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Handbook of Stochastic Methods

for Physics, Chemistry and the Natural Sciences

Second Edition
With 29 Figures



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