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Neutral stability in Josephson junction arrays with arbitrary lattice geometry

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Abstract

We consider DC-biased arrays of overdamped Josephson junctions with different lattice geometries, and demonstrate that, with suitable choice of bias currents, it is possible for the in-phase state of the array to exhibit so-called "neutral stability". This extends the finding of Wiesenfeld et al. (J. Appl. Phys. 76 (1994) 3835) for two-dimensional rectangular lattices to arbitrary lattice types. © 2001 Published by Elsevier Science B.V.

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1. Introduction

Though the behavior of an individual Josephson junction is well understood, it has been observed that large groups of interacting Josephson junctions can display interesting new collective behaviors (e.g., [1–4]). A prime example of this is the phenomenon of "neutral stability", an unusual mathematical property referring to systems which, in a certain sense, sit poised on the boundary between stability and instability (i.e., they possess some Floquet exponents equal to zero). In 1994, Wiesenfeld et al. [1], extending key work by Hadley [5], demonstrated a remarkable result: in an $N \times M$ RSJ Josephson junction array with a rectangular lattice geometry, the in-phase state is (globally) neutrally stable, and possesses an astonishing $N-1$ zero Floquet exponents. The significance of this result is twofold: First, on the applied math side, it raises an intriguing issue: The presence of a large number of zero Floquet exponents in a dynamical system is a rare occurrence, typically found only in systems possessing obvious global symmetries; it is not at all apparent why neutral stability should make an appearance here. Second, on the applied physics side, neutral stability renders a Josephson junction array highly sensitive to small perturbations which can potentially disrupt the synchronous functioning of the array. Since achieving synchronization is of crucial importance for many device applications, the potentially desynchronizing effect of neutral stability has potentially important practical design implications (e.g., [6]).

If this phenomenon were restricted solely to rectangular Josephson junction arrays, then perhaps neutral stability would be of somewhat limited import in terms of array design. However, as we will demonstrate in this Letter, the neutral-stability property observed by Wiesenfeld et al. is not limited to 2-D rectangular lattices (including those with periodic boundary

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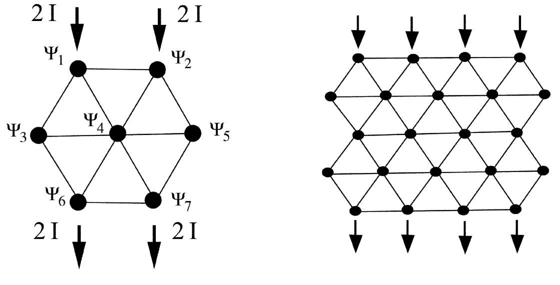

(a) (b)

Fig. 1. (a) A simple triangular lattice. The filled circles show the location of the superconducting nodes; the Ψ_i's denote their macroscopic phases. A line between two nodes indicates that they are connected via a Josephson junction. (b) A larger triangular lattice.

conditions [7]), but in fact arises in Josephson junction arrays with arbitrary lattice geometries, for suitable choices of the bias currents. We first demonstrate this explicitly for the case of a triangular lattice of Josephson junctions, and then describe a criterion for the existence of neutral stability in arbitrary lattice geometries.

2. A triangular lattice

We begin by considering an array with a triangular lattice geometry, as depicted in Fig. 1(a). The array is comprised of identical, overdamped (i.e., RSJ [8]) junctions. DC-bias current is injected uniformly along the upper boundary of the array, and removed along the lower boundary. The Ψ_i's denote the so-called "node variables", representing the macroscopic phase at each superconducting site. Letting $\psi_{ij} \equiv \Psi_i - \Psi_j$ denote the phase differences between the nodes, the basic circuit equations for the array are

$$\begin{aligned}
&\dot\psi_{21} + I_c\sin(\psi_{21}) + \dot\psi_{31} + I_c\sin(\psi_{31}) \\
&\quad + \dot\psi_{41} + I_c\sin(\psi_{41}) = 2I, \\
&-\dot\psi_{21} - I_c\sin(\psi_{21}) + \dot\psi_{42} + I_c\sin(\psi_{42}) \\
&\quad + \dot\psi_{52} + I_c\sin(\psi_{52}) = 2I, \\
&-\dot\psi_{31} - I_c\sin(\psi_{31}) + \dot\psi_{43} + I_c\sin(\psi_{43}) \\
&\quad + \dot\psi_{63} + I_c\sin(\psi_{63}) = 0, \\
&-\dot\psi_{41} - I_c\sin(\psi_{41}) - \dot\psi_{42} - I_c\sin(\psi_{42}) \\
&\quad - \dot\psi_{43} - I_c\sin(\psi_{43}) + \dot\psi_{54} + I_c\sin(\psi_{54}) \\
&\quad + \dot\psi_{64} + I_c\sin(\psi_{64}) + \dot\psi_{74} + I_c\sin(\psi_{74}) = 0, \\
&-\dot\psi_{52} - I_c\sin(\psi_{52}) - \dot\psi_{54} - I_c\sin(\psi_{54}) \\
&\quad + \dot\psi_{75} + I_c\sin(\psi_{75}) = 0, \\
&-\dot\psi_{63} - I_c\sin(\psi_{63}) - \dot\psi_{64} - I_c\sin(\psi_{64}) \\
&\quad + \dot\psi_{76} + I_c\sin(\psi_{76}) = -2I, \\
&-\dot\psi_{74} - I_c\sin(\psi_{74}) - \dot\psi_{75} - I_c\sin(\psi_{75}) \\
&\quad - \dot\psi_{76} - I_c\sin(\psi_{76}) = -2I,
\end{aligned} \tag{1}$$

where the overdot on the ψ_{ij}'s denotes the time derivative. For convenience, $\hbar/2er$ has been set equal to one in the above equations.

One may readily verify that there exists a so-called "in-phase" solution to these equations, given by

$$\Psi_1 = \Psi_2 = \phi(t),$$
$$\Psi_3 = \Psi_4 = \Psi_5 = 2\phi(t),$$
$$\Psi_6 = \Psi_7 = 3\phi(t),$$

where the function $\phi(t)$ satisfies

$$\dot\phi + I_c\sin(\phi) = I. \tag{2}$$

The in-phase state describes a state of perfect synchrony for the array, and represents the ideal configuration for many practical device applications. Our goal here is to show that, for this triangular lattice, the in-phase state is neutrally stable with respect to certain types of perturbations. This proves to be remarkably straightforward.

There are two basic approaches one can adopt. The first involves a linear stability analysis of the in-phase state of (1), from which one can deduce the existence of several Floquet exponents which are equal to zero for this array. This method was used in [1,5] for the case of rectangular lattices. The drawback of this first approach is that since it relies on a linear analysis of the equations, it only demonstrates the existence of *linear* neutral stability. The second approach is based on a more general understanding of neutral stability, as described in [1] (see also [6]). As will be described below, the basic idea here is to show that the in-phase solution is really just one element of a multi-parameter family of solutions. Since these parameters can be varied continuously and independently, the in-phase state is not merely linearly neutrally stable, but rather nonlinearly (i.e., globally) neutrally stable. We will follow this second approach, which is not only more powerful, but also significantly easier to implement.

To do so, we must explicitly construct the multi-parameter family of solutions to which the in-phase state of the triangular lattice belongs. We find that the desired family of solutions takes the form

$$\begin{aligned} &\Psi_1 = \Psi_2 = \phi(t+\delta_1), \\ &\Psi_3 = \Psi_4 = \Psi_5 = \phi(t+\delta_1) + \phi(t+\delta_2), \\ &\Psi_6 = \Psi_7 = \phi(t+\delta_1) + \phi(t+\delta_2) + \phi(t+\delta_3), \qquad (3) \end{aligned}$$

where $\delta_1, \delta_2, \delta_3$ are three arbitrary parameters, and the function $\phi(t)$ satisfies (2), as before. Note that, for $\delta_1 = \delta_2 = \delta_3 = 0$, we recover the in-phase solution. By direct substitution into (1), one may verify that (3) is indeed a family of solutions to the array equations. Now, since $\delta_1, \delta_2, \delta_3$ can be varied continuously (and independently), it immediately follows from standard stability theory that there exist three independent neutrally stable "directions" in the phase space of the array, thus proving our desired result for a simple triangular lattice. (We mention here that only two of these three directions can be considered nontrivial, since the array equations are autonomous and thereby possess an overall time-shift symmetry.) We also note that one can readily extend this result to larger triangular lattices (Fig. 1(b)), in which case one finds that there is one neutrally stable direction associated with each "row" of the triangular lattice (although again one of these directions is trivial). This is entirely analogous to what was found previously for the case of a rectangular lattice.

Hence, we have demonstrated (through a relatively simple argument) that a very common lattice type — a triangular lattice — also exhibits global neutral stability, thereby showing that the previous finding of [1,5] is not restricted exclusively to rectangular Josephson junction arrays. We next demonstrate that this phenomenon actually extends well beyond the simple cases of triangular and rectangular lattices, and indeed can occur in arrays with arbitrary geometries. To do so, we use a generalization of the above method, as we now describe.

3. A generalized lattice

To begin, consider a general lattice (of overdamped Josephson junctions) consisting of an arbitrary number of superconducting nodes (with no restrictions on network connectivity or dimension). As before, we label the superconducting nodes in the lattice by i, and the macroscopic phase at the ith site by Ψ_i. A representative example is shown Fig. 2. Now, to demonstrate that neutral stability is possible in such an array, it is necessary to choose the bias currents appropriately, since neutral stability will not appear for arbitrary choices of bias current. Unlike the previous example in which the triangular array was fed by a uniform bias current, we must now allow for the possibility that the injected dc current may vary from one superconducting node to the next. This is accomplished in the following manner: To each site i in the lattice we assign an integer n_i. The assignment of integers to the nodes is arbitrary, save one restriction: if two nodes in the array are "neighbors" (i.e., if they are connected to one another via a junction), then their assigned integer values must either be the same, or differ at most by one. An example of a permissible choice of integers is shown (by the encircled numbers) in Fig. 2. For convenience we will always assume that the smallest of the n_i's is 1. Two points are worthy of note here. First, the choice of integer labels $\{n_i\}$ is not unique. Second, for *any* lattice geometry there always exists at least one (nontrivial) allowable integer labeling. (For example, one can randomly assign a 1 or 2 to each site in the lattice; this choice clearly satisfies the labeling criterion.)

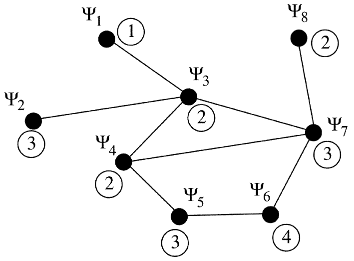

Fig. 2. A Josephson junction array with arbitrary lattice geometry. The filled circles show the location of the superconducting nodes; the Ψ_i's denote their macroscopic phases. The encircled numbers indicate the particular integer (n_i) assigned to each node.

The bias current injected into (or extracted from) the ith node is then chosen according to the following prescription:

$$I_i = \sum_{\{j\}_i} (n_j - n_i) I. \qquad (4)$$

Here, the notation $\{j\}_i$ indicates that the summation is over all sites j that are neighbors with (i.e., connected to) site i, and I may be regarded as the basic unit of current flowing through the individual Josephson junctions in the array (i.e., the external bias current at any superconducting node is taken to be an integer multiple of I). Defining $\psi_{ij} \equiv \Psi_i - \Psi_j$ as before, we may write the circuit equations for the array as

$$\sum_{\{j\}_i} \dot{\psi}_{ji} + I_c \sin(\psi_{ji}) = \sum_{\{j\}_i} (n_j - n_i) I. \tag{5}$$

To show that the in-phase state of this array has multiple Floquet exponents equal to zero, we will, as before, explicitly construct a multi-parameter family of solutions to the circuit equations which contains the in-phase solution. The existence of this family of solutions is sufficient to prove the result. (We note that the precise number of neutrally stable directions for the generalized array will turn out to be equal to the largest integer in the set $\{n_i\}$.)

We claim that this desired family of solutions to Eqs. (5) is given by

$$\Psi_i = \sum_{k=1}^{k=n_i} \phi(t + \delta_k), \tag{6}$$

where the function $\phi(t)$ again satisfies the relation $\dot{\phi} + I_c \sin(\phi) = I$, and the δ_k's denote the freely adjustable parameters. We must now show that (6) is indeed a solution to (5). We start by observing that the summation in (5) is only over sites j which are neighbors to site i. Hence, by design, the value of n_j for any of these neighbors can differ from n_i by at most one, i.e., $n_j = \{n_i - 1, n_i, n_i + 1\}$. Thus, from (6) we see that

$$\psi_{ji} = \begin{cases} \phi(t + \delta_{n_j}) & \text{if } n_j = n_i + 1, \\ 0 & \text{if } n_j = n_i, \\ -\phi(t + \delta_{n_i}) & \text{if } n_j = n_i - 1. \end{cases} \tag{7}$$

Substituting these ψ_{ji}'s into the left-hand side of (5), and letting N_i^+ denote the total number of neighboring sites to i that have $n_j = n_i + 1$, and N_i^- the number of neighbors with $n_j = n_i - 1$, then the left-hand side of (5) may be re-expressed as

$$N_i^+ \big[\dot{\phi}(t + \delta_{n_i+1}) + I_c \sin\big(\phi(t + \delta_{n_i+1})\big)\big] - N_i^- \big[\dot{\phi}(t + \delta_{n_i}) + I_c \sin\big(\phi(t + \delta_{n_i})\big)\big]. \tag{8}$$

Observe, however, that the terms in brackets in the above expression are in fact equal to the basic unit of current I (this follows from the defining equation for $\phi(t)$). Thus, the left-hand side of (5) reduces to $N_i^+ I - N_i^- I$. Comparing this to the right-hand side of (5), we see that the two expressions are identical, which demonstrates that (6) represents a multi-parameter family of solutions to the original array equations. Hence, by means of a relatively simple criterion, we have proven that the in-phase state of the array exhibits the neutral-stability property, and have developed guidelines for recognizing its existence in an array with arbitrary geometry.

4. Discussion

In summary, we have shown that a Josephson junction array of any lattice geometry can possess the neutral stability property (for suitably chosen bias currents), and that this feature is not unique to the particular two-dimensional rectangular array studied by [1,5]. Several comments are in order.

First, in analysing the stability of the in-phase solution of a general array, we have found the existence of multiple neutrally stable 'directions' in the phase space of the array — the exact number being equal to the largest value in the set of integers $\{n_i\}$. We emphasize, however, that our analysis does not allow us to say anything about the stability of the in-phase solution in the remaining directions of phase space. (To do so would require a detailed stability analysis that would depend on the peculiarities of an array's geometry; it would seem unlikely that any sort of general result for arbitrary lattices could be obtained.)

Secondly, to demonstrate neutral stability in an array with an arbitrary lattice geometry, we had to rely on a somewhat nontraditional current-biasing scheme (i.e., different amounts of current were fed into the individual junctions). Now, from the point of view of experiment, this might at first seem unfortunate, since in many experimental applications one would naturally feed a *uniform* bias current into the array, and moreover, to apply this current only to those superconducting nodes which lie on the array's boundaries (not the interior). However, by examining the more general case as we have, it becomes rather straightforward to determine if a traditional biasing scheme in an array would produce neutral stability. For instance, if we now return to our first example — the special case of a triangular lattice being fed by a uniform bias current applied along the top boundary and removed along the lower boundary — it becomes easy to recognize that this system meets all the criteria described in Section 3, and hence will exhibit neutral stability (i.e., for the integers $\{n_i\}$, one simply assigns a value of 1 to all nodes on the top row, 2 to all nodes in the second row, 3 to those in the third row, etc.) Likewise, it becomes just as straightforward to show, for instance, that a cubic lattice (fed by a uniform bias current applied along its top face and removed along its bottom face), will exhibit neutral stability as well.

Lastly, we note that in an effort to isolate the relationship between lattice geometry and neutral stability, we have focused exclusively on the simplest type of Josephson junction array, wherein the individual junctions were assumed to be overdamped and identical. In particular, we have neglected other potentially complicating effects such as self and mutual inductances, disorder, external signals, etc. which are sometimes present. While this simplification imposes certain limitations, it nonetheless offers one significant virtue; namely, if one determines that neutral stability is present in this simplest type of array, then these other influences — even if they are very small in absolute size — will likely have a severe impact on the array's dynamical behavior. In other words, the presence of neutral stability renders the array highly sensitive to external or internal perturbations, regardless of how weak these perturbations might be. (For instance, it has been shown that adding even a small amount of disorder to a rectangular Josephson junction array results in a broadening of the array's linewidth — no such broadening would be seen had neutral stability not been present [9].) Hence, the criteria for recognizing the presence of neutral stability which emerges from our analysis can serve as a cautionary indicator of when seemingly small collateral influences might have an unexpectedly pronounced influence on an array's behavior.

Acknowledgements

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