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(α_o-λ_o)-Contractive Mapping in Multiplicative Metric Space and Fixed Point Results

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Abstract In this manuscript we introduce new type of contraction mapping in the framework of multiplicative metric space and some fixed point results. Also some example for the support of our constructed results.

Keywords: *complete multiplicative metric space, multiplicative contraction mapping, multiplicative (α_o-λ_o)-contraction, fixed point*

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1. Introduction and Preliminaries

The Banach-contraction principal was introduced by Banach [1]. It is one of the important results for metric fixed point theory and also vast applicability in mathematical analysis, like used to establish the existence of solution of integral equation. After Banach contraction mapping, a new type of contraction mapping was introduced by Kannan [5,6], which is known as Kannan-contraction. Many researcher work on the generalization and fixed point theory of Kannan-contraction mapping like in [4,8,9,11]. Like Kannan, Chatterjea [3] also introduced a similar contractive condition and fixed point theorems in metric space. After that, in 2008, a new concept of multiplicative distance was introduced by Bashirov [2].

Definition 1.1 *Let* $\mathcal{X}_0$ *be a non-empty set, then multiplicative metric is a mapping* $\mathcal{M}:\mathcal{X}_0\times\mathcal{X}_0\to\mathbb{R}$ *satisfying the following conditions*:

(1) $\mathcal{M}(x_0,y_0)\geq 1$ *for all* $x_0,y_0\in\mathcal{X}_0$,

(2) $\mathcal{M}(x_0,y_0)=1$ *if and only if* $x_0=y_0$,

(3) $\mathcal{M}(x_0,y_0)=\mathcal{M}(y_0,x_0)$,

(4) $\mathcal{M}(x_0,z_0)\leq\mathcal{M}(x_0,y_0).\mathcal{M}(y_0,z_0)$ *for all* $x_0,y_0,z_0\in\mathcal{X}_0$.

The pair $(\mathcal{X}_0,\mathcal{M})$ *is known as multiplicative metric space*.

Ozavsar and Cevikel [10] studied multiplicative metric space and its topological properties, they also introduce the concepts of Banach-contraction, Kannan-contraction and Chatterjea-contraction mappings in the framework of multiplicative metric space and proved fixed point results on complete multiplicative metric space.

Definition 1.2 [10] *Let* $(\mathcal{X}_0,\mathcal{M})$ *be a multiplicative metric space then the mapping* $T_0:\mathcal{X}_0\to\mathcal{X}_0$ *is multiplicative Banach-contraction if*

$$\mathcal{M}(T_0x_0,T_0y_0)\leq\mathcal{M}(x_0,y_0)^k, \tag{1.1}$$

for all $x_0,y_0\in\mathcal{X}_0$, *where* $k\in[0,1)$.

Definition 1.3 [10] *Let* $(\mathcal{X}_0,\mathcal{M})$ *be a multiplicative metric space then the mapping* $T_0:\mathcal{X}_0\to\mathcal{X}_0$ *is multiplicative Kannan-contraction if*

$$\mathcal{M}(T_0x_0,T_0y_0)\leq\left[\mathcal{M}(T_0x_0,x_0).\mathcal{M}(T_0y_0,y_0)\right]^k, \tag{1.2}$$

for all $x_0,y_0\in\mathcal{X}_0$, *where* $k\in\left[0,\frac{1}{2}\right)$.

Definition 1.4 [10] *Let* $(\mathcal{X}_0,\mathcal{M})$ *be a multiplicative metric space then the mapping* $T_0:\mathcal{X}_0\to\mathcal{X}_0$ *is multiplicative Chatterjea-contraction if*

$$\mathcal{M}(T_0x_0,T_0y_0)\leq\left[\mathcal{M}(T_0x_0,y_0).\mathcal{M}(T_0y_0,x_0)\right]^k, \tag{1.3}$$

for all $x_0,y_0\in\mathcal{X}_0$, *where* $k\in\left[0,\frac{1}{2}\right)$.

The concept of α_o-admissible mapping was introduced by B. Samet, C. Vetro and P. Vetro [7]:

Definition 1.5 *Suppose* $\mathcal{X}_0\neq\emptyset$, *and let* $\alpha_0:\mathcal{X}_0\times\mathcal{X}_0\to[0,\infty)$ *be a mapping. Then* $T_0:\mathcal{X}_0\to\mathcal{X}_0$ *is said to be* α_0*-admissible mapping if:*

for all $x_0,y_0\in\mathcal{X}_0$ *for which*

$$\alpha_o(x_0,y_0)\geq 1\Rightarrow\alpha_o(T_0x_0,T_0y_0)\geq 1.$$

2. Multiplicative (α_o, λ_o)-contraction and Fixed Point Results

Now we will introduce (α_o, λ_o)-contraction mapping in the framework of multiplicative metric space.

Let Ω_{T_0} be the class of functions for which $\lambda_o(T_0(x_0)) \leq \lambda_o(x_0)$ for all $x_0 \in \mathcal{X}_0$, where $\mathcal{X}_0 \neq \emptyset$, and $T_0 : \mathcal{X}_0 \to \mathcal{X}_0$ is self-mapping.

Definition 2.1 *Suppose $(\mathcal{X}_0, \mathcal{M})$ be a multiplicative metric space and let a mapping $T_0 : \mathcal{X}_0 \to \mathcal{X}_0$, then T_0 is said to be multiplicative (α_o, λ_o)-Banach-contraction if there exists $\alpha_o : \mathcal{X}_0 \times \mathcal{X}_0 \to [0,\infty)$ and $\lambda_o \in \Omega_{T_0}$ such that*

$$\alpha_o(x_0, y_0)\mathcal{M}(T_0x_0, T_0y_0) \leq \mathcal{M}(x_0, y_0)^{\lambda_o(x_0)}, \quad (2.1)$$

for all $x_0, y_0 \in \mathcal{X}_0$, where $\lambda_o \in [0,1)$.

Remark 2.2 *When $\alpha_o(x_0, y_0) = 1$ for all $x_0, y_0 \in \mathcal{X}_0$ and $\lambda_o(x) = k$ for all $x_0 \in \mathcal{X}_0$, where $k \in [0,1)$, then multiplicative (α_o, λ_o)-contraction mapping reduces to multiplicative Banach-contraction mapping.*

Example 2.3 *$(\mathcal{X}_0, \mathcal{M}_*)$ is multiplicative metric space, where $\mathcal{X}_0 = [0.1, \infty)$ and $\mathcal{M}_* : \mathcal{X}_0 \times \mathcal{X}_0 \to \mathbb{R}$ be defined as follows:*

$$\mathcal{M}_*(x_0, y_0) = \left|\frac{x_0}{y_0}\right|_*$$

for all $x_0, y_0 \in \mathcal{X}_0$, where $|.|_* : \mathcal{X}_0 \to \mathcal{X}_0$ is defined by

$$|d|_* = \begin{cases} d, & d \geq 1; \\ \frac{1}{d}, & d < 1, \end{cases} \quad (2.2)$$

we define the mapping $\alpha_o : \mathcal{X}_0 \times \mathcal{X}_0 \to [0,\infty)$ and $T_0 : \mathcal{X}_0 \to \mathcal{X}_0$ as follows:

$$\alpha_o(x_0, y_0) = \begin{cases} 1, & x_0, y_0 \in [0.1, 0.9]; \\ 0, & \text{otherwise}, \end{cases} \quad (2.3)$$

and

$$T_0x_0 = \begin{cases} e^{x_0 - 1 - \frac{x_0^3}{10}}, & x_0, y_0 \in [0.1, 0.9]; \\ \frac{2x_0 - 1}{3}, & x_0 \in (0.9, \infty). \end{cases} \quad (2.4)$$

for all $x_0, y_0 \in \mathcal{X}_0$, where $\lambda_o : \mathcal{X}_0 \to [0,1)$ is defined by $\lambda_o(x_0) = 0.67$. T_0 is multiplicative (α_o, λ_o)-Banach-contraction mapping.

NOTE. In the above example T_0 is not multiplicative Banach-contraction mapping: that is, for $x_0 = 3$ and $y_0 = 6$, we have

$$\mathcal{M}_*(T_0x_0, T_0y_0) = \left|\frac{T_0x_0}{T_0y_0}\right|_* = \left|\frac{5}{11}\right|_* > \left|\frac{3}{6}\right|_* = \mathcal{M}_*(x_0, y_0),$$

for all $\lambda_o \in [0,1)$.

So this mapping is said to be extension of multiplicative Banach-contraction mapping.

Now we prove some fixed point results for Multiplicative (α_o, λ_o)-Banach-contraction mapping.

Theorem 2.4 *Let $(\mathcal{X}_0, \mathcal{M})$ be a complete multiplicative metric space and assume that $T_0 : \mathcal{X}_0 \to \mathcal{X}_0$ be (α_o, λ_o)-Banach-contraction mapping satisfying the conditions:*

1. *there exist $\hat{x}_0 \in \mathcal{X}_0$ such that $\alpha_o(\hat{x}_0, T_0\hat{x}_0) \geq 1$;*
2. *T_0 is α_o-admissible;*
3. *one of the conditions holds;*

(a) *is continuous;*

(b) if a sequence $\{\hat{x}_n\} \in \mathcal{X}_0$ such that $\alpha_o(\hat{x}_n, \hat{x}_{n+1}) \geq 1$ for all $n \in \mathcal{N}$ and $\hat{x}_n \to x_0 \in \mathcal{X}_0$ as $n \to \infty$, then $\alpha_o(\hat{x}_n, x_0) \geq 1$.

Then T_0 has a fixed point.

(A1) If $\alpha_o(c,d) \geq 1$ for all fixed point $c, d \in \mathcal{X}_0$,

(A2) there exist $z_0 \in \mathcal{X}_0$ such that $\alpha_o(x_0, z_0) \geq 1$ and $\alpha_o(y_0, z_0) \geq 1$ for all $x_0, y_0 \in \mathcal{X}_0$,

then T_0 has a unique fixed point.

Proof. Assume $\hat{x}_0 \in \mathcal{X}_0$ such that $\alpha_o(\hat{x}_0, T_0\hat{x}_0) \geq 1$.

Define the sequence $\{\hat{x}_n\} \in \mathcal{X}_0$ such that for all $n \in \mathcal{N}$

$$\hat{x}_n = T_0\hat{x}_{n-1}. \quad (2.5)$$

Assume that $\hat{x}_n \neq \hat{x}_{n-1}$.

Since T_0 is α_o-admissible and $\alpha_o(\hat{x}_0, \hat{x}_1) = \alpha_o(\hat{x}_0, T_0\hat{x}_0) \geq 1$ and similarly by induction, we get

$$\alpha_o(\hat{x}_{n-1}, \hat{x}_n) \geq 1 \quad \text{for all } n \in \mathcal{N}.$$

Applying Inequality (2.1) with $x_0 = \hat{x}_0$ and $y_0 = \hat{x}_1$, we have

$$\begin{aligned}\mathcal{M}(\hat{x}_1, \hat{x}_2) &= \mathcal{M}(T_0\hat{x}_0, T_0\hat{x}_1), \\ &\leq \alpha_o(\hat{x}_0, \hat{x}_1)\mathcal{M}(T_0\hat{x}_0, T_0\hat{x}_1), \\ &\leq \mathcal{M}(\hat{x}_0, \hat{x}_1)^{\lambda_0(\hat{x}_0)}.\end{aligned}$$

Again, using inequality (2.1) with $x_0 = \hat{x}_1$ and $y_0 = \hat{x}_2$, we have

$$\mathcal{M}\left(\hat{x}_2,\hat{x}_3\right) = \mathcal{M}\left(T_0\hat{x}_1, T_0\hat{x}_2\right),$$
$$\leq \alpha_o\left(\hat{x}_1,\hat{x}_2\right)\mathcal{M}\left(T_0\hat{x}_1, T_0\hat{x}_2\right),$$
$$= \mathcal{M}\left(\hat{x}_1,\hat{x}_2\right)^{\lambda_o\left(\hat{x}_1\right)},$$
$$\leq \mathcal{M}\left(\hat{x}_1,\hat{x}_2\right)^{\lambda_o\left(T_0\hat{x}_0\right)},$$
$$\leq \mathcal{M}\left(\hat{x}_1,\hat{x}_2\right)^{\lambda_o\left(\hat{x}_0\right)},$$
$$\leq \mathcal{M}\left(\hat{x}_0,\hat{x}_1\right)^{\lambda_o\left(\hat{x}_0\right)^2}.$$

By continuing this process, we get

$$\mathcal{M}\left(\hat{x}_n,\hat{x}_{n-1}\right) \leq \mathcal{M}\left(\hat{x}_0,\hat{x}_1\right)^{\lambda_o\left(\hat{x}_0\right)^n}. \quad (2.6)$$

As $\lambda_o\left(\hat{x}_0\right) \in [0,1)$, we get $\left\{\hat{x}_n\right\}$ is multiplicative Cauchy sequence in $\mathcal{X}_0$.

From the completeness of $\mathcal{X}_0$, there exist $x^* \in \mathcal{X}_0$ such that $\hat{x}_n \to x^*$ as $n \to \infty$.

We suppose that T_0 is continuous from condition(3a), so

$$x^* = \lim_{n\to\infty} \hat{x}_{n+1} = \lim_{n\to\infty} T_0\hat{x}_n = T_0\left(\lim_{n\to\infty}\hat{x}_n\right)T_0x^*.$$

Now, we suppose that condition(3b) holds: As $\left\{\hat{x}_n\right\}$ is multiplicative Cauchy sequence. So, there exist $x^* \in \mathcal{X}_0$ such that $\hat{x}_n \to x^*$ as $n \to \infty$.

From condition(3b), we have

$$\alpha_o\left(\hat{x}_n, x^*\right) \geq 1 \quad \textit{for all } n \in \mathcal{N}.$$

And

$$\mathcal{M}\left(T_0x^*, x^*\right) \leq \mathcal{M}\left(T_0x^*, \hat{x}_n\right).\mathcal{M}\left(T_0\hat{x}_n, x^*\right),$$
$$= \mathcal{M}\left(T_0\hat{x}_n, x^*\right).\mathcal{M}\left(T_0\hat{x}_n, T_0x^*\right),$$
$$\leq \mathcal{M}\left(T_0\hat{x}_n, x^*\right).\alpha_o\left(\hat{x}_n, x^*\right)\mathcal{M}\left(T_0\hat{x}_n, T_0x^*\right),$$
$$\leq \mathcal{M}\left(\hat{x}_{n+1}, x^*\right).\mathcal{M}\left(\hat{x}_n, x^*\right)^{\lambda_o\left(\hat{x}_0\right)}.$$

As $\lambda_o\left(\hat{x}_n\right) \leq \lambda_o\left(\hat{x}_0\right)$ for all $n \in \mathcal{N}$. Therefore, we have

$$\mathcal{M}\left(T_0x^*, x^*\right) \leq \mathcal{M}\left(\hat{x}_{n+1}, x^*\right).\mathcal{M}\left(\hat{x}_n, x^*\right)^{\lambda_o\left(\hat{x}_0\right)}.$$

Assume that $n \to \infty$ in the above inequality, we get $\mathcal{M}\left(T_0x^*, x^*\right) = 1$, that is $x^* = T_0x^*$, which shows that x^* is fixed point of T_0.

To show uniqueness of x^*, let y^* is another fixed point of T_0, if condition(A1) holds, then the fixed point is unique from (2.1). Now we have to show that condition(A2) holds. From(A2), we have $z_0 \in \mathcal{X}_0$ such that

$$\alpha_o\left(x^*, z_0\right) \geq 1, \alpha_o\left(y^*, z_0\right) \geq 1. \quad (2.7)$$

As T_0 is α_o-admissible, from (2.7), we have

$$\alpha_o\left(x^*, T_0^n z_0\right) \geq 1, \alpha_o\left(y^*, T_0^n z_0\right) \geq 1. \quad (2.8)$$

So

$$\mathcal{M}\left(x^*, T_0^n z_0\right) = \mathcal{M}\left(T_0x^*, T_0\left(T_0^{n-1}z_0\right)\right),$$
$$\leq \alpha_o\left(x^*, T_0^{n-1}z_0\right)\mathcal{M}\left(T_0x^*, T_0\left(T_0^{n-1}z_0\right)\right),$$
$$\leq \mathcal{M}\left(x^*, T_0^{n-1}z_0\right)^{\lambda_o\left(x^*\right)},$$
$$\leq \mathcal{M}\left(x^*, z_0\right)^{\lambda_o\left(x^*\right)^n} \quad \textit{for all } n \in \mathcal{N}.$$

Taking $\lim_{n\to\infty}$ in the above inequality, we have

$$\lim_{n\to\infty} T_0^n z_0 = x^*,$$

and similarly

$$\lim_{n\to\infty} T_0^n z_0 = y^*.$$

By uniqueness of limit, we have $x^* = y^*$, which shows the uniqueness of fixed point.

Definition 2.5 *Suppose* $\left(\mathcal{X}_0, \mathcal{M}\right)$ *be a multiplicative metric space and let a mapping* $T_0 : \mathcal{X}_0 \to \mathcal{X}_0$, *then* T_0 *is said to be multiplicative* $\left(\alpha_o, \lambda_o\right)$*-Kannan-contraction if there exists* $\alpha_o : \mathcal{X}_0 \times \mathcal{X}_0 \to [0,\infty)$ *and* $\lambda_o \in \Omega_{T_0}$ *such that*

$$\alpha_o(x_0,y_0)\mathcal{M}(T_0x_0,T_0y_0) \le \left[\mathcal{M}(T_0x_0,x_0).\mathcal{M}(T_0y_0,y_0)\right]^{\lambda_o(x_0)}, \quad (2.9)$$

for all $x_0,y_0 \in \mathcal{X}_0$, *where* $\lambda_o \in \left[0,\frac{1}{2}\right)$.

Definition 2.6 *Suppose* $(\mathcal{X}_0,\mathcal{M})$ *be a multiplicative metric space and let a mapping* $T_0:\mathcal{X}_0 \to \mathcal{X}_0$, *then* T_0 *is said to be multiplicative* (α_o,λ_o)*-Chatterjea-contraction if there exists* $\alpha_o:\mathcal{X}_0\times\mathcal{X}_0 \to [0,\infty)$ *and* $\lambda_o \in \Omega_{T_0}$ *such that*

$$\alpha_o(x_0,y_0)\mathcal{M}(T_0x_0,T_0y_0) \le \left[\mathcal{M}(T_0x_0,y_0).\mathcal{M}(T_0y_0,x_0)\right]^{\lambda_o(x_0)}, \quad (2.10)$$

for all $x_0,y_0 \in \mathcal{X}_0$, *where* $\lambda_o \in \left[0,\frac{1}{2}\right)$.

Theorem 2.7 *Let* $(\mathcal{X}_0,\mathcal{M})$ *be a complete multiplicative metric space and assume that* $T_0:\mathcal{X}_0 \to \mathcal{X}_0$ *be* (α_o,λ_o) *-Kannan-contraction mapping satisfying the conditions*:

1. *there exist* $\hat{x}_0 \in \mathcal{X}_0$ *such that* $\alpha_o\left(\hat{x}_0,T_0\hat{x}_0\right)\ge 1$*;*
2. T_0 *is* α_o*-admissible;*
3. *one of the conditions holds;*

(a) T_0 *is continuous;*

(b) if a sequence $\left\{\hat{x}_n\right\}\in\mathcal{X}_0$ *such that* $\alpha_o\left(\hat{x}_n,\hat{x}_{n+1}\right)\ge 1$ *for all* $n\in\mathcal{N}$ *and* $\hat{x}_n \to x_0 \in \mathcal{X}_0$ *as* $n\to\infty$, *then* $\alpha_o\left(\hat{x}_n,x_0\right)\ge 1$.

Then T_0 has a fixed point.

(B1) If $\alpha_o(c,d)\ge 1$ *for all fixed point* $c,d\in\mathcal{X}_0$,

(B2) there exist $z_0\in\mathcal{X}_0$ *such that* $\alpha_o(x_0,z_0)\ge 1$ *and* $\alpha_o(y_0,z_0)\ge 1$ *for all* $x_0,y_0\in\mathcal{X}_0$,

then T_0 *has a unique fixed point.*

Proof. Assume $\hat{x}_0\in\mathcal{X}_0$ such that $\alpha_o\left(\hat{x}_0,T_0\hat{x}_0\right)\ge 1$.

Define the sequence $\left\{\hat{x}_n\right\}\in\mathcal{X}_0$ such that for all $n\in\mathcal{N}$

$$\hat{x}_n = T_0\hat{x}_{n-1}. \quad (2.11)$$

Assume that $\hat{x}_n \doteq \hat{x}_{n-1}$.

Since T_0 is α_o -admissible and $\alpha_o\left(\hat{x}_0,\hat{x}_1\right)=\alpha_o\left(\hat{x}_0,T_0\hat{x}_0\right)\ge 1$ and similarly by induction, we get

$$\alpha_o\left(\hat{x}_{n-1},\hat{x}_n\right)\ge 1 \text{ for all } n\in\mathcal{N}.$$

Applying Inequality (2.9) with $x_0=\hat{x}_0$ and $y_0=\hat{x}_1$, we have

$$\begin{aligned}\mathcal{M}\left(\hat{x}_n,\hat{x}_{n+1}\right) &= \mathcal{M}\left(T_0\hat{x}_{n-1},\hat{x}_n\right),\\ &\le \alpha_o\left(\hat{x}_{n-1},\hat{x}_n\right)\mathcal{M}\left(T_0\hat{x}_{n-1},\hat{x}_n\right),\\ &\le \left[\mathcal{M}\left(T_0\hat{x}_{n-1},\hat{x}_{n-1}\right).\mathcal{M}\left(T_0\hat{x}_n,\hat{x}_n\right)\right]^{\lambda_0\left(\hat{x}_n\right)},\\ &\le \left[\mathcal{M}\left(T_0\hat{x}_{n-1},\hat{x}_{n-1}\right).\mathcal{M}\left(T_0\hat{x}_n,\hat{x}_n\right)\right]^{\lambda_0\left(\hat{x}_0\right)},\end{aligned}$$

for all $n\in\mathcal{N}$,

and so

$$\mathcal{M}\left(\hat{x}_n,\hat{x}_{n+1}\right)\le\mathcal{M}\left(\hat{x}_{n-1},\hat{x}_n\right)^{w\left(\hat{x}_0\right)}$$

$$\text{where } w\left(\hat{x}_0\right)=\frac{\lambda_o\left(\hat{x}_0\right)}{1-\lambda_o\left(\hat{x}_0\right)}<1.$$

Suppose $m,n\in\mathcal{N}$ such that $m<n$, we have

$$\begin{aligned}&\mathcal{M}\left(\hat{x}_m,\hat{x}_n\right)\\ &\le \mathcal{M}\left(\hat{x}_m,\hat{x}_{m+1}\right).\mathcal{M}\left(\hat{x}_{m+1},\hat{x}_{m+2}\right)....\mathcal{M}\left(\hat{x}_{n-1},\hat{x}_n\right),\\ &\le \left[\mathcal{M}\left(\hat{x}_0,\hat{x}_1\right)\right]^{w\left(\hat{x}_0\right)^m+w\left(\hat{x}_0\right)^{m+1}+...+w\left(\hat{x}_0\right)^{n-1}},\\ &\le \left[\mathcal{M}\left(\hat{x}_0,\hat{x}_1\right)\right]^{\frac{w\left(\hat{x}_0\right)^m}{1-w\left(\hat{x}_0\right)}}.\end{aligned}$$

Taking $\lim_{m,n\to\infty}$, we get $\left[\mathcal{M}\left(\hat{x}_m,\hat{x}_n\right)\right]\to 1$ and $\left\{\hat{x}_n\right\}$ is multiplicative Cauchy sequence in $\mathcal{X}_0$.

From the completeness of $\mathcal{X}_0$, there exist $x^*\in\mathcal{X}_0$ such that $\hat{x}_n\to x^*$ as $n\to\infty$.

We suppose that T_0 is continuous from condition(3a), so

$$x^* = \lim_{n\to\infty} \hat{x}_{n+1} = \lim_{n\to\infty} T_0 \hat{x}_n = T_0\left(\lim_{n\to\infty} \hat{x}_n\right) = T_0 x^*.$$

Now, we suppose that condition(3b) holds: As $\left\{\hat{x}_n\right\}$ is multiplicative Cauchy sequence. So, there exist $x^* \in \mathcal{X}_0$ such that $\hat{x}_n \to x^*$ as $n \to \infty$.

From condition(3b), we have

$$\alpha_o\left(\hat{x}_n, x^*\right) \geq 1 \text{ for all } n \in \mathcal{N}.$$

And

$$\mathcal{M}\left(T_0 x^*, x^*\right) \leq \mathcal{M}\left(T_0 x^*, T_0 \hat{x}_n\right).\mathcal{M}\left(T_0 \hat{x}_n, x^*\right),$$

$$= \mathcal{M}\left(T_0 \hat{x}_n, x^*\right).\mathcal{M}\left(T_0 \hat{x}_n, T_0 x^*\right),$$

$$\leq \mathcal{M}\left(T_0 \hat{x}_n, x^*\right).\alpha_o\left(\hat{x}_n, x^*\right)\mathcal{M}\left(T_0 \hat{x}_n, T_0 x^*\right),$$

$$\leq \mathcal{M}\left(\hat{x}_{n+1}, x^*\right).\mathcal{M}\left(\hat{x}_n, x^*\right)^{\lambda_o\left(\hat{x}_n\right)}.$$

As $\lambda_o\left(\hat{x}_n\right) \leq \lambda_o\left(\hat{x}_0\right)$ for all $n \in \mathcal{N}$. Therefore, we have

$$\mathcal{M}\left(T_0 x^*, x^*\right) \leq \mathcal{M}\left(\hat{x}_{n+1}, x^*\right).\mathcal{M}\left(\hat{x}_n, x^*\right)^{\lambda_o\left(\hat{x}_0\right)}.$$

Assume that $n \to \infty$ in the above inequality, we get $\mathcal{M}\left(T_0 x^*, x^*\right) = 1$, that is $x^* = T_0 x^*$, which shows that x^* is fixed point of T_0.

To show uniqueness of x^*, let y^* is another fixed point of T_0, if condition(B1) holds, then the fixed point is unique from (2.9). Now we have to show that condition(B2) holds. From(B2), we have $z_0 \in \mathcal{X}_0$ such that

$$\alpha_o\left(x^*, z_0\right) \geq 1, \alpha_o\left(y^*, z_0\right) \geq 1. \tag{2.12}$$

As T_0 is α_o-admissible, from (2.12), we have

$$\alpha_o\left(x^*, T_0^n z_0\right) \geq 1, \ \alpha_o\left(y^*, T_0^n z_0\right) \geq 1. \tag{2.13}$$

So

$$\mathcal{M}\left(x^*, T_0^n z_0\right) = \mathcal{M}\left(T_0 x^*, T_0\left(T_0^{n-1} z_0\right)\right)$$

$$\leq \alpha_o\left(x^*, T_0^{n-1} z_0\right)\mathcal{M}\left(T_0 x^*, T_0\left(T_0^{n-1} z_0\right)\right),$$

$$\leq \mathcal{M}\left(x^*, T_0^{n-1} z_0\right)^{\lambda_o\left(x^*\right)},$$

$$\leq \mathcal{M}\left(x^*, z_0\right)^{\lambda_o\left(x^*\right)^n} \text{ for all } n \in \mathcal{N}.$$

Taking $\lim_{n\to\infty}$ in the above inequality, we have

$$\lim_{n\to\infty} T_0^n z_0 = x^*,$$

and similarly

$$\lim_{n\to\infty} T_0^n z_0 = y^*.$$

By uniqueness of limit, we have $x^* = y^*$, which shows the uniqueness of fixed point.

Theorem 2.8 *Let $\left(\mathcal{X}_0, \mathcal{M}\right)$ be a complete multiplicative metric space and as-sume that $T_0 : \mathcal{X}_0 \to \mathcal{X}_0$, be $\left(\alpha_o, \lambda_o\right)$ -Chatterjea-contraction mapping satisfying the conditions*:

1. there exist $\hat{x}_0 \in \mathcal{X}_0$ such that $\alpha_o\left(\hat{x}_0, T_0 \hat{x}_0\right) \geq 1$;

2. T_0 is α_o -admissible;

3. one of the conditions holds;

(a) T_0 is continuous;

(b) if a sequence $\left\{\hat{x}_n\right\} \in \mathcal{X}_0$ such that $\alpha_o\left(\hat{x}_n, \hat{x}_{n+1}\right) \geq 1$ for all $n \in \mathcal{N}$ and $\hat{x}_n \to x_0 \in \mathcal{X}_0$ as $n \to \infty$, then $\alpha_o\left(\hat{x}_n, x_0\right) \geq 1$.

Then T_0 has a fixed point.

(C1) If $\alpha_o\left(c, d\right) \geq 1$ for all fixed point $c, d \in \mathcal{X}_0$,

(C2) there exist $z_0 \in \mathcal{X}_0$ such that $\alpha_o\left(x_0, z_0\right) \geq 1$ and $\alpha_o\left(y_0, z_0\right) \geq 1$ for all $\left(x_0, x_0\right) \in \mathcal{X}_0$,

then T_0 has a unique fixed point.

Proof. Assume $\hat{x}_0 \in \mathcal{X}_0$ such that $\alpha_o\left(\hat{x}_0, T_0 \hat{x}_0\right) \geq 1$.

Define the sequence $\left\{\hat{x}_n\right\} \in \mathcal{X}_0$ such that for all $n \in \mathcal{N}$

$$\hat{x}_n = T_0 \hat{x}_{n-1}. \tag{2.14}$$

Assume that $\hat{x}_n \neq \hat{x}_{n-1}$.

Since T_0 is α_o -admissible and $\alpha_o\left(\hat{x}_0, \hat{x}_1\right) = \alpha_o\left(\hat{x}_0, T_0 \hat{x}_0\right) \geq 1$ and similarly by induction, we get

$$\alpha_o\left(\hat{x}_{n-1}, \hat{x}_n\right) \geq 1 \text{ for all } n \in \mathcal{N}.$$

Applying Inequality (2.10) with $x_0 = \hat{x}_0$ and $y_0 = \hat{x}_1$, we have

$$\mathcal{M}\left(\hat{x}_n,\hat{x}_{n+1}\right)=\mathcal{M}\left(T_0\hat{x}_{n-1},T_0\hat{x}_n\right),$$
$$\le\alpha_0\left(\hat{x}_{n-1},\hat{x}_n\right)\mathcal{M}\left(T_0\hat{x}_{n-1},T_0\hat{x}_n\right),$$
$$\le\left[\mathcal{M}\left(T_0\hat{x}_{n-1},\hat{x}_n\right).\mathcal{M}\left(T_0\hat{x}_0,\hat{x}_{n-1}\right)\right]^{\lambda_o\left(\hat{x}_n\right)},$$
$$=\mathcal{M}\left(\hat{x}_{n-1},\hat{x}_{n+1}\right)^{\lambda_o\left(\hat{x}_0\right)}$$
$$\le\left[\mathcal{M}\left(\hat{x}_{n-1},\hat{x}_n\right).\mathcal{M}\left(\hat{x}_n,\hat{x}_{n-1}\right)\right]^{\lambda_o\left(\hat{x}_0\right)},$$
for all $n\in\mathcal{N}$.

and so

$$\mathcal{M}\left(\hat{x}_n,\hat{x}_{n+1}\right)\le\mathcal{M}\left(\hat{x}_{n-1},\hat{x}_n\right)^{w\left(\hat{x}_0\right)}$$
$$\text{where } w\left(\hat{x}_0\right)=\frac{\lambda_o\left(\hat{x}_0\right)}{1-\lambda_o\left(\hat{x}_0\right)}\le 1.$$

Suppose $m,n\in\mathcal{N}$ such that $m<n$, we have

$$\mathcal{M}\left(\hat{x}_m,\hat{x}_n\right)$$
$$\le\mathcal{M}\left(\hat{x}_m,\hat{x}_{m+1}\right).\mathcal{M}\left(\hat{x}_{m+1},\hat{x}_{m+2}\right)....\mathcal{M}\left(\hat{x}_{n-1},\hat{x}_n\right),$$
$$\le\left[\mathcal{M}\left(\hat{x}_0,\hat{x}_1\right)\right]^{w\left(\hat{x}_0\right)^m+w\left(\hat{x}_0\right)^{m+1}+...+w\left(\hat{x}_0\right)^{n-1}},$$
$$\le\left[\mathcal{M}\left(\hat{x}_0,\hat{x}_1\right)\right]^{\frac{w\left(\hat{x}_0\right)^m}{1-w\left(\hat{x}_0\right)}}.$$

Taking $\lim_{m,n\to\infty}$, we get $\left[\mathcal{M}\left(\hat{x}_m,\hat{x}_n\right)\right]\to 1$ and $\left\{\hat{x}_n\right\}$ is multiplicative cauchy sequence in $\mathcal{X}_0$.

From the completeness of $\mathcal{X}_0$, there exist $x^*\in\mathcal{X}_0$ such that $\hat{x}_n\to x^*$ as $n\to\infty$.

We suppose that T_0 is continuous from condition(3a), so

$$x^*=\lim_{n\to\infty}\hat{x}_{n+1}=\lim_{n\to\infty}T_0\hat{x}_n=T_0\left(\lim_{n\to\infty}\hat{x}_n\right)=T_0x^*.$$

Now, we suppose that condition(3b) holds: As $\left\{\hat{x}_n\right\}$ is multiplicative Cauchy sequence. So, there exist $x^*\in\mathcal{X}_0$ such that $\hat{x}_n\to x^*$ as $n\to\infty$.

From condition(3b), we have

$$\alpha_o\left(\hat{x}_n,x^*\right)\ge 1 \text{ for all } n\in\mathcal{N}.$$

And

$$\mathcal{M}\left(T_0x^*,x^*\right)\le\mathcal{M}\left(T_0x^*,T_0\hat{x}_n\right).\mathcal{M}\left(T_0\hat{x}_n,x^*\right),$$
$$=\mathcal{M}\left(T_0\hat{x}_n,x^*\right).\mathcal{M}\left(T_0\hat{x}_n,T_0x^*\right),$$
$$\le\mathcal{M}\left(T_0\hat{x}_n,x^*\right).\alpha_o\left(\hat{x}_n,x^*\right)\mathcal{M}\left(T_0\hat{x}_n,T_0x^*\right),$$
$$\le\mathcal{M}\left(\hat{x}_{n+1},x^*\right).\mathcal{M}\left(\hat{x}_n,x^*\right)^{\lambda_o\left(\hat{x}_n\right)}.$$

As $\lambda_o\left(\hat{x}_n\right)\le\lambda_o\left(\hat{x}_0\right)$ for all $n\in\mathcal{N}$. Therefore, we have

$$\mathcal{M}\left(T_0x^*,x^*\right)\le\mathcal{M}\left(\hat{x}_{n+1},x^*\right).\mathcal{M}\left(\hat{x}_n,x^*\right)^{\lambda_o\left(\hat{x}_0\right)}.$$

Assume that $n\to\infty$ in the above inequality, we get $\mathcal{M}\left(T_0x^*,x^*\right)=1$, that is $x^*=T_0x^*$, which shows that x^* is fixed point of T_0.

To show uniqueness of x^*, let y^* is another fixed point of T_0, if condition(C1) holds, then the fixed point is unique from (2.9). Now we have to show that condition(C2) holds. From(C2), we have $z_0\in\mathcal{X}_0$ such that

$$\alpha_o\left(x^*,z_0\right)\ge 1,\ \alpha_o\left(y^*,z_0\right)\ge 1. \tag{2.15}$$

As T_0 is α_o-admissible, from (2.15), we have

$$\alpha_o\left(x^*,T_0^nz_0\right)\ge 1,\ \ \alpha_o\left(y^*,T_0^nz_0\right)\ge 1. \tag{2.16}$$

So

$$\mathcal{M}\left(x^*,T_0^nz_0\right)=\mathcal{M}\left(T_0x^*,T_0\left(T_0^{n-1}z_0\right)\right)$$
$$\le\alpha_o\left(x^*,T_0^{n-1}z_0\right)\mathcal{M}\left(T_0x^*,T_0\left(T_0^{n-1}z_0\right)\right),$$
$$\le\mathcal{M}\left(x^*,T_0^{n-1}z_0\right)^{\lambda_o\left(x^*\right)},$$
$$\le\mathcal{M}\left(x^*,z_0\right)^{\lambda_o\left(x^*\right)^n} \text{ for all } n\in\mathcal{N}.$$

Taking $\lim_{n\to\infty}$ in the above inequality, we have

$$\lim_{n\to\infty} T_0^n z_0 = x^*,$$

and similarly

$$\lim_{n\to\infty} T_0^n z_0 = y^*.$$

By uniqueness of limit, we have $x^* = y^*$, which shows the uniqueness of fixed point.

Remark 2.9 *The multiplicative* (α_o, λ_o) *-Banach-contraction mapping, multiplicative* (α_o, λ_o) *-Kannan-contraction mapping and multiplicative* (α_o, λ_o) *-Chatterjea-contraction mapping is the generalization of multiplicative Banach-contraction mapping, multiplicative Kannan-contraction mapping and multiplicative Chatterjea-contraction mapping respectively, i.e by simply putting* $\alpha_o(x_0) = k \in [0,1)$ *in Definition 2.1,* $\lambda_o(x_0) = k \in \left[0, \frac{1}{2}\right)$ *in Definition 2.5 and2.6 with* $\alpha_o(x_0, y_0) = 1$ *we obtain multiplicative Banach-contraction mapping, multiplicative Kannan-contraction mapping and multiplicative Chatterjea-contraction mapping respectively*.

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