Sidney J. Kolpas

Let Your Fingers Do the Multiplying

In the movie *Stand and Deliver* (Warner Brothers 1988), Edward James Olmos, portraying mathematics teacher Jaime Escalante, confronts Chuco, a defiant gang member, during Olmos's class at Garfield High School in East Los Angeles:

> *Escalante.* Ohh. You know the times tables?
>
> *Chuco.* I know the ones . . . twos . . . three.
>
> [*On "three" Chuco flips the bird to Escalante.*]
>
> *Escalante.* Finger Man. I heard about you. Are you The Finger Man? I'm the Finger Man, too. Do you know what I can do? I know how to multiply by nine! Nine times three. What you got? Twenty-seven. Six times nine. One, two, three, four, five, six. What you got? Fifty-four. You wanna hard one. How about eight times nine? One, two, three, four, five, six, seven, eight. What do you got? Seventy-two. (Warner Brothers 1988, p. 9)

Escalante was using a well-known finger-multiplication trick to capture Chuco's attention

To capture Chuco's attention, Escalante was using a well-known finger-multiplication trick: multiplying by nines by counting on his fingers. Just as this trick captured Chuco's interest, it can also capture the interest of high school and college students when they investigate the mathematics behind the reasons that it works.

Students are often admonished that they should not count on their fingers. This article investigates two methods of finger multiplication that can, however, help students. Although students may have first learned these methods in elementary school, high school and college students can apply their knowledge of number theory, algebra, and problem-solving strategies to prove why these finger techniques "work." The two methods can also serve as catalysts for extension activities and multicultural projects on the history of finger reckoning. Suggested activities and projects are found at the end of this article.

The first method—the one that Escalante used with Chuco for finding multiples of nine up to 90—requires that the students hold both hands up, with palms facing the student. The student counts his or her thumbs and fingers consecutively from left to right, with the thumb on the left hand representing the number 1 and the thumb on the right hand representing the number 10. To multiply n times 9, where $1 \le n \le 10$ and where n is a whole number, the student bends down the nth finger. The number of fingers to the left of the bent finger represents the tens-place digit of the product, whereas the number of fingers to the right of the bent finger represents the ones-place digit of the product. **Figures 1** and **2** illustrate 3×9 and 7×9, respectively. But why does Escalante's trick work?

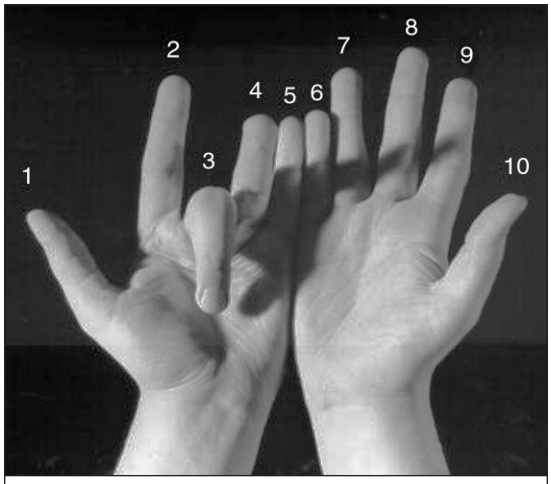

Fig. 1
$3 \times 9 = 27$;
two fingers to the left of the bent finger = 20;
seven fingers to the right of the bent finger = 7;
20 + 7 = 27.

PROOF OF ESCALANTE'S TRICK FOR MULTIPLYING BY 9

Escalante's trick works because it is based on the theorem from number theory that states that if a number is a multiple of 9, then the sum of its digits is a multiple of 9. We prove that in the special case of multiples of 9 no larger than 10×9, the sum of the digits is exactly 9 and that the digit in the tens

Sidney Kolpas, skolpas@glendale.cc.ca.us, has taught for thirty-one years, the last eleven years at Glendale Community College in Glendale, CA 91208. He is interested in mathematics history, mathematical magic, and the use of technology in teaching.

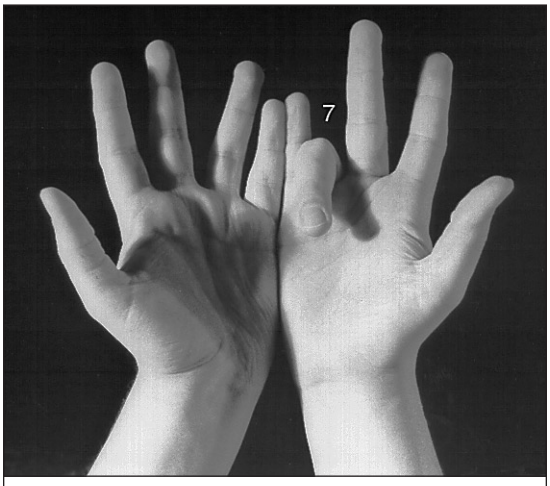

Fig. 2
$7 \times 9 = 63$;
six fingers to the left of the bent finger = 60;
three fingers to the right of the bent finger = 3;
$60 + 3 = 63$.

place is one less than n. Thus, bending down the nth finger ensures that the number of fingers to its left is one less than n and that the number of fingers to its right is the number that remains to make the digits sum to 9. Discussing with students the reasoning behind the following proof should increase their understanding of, and appreciation for, both Escalante's finger trick and number theory.

To show that if a two-digit product is a multiple of nine, the sum of its digits is a multiple of nine, we let the two-digit product be represented by $p = 10a + b$, where a and b are whole numbers and where $1 \le a \le 9$ and $0 \le b \le 9$. In this place-value notation, a and b are, respectively, the tens-place digit and the units-place digit of the product p. Since we are given $9 \mid p$, that is, p is a multiple of 9, we know that $9 \mid (p - 9a)$. But

$$\begin{aligned}(p - 9a) &= 10a + b - 9a \\ &= a + b,\end{aligned}$$

the sum of the digits of the product. (The students should verify that fact.) Therefore, $9 \mid (a + b)$, the sum of the digits of the product. Thus, we have established that if a two-digit number is a multiple of 9, then the sum of its digits is a multiple of 9. We next need to show that the sum of the digits is exactly 9 for multiples of 9 no larger than 90. We know that multiples of 9 no larger than 90 can be written as $9n$, where $1 \le n \le 10$. To be able to prove that the sum of the digits is exactly 9, we need to rewrite the product $9n$ in place-value form to identify its digits. In place-value form, $9n = (n - 1)10 + [9 - (n - 1)]$. Moreover, since $1 \le n \le 10$, $0 \le (n - 1) \le 9$ and $0 \le [9 - (n - 1)] \le 9$. This place-value form reveals that $(n - 1)$ and $[9 - (n - 1)]$ are the tens digit and units digit, respectively, of the product. Therefore, for multiples of 9 no larger than 90, the tens digit is one less than n. Moreover, the two digits sum to exactly 9: $(n - 1) + [9 - (n - 1)] = 9$. The reason that Escalante's trick works is no longer a mystery.

To further test students' understanding of the theory behind Escalante's trick, the teacher might ask them to prove that if a three-digit number is a multiple of 9, then the sum of its digits is a multiple of 9; their proof would parallel the previous proof. Advanced students can try to prove the more difficult general case.

Escalante's trick works only for multiplication from 1×9 to 10×9. A second, more general, technique for obtaining products by counting on our fingers also exists. The roots of the technique date back to the Middle Ages, when finger counting was a regular means of communicating arithmetic information. We can use multiplication of digits between 6×6 and 10×10 to introduce this technique; this article presents and proves an algorithm for 6×6 and 10×10. The technique is then extended to two different cases of algorithms that cover larger ranges of multipliers and multiplicands. Finally, the article presents and proves two general algorithms, one for each case.

Finger-multiplication techniques were universal

TOWARD MORE GENERAL ALGORITHMS

6×6 to 10×10

The following method for multiplying digits from 6×6 to 10×10 is sometimes called *European peasant multiplication.* The method assumes knowledge of multiplication facts from 0×0 to 4×4, as well as knowing how to multiply by 10, as do all the remaining methods discussed in this article. Commonly used during the Middle Ages, the method was still used by Russian and French peasants in the early twentieth century (NCTM 1989, p. 122). In general, finger-arithmetic methods became popular during the Middle Ages because the Hindu-Arabic number system was not yet in widespread use and because paper and writing implements were scarce. Moreover, finger-multiplication techniques were universal, since they were not based on any particular number system or language; finger arithmetic "had the advantage of transcending language differences" (Eves 1969, p. 8). Even today, finger numeration and computation is still used for bargaining in marketplaces around the world.

To use the technique for digits from 6×6 to 10×10, the students hold up both hands, with palms facing the student. On each hand, thumb to pinkie, respectively, represent 6 through 10. To multiply x times y, where $6 \le x \le 10$ and $6 \le y \le 10$, we use the following algorithm:

- We touch the left hand's finger that corresponds to the multiplier to the right hand's finger that represents the multiplicand. ➔

- We bend down the fingers below the two that are touching, that is, those fingers that represent numbers greater than those that we are multiplying. (*Below* is used in this manner throughout the remainder of this article.)
- We multiply the total number of unbent fingers by 10.
- We multiply the number of bent fingers on the left hand by the number of bent fingers on the right hand.
- The sum of the previous two results is the product of x and y.

We can expand the European peasant technique for larger products

Figure 3 illustrates 9×8. We next prove why this algorithm works.

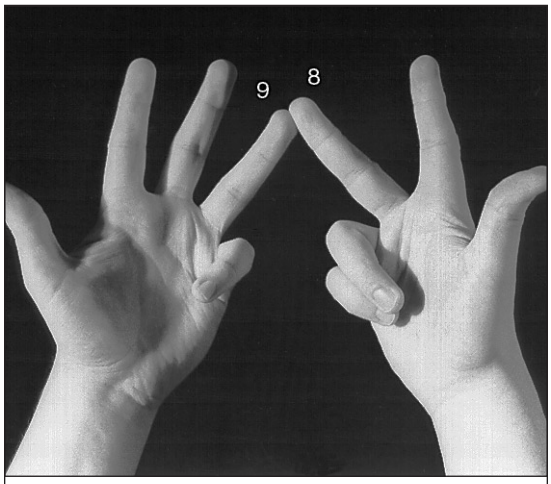

Fig. 3
$9 \times 8 = 7 \times 10 + 1 \times 2$
$= 72.$

PROOF: If $6 \le x \le 10$ and $6 \le y \le 10$, the previous algorithm gives the product xy. To begin, we verify that if the finger that represents x on the left hand is touching the finger that represents y on the right hand, then we can algebraically label—

- $(x - 5)$ as the number of unbent fingers on the left hand,
- $(y - 5)$ as the number of unbent fingers on the right hand,
- $(10 - x)$ as the number of bent fingers on the left hand, and
- $(10 - y)$ as the number of bent fingers on the right hand.

Following the previous algorithm, using the algebraic labels, and simplifying, we find that the total number of unbent fingers times 10 plus the product of the number of bent fingers on each hand is equal to

$$10[(x - 5) + (y - 5)] + (10 - x)(10 - y)$$
$$= 10[x + y - 10] + (10 - x)(10 - y)$$
$$= 10x + 10y - 100 + 100 - 10y - 10x + xy$$
$$= xy,$$

which is the product of the numbers.

Students can follow specific instances of the proof and can thus practice the algorithm by counting on their fingers.

We can expand the European peasant technique for even larger products. However, a restriction exists. The new techniques work only with products that result when both multiplier and multiplicand belong to the same set of five consecutive whole numbers, for example, the product xy, where $11 \le x \le 15$ and $11 \le y \le 15$. Two cases arise. In case 1, the units digit of both numbers ranges from 1 through 5; and in case 2, the units digit of both numbers ranges from 6 through 0. The European peasant multiplication just developed was an example of case 2. We next discuss an algorithm for a case 1 example, the digits from 11×11 to 15×15, followed by a proof showing why it works.

11×11 to 15×15 (case 1)

The students hold up both hands, with palms facing the student. On each hand, thumb to pinkie, respectively, represent 11 through 15. To multiply x times y, where $11 \le x \le 15$ and $11 \le y \le 15$, we use the following algorithm, which differs from the one that we previously used.

- We touch the left hand's finger that corresponds to the multiplier to the right hand's finger that represents the multiplicand.
- We bend down the fingers below the two that are touching.
- We multiply the total number of unbent fingers by 10.
- We multiply the number of unbent fingers on the left hand by the number of unbent fingers on the right hand.
- We add 100 to the sum of the two previous results.
- The answer to the previous step is the product of x and y.

Figure 4 illustrates 13×12. We next prove why this algorithm works.

PROOF: If $11 \le x \le 15$ and $11 \le y \le 15$, the previous algorithm gives the product xy. To begin, we verify that if the finger that represents x on the left hand is touching the finger that represents y on the right hand, then we can algebraically label $(x - 10)$ as the number of unbent fingers on the left hand and $(y - 10)$ as the number of unbent

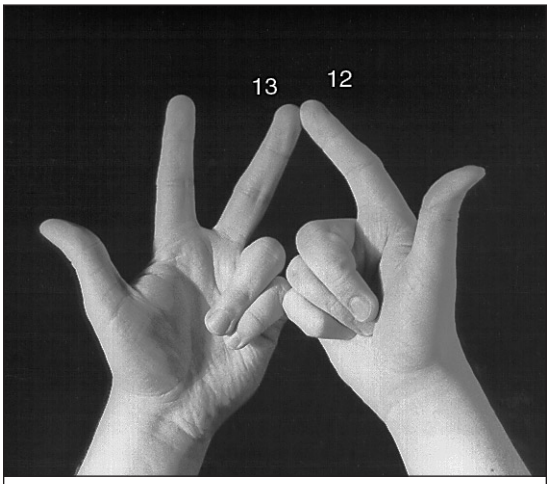

Fig. 4
$13 \times 12 = 5 \times 10 + 3 \times 2 + 100$
$= 156.$

fingers on the right hand. For this algorithm, we do not use the number of bent fingers on each hand.

Following the previous algorithm, using the algebraic labels, and simplifying, we find that the total number of unbent fingers times 10 plus the product of the number of unbent fingers on each hand plus 100 equals

$$\begin{aligned} &10[(x-10)+(y-10)] + (x-10)(y-10) &&+ 100 \\ &= 10[x+y-20] &&+ (x-10)(y-10) &&+ 100 \\ &= 10x + 10y - 200 + xy - 10x - 10y &&+ 100 + 100 \\ &= xy, \end{aligned}$$

which is the product of the numbers.

Students can follow specific instances of the proof and can practice the algorithm by counting on their fingers. We now consider the next set of five consecutive whole numbers, 16×16 to 20×20, a case 2 example.

16×16 to 20×20 (case 2)

Students hold up both hands, palms facing the student. On each hand, thumb to pinkie, respectively, represent 16 through 20. To multiply x times y, where $16 \le x \le 20$ and $16 \le y \le 20$, we use the following algorithm, which differs from the previous algorithms.

- We touch the left hand's finger that corresponds to the multiplier to the right hand's finger that corresponds to the multiplicand.
- We bend down the fingers below the two that are touching.
- We multiply the total number of unbent fingers by 20.
- We multiply the number of bent fingers on the left hand by the number of bent fingers on the right hand.
- We add 200 to the sum of the two previous results.
- The answer to the previous step is the product of x and y.

Figure 5 illustrates 16×19. We next prove why this algorithm works.

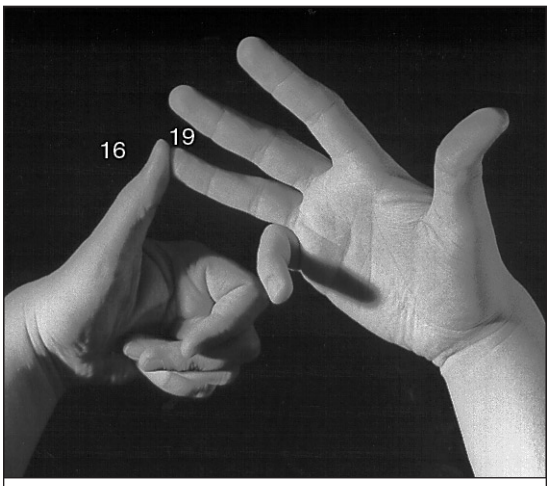

Fig. 5
$16 \times 19 = 5 \times 20 + 4 \times 1 + 200$
$= 304.$

PROOF: If $16 \le x \le 20$ and $16 \le y \le 20$, the previous algorithm gives the product xy. To begin, we verify that if the finger that represents x on the left hand is touching the finger that represents y on the right hand, then we can algebraically label—

- $(x - 15)$ as the number of unbent fingers on the left hand,
- $(y - 15)$ as the number of unbent fingers on the right hand,
- $(20 - x)$ as the number of bent fingers on the left hand, and
- $(20 - y)$ as the number of bent fingers on the right hand.

Following the previous algorithm, using the algebraic labels, and simplifying, we find that the total number of unbent fingers times 20 plus the product of the number of bent fingers on each hand plus 200 equals

$$\begin{aligned} &20[(x-15)+(y-15)] + (20-x)(20-y) &&+ 200 \\ &= 20[x+y-30] &&+ (20-x)(20-y) &&+ 200 \\ &= 20x + 20y - 600 + 400 - 20y - 20x + xy &&+ 200 \\ &= xy, \end{aligned}$$

which is the product of the numbers. ➔

Students can practice the algorithm by counting on their fingers

Students can follow specific instances of the proof and can thus practice the algorithm by counting on their fingers.

Using similar approaches, we can extend the finger-multiplication process to still larger numbers in which the multiplier and multiplicand belong to the same set of five consecutive whole numbers (case 1, case 2). Before developing the generalization of each case, the teacher should determine whether the students can derive the algorithm on their own for the case 1 example 21×21 to 25×25 and then prove why it works. They should use the previous case 1 algorithm and proof as their guide. The generalizations of each case are presented below, with proof.

THE GENERAL ALGORITHM

Discovering the general algorithm behind each of the preceding cases is a challenging lesson for students. The process seems to have a pattern, and definite differences occur in the algorithms for sets ending with units digit 5 (case 1) and sets ending with units digit 0 (case 2). To explore this result, we redefine case 1 and 2.

We let d be a nonnegative multiple of ten. Then we reexamine each case, as follows:

Case 1

The five consecutive whole numbers start at units digit 1 and end at units digit 5. In general, this set is $d + 1$, $d + 2$, $d + 3$, $d + 4$, and $d + 5$. For example, if $d = 20$, then the set is 21, 22, 23, 24, 25. We call d the *decade* for this set. In the previous example, the set's decade is 20. The teacher can ask the students to review the case 1 algorithms and determine whether they can discover a general algorithm in terms of the set's decade.

Discovering the general algorithm behind each of the preceding cases is a challenging lesson for students

Case 2

The five consecutive whole numbers start at units digit 6 and end at units digit 0. In general, this set is $d + 6$, $d + 7$, $d + 8$, $d + 9$, and $d + 10$. We define this set's *low decade* to be d and its *high decade* to be $d + 10$. For example, if $d = 10$, then the set is 16, 17, 18, 19, 20, with a low decade of 10 and a high decade of 20. If $d = 0$, then the set is 6, 7, 8, 9, 10, with a low decade of 0 and a high decade of 10. The students should review the case 2 algorithms and determine whether they can discover a general algorithm in terms of the set's low decade and high decade.

THE GENERAL CASE 1 AND CASE 2 ALGORITHMS

In both cases, if x represents the multiplier (left hand) and y represents the multiplicand (right hand), we can verify the following by counting on our fingers:

- The number of unbent fingers on the left hand is x minus one less than the lowest number in the set. For case 1, it is $x - d$; for case 2, it is $x - (d + 5)$.
- The number of unbent fingers on the right hand is y minus one less than the lowest number in the set. For case 1, it is $x - d$; for case 2, it is $y - (d + 5)$.
- The number of bent fingers on the left hand is the highest number in the set minus x. For case 1, it is $d + 5 - x$; for case 2, it is $d + 10 - x$.
- The number of bent fingers on the right hand is the highest number in the set minus y. For case 1, it is $d + 5 - y$; for case 2, it is $d + 10 - y$.

We next consider the case 1 general algorithm and proof.

Case 1 general algorithm and proof

The set begins with units digit 1 and ends in units digit 5. Then the set is $d + 1$, $d + 2$, $d + 3$, $d + 4$, $d + 5$, with decade d.

- We multiply the total number of unbent fingers by the set's decade.
  $d[(x - d) + (y - d)]$
- We multiply the number of unbent fingers on each hand.
  $(x - d)(y - d)$
- We square the set's decade.
  d^2
- The sum of the previous three results gives the product xy:

$$d[(x - d) + (y - d)] + (x - d)(y - d) + d^2$$
$$= dx + dy - 2d^2 + xy - dx - dy + d^2 + d^2$$
$$= xy$$

Students should verify that this algorithm generalizes the case 1 algorithms.

Students can use a similar approach to supply the proof and to verify that the following algorithm generalizes the case 2 algorithms.

Case 2 general algorithm

The set begins with units digit 6 and ends in units digit 0. Then the set is $d + 6$, $d + 7$, $d + 8$, $d + 9$, $d + 10$, with low decade d and high decade $d + 10$.

- We multiply the total number of unbent fingers by the set's high decade.
- We multiply the number of bent fingers on each hand.
- We find the product of the set's low decade and its high decade.
- The sum of the previous three results gives the product xy.

QUESTIONS AND PROJECTS

From classroom experience, I have found that many extension activities arise naturally when presenting this material. My students have been highly motivated to work on the following questions and projects:

1. Develop algorithms and proofs for 26×26 to 30×30 and for 31×31 to 35×35.
2. Find one general algorithm that covers both case 1 and case 2.
3. Verify that 1×1 to 5×5 fits into the general algorithm for case 1. Hint: The set's decade is 0.
4. Tell how the commutative property of multiplication is implied in the algorithms.
5. Describe how you would you handle a problem such as 22×58.
6. Find algorithms for such sets of five numbers as 23, 24, 25, 26, 27 that do not fit into case 1 or case 2.
7. Write a computer program, with graphics if possible, that simulates the two multiplication techniques.
8. Research the history of finger-computing techniques, including the one used by Jaime Escalante in the movie *Stand and Deliver.*
9. Research Vedic multiplication on the Internet, and compare it with finger-multiplication techniques. Striking similarities exist between the Vedic algorithm called *vertically and crosswise* and European peasant multiplication. Two excellent Web sites are www.vedicmaths.org /group_files/tutorial/Tutorial%20menus.htm and members.aol.com/vedicmaths/vm.htm.
10. Research the Korean method of finger computation called *chisenbop.* A good Web site is klingon .cs.iupui.edu/~aharris/chis/chis.html. Discuss this method's similarities with the algorithms discussed in this article.
11. Research the finger-arithmetic techniques of Mohammad Abu'l-Wafa Al-Buzjani (940–998). A good Web site is www-groups.dcs.st-and.ac.uk /~history/ Mathematicians/Abu'l-Wafa.html.
12. Develop finger-multiplication algorithms that are based on six-fingered hands.
13. Look up the origin of the word *digit,* and tell how its origin relates to finger computation.

CONCLUSION

The two different finger-multiplication methods discussed in this article truly provide hands-on activities for students at the secondary and college levels. Not only are these methods fun to practice, but they also lead to an investigation of why they work. Students practice algebraic concepts, number theory, inductive thinking, and deductive proof. Additionally, many interesting follow-up activities, including learning the history of finger-reckoning techniques, are possible. Counting on one's fingers is sometimes OK, especially when students let their fingers do the multiplying.

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The author would like to thank Susan Cisco for preparing all figures used in this article, as well as the author's colleagues Gary Massion and Peter Stathis for their suggestions.