Homotopy Equivalence of Hilbert *C**-modules

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Abstract

In this paper we introduce the concept of homotopy equivalence for Hilbert *C**-modules and investigate some properties of this equivalence relation. We then present the homotopy equivalence in the context of Fredholm operators on Hilbert *C**-modules and classify these operators in terms of their index.

Keywords: Hilbert *C**-module; Homotopy equivalence; Fredholm operator.

1. Introduction

One of the main ideas of algebraic topology is to consider two spaces to be equivalent if they have 'the same shape' in a sense that is much broader than homeomorphism. Homotopy theory studies topological objects up to homotopy equivalence. Let X be a topological space. Then $x, y \in X$ are *homotopy* in X, denoted $x \sim_h y$, if there exists a continuous map $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$. Two topological spaces X and Y are called *homotopy equivalent* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that

$$g \circ f : X \to X \quad \text{and} \quad f \circ g : Y \to Y$$

are homotopic to the corresponding identities I_X and I_Y. Homotopy equivalence is a weaker relation than topological equivalence, i.e. homotopy classes of spaces are larger than homeomorphism classes.

It is interesting to point out that in order to define the homotopy equivalence, a relation between *C**-algebras. Basic homotopy theory for *C**-algebras can be developed in an analogous way to the homotopy theory for topological spaces, using the Gelfand-Naimark duality between pointed compact Hausdorff spaces and abelian *C**-algebras. All the below concepts are the non-commutative generalizations of the usual topological ones. First, we need to consider a certain relation between morphisms as follows:

Let $\mathcal{A}$ and $\mathcal{B}$ be *C**-algebras. Two morphisms $\phi, \psi : \mathcal{A} \to \mathcal{B}$ are said to be *homotopic*, denoted $\phi \sim_h \psi$, if there exist $*$-homomorphisms $\gamma_t : \mathcal{A} \to \mathcal{B}, 0 \le t \le 1$, such that $\gamma_0 = \phi$ and $\gamma_1 = \psi$ and for every fixed element $a \in \mathcal{A}$, the map $[0, 1] \to \mathcal{B}, t \to \gamma_t(a)$, is continuous from the usual topology on $[0, 1]$ to the norm topology on $\mathcal{B}$. In other words, there should exist $*$- homomorphisms which give norm continuous paths from $\phi(a)$ to $\psi(a)$, for every $a \in \mathcal{A}$.

Now, we can define the concept of homotopy for *C**-algebras. Two *C**-algebras $\mathcal{A}$ and $\mathcal{B}$ are *homotopy equivalent* if there exist $*$-homomorphisms $\phi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{A}$ such that $\psi \circ \phi$ is homotopic to $I_{\mathcal{A}}$ and $\phi \circ \psi$ is homotopic to $I_{\mathcal{B}}$.

In the next section, we will introduce the concept of homotopy equivalence of Hilbert *C**-modules and check some properties of this relation. For this end, let us recall some elementary notations of Hilbert *C**-modules.

The concept of Hilbert modules was introduced by (Paschke, 1973) and (Rieffel, 1974) for the first time in a non-commutative context. Hilbert modules are a straightforward generalization of Hilbert spaces where the scalar field $\mathbb{C}$ is replaced by a *C**-algebra. The origin of Hilbert modules is in operator theory, where they constitute an important tool in areas like K-theory, quantum groups and several other areas.

A (right) Hilbert *C**-module E over a *C**-algebra $\mathcal{A}$ (or a Hilbert $\mathcal{A}$-module) is a linear space that is also a right $\mathcal{A}$-module, equipped with an $\mathcal{A}$-valued inner product $\langle ., . \rangle$ that is $\mathbb{C}$- and $\mathcal{A}$-linear in the second variable and conjugate linear in the first variable such that E is complete with the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$.

If the closed bilateral $*$-sided ideal $\langle E, E \rangle$ of $\mathcal{A}$ generated by $\{\langle x, y \rangle; x, y \in E\}$ coincides with $\mathcal{A}$, we say that E is *full*. We denote by $\mathcal{L}_{\mathcal{A}}(E)$ the *C**-algebra of all adjointable operators on E (i.e. of all maps $T : E \to E$ such that there exists $T^* : E \to E$ with the property $\langle Tx, y \rangle = \langle x, T^*y \rangle$, for all $x, y \in E$).

Given elements $x, y \in E$, we define $\theta_{x,y} : E \to E$ by $\theta_{x,y}(z) = x\langle y, z \rangle$ for each $z \in E$, then $\theta_{x,y} \in \mathcal{L}_{\mathcal{A}}(E)$, with $\theta_{x,y}^* = \theta_{y,x}$. The closure of the span of $\{\theta_{x,y} : x, y \in E\}$ in $\mathcal{L}_{\mathcal{A}}(E)$ is denoted by $\mathcal{K}_{\mathcal{A}}(E)$ and elements from this set will be called compact

operators.

A *morphism of Hilbert C*-modules* from a Hilbert C^*-module E over $\mathcal{A}$ to a Hilbert C^*-module F over $\mathcal{B}$ is a map $\Phi : E \rightarrow F$ with the property that there is a C^*-morphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$ such that

$$\langle \Phi(x), \Phi(y) \rangle = \phi(\langle x, y \rangle)$$

for all $x, y \in E$. Two Hilbert C^*-modules E and F, respectively, over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* if there is a bijective map $\Phi : E \rightarrow F$ such that Φ and Φ^{-1} are morphisms of Hilbert C^*-modules. The basic theory of Hilbert C^*-modules can be found in (Lance, 1995).

2. Homotopy equivalence of Hilbert *C*-modules*

We begin this section with the following essential definitions and then some properties will be presented.

Definition 1. Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras and E and F be two Hilbert C^*-modules over $\mathcal{A}$ and $\mathcal{B}$, respectively. Two morphisms $\Phi, \Psi : E \rightarrow F$ are said to be *homotopic*, denoted by $\Phi \sim_h \Psi$, if there exist morphisms $\Gamma_t : E \rightarrow F, 0 \leq t \leq 1$, such that $\Gamma_0 = \Phi$ and $\Gamma_1 = \Psi$ and the map $[0, 1] \ni t \mapsto \Gamma_t(x)$ is norm continuous for each $x \in E$.

Definition 2. Two Hilbert C^*-modules E and F, respectively, over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are *homotopy* equivalent, if there exist morphisms $\Phi : E \rightarrow F$ and $\Psi : F \rightarrow E$ such that $\Psi o \Phi \sim_h I_E$ and $\Phi o \Psi \sim_h I_F$.

It is not difficult to check that the above relations are equivalence relations. Now, we state and prove some propositions for homotopy equivalence on Hilbert modules. The next proposition shows that homotopy equivalence of Hilbert C^*-modules is a weaker relation than module isomorphism.

Proposition 3. Let E and F be two Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. If E and F are isomorphic, then $E \sim_h F$.

Proof. Since E and F are isomorphic, there exists a bijective map $\Omega : E \rightarrow F$ such that Ω and Ω^{-1} are morphisms. Now put $\Phi = \Omega$ and $\Psi = \Omega^{-1}$. Then $\Psi o \Phi = I_E \sim_h I_E$ and similarly $\Phi o \Psi = I_F \sim_h I_F$. Hence $E \sim_h F$. $\square$

Proposition 4. Let E and F be two full Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. If $\Phi, \Psi : E \rightarrow F$ are morphisms corresponding to C^*-morphisms $\phi, \psi : \mathcal{A} \rightarrow \mathcal{B}$, respectively, and $\Phi \sim_h \Psi$, then $\phi \sim_h \psi$.

Proof. Since $\Phi \sim_h \Psi$, there exist morphisms $\Gamma_{t \in [0,1]} : E \rightarrow F$ such that $\Gamma_0 = \Phi$ and $\Gamma_1 = \Psi$. Now, consider the C^*-morphisms $\gamma_{t \in [0,1]} : \mathcal{A} \rightarrow \mathcal{B}$ corresponding to $\Gamma_{t \in [0,1]}$. By definition, for every $x, y \in E$, we have:

$$\gamma_t \langle x, y \rangle = \langle \Gamma_t(x), \Gamma_t(y) \rangle,$$

so

$$\phi(\langle x, y \rangle) = \langle \Phi(x), \Phi(y) \rangle = \langle \Gamma_0(x), \Gamma_0(y) \rangle = \gamma_0(\langle x, y \rangle).$$

Since E is a full Hilbert $\mathcal{A}$-module, we conclude that $\gamma_0 = \phi$. By the same argument, we obtain that $\gamma_1 = \psi$. To complete the proof, it is enough to show that the map $t \mapsto \gamma_t(a)$ is continuous for each $a \in \mathcal{A}$. But both continuity of $(\Gamma_t)_{t \in [0,1]}$ and

$$\begin{aligned}
&\|\gamma_t \langle x, y \rangle - \gamma_{t'} \langle x, y \rangle\|_{\mathcal{B}} \\
&= \|\langle \Gamma_t(x), \Gamma_t(y) \rangle - \langle \Gamma_{t'}(x), \Gamma_{t'}(y) \rangle\|_{\mathcal{B}} \\
&= \|\langle \Gamma_t(x), \Gamma_t(y) \rangle - \langle \Gamma_{t'}(x), \Gamma_t(y) \rangle + \langle \Gamma_{t'}(x), \Gamma_t(y) \rangle - \langle \Gamma_{t'}(x), \Gamma_{t'}(y) \rangle\|_{\mathcal{B}} \\
&\leq \|\langle \Gamma_t(x) - \Gamma_{t'}(x), \Gamma_t(y) \rangle\|_{\mathcal{B}} + \|\langle \Gamma_{t'}(x), \Gamma_t(y) - \Gamma_{t'}(y) \rangle\|_{\mathcal{B}} \\
&\leq \|\Gamma_t(x) - \Gamma_{t'}(x)\|_{\mathcal{B}} \|\Gamma_t(y)\|_{\mathcal{B}} + \|\Gamma_t(y) - \Gamma_{t'}(y)\|_{\mathcal{B}} \|\Gamma_{t'}(x)\|_{\mathcal{B}},
\end{aligned}$$

imply that $(\gamma_t)_{t \in [0,1]}$ is a continuous path from ϕ to ψ and therefore $\phi \sim_h \psi$. $\square$

Corollary 5. Let E and F be two full Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. If $E \sim_h F$, then $\mathcal{A} \sim_h \mathcal{B}$.

Proof. Since $E \sim_h F$, there exist morphisms $\Phi : E \rightarrow F$ and $\Psi : F \rightarrow E$ such that $\Psi o \Phi \sim_h I_E$ and $\Phi o \Psi \sim_h I_F$. Now from proposition 4, we conclude $\psi o \phi \sim_h I_{\mathcal{A}}$ and $\phi o \psi \sim_h I_{\mathcal{B}}$. Hence $\mathcal{A} \sim_h \mathcal{B}$. $\square$

Remark 6. It is well known that any C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module in a natural way, so if $\mathcal{A} \sim_h \mathcal{B}$ as two Hilbert modules, then they are homotopy equivalent as C^*-algebras. Conversely, if the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are homotopy equivalent as C^*-algebras, then they are homotopy equivalent as Hilbert modules since it is enough to consider the C^*-morphisms as module morphisms between $\mathcal{A}$ and $\mathcal{B}$.

Example 7. Let X be a contractible, compact and Hausdorff space and E be a Hilbert module over C^*-algebra $\mathcal{A}$. Recall that $C(X, E)$ has a $C(X, \mathcal{A})$ module structure by the action $(f.F)(x) = f(x)F(x)$ and the inner product

$$\big(\langle f, g \rangle\big)(x) = \langle f(x), g(x) \rangle$$

for every $x \in X$, $f, g \in C(X, E)$ and $F \in C(X, \mathcal{A})$. We show that $C(X, E)$ and E are homotopy equivalent as two Hilbert modules over C^*-algebras $C(X, \mathcal{A})$ and $\mathcal{A}$, respectively.

First, assume that X is contractible. So, by definition, there exist $x_0 \in X$ and continuous maps $\alpha_t : X \to X$ such that $\alpha_0(x) = x_0$ and $\alpha_1(x) = x$ for every $x \in X$.

Define, for each $t \in [0, 1]$, the map $\Phi_t : C(X, E) \to C(X, E)$ by $\Phi_t(f) = f \ o \ \alpha_t$. Then, obviously Φ_t are module morphisms, because by considering $*$-homomorphisms $\phi_t : C(X, \mathcal{A}) \to C(X, \mathcal{A})$ which are defined by $\phi_t(F) = F \ o \ \alpha_t$, for every $F \in C(X, \mathcal{A})$, we have

$$
\begin{aligned}
\langle \Phi_t(f), \Phi_t(g) \rangle(x) &= \langle f \ o \ \alpha_t, g \ o \ \alpha_t \rangle(x) \\
&= \langle f \ o \ \alpha_t(x), g \ o \ \alpha_t(x) \rangle \\
&= \langle f, g \rangle(\alpha_t(x)) \\
&= \phi_t(\langle f, g \rangle)(x).
\end{aligned}
$$

Furthermore $\Phi_0(f)(x) = f(x_0)$ and $\Phi_1(f)(x) = f(x) = I_{C(X,E)}(x)$. Moreover it is easy to check that the map $t \mapsto \Phi_t(f)$ is continuous for each $f \in C(X, E)$. This shows that, $\Phi_0 \sim_h I_{C(X,E)}$ as two module morphisms.

Now, define two module morphisms $\Phi : C(X, E) \to E$ and $\Psi : E \to C(X, E)$ by $\Phi(f) := f(x_0)$ and $\Psi(e) := f_e$, respectively, where $f_e(x) = e$ is the constant map. Then

$$\Phi \ o \ \Psi(e) = \Phi(f_e) = f_e(x_0) = e = I_E.$$

On the other hand

$$\Psi \ o \ \Phi(f) = \Psi(f(x_0)) = f_{f(x_0)}(x) = f(x_0) = \Phi(f),$$

so $\Psi \ o \ \Phi \sim_h I_{C(X,E)}$ and hence the result holds.

Corollary 8. Let X be a contractible, compact and Hausdorff space. Then

- $C(X, \mathcal{A}) \sim_h \mathcal{A}$ for any C^*-algebra $\mathcal{A}$.

- Let E and F be two Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. If $E \sim_h F$, then $C(X, E) \sim_h C(X, F)$.

Given two Hilbert C^*-modules E and F over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, recall that the exterior tensor product $E \otimes F$ of E and F is a Hilbert C^*-module over the injective tensor product $\mathcal{A} \otimes \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$. Also, the morphism $\Phi_1 \otimes \Phi_2$ acts on $E_1 \otimes E_2$ as $(\Phi_1 \otimes \Phi_2)(x \otimes y) = \Phi_1(x) \otimes \Phi_2(y)$ for each x and y in E_1 and E_2, respectively (see Lance, 1995). Using these concepts, we have the following proposition.

Proposition 9. Let E_1, E_2, F_1 and F_2 be four Hilbert C^*-modules over C^*-algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1$ and $\mathcal{B}_2$, respectively. If $E_1 \sim_h F_1$ and $E_2 \sim_h F_2$, then $E_1 \otimes E_2 \sim_h F_1 \otimes F_2$.

Proof. Since $E_i \sim_h F_i, (i = 1, 2)$, there exist morphisms $\Phi_i : E_i \to F_i$ and $\Psi_i : F_i \to E_i$ such that $\Psi_i o \Phi_i \sim_h I_{E_i}$ and $\Phi_i o \Psi_i \sim_h I_{F_i}$. Hence $(\Psi_1 o \Phi_1) \otimes (\Psi_2 o \Phi_2) \sim_h I_{E_1} \otimes I_{E_2}$. But the morphisms $(\Psi_1 o \Phi_1) \otimes (\Psi_2 o \Phi_2)$ and $(\Psi_1 \otimes \Psi_2) o (\Phi_1 \otimes \Phi_2)$ are equal. Moreover the morphisms $I_{E_1} \otimes I_{E_2}$ and $I_{E_1 \otimes E_2}$ are equal too. From these facts, we conclude that $(\Psi_1 \otimes \Psi_2) o (\Phi_1 \otimes \Phi_2) \sim_h I_{E_1 \otimes E_2}$. In the same manner, one can show that $(\Phi_1 \otimes \Phi_2) o (\Psi_1 \otimes \Psi_2) \sim_h I_{F_1 \otimes F_2}$. Therefore $E_1 \otimes E_2 \sim_h F_1 \otimes F_2$. $\square$

3. Fredholm Operators and Homotopy Equivalence

We shall now study homotopy equivalence between Fredholm operators on Hilbert C^*-modules. As before, we start by reminding some basic notations. More related discussion can be found in (Exel, 1993).

Definition 10. Let E and F be two Hilbert C^*-modules over C^*-algebra $\mathcal{A}$ and let T be in $\mathcal{L}_{\mathcal{A}}(E, F)$. Suppose there is S in $\mathcal{L}_{\mathcal{A}}(F, E)$ such that $I_E - ST$ is in $\mathcal{K}_{\mathcal{A}}(E)$ and $I_F - TS$ is in $\mathcal{K}_{\mathcal{A}}(F)$. Then T is said to be a *Fredholm* operator. If T is a Fredholm operator, then the Fredholm index of T is defined by

$$ind(T) = rank(Ker(T)) - rank(Ker(T^*)).$$

If T is a Fredholm operator, then the Fredholm index of T is an element of $K_0(\mathcal{A})$, indeed for any α in $K_0(\mathcal{A})$ there is a Fredholm operator T such that $ind(T) = \alpha$. Recall that if $\mathcal{A}$ is a unital C^*-algebra, $\mathcal{P}_n(\mathcal{A}) = \mathcal{P}(M_n(\mathcal{A}))$ (the set of all projections of matrix algebra) and $\mathcal{P}_\infty(\mathcal{A}) = \bigcup_n \mathcal{P}_n(\mathcal{A})$, then $K_0(\mathcal{A})$ is defined as Grothendieck group $G(\mathcal{D}(\mathcal{A}))$ where $\mathcal{D}(\mathcal{A})$ is the abelian semigroup $(\mathcal{P}_\infty(\mathcal{A})/ \sim_\circ, \oplus)$ and

$$p \sim_\circ q \quad \text{iff} \quad \exists \; v \in M_{m,n}(\mathcal{A}), \quad p = v^*v \; \text{ and } \; q = vv^*.$$

For more details about K-theory for C^*-algebras, see (Wegge-Olsen, 1993).

Elementary properties of the Fredholm index are collected in the next proposition.

Proposition 11. (Exel, 1993) If T in $\mathcal{L}_\mathcal{A}(E, F)$ is a Fredholm operator, then

- $ind(T) = -ind(T^*)$

- If $U \in \mathcal{L}_\mathcal{A}(X, E)$ and $V \in \mathcal{L}_\mathcal{A}(Y, F)$ are invertible, then $ind(VTU) = ind(T)$.

- If $T' \in \mathcal{L}_\mathcal{A}(E, F)$ is such that $T' - T \in \mathcal{K}_\mathcal{A}(E, F)$, then T' is also Fredholm and $ind(T') = ind(T)$.

- If $S \in \mathcal{L}_\mathcal{A}(F, E)$ is such that $I_E - ST \in \mathcal{K}_\mathcal{A}(E)$, then $ind(S) = -ind(T)$.

- If $T_1 \in \mathcal{L}_\mathcal{A}(E_1, F_1)$ is Fredholm, then $ind(T \oplus T_1) = ind(T) + ind(T_1)$.

Proposition 12. (Exel, 1993) If $T_1 \in \mathcal{L}_\mathcal{A}(E, F)$ and $T_2 \in \mathcal{L}_\mathcal{A}(F, P)$ are two Fredholm operators, then T_2T_1 is Fredholm and $ind(T_2T_1) = ind(T_2) + ind(T_1)$.

One can also introduce homotopy equivalence for Fredholm operators as follows. Using this relation, we will be able to classify Fredholm operators in terms of their index.

Definition 13. If $T, S \in \mathcal{L}_\mathcal{A}(E, F)$ be two Fredholm operators, we say that they are *homotopic*, denoted $T \sim_h S$, if there exists a norm continuous path from T to S consisting of Fredholm operators.

Before expressing the main proposition, we need the following two useful lemmas, the first ones shows that the index map is locally constant and continuous in norm.

Lemma 14. (Exel, 1993) Let T in $\mathcal{L}_\mathcal{A}(E, F)$ be a Fredholm operator. Then There is a positive real number ϵ such that any T' satisfying $\|T' - T\| < \epsilon$ is also Fredholm with $ind(T') = ind(T)$.

Lemma 15. (Exel, 1993) Let $T \in \mathcal{L}_\mathcal{A}(E, F)$ be a Fredholm operator with $ind(T) = 0$. then there exists an integer n such that $T \oplus I_{\mathcal{A}^n} : E \oplus \mathcal{A}^n \to F \oplus \mathcal{A}^n$ is an compact perturbation of an invertible operator.

We are now in a position to state and prove our main proposition.

Proposition 16. Let E and F be two Hilbert $\mathcal{A}$-modules and $T, S \in \mathcal{L}_\mathcal{A}(E, F)$ be Fredholm operators. Then T and S are homotopic if and only if they have the same index.

Proof. Suppose that T and S are homotopic Fredholm operators, and let $t \mapsto V_t$ be a continuous path of Fredholm operators from S to T. Then by lemma 14, the map $t \mapsto ind(V_t)$ is continuous and hence constant.

To show the converse, we first observe that every Fredholm operator U with $ind(U) = 0$ is homotopic to identity. Indeed, since $ind(U) = 0$, by lemma 15, there is an invertible operator V such that $K = V - U$ is compact. Then the norm continuous path $t \mapsto U + tK, t \in [0, 1]$, consists of Fredholm operators and connects U to V. Now, by (Wegge-Olsen, 1993, Lemma 4.2.3), V is connected to $V|V|^{-1}$ where $|V|$ is the polar decomposition of V. Also, by using the Kuiper result (cf. Wegge-Olsen, 1993), $V|V|^{-1}$ can be connected to identity and hence U is homotopic to identity.

Now, suppose that $ind(S) = ind(T)$. Then both ST^* and T^*T have index 0 and thus are homotopic to identity. Consequently, the operators $S, S(T^*T) = (ST^*)T$ and T are homotopic. $\square$

The following two corollaries are direct consequences from propositions 11 and 16.

Corollary 17. Suppose that E and F are two Hilbert $\mathcal{A}$-modules. If $T, T' \in \mathcal{L}_\mathcal{A}(E, F)$ be two Fredholm operators such that $T - T' \in \mathcal{K}_\mathcal{A}(E, F)$, then $T \sim_h T'$.

Corollary 18. Let E and F be two Hilbert $\mathcal{A}$-modules, $T \in \mathcal{L}_\mathcal{A}(E, F)$ and $S \in \mathcal{L}_\mathcal{A}(F, E)$. If $I_E - ST \in \mathcal{K}_\mathcal{A}(E)$, then $S \sim_h T^*$.

Moreover we have the following important result.

Corollary 19. Let E_1, E_2, F_1 and F_2 be four Hilbert C^*-modules over a C^*-algebra $\mathcal{A}$ and $T_i \in \mathcal{L}_{\mathcal{A}}(E_i, F_i), (i = 1, 2)$ be two Fredholm operators. If $T_1 \sim_h T_2$, then for some integer n, the operator

$$T_1 \oplus T_2^* \oplus I_{\mathcal{A}^n} : E_1 \oplus F_2 \oplus \mathcal{A}^n \to F_1 \oplus E_2 \oplus \mathcal{A}^n$$

is a compact perturbation of an invertible operator.

Proof. Follows immediately from (Exel, 1993, Proposition 3.16) and proposition 16.□

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