_L$ to be the supremum of the total bending angle in any open subsegment of γ of length L.

FIGURE 2. $G(L)$ and the optimal $2\sin^{-1}(\tanh(L/2))$ for $L \in [0, 2\sinh^{-1}(1)]$

If $t \neq t_i$ for any i, then let $\theta(t)$ be the angle between the ray from $\gamma(0)$ to $\gamma(t)$ and the tangent vector $\gamma'(t)$. For $i = 1, \dots, n$, we define

$$\gamma'_+(t_i) = \lim_{t \to t_i^+} \gamma'(t) \quad \text{and} \quad \gamma'_-(t_i) = \lim_{t \to t_i^-} \gamma'(t).$$

We then choose $\theta^{\pm}(t_i)$ to be the angle between the ray from $\gamma(0)$ to $\gamma(t)$ and the vector $\gamma'_{\pm}(t)$. (Equivalently, we could have defined $\theta^{\pm}(t_i)$ to be the angle between the ray from $\gamma(0)$ to $\gamma(t)$ and the geodesic segment $\gamma([t_i, t_{i\pm1}])$.) Notice that $\theta(t)$ decreases smoothly on (t_i, t_{i+1}) for all i and that

$$|\theta^+(t_i) - \theta^-(t_i)| = \phi_i$$

for all i.

If $0 \in J$ and $t \neq t_i$ for any i, then Lemma 4.4 in Epstein-Marden-Markovic [11] gives that

$$(1) \qquad s'(t) = \cos(\theta(t)) \quad \text{and} \quad \theta'(t) = -\frac{\sin(\theta(t))}{\tanh(s(t))} < -\sin(\theta(t)).$$

4.2. **The hill function of Epstein and Jerrard.** A key tool in Epstein and Jerrard's work is the following *hill function*

$$h : \mathbb{R} \to (0, \pi) \text{ given by } h(x) = \cos^{-1}(\tanh(x)).$$

The defining features of the hill function are that

$$h'(x) = -\operatorname{sech}(x) = -\sin(h(x)) \quad \text{and} \quad h(0) = \frac{\pi}{2}.$$

In particular, h is a decreasing homeomorphism.

For fixed $L > 0$, we consider solutions to the equation

$$h'(x) = \frac{h(x) - h(x - L)}{L}.$$

Geometrically, we are finding the point on the graph of h such that the tangent line at $(x, h(x))$ intersects the graph at the point $(x - L, h(x - L))$ (see Figure 3). We will show that there is a unique solution $x = c(L)$ and that $c(L) \in (0, L)$.

Given $x \in \mathbb{R}$, the tangent line at $(x, h(x))$ to the graph of h intersects the graph in two points $(x, h(x))$ and $(f(x), h(f(x)))$ (except at $x = 0$ where the points are equal). The function f is continuously differentiable and odd. We let $A(x) = x - f(x)$ be the difference in their x-values, so A is also continuously differentiable and odd. Since A is odd, to show that A is strictly increasing, it suffices to show that it is strictly increasing on $[0, \infty)$. Suppose that $0 \leq x_1 < x_2$, and that T_1 and T_2 are the tangent lines to h at x_1 and x_2. Since h is convex on $[0, \infty)$, $T_1 \cap T_2 = (x_0, y_0)$ lies below the graph of h and $x_1 < x_0 < x_2$. Thus T_2 intersects the graph of h to the left of the point of intersection of T_1 with the graph of h. Therefore, $f(x_2) < f(x_1) \leq f(0) = 0$ and f is decreasing. It follows that $A(x) = x - f(x)$ is increasing and that $A(x) > x$ for all $x \in (0, \infty)$. The function c is the inverse of A, so c is also continuous differentiable and strictly increasing. Since $A(x) > x$ for $x > 0$, $c(L) \in (0, L)$.

We now define
$$\Theta(L) = h(c(L)) \quad \text{and} \quad G(L) = h(c(L) - L) - h(c(L)) = -Lh'(c(L)).$$

To show G is monotonic, we define $B(x) = h(f(x)) - h(x)$, the difference of the y-values of the intersection points of the tangent line at $(x, h(x))$ with the graph of h. As h and f are both strictly decreasing continuous functions, B is strictly increasing and continuous. Since $G(L) = B(c(L))$, G is a strictly increasing continuous function.
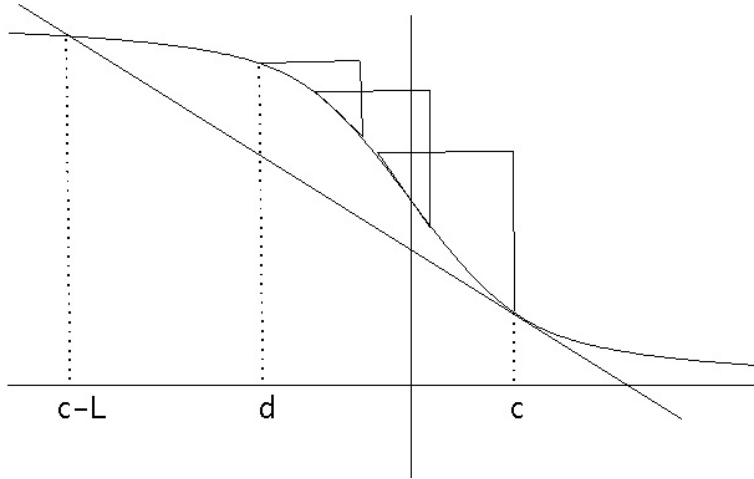

FIGURE 3. Jumps and slides on the graph of h

We note that
$$\Theta(L) + G(L) = h(c(L) - L) < \pi.$$

The following lemma is the key estimate in the proof of Theorem 4.1.

Lemma 4.4. *If $\gamma : [0, \infty) \to \mathbb{H}^3$ is piecewise geodesic, $L > 0$ and*

$$||\gamma||_L \leq G(L),$$

then

$$\theta^+(t) \leq \Theta(L) + G(L) < \pi$$

for all $t > 0$.

Proof. We define maps $P^\pm : (0, \infty) \to \mathbb{R}^2$ which are continuous except at the bending points $\{t_i\}$ and whose image lies on the graph of h. Since h is a homeomorphism onto $[0, \pi]$, given $t \in (0, \infty)$, we can find a unique $g^\pm(t) \in \mathbb{R}$, such that

$$h(g^\pm(t)) = \theta^\pm(t).$$

We then define

$$P^\pm(t) = (P_1^\pm(t), P_2^\pm(t)) = (g^\pm(t), h(g^\pm(t))) = (g^\pm(t), \theta^\pm(t)).$$

Note that the functions P^+ and P^- agree except at the bending points. In the intervals, we denote the common functions by $P(t)$, $g(t)$, and $\theta(t)$.

Notice that as one moves along the geodesic ray γ, the functions $\theta^\pm(t)$ decrease on each interval (t_i, t_{i+1}) and jump by ϕ_i at each t_i, i.e.

$$|\theta^+(t_i) - \theta^-(t_i)| = \phi_i.$$

Correspondingly, the point $P^\pm(t)$ move along the graph of h by sliding rightward (and downward) along (t_i, t_{i+1}) and jumping vertically, either upwards or downwards, by ϕ_i at t_i, see Figure 3.

We argue by contradiction. Let $c = c(L)$, $G = G(L)$, and $\Theta(L) = \Theta$. Suppose there exists $T > 0$ so that $\theta^+(T) > \Theta + G$. Let

$$s_0 = \sup\{s \in (0, T] \mid \theta^-(s) \leq \Theta\}.$$

Notice that if $s_0 = T$, then, since $|\theta^+(s_0) - \theta^-(s_0)| < G$,

$$\theta^+(T) \leq \theta^-(T) + G \leq \Theta + G$$

which would be a contradiction.

Also notice that $s_0 = t_i$ for some i, since otherwise θ^- is continuous and non-increasing at s_0, which would contradict the choice of s_0.

If $T - s_0 < L$, then since θ can only increase at the bending points and the total bending in the region $[s_0, T]$ is at most G, again

$$\theta^+(T) \leq \theta^-(s_0) + G \leq \Theta + G$$

which is a contradiction.

So, we may assume that $T - s_0 \geq L$. We will use the assumption that $\theta^-(t) > \Theta$ on $(s_0, s_0 + L]$ to arrive at a contradiction and complete the proof of the lemma.

We will show that under our hypotheses, $P(T)$ cannot lie to the left of $(c(L) - L, h(c(L) - L))$.

The key observation in the proof is that

$$h'(g(t))g'(t) = \theta'(t) < -\sin(\theta(t)) = -\sin(h(g(t))) = h'(g(t))$$

where the middle inequality follows from equation (1). Since $h'(g(t)) < 0$, we conclude that $g'(t) > 1$ for all $t \in (t_i, t_{i+1})$. Therefore,

$$(2) \qquad g(t_{i+1}) - g(t_i) = g^-(t_{i+1}) - g^+(t_i) > t_{i+1} - t_i$$

for all i.

Let $\{s_0 = t_j, t_{j+1}, \ldots, t_{j+m}\}$ be the bending points in the interval $[s_0, s_0 + L)$. For convenience, we redefine $t_{j+m+1} = s_0 + L$. Since $||\gamma||_L \leq G$, the total vertical jump in the region $[s_0, s_0 + L)$ is at most G, i.e.

$$\sum_{i=j}^{j+m} |\theta^+(t_i) - \theta^-(t_i)| \leq G,$$

Since θ^+ is non-increasing on each interval (t_i, t_{i+1}) and $\theta^-(s_0) \leq \Theta$, it follows that

$$\theta^+(t) \leq \Theta + G$$

for all $t \in [s_0, s_0 + L)$.

Let $d = \min\{g^+(t) \mid t \in [s_0, s_0 + L)\}$. Notice that as g^+ is non-decreasing on (t_i, t_{i+1}) for all i, there exists a largest $k \in \{j, \ldots, j+m\}$ so that $g^+(t_k) = d$. We further note that $d \in [c - L, c]$ since $\theta^+(t) \in [\Theta, \Theta + G]$ for all $t \in [s_0, s_0 + L)$.

If $d \in [-c, c]$, then $g^+([s_0, s_0 + L]) \subseteq [-c, c]$. Since $\theta^-(t) \geq \Theta$ on $(s_0, s_0 + L]$, we have $g^-((s_0, s_0+L]) \subseteq [-c, c]$. Notice that, since $h'(x) = -\sin(h(x))$ and h is decreasing, if $x \in [-c, c]$, then

$$h'(x) \leq h'(c) = -\frac{G}{L}.$$

Therefore, applying (2), we see that

$$\theta^-(t_{i+1}) - \theta^+(t_i) \leq h'(c)(g^-(t_{i+1}) - g^+(t_i)) = -\frac{G}{L}(g^-(t_{i+1}) - g^+(t_i)) \leq -\frac{G}{L}(t_{i+1} - t_i)$$

for all $i = j, \ldots, j+m$. Thus,

$$
\begin{aligned}
\theta^-(s_0 + L) - \theta^-(s_0) &= \left(\sum_{i=j}^{j+m} \theta^+(t_i) - \theta^-(t_i) \right) + \left(\sum_{i=j}^{j+m} \theta^-(t_{i+1}) - \theta^+(t_i)| \right) \\
&\leq \left(\sum_{i=j}^{j+m} |\theta^+(t_i) - \theta^-(t_i)| \right) - \left(\sum_{i=1}^{j+m} \frac{G}{L}(t_{i+1} - t_i) \right) \\
&\leq G - \frac{G}{L} \sum_{i=1}^{j+m} (t_{i+1} - t_i) = 0
\end{aligned}
$$

This implies that $\theta^-(s_0 + L) \leq \Theta$, which again contradicts the choice of s_0.

Therefore, $d \in [c - L, -c)$, so

$$|h'(g(t))| \geq |h'(d)|$$

for all $t \in [s_0, s_0 + L]$. So,

$$(3) \quad (\theta^+(t_i) - \theta^-(t_{i+1})) \geq |h'(d)|(g^-(t_{i+1}) - g^+(t_i)) \geq |h'(d)|(t_{i+1} - t_i)$$

for all $i = j, \ldots, j + m$. It follows that

$$\sum_{i=i}^{k-1} \left(\theta^+(t_i) - \theta^-(t_{i+1})\right) \geq |h'(d)|(t_k - s_0).$$

Thus, since $\theta^+(t_k) = h(d)$ and $\theta^-(t_j) \leq \Theta$,

$$\sum_{i=j}^{k} \left(\theta^+(t_i) - \theta^-(t_i)\right) \geq (h(d) - \Theta) + |h'(d)|(t_k - s_0)$$

and so, since the total jump on the interval $[s_0, s_0 + L)$ is at most G,

$$\sum_{i=k+1}^{j+m} \theta^+(t_i) - \theta^-(t_i) \leq G - (h(d) - \Theta) - |h'(d)|(t_k - s_0) = h(c - L) - h(d) - |h'(d)|(t_k - s_0).$$

Since $g^+(t_k) = d$,

$$g^-(s_0 + L) = d + \left(\sum_{i=k}^{j+m} g^-(t_{i+1}) - g^+(t_i)\right) - \left(\sum_{i=k+1}^{j+m} g^-(t_i) - g^+(t_i)\right).$$

Applying inequalities (2) and (3), we see that

$$g^-(s_0 + L) > d + \left(\sum_{i=k}^{j+m} t_{i+1} - t_i\right) - \frac{1}{|h'(d)|} \left(\sum_{i=k+1}^{j+m} \theta^+(t_i) - \theta^-(t_i)\right)$$

$$> d + (s_0 + L - t_k) - \frac{1}{|h'(d)|} \left(h(c - L) - h(d) - |h'(d)|(t_k - s_0)\right)$$

$$= d + L - \left(\frac{h(c - L) - h(d)}{|h'(d)|}\right).$$

Taking the tangent line at d we note that, since h' is negative and decreasing on the interval $[c - L, d]$, we have

$$h(c - L) \leq h(d) + h'(d)(c - L - d)$$

which implies that

$$\frac{1}{h'(d)}(h(c - L) - h(d)) \geq c - L - d.$$

Therefore,

$$g^-(s_0 + L) > d + L + \frac{1}{h'(d)}(h(c - L) - h(d)) \geq c,$$

so, $\theta^-(s_0 + L) \leq \Theta$ contradicting the definition of s_0. This final contradiction completes the proof. $\qquad \square$

As a nearly immediate corollary, we obtain an embeddedness criterion for piecewise geodesics.

Corollary 4.5. *If $\gamma : [0, \infty) \to \mathbb{H}^3$ is a piecewise geodesic, and $||\gamma||_L \leq G(L)$ for some $L > 0$, then γ is an embedding.*

Proof. Notice that if the corollary fails, then there exists a piecewise geodesic ray $\gamma : [0, \infty) \to \mathbb{H}^3$ such that $||\gamma||_L \leq G(L)$ and $\gamma(0) = \gamma(b)$ for some $b > 0$. (Since if $\gamma(p) = \gamma(q)$ for some $0 \leq p < q$, we can instead consider the piecewise geodesic ray $\gamma_1 : [0, \infty) \to \mathbb{H}^3$ where $\gamma_1(t) = \gamma(t - p)$.) There must exist $t_i \in (0, b)$ so that γ is geodesic on $[t_i, b]$. Then, $\theta^+(t) = \pi$ on (t_i, b), contradicting Lemma 4.4 above. $\qquad\square$

If μ is a finite-leaved measured lamination on $\mathbb{H}^2$ and $\alpha : [0, \infty)$ is any geodesic ray in $\mathbb{H}^3$, then $\gamma = P_\mu \circ \alpha$ is a piecewise geodesic and $||\gamma||_L \leq ||\mu||_L$. Since any two points in $\mathbb{H}^3$ can be joined by a geodesic ray, we immediately obtain an embeddedness criterion for pleated planes.

Corollary 4.6. *If μ is a finite-leaved measured lamination on $\mathbb{H}^2$ and $||\mu||_L \leq G(L)$ for some $L > 0$, then $P_\mu : \mathbb{H}^2 \to \mathbb{H}^3$ is an embedding.*

4.3. **Uniformly bilipschitz embeddings.** We next prove that if $\gamma : \mathbb{R} \to \mathbb{H}^3$ is a piecewise geodesic and $||\gamma||_L < G(L)$, then γ is uniformly bilipschitz. We note that since γ is 1-Lipschitz, we only have to prove a lower bound. This will immediately imply that if μ is a finite-leaved lamination on $\mathbb{H}^2$ and $||\mu_L|| < G(L)$, then P_μ is a K-bilipschitz embedding.

Proposition 4.7. *Let $\gamma : \mathbb{R} \to \mathbb{H}^3$ is a piecewise geodesic such that*

$$||\gamma||_L < G(L),$$

then γ is K-bilipschitz where K depends only on L and $||\gamma||_L$.

Proof. We first set our notation. We may assume, without loss of generality, that 0 is not a bending point of γ. Let $t_0 = 0$ and assume that the bending points in $(0, \infty)$ are indexed by an interval of positive integers beginning with 1 and the ending points in $(-\infty, 0)$ are indexed by an interval of negative integers ending with -1. Let ϕ_i be the bending angle of γ at t_i.

The following lemma will allow us to reduce to the planar setting.

Lemma 4.8. *There exists an embedded piecewise geodesic $\alpha : \mathbb{R} \to \mathbb{H}^2$ with the same bending points as γ such that*

(1) *if the bending angle of α at a bending point t_i is given by ϕ_i', then $\phi_i' \leq \phi_i$,*
(2) *$d(\alpha(0), \alpha(t)) = d(\gamma(0), \gamma(t))$ for all t, and*
(3) *there exists a non-decreasing function $\Psi : \mathbb{R} \to (-\pi, \pi)$ such that if $t > 0$, then $\Psi(t)$ is the angle between $\alpha([0, t_1])$ and the geodesic joining $\alpha(0)$ to $\alpha(t)$, while if $t < 0$, then $\Psi(t)$ is the angle between $\alpha([-t_1, 0])$ and the geodesic joining $\alpha(0)$ to $\alpha(t)$.*
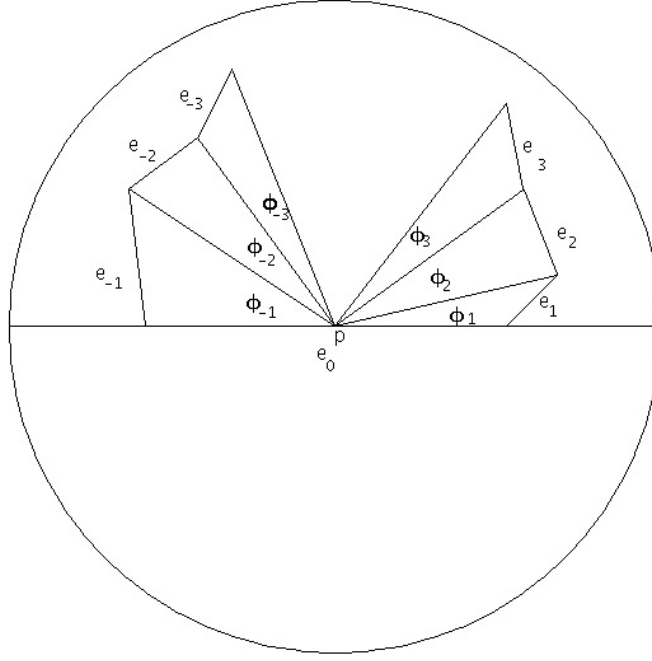
FIGURE 4. The curve α

Proof. Let f_i be the geodesic arc from $\gamma(0)$ to $\gamma(t_i)$ and let T_i be the hyperbolic triangle with vertices $\gamma(0)$, $\gamma(t_i)$, and $\gamma(t_{i+1})$ and edges f_i, $\gamma([t_i, t_{i+1}])$ and f_{i+1}. We construct α by first placing an isometric copy of T_0 in $\mathbb{H}^2$, so that f_1 is counterclockwise from f_0. We then iteratively placing a copy of T_i adjacent to a copy of T_{i-1}(so that their interiors are disjoint) along the image of f_i for all positive t_i. We then place a copy of T_{-1} in $\mathbb{H}^2$ so that T_{-1} and T_0 intersect along the image of $\gamma(0)$, so that the images of f_1 and f_{-1} lie in a geodesic and the image of f_{-2} is clockwise from f_{-1}. We then iteratively place a copy of T_{-i-1} next to the copy of T_{-i} for all negative t_{-i} (see Figure 4).

Let $\alpha : \mathbb{R} \to \mathbb{H}^2$ be the piecewise geodesic traced out by the images of pieces of γ. Then α has the same bending points as γ by construction. Moreover, since $d(\alpha(0), \alpha(t))$ is realized in the isometric copy of T_n when $t \in [t_n, t_{n+1}]$, it is also immediate that $d(\alpha(0), \alpha(t)) = d(\gamma(0), \gamma(t))$ for all t.

We next check that the bending angle ϕ_i' of α at t_i is at most ψ_i. We consider the vectors $v_n^- = \gamma_-'(t_i)$ and $v_n^+ = \gamma_+'(t_i)$ at $\gamma(t_i)$. Then the exterior angle ϕ_i is the distance between v_i^- and v_i^+ in the unit tangent sphere at $\gamma(t_i)$. The edge f_n defines an axis in the unit sphere. The possibilities for gluing T_n to T_{n-1} are given by the one-parameter family of triangles obtained by rotating T_i about f_i. It is then easy to see that the distance is shortest when T_i lies in the same plane as T_{i-1} and has disjoint interior

Therefore, $\phi_n' \leq \phi_n$. Since

$$||\alpha||_L \leq ||\gamma||_L < G(L),$$

Corollary 4.5 implies that α is an embedding.

We can now define a continuous non-decreasing function $\Psi : \mathbb{R} \to \mathbb{R}$ so that $\Psi(0) = 0$ and, if $t > 0$, then $\Psi(t)$ is the angle, modulo 2π, between $\alpha([0,t_1])$ and the geodesic joining $\alpha(0)$ to $\alpha(t)$, while if $t < 0$, then $\Psi(t)$ is the angle between $\alpha([-t_1,0])$ and the geodesic joining $\alpha(0)$ to $\alpha(t)$.

We next show that $\Psi(t) < \pi$ for all $t > 0$. If not, then γ intersects the line g_0 containing $\alpha([0,t_1])$. Suppose that $\alpha(b) \in g_0$ for some $b > 0$. Then, consider the piecewise geodesic $\hat{\alpha}$ which first traces $\alpha([0,b])$ backwards and then continues along g_0 forever. Notice that $\hat{\alpha}$ is not an embedding. However,

$$||\hat{\alpha}||_L \leq ||\alpha||_L < G(L),$$

so Corollary 4.5 implies that $\hat{\alpha}$ is an embedding, which is a contradiction. Similarly, $\Psi(t) > -\pi$ for all $t < 0$. This completes the proof of (3).   □

We notice that it suffices to show that there exists K depending only on L and $||\gamma||_L$, so that

$$s(t) = d(\gamma(0), \gamma(t)) = d(\alpha(0), \alpha(t)) \geq K|t|$$

for all $t \in \mathbb{R}$. Since, if we suppose that $\gamma : \mathbb{R} \to \mathbb{H}^3$ is any piecewise geodesic with $||\gamma||_L < G(L)$ and $r_1 < r_2$, then we can consider the new piecewise geodesic $\gamma_{r_1} : \mathbb{R} \to \mathbb{H}^3$ given by $\gamma_{r_1}(t) = \gamma(t - r_1)$. Then $||\gamma_{r_1}||_L = ||\gamma||_L$ and

$$s_{r_1}(t) = d(\gamma_{r_1}(t), \gamma_{r_1}(0)) \geq K|t|.$$

It follows that

$$d(\gamma(r_1), \gamma(r_2)) = s_{r_1}(r_2 - r_1) \geq K|r_2 - r_1|$$

Since γ is 1-Lipschitz by definition, it follows immediately that γ is a K-bilipschitz embedding.

Since Ψ is monotone and bounded we may define

$$\Psi_{+\infty} = \lim_{t \to \infty} \Psi(t) \quad \text{and} \quad \Psi_{-\infty} = \lim_{t \to -\infty} \Psi(t).$$

We now show that α is proper. The basic idea is that, since Ψ is monotonic, then $\alpha([0, \infty)$ can only accumulate on the geodesic ray $\vec{r}_+$ emanating from $\alpha(0)$ and making angle Ψ_+ with $\alpha([0, t_1])$. If it accumulate at q, then there must be infinitely many segments of α running nearly parallel to $\vec{r}_+$ and accumulating at some point q on $\vec{r}$. However, by Lemma 4.4, no segment of α can be pointing nearly straight back to $\alpha(0)$, so the total length of these segments which are "pointing towards" $\alpha(0)$ is finite. This will allow us to arrive at a contradiction.

If α is not proper, then either $\alpha|_{[0,\infty)}$ or $\alpha|_{(-\infty,0]}$ is not proper. We may assume ray $\alpha|_{[0,\infty)}$ is not proper. We recall that if t is not a bending point,

then $\theta(t)$ is the angle between $\alpha'(t)$ and the geodesic segment joining $\alpha(0)$ to $\alpha(t)$. Lemma 4.4 implies that

$$\theta(t) \leq \Theta_0 = \Theta(L) + G(L) < \pi$$

for all t. Since $\alpha|_{[0,\infty)}$ is not proper, there is an accumulation point q of $\alpha|_{[0,\infty)}$ on the ray $\vec{r}_+$ emanating from $\alpha(0)$ which makes an angle $\Phi_{+\infty}$ with $\alpha([0,t_1])$.

We may work in the disk model and assume that $\alpha(0) = 0$ and $\alpha([0,t_1])$ lies in the positive real axis. If $\epsilon > 0$ is small enough, we can consider the region given in hyperbolic polar coordinates $(\bar{r}, \bar{\theta})$ by

$$B_\epsilon = [r(q) - \epsilon, r(q) + \epsilon] \times [\theta(q) - \epsilon, \theta(q)] \subset D^2.$$

We notice that that the taxicab metric on B_ϵ, with respect to hyperbolic polar coordinates, is bilipschitz to the hyperbolic metric. If $J = \alpha^{-1}(B_\epsilon)$, then J is a countable collection of disjoint arcs. Notice that $\alpha(J) = \alpha([0,\infty)) \cap B_\epsilon$.

Since Ψ is monotonic, the $\bar{\theta}$ coordinate of α is monotonic, so the the total length of $\alpha(J)$ in the $\bar{\theta}$ direction is bounded above by ϵ. Also the signed length of $\alpha(J)$ in the r direction is bounded above by 2ϵ. Since $\theta(t) \leq \Theta_0$, at all non-bending points, the total length in the negative r-direction is bounded above by $\epsilon \tan(\Theta_0)$. Therefore, the total length in the positive r-direction is bounded above by $\epsilon + \epsilon \tan(\Theta_0)$. It follows that $\alpha(J)$ has finite length in the taxicab metric on B_ϵ. We choose $\bar{t} \in J$, so that $\alpha(J \cap [\bar{t}, \infty))$ has length, in the taxicab metric, less than $\epsilon/4$ and $d_{B_\epsilon}(\alpha(\bar{t}), q) < \epsilon/4$. Therefore, $\alpha(J \cap [\bar{t}, \infty)) \subset B_{\epsilon/2}(q)$ and $\overline{B_{\epsilon/2}(q)} \subset B_\epsilon$ (where $B_{\epsilon/2}(q)$ is the neighborhood of radius $\epsilon/2$ of q in the taxicab metric on B_ϵ). It follows that $[\bar{t}, \infty) \subset J$, which contradicts the fact that $\alpha([\bar{t}, \infty))$ has infinite length. Therefore, α must be proper.

Since α is proper and Ψ is monotone, α has two unique limit points ξ^- and ξ^+ in $\mathbb{S}^1$ which are endpoints of the geodesic rays from $\alpha(0)$ which make angles $\Psi_{-\infty}$ and $\Psi_{+\infty}$ with $\alpha([t_{-1}, t_1])$. Thus, since α is embedded,

$$\Psi_{+\infty} - \Psi_{+\infty} \leq \pi.$$

Let

$$B = \frac{G(L) - ||\mu||_L}{2}$$

We further observe that

$$\Psi_{+\infty} - \Psi_{-\infty} \leq \pi - B$$

If not, we construct a new piecewise geodesic $\alpha_1 : \mathbb{R} \to \mathbb{H}^2$ which has a bend of angle $\frac{3(G(L) - ||\mu||_L)}{4}$ at 0. One then checks that

$$||\alpha_1||_L \leq ||\alpha||_L + 3/4 \left(G(L) - ||\mu||_L \right) < G(L)$$

but α_1 is not an embedding, which would contradict Corollary 4.5.

Let g be the geodesic joining ξ^- to ξ^+. Since $\Psi_{+\infty} - \Psi_{-\infty} \leq \pi - B$, the visual distance between ξ^+ and ξ^-, as viewed from $\alpha(0)$ is at least B. It follows that there exists C, depending only on B, so that $d(\alpha(0), g)) \leq C$.

In fact, one may apply Theorem 7.9.1 in Beardon [2] to check that we may choose

$$C = \cosh^{-1}\left(\frac{1}{\sin(B/2)}\right).$$

Notice that, by considering a reparameterization of α, we can see that the visual distance between ξ^+ and ξ^- is at least B as viewed from $\alpha(t)$ for any $t \in \mathbb{R}$, and thus that $\alpha(t)$ lies within C of g for any $t \in \mathbb{R}$.

We next claim there exists $K > 0$ such that if $p : \mathbb{H}^2 \to g$ is orthogonal projection, then $p \circ \alpha$ is a 1-Lipschitz, K-bilipschitz orientation-preserving embedding. The fact that $p \circ \alpha$ is 1-Lipschitz follows immediately from the fact that both p and α are 1-Lipschitz. Let ν_0 be the angle between the orthogonal geodesic h_0 to g through α_0 and the geodesic segment $\alpha([t_{-1}, t_1])$ chosen so that $\nu_0 > 0$ if $\alpha(t_1)$ lies on the same side of h_0 as ξ^+. Notice that

$$\frac{B}{2} \leq \nu_0 \leq \pi - \frac{B}{2}$$

since otherwise $\Psi_{+\infty} - \Psi_{-\infty} \geq \pi - B$. Therefore, the restriction of $p \circ \alpha$ to $[t_{-1}, t_1]$ is an orientation-preserving embedding. We let v_0 be a unit tangent vector at $\alpha(0)$ perpendicular to g. Then

$$\|p'(\alpha(0))(v)\| = \frac{1}{\cosh(d(\alpha(0), g))} \geq \frac{1}{\cosh(C)} = \sin(B/2)$$

As $\alpha'(0)$ makes an angle at most $B/2$ with v

$$\|(p \circ \alpha)'(0)\| \geq \frac{\sin(B/2)}{\cosh(C)} = \sin^2(B/2) = \frac{1}{K}.$$

Again, by reparameterizing, we may check that if t is a non-bending point, then $p \circ \alpha$ is an orientation-preserving local homeomorphism at t and that

$$\|(p \circ \alpha)'(t)\| \geq \frac{1}{K}.$$

It follows that, for all t,

$$d(p(\gamma(0)), p(\gamma(t))) \geq \frac{1}{K} t.$$

Therefore, since p is 1-Lipschitz,

$$s(t) = d(\alpha(0), \alpha(t)) \geq d(p(\gamma(0)), p(\gamma(t))) \geq \frac{t}{K}$$

We observed earlier that this is enough to guarantee that γ is K-bilipschitz. $\square$

As an immediate corollary, we obtain a version of Theorem 4.1 for finite-leaved laminations.

Corollary 4.9. *If μ is a finite-leaved measured lamination on $\mathbb{H}^2$ such that*

$$\|\mu\|_L < G(L),$$

then P_μ is a K-bilipschitz embedding, where K depends only on L and $\|\mu\|_L$.

4.4. **Proof of Theorem 4.1.** Suppose that μ is a measured lamination on $\mathbb{H}^2$ with $||\mu||_L < G(L)$. By Lemma 4.6 in Epstein-Marden-Markovic [12], there exists a sequence $\{\mu_n\}$ of finite-leaved measured laminations which converges to μ such that $||\mu_n||_L = ||\mu||_L$ for all n. Corollary 4.9 implies that each P_{μ_n} is a K-bilipschitz embedding where K depends only on L and $||\mu||_L$. The maps $\{P_{\mu_n}\}$ converges uniformly on compact sets to P_μ (see [10, Theorem III.3.11.9]), so P_μ is also a K-bilipschitz embedding. Therefore, P_μ extends continuously to $\hat{P}_\mu : \mathbb{H}^2 \cup \mathbb{S}^1_\infty \to \mathbb{H}^3 \cup \mathbb{S}^2_\infty$ and $\hat{P}_\mu(\mathbb{S}^1)$ is a quasi-circle.
□

5. COMPLEX EARTHQUAKES

In this section, we use Theorem 4.1 to give improved bounds in results of Epstein-Marden-Markovic which will lead to the improved bound obtained in our main result. We first obtain new bounds guaranteeing that complex earthquakes extend to homeomorphisms at infinity, see Corollaries 5.2 and 5.3. Once we have done so, we obtain a generalization of [11, Theorem 4.14] which produces a family of conformally natural quasiconformal maps associated to complex earthquakes with the same support μ which satisfy the bounds obtained in Corollary 5.2 or Corollary 5.3. Finally, we give a version of [12, Theorem 4.3] which gives rise to a family of quasiregular maps associated to all complex earthquakes with positive bending along μ.

If μ is a measured lamination on $\mathbb{H}^2$, we define $E_\mu : \mathbb{H}^2 \to \mathbb{H}^2$ to be the earthquake map defined by fixing a component of the complement of μ and left-shearing all other components by an amount given by the measure on μ. An earthquake map is continuous except on leaves of μ with discrete measure and extends to a homeomorphism of $\mathbb{S}^1$. Therefore, any measured lamination λ on $\mathbb{H}^2$ is mapped to a well-defined measured lamination on $\mathbb{H}^2$ which we denote $E_\mu(\lambda)$.

Given a measured lamination μ on $\mathbb{H}^2$ and $z = x + iy \in \mathbb{C}$, we define the *complex earthquake*

$$\mathbb{C}E_z = P_{yE_{x\mu}} \circ E_{x\mu} : \mathbb{H}^2 \to \mathbb{H}^3$$

to be the composition of earthquaking along $x\mu$ and then bending along the lamination $yE_{x\mu}(\mu)$. The sign of y determines the direction of the bending. By linearity,

$$||yE_{x\mu}(\mu)||_L = |y| \ ||E_{x\mu}(\mu)||_L.$$

(See Epstein-Marden [10, Chapter 3] or Epstein-Marden-Markovic [11, Section 3] for a detailed discussion of complex earthquakes.)

The following estimate allows one to bound $||E_{x\mu}(\mu)||_L$.

Theorem 5.1. (Epstein-Marden-Markovic [11, Theorem 4.12]) *Let ℓ_1 and ℓ_2 be distinct leaves of a measured lamination μ on $\mathbb{H}^2$. Suppose that α is a closed geodesic segment with endpoints on ℓ_1 and ℓ_2 and let $x = i(\alpha, \mu)$. Let*

ℓ'_1 and ℓ'_2 be the images of ℓ_1 and ℓ_2 under the earthquake E_μ. Then

$$\sinh(d(\ell'_1, \ell'_2)) \leq e^x \sinh(d(\ell_1, \ell_2)) \quad \text{and} \quad d(\ell'_1, \ell'_2) \leq e^{x/2} d(\ell_1, \ell_2).$$

Furthermore,

$$\sinh(d(\ell_1, \ell_2)) \leq e^x \sinh(d(\ell'_1, \ell'_2)) \quad \text{and} \quad d(\ell_1, \ell_2) \leq e^{x/2} d(\ell'_1, \ell'_2).$$

Motivated by this result, Epstein, Marden, and Markovic define the function

$$f(L, x) = \min\left(Le^{|x|/2}, \sinh^{-1}(e^{|x|} \sinh(L)) \right).$$

Corollary 4.13 in [11] generalizes to give:

Corollary 5.2. *If μ is a measured lamination on $\mathbb{H}^2$, $z = x + iy \in \mathbb{C}$, and $L > 0$, then*

$$||E_{x\mu}(\mu)||_L \leq \left\lceil \frac{f(L, x)}{L} \right\rceil ||\mu||_L.$$

Furthermore, if

$$|y| < \frac{G(L)}{\left\lceil \frac{f(L,x)}{L} \right\rceil ||\mu||_L},$$

then $\mathbb{C}E_z$ extends to an embedding of $\mathbb{S}^1$ into $\hat{\mathbb{C}}$.

We similarly define

$$g(L, x) = \max\left(Le^{-|x|/2}, \sinh^{-1}(e^{-|x|} \sinh(L)) \right).$$

We will show later, see Lemma 7.1, that if $2\tanh(L) > L$ then $g(L, x) = Le^{-|x|/2}$.

Theorem 5.1 and Theorem 4.1 combine to give the following:

Corollary 5.3. *If μ is a measured lamination on $\mathbb{H}^2$, $z = x + iy \in \mathbb{C}$, and $L > 0$, then*

$$||E_{x\mu}(\mu)||_{g(L,x)} \leq ||\mu||_L.$$

Furthermore, if

$$|y| < \frac{G(g(L, x))}{||\mu||_L},$$

then $P_{yE_{x\mu}}$ is a bilipschitz embedding and $\mathbb{C}E_z$ extends to an embedding of $\mathbb{S}^1$ into $\hat{\mathbb{C}}$.

Proofs: The proofs of Corollaries 5.2 and 5.3 both follow the same outline as the proof of [11, Corollary 4.13]. Let μ be a measured lamination on $\mathbb{H}^2$, $z = x + iy \in \mathbb{C}$, and $L > 0$.

Suppose that $A > 0$ and that α is an open geodesic arc in $\mathbb{H}^2$ of length A which is transverse to $E_{x\mu}(\mu)$. Theorem 5.1 guarantees that one can choose an open geodesic arc β in $\mathbb{H}^2$ which intersects exactly the leaves of μ which correspond to leaves of $E_{x\mu}$ which intersect α and has total length at most $f(A, x)$. Therefore,

$$i(\alpha, E_{x\mu}(\mu)) = i(\beta, \mu) \leq ||\mu||_{f(A,x)},$$

so

$$(4) \qquad ||E_{x\mu}(\mu)||_A \le ||\mu||_{f(A,x)}.$$

We begin with the proof of Corollary 5.3. Inequality 4 immediately implies that

$$||E_{x\mu}(\mu)||_{g(L,x)} \le ||\mu||_{f(g(L,x))} = ||\mu||_L.$$

So, if

$$|y| < \frac{G(g(L,x))}{||\mu||_L},$$

then

$$||y \ E_{x\mu}||_{g(L,x)} < G(g(L,x)).$$

Theorem 4.1 then implies that $P_{yE_{x\mu}}$ is a bilipschitz embedding which extends to an embedding of $\mathbb{S}^1$ into $\hat{\mathbb{C}}$. Since $E_{x\mu}$ extends to a homeomorphism of $\mathbb{S}^1$, it follows that $\mathbb{C}E_z$ extends to an embedding of $\mathbb{S}^1$ into $\hat{\mathbb{C}}$. This completes the proof of Corollary 5.3.

We now turn to the proof of Corollary 5.2. We can divide a half open geodesic arc in $\mathbb{H}^2$ of length $f(L,x)$ into $\lceil f(L,x)/L \rceil$ half open geodesic arcs of length less than or equal to L, so

$$||E_{x\mu}(\mu)||_L \le ||\mu||_{f(L,x)} \le \left\lceil \frac{f(L,x)}{L} \right\rceil ||\mu||_L.$$

Therefore, if

$$|y| < \frac{G(L)}{\left\lceil \frac{f(L,x)}{L} \right\rceil ||\mu||_L},$$

then

$$||y \ E_{x\mu}(\mu)||_L < G(L).$$

and we may again use Theorem 4.1 to complete the proof of Corollary 5.2.
$\square$

For all $L > 0$, we define

$$Q(L,x) = \max \left(\frac{G(L)}{\left\lceil \frac{f(L,x)}{L} \right\rceil}, G(g(L,x)) \right)$$

and

$$\mathcal{T}_0^L = \mathrm{int}(\{x + iy \mid |y| < Q(L,x)\}).$$

The following theorem is a direct generalization of Theorem 4.14 in Epstein-Marden-Markovic [11]. In its proof, we simply replace their use of Corollary 4.13 in [11] with our Corollaries 5.2 and 5.3.

Theorem 5.4. *Suppose that $L > 0$ and μ is a measured lamination on $\mathbb{H}^2$ such that $||\mu||_L = 1$. Then, for $z \in \mathcal{T}_0^L$,*

(1) *$\mathbb{C}E_z$ extends to an embedding $\phi_z : \mathbb{S}^1 \to \hat{\mathbb{C}}$ which bounds a region Ω_z.*

(2) *There is a quasiconformal map $\Phi_z : \mathbb{D}^2 \to \Omega_z$ with domain the unit disk and quasiconformal dilatation K_z bounded by*

$$K_z \leq \frac{1 + |h(z)|}{1 - |h(z)|}$$

where $h : \mathcal{T}_0 \to \mathbb{H}^2$ is a Riemann map taking 0 to 0.
   Moreover, $\Phi_z \cup \phi_z : \mathbb{D}^2 \cup \mathbb{S}^1 \to \hat{\mathbb{C}}$ is continuous.
(3) *If G is a group of Möbius transformations preserving μ, then Φ_z can be chosen so that there is a homomorphism $\rho_z : G \to G_z$ where G_z is also a group of Möbius transformations and*

$$\Phi_z \circ g = \rho_z(g) \circ \Phi_z$$

for all $g \in G$.

Epstein, Marden and Markovic [12] introduce the theory of complex angle scaling maps and use them to produce a family of quasiregular mappings indexed by

$$\mathcal{S}^L = \text{int} \left\{ x + iy \in \mathbb{C} \mid y > -\frac{0.73}{f(1, x)} \right\}$$

so that if $|\text{Im}(t)| < \frac{0.73}{f(1,x)}$, then Φ_t is quasiconformal. (See also the discussion in [7, Section 3.4].)

We consider the enlarged region

$$\mathcal{T}^L = \text{int} \left\{ x + iy \in \mathbb{C} \mid y > -Q(x, L) \right\}.$$

Given Theorems 4.1 and 5.4, their proof of Theorem 4.3 extends immediately to give:

Theorem 5.5. ([12, Theorem 4.3]) *Suppose that $L > 0$, μ is a measured lamination on $\mathbb{H}^2$ with $\|\mu\|_L = 1$, $v_0 > 0$ and $t_0 = iv_0 \in \mathcal{T}_0^L$. If $t \in \mathcal{T}_0^L$, let Ω_t be the the image of $\mathbb{D}^2$ under the map Φ_t given by Theorem 5.4. Then there exists a continuous map $\Psi : \mathbb{U} \times \Omega_{t_0} \to \hat{\mathbb{C}}$ such that*

(1) $\Psi_{t_0} = id.$
(2) *For each $z \in \Omega_{t_0}$, $\Psi(t, z)$ depends holomorphically on t.*
(3) *For each $t \in \mathcal{T}_0^L$, Ψ_t can be continuously extended to $\partial \Omega_{t_0}$ such that*

$$\Psi_t \circ \Phi_{t_0}|_{\mathbb{S}^1} = \Phi_t|_{\mathbb{S}^1}.$$

   In particular $\Psi_0 : \partial \Omega_{t_0} \to \mathbb{S}^1$ and $\Phi_{t_0} : \mathbb{S}^1 \to \partial \Omega_0$ are inverse homeomorphisms.
(4) *If $t \in \mathcal{T}_0^L$ and $\text{Im}(t) > 0$, then Ψ_t is injective and $\Psi_t(\Omega_0) = \Phi_t(\mathbb{D}^2) = \Omega_t$.*
(5) *When $t \in \mathbb{U}^2$, $t = u + iv$, then Ψ_t is locally injective K_t-quasiregular mapping where*

$$K_t - \frac{1 + |\kappa(t)|}{1 - |\kappa(t)|}, \qquad |\kappa(t)| = \frac{\sqrt{u^2 + (v - v_0)^2}}{\sqrt{u^2 + (v + v_0)^2}}$$

(6) *If G is a group of Möbius transformations preserving Ω_0, then there is a homomorphism $\rho_t : G \to G_t$ where G_t is also a group of Möbius transformations, such that*

$$\Psi_t \circ g = \rho_t(g) \circ \Psi_t$$

for all $g \in G$.

6. Quasiconfomal bounds

One can now readily adapt the techniques of proof of Epstein-Marden-Markovic [12, Theorem 6.11] to establish:

Theorem 6.1. *If Ω is a simply connected hyperbolic domain in $\hat{\mathbb{C}}$ and $L > 0$, then there is a conformally natural K-quasiconformal map $f : \Omega \to \mathrm{Dome}(\Omega)$ which extends to the identity on $\partial\Omega \subset \hat{\mathbb{C}}$ such that*

$$\log(K) \leq d_{\mathcal{T}^L}(ic_1(L), 0)$$

where $d_{\mathcal{T}^L}$ is the Poincaré metric on the domain $\mathcal{T}^L$ and $c_1(L) = 2\cos^{-1}\left(-\sinh\left(\frac{L}{2}\right)\right)$.

We offer a brief sketch of the proof in order to indicate where our new bounds, as given in Theorems 3.1, 5.4 and 5.5, are used in the argument.

We recall that universal Teichmüller space $\mathcal{U}$ is the space of quasisymmetric homeomorphisms of the unit ciricle $\mathbb{S}^1$, modulo the action of Möbius transformations by post-composition (see, for example, Ahlfors [1, Chapter VI]). The Teichmüller metric on the space $\mathcal{U}$ is defined by

$$d_{\mathcal{U}}(f, g) = \log\inf K(\hat{f}^{-1} \circ \hat{g}))$$

where the infimum is over all quasiconformal extensions $\hat{f}$ and $\hat{g}$ of f and g to maps from the unit disk to itself and $K(\hat{f}^{-1} \circ \hat{g})$ is the quasiconformal dilatation of $\hat{f}^{-1} \circ \hat{g}$. If Γ is a group of conformal automorphisms of $\mathbb{D}^2$, we define $\mathcal{U}(\Gamma) \subseteq \mathcal{U}$ to be the quasisymmetric homeomorphisms which conjugate the action of Γ to the action of an isomorphic group of conformal automorphisms. The Teichmüller metric on $\mathcal{U}(\Gamma)$ is defined similarly by considering extensions which conjugate Γ to a group of conformal automorphisms.

Let $g : \mathbb{D}^2 \to \hat{\mathbb{C}}$ be a locally injective quasiregular map, i.e. $g = h \circ f$ where f is a quasiconformal homeomorphism and h is locally injective and holomorphic on the image of f. We may define a complex structure C_g on $\mathbb{D}^2$ by pulling back the complex structure on $\hat{\mathbb{C}}$ via g. The identity map defines a quasiconformal homeomorphism $\hat{g} : \mathbb{D}^2 \to C_g$. We then uniformize C_g by a conformal map $R : C_g \to \mathbb{D}^2$ and consider the quasiconformal map $R \circ \hat{g} : \mathbb{D}^2 \to \mathbb{D}^2$. This map extends to the boundary to give a quasisymmetric map $qs(g) : \mathbb{S}^1 \to \mathbb{S}^1$.

Choose μ so that $\mathrm{Dome}(\Omega) = P_{c\mu}(\mathbb{D}^2)$ where $||\mu||_L = 1$ and $c > 0$. We use Theorem 5.4 to define a map

$$F : \mathcal{T}_0^L \to \mathcal{U}(\Gamma),$$

where Γ is the group of conformal automorphisms of $\mathbb{H}^2$ preserving μ. If $t \in \mathcal{T}_0^L$, let $F(t) = qs(\Phi_t)$.

Similarly, we may use Theorem 5.5 to define a map

$$G : \{z \in \mathbb{C} \mid Im(z) > 0\} \to \mathcal{U}(\Gamma)$$

by letting $G(t) = qs(\Psi_t \circ \Phi_{t_0})$.

If t lies in the intersection of the domains of F and G, then even though Φ_t and $\Psi_t \circ \Phi_{t_0}$ need not agree on $\mathbb{D}^2$, Theorem 5.5 implies that they have the same boundary values and quasi disk image Ω_t. Therefore F and G agree on the overlap $\{z \in \mathcal{T}_0^L \mid Im(z) > 0\}$ of their domains. We may combine the function to obtain a well-defined function

$$\bar{F} : \mathcal{T}^L \to \mathcal{U}(\Gamma).$$

Epstein, Marden, Markovic further show that $\bar{F}$ is holomorphic (see [12, Theorem 6.5 and Proposition 6.9]).

The Kobayashi metric on a complex manifold M is defined to be the largest metric on M with the property that for any $f : \mathbb{D}^2 \to M$ holomorphic, f is 1-Lipschitz with respect the hyperbolic metric on $\mathbb{D}^2$. Therefore, holomorphic maps between complex manifolds are 1-Lipschitz with respect to their Kobayashi metrics. The Teichmüller metric agrees with the Kobayashi metric on $\mathcal{U}$ and $\mathcal{U}(\Gamma)$ (see [14, Chapter 7]). Morever, the Poincaré metric on any simply connected domain, in particular $\mathcal{T}^L$, agrees with its Kobayashi metric. It follows then that for any $t \in \mathcal{T}^L$,

$$d_{\mathcal{U}(\Gamma)}(\bar{F}(t), \bar{F}(0)) \leq d_{\mathcal{T}^L}(t, 0).$$

Theorem 3.1 implies that

$$c \leq c_1(L) = 2\cos^{-1}\left(-\sinh\left(\frac{L}{2}\right)\right)$$

so

$$d_{\mathcal{U}(\Gamma)}(\bar{F}(ic), \bar{F}(0)) \leq d_{\mathcal{T}^L}(ic, 0) \leq d_{\mathcal{T}^L}(ic_1(L), 0).$$

Since $\mathbb{C}E_{ic} = P_{c\mu}$ and $\Omega = \Omega_{ic}$ is simply connected, the map $g_{ic} = \Psi_{ic} \circ \Phi_{t_0}$ is a conformally natural quasiconformal mapping with image Ω. Moreover, $P_{c\mu} \circ g_{ic}^{-1} : \Omega \to \text{Dome}(\Omega)$ extends to the identity on $\partial\Omega = \partial\text{Dome}(\Omega)$. (For more details, see the discussion in the proofs of [12, Theorem 6.11] or [7, Theorem 1.1].)

We have that $\bar{F}(ic) = qs(g_{ic}) = (R \circ g_{ic})|_{\mathbb{S}^1}$ where $R : \Omega \to \mathbb{D}^2$ is a uniformization map. Therefore,

$$d_{\mathcal{U}(\Gamma)}(\bar{F}(ic), \bar{F}(0)) = d_{\mathcal{U}(\Gamma)}(\bar{F}(ic), Id) = \log\inf K(h)$$

where the infimum is taken over all quasiconformal maps from $\mathbb{D}^2$ to $\mathbb{D}^2$ extending $(R \circ g_{ic})|_{\mathbb{S}^1}$ and conjugating Γ to a group of conformal automorphisms. By basic compactness results for families of quasiconformal maps, this infimal quasiconformal dilatation is achieved by a quasiconformal map $h : \mathbb{D}^2 \to \mathbb{D}^2$. If $f : \Omega \to \mathbb{D}^2$ is given by $f = h^{-1} \circ R$, then

$$K(f) = K(h) = d_{\mathcal{U}(\Gamma)}(\bar{F}(ic), \bar{F}(0)) \leq d_{\mathcal{T}^L}(ic_1(L), 0).$$

Since h and $R \circ g_{ic}$ are quasiconformal maps with the same extension to $\partial \mathbb{H}^2$, they are boundedly homotopic (see, e.g., [12, Lemma 5.10]). So, f is boundedly homotopic to g_{ic}^{-1}. Thus, $P_{c\mu} \circ f : \Omega \to \text{Dome}(\Omega)$ is boundedly homotopic to $P_{c\mu} \circ g_{ic}^{-1}$. Since $P_{c\mu} \circ g_{ic}^{-1}$ extends to the identity on $\partial \Omega$, it follows that $P_{c\mu} \circ f$ also extends to the identity on $\partial \Omega$. Therefore, $P_{c\mu} \circ f : \Omega \to \text{Dome}(\Omega)$ is the desired conformally natural K-quasiconformal map which extends to the identity on $\partial \Omega$ such that

$$\log(K) \le d_{\mathcal{T}^L}(ic_1(L), 0).$$

This completes the sketch of the proof of Theorem 6.1.

Remark: Epstein, Marden and Markovic showed that if Ω is simply connected, then a quasiconformal map between Ω and $\text{Dome}(\Omega)$ extends to the identity on $\partial \Omega$ if and only if it is boundedly homotopic to the nearest point retraction from Ω to $\text{Dome}(\Omega)$ (see [12, Theorem 5.9]).

7. Derivation of main theorem

In order to complete the proof of our main theorem, Theorem 1.3, it suffices to show that one can choose $L > 0$ such that

$$d_{\mathcal{T}^L}(ic_1(L), 0) < 7.1695.$$

Motivated by computer calculations for various value of L, we choose $L = 1.48$.

First, we construct a polygonal approximation for the region $\mathcal{T}^L$ from within, see figure 5. The approximation is constructed using MATLAB's
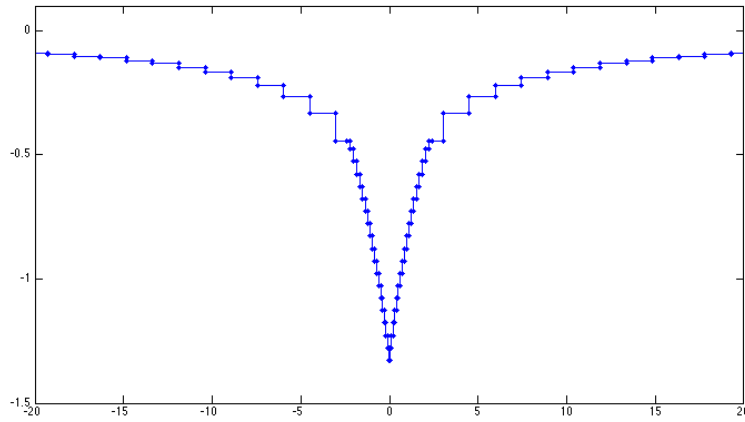

FIGURE 5. Polygonal approximation of $\mathcal{T}^L$

Symbolic Math Toolbox and variable precision arithmetic. Variable precision arithmetic allows us to compute vertex positions to arbitrary precision. In particular, we can deduce sign changes to find intervals containing intersection points.

We build a step function $s(x) \leq Q(L, x)$ as follows; We recall that

$$Q(L, x) = \max \left(\frac{G(L)}{\left\lceil \frac{f(L,x)}{L} \right\rceil}, G(g(L, x)) \right).$$

We first locate intervals where $\frac{G(L)}{\left\lceil \frac{f(L,x)}{L} \right\rceil}$ and $G(g(L, x))$ intersect. For values where $\frac{G(L)}{\lceil f(L,x)/L \rceil}$ dominates, we bound $Q(L, x)$ by truncated decimal expansions (i.e. lower bounds) of values of $\frac{G(L)}{\lceil f(L,x)/L \rceil}$, which we compute using variable precision arithmetic.

For parts dominated by $G(g(L, x))$, we simplify our computation by using the following lemma.

Lemma 7.1. *Let $L_0 > 0$ be the unique positive solution to $2 \tanh(L) = L$. If $L < L_0 \approx 1.91501$, then $g(L, x) = Le^{-|x|/2}$.*

Proof. Recall that

$$g(L, x) = \max \left(Le^{-|x|/2}, \sinh^{-1}(e^{-|x|} \sinh(L)) \right).$$

Let $L < L_0$ and consider the function $j(x) = e^x \sinh(Le^{-x/2})$. It has a critical point precisely when

$$2 \tanh(Le^{-x/2}) = Le^{-x/2}.$$

Since $L < L_0$, we have $Le^{-x/2} < L_0$ when $x \geq 0$, so j has no critical points in the interval $[0, \infty)$. Since $j'(0) = \sinh L - \frac{L}{2} \cosh L > 0$, j is increasing on the interval $[0, \infty)$. Therefore,

$$j(x) = e^x \sinh(Le^{-x/2}) \geq \sinh(L) = j(0)$$

for all $x \geq 0$, so

$$Le^{-x/2} \geq \sinh^{-1}(e^{-x} \sinh(L))$$

for all $x \geq 0$. Thus, $g(L, x) = Le^{-|x|/2}$ for all x. $\qquad\square$

From our initial analysis of the hill function, we know that $G(t)$ is an increasing function on $t \in [0, \infty)$. It follows that $G(g(L, x))$ is a a decreasing function for $x \in [0, \infty)$. Therefore, we can approximate $G(g(L, x))$ by a step function from below.

To compute the values of $G(g(L, x))$, recall that $G(t) = h(c(t) - t) - h(c(t))$. The function $c(t)$ can be computed to arbitrary precision from the equation $t\,h'(c(t)) = h(c(t)) - h(c(t) - t)$. In particular, variable precision arithmetic can give us truncated decimal expansions of values of $G(g(L, x))$. We sample at a collection of points to obtain a step function where $G(g(L, x))$ dominates.

We use these computations to build $s(x) \leq Q(L, x)$ on some interval $[-a, a]$. Outside of that interval, we set $s(x) = 0$. The graph of $-s(x)$ gives us the boundary of a polygonal region contained in $\mathcal{T}^L$.

Using the Schwarz-Christoffel mapping toolbox developed by Toby Driscoll [8], the images of the points 0 and $2\cos^{-1}\left(-\sinh\left(\frac{L}{2}\right)\right)i$ are computed under a Riemann mapping of the approximation of $\mathcal{T}^L$ to the upper half plane. Computing the hyperbolic distance between the images provides the result. The Schwarz-Christoffel mapping toolbox provides precision and error estimates. The error bounds are on the order of 10^{-5}.

We found that the optimal bound is given when L is approximately 1.48. Using $L = 1.48$, the point

$$B = c_1(L)i = 2\cos^{-1}\left(-\sinh\left(\frac{L}{2}\right)\right)i \approx 5.027888826784i$$

and $exp(d_{\mathcal{T}^L}(ic_1(L), 0)) \approx 7.1695$. A truncated version of the output provides the values of $G(L)$, $HPL(0)$, and $HPL(B)$, where $HPL : \mathcal{T}^L \to \mathbb{H}^2$ is a Riemann mapping from $\mathcal{T}^L$ to the upper half-plane. We also have $H(L) = d_{\mathbb{H}^2}(HPL(0), HPL(B))$ and $K(L) = exp(d_{\mathcal{T}^L}(ic_1(L), 0))$.

```
L=1.48
G(L) = 1.327185362837166
HPL(0) = 0.000007509959438 + 0.009347547230674i
HPL(B) =  0.000009420062234 + 0.067016970686742i
H(L) = 1.969831901361628
K(L) = 7.169471208698489
```

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