Sensitivity studies of friction-induced vibration

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Abstract: Friction-induced vibration is notoriously twitchy. This paper examines the origin of the sensitivity, using a model with two linear systems coupled at a single-point sliding contact where a general linearised model for dynamic frictional force is allowed. Sensitivity and convergence studies show that system uncertainty is significant enough to affect the stability of predictions and that modes neglected from the model can sensitively affect predictions. Some key results from a large-scale experimental study are presented. The integration of the uncertainty and sensitivity analysis with data-processing techniques to extract reliable data allows critical evaluation of the modelling details.

Keywords: brake squeal; brake noise; friction-induced vibration; friction excited vibration; sensitivity; uncertainty; perturbation analysis; convergence; minimal models; reduced order models; low order models.

Reference to this paper should be made as follows: Butlin, T. and Woodhouse, J. (2009) 'Sensitivity studies of friction-induced vibration', *Int. J. Vehicle Design*, Vol. 51, Nos. 1/2, pp.238–257.

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Jim Woodhouse studied Mathematics at the University of Cambridge (BA 1972), and stayed on there for his PhD (1977) and postdoctoral studies, both related to the acoustics of the violin. After a spell working for a consultancy company, in 1985 he took up a post in the Engineering Department in Cambridge, and has been successively a Lecturer, Reader and Professor there.

1 Introduction

Friction-induced vibration is a phenomenon that occurs across a diverse range of scales and contexts, including wheel-rail noise, machine tool vibration, stringed instruments and vehicle brake squeal. Vehicle brake squeal in particular remains difficult to predict despite decades of research. Part of this difficulty is due to its well-known twitchy nature: consecutive experiments under nominally identical conditions frequently produce inconsistent behaviour. Until recently this twitchiness has been regarded simply as a nuisance, preventing clean experimental calibration and testing of theoretical models. It is now beginning to be seen as an essential ingredient of the phenomenon and to be studied in its own right. This paper presents recent work in this emerging subject area.

There are many mechanisms that can lead to structural vibration mediated by interfacial friction of some kind. Thermo-elastic instability, as first described by Barber (1969), can lead to unstable 'hot spots' in the friction material: areas of increased temperature cause local expansion of the pad, which in turn increases the local contact pressure and causes further heating. Another possible cause of brake 'judder' (low-frequency vibration) is disk thickness variation, which can lead to forced vibration (e.g., Jacobsson, 2003). The initial conditions of a sliding contact system also play a role in vibration: it is often commented that if rotation begins after a contact pressure has been applied then squeal is more likely. The effect of the pad acting as a moving load across the surface of the disk can result in instabilities, by a mechanism investigated by Ouyang et al. (2003). The effect of friction behaving as a 'follower force', always aligned with the tangent to the disk even when vibration causes rotational motion of the disk, can also result in self-excited vibration. A survey of studies into this mechanism is provided by Langthjem and Sugiyama (2000); but the sceptical discussion of the physical importance of follower forces by Elishakoff (2005) should also be noted.

However, the simplest mechanism with by far the greatest representation within the literature is that of linear instability: the coupled system dynamics are linearised about the steady-sliding equilibrium and the poles of the sliding-coupled system determine the stability of this operating point. Unstable poles describe exponentially-growing vibration starting from arbitrarily small amplitude (so that the linearising assumption should be justified). Models of this kind vary in complexity from lumped parameter models (e.g., Hoffmann, 2002) to detailed finite element models (e.g., Lee et al., 2003a, 2003b, 2003c). Although lumped parameter models allow relatively transparent analysis they are difficult to correlate quantitatively with real systems. Conversely, finite element models can readily be associated with physical counterparts, but understanding how predictions relate to system parameters is difficult and experimental results often give unconvincing agreement. In their review, Ouyang et al. (2005) highlight this commonly encountered difficulty.

It is frequently observed within the literature that experimental validation of the predictions of such linearised instability models is hindered by the difficulty in obtaining repeatable results (e.g., Bergman et al., 2000). In the first of a series of studies, Giannini et al. (2006) design a test rig specifically to tackle this difficulty. With significant care taken to control the system parameters, their experimental results indicate improved repeatability. Nevertheless, changing behaviour was observed and there was still a fairly high level of scatter within the data, which was primarily attributed to wear of the pad. The non-repeatability of results is often taken to imply that the sliding contact behaviour is highly sensitive to parameters beyond an experimenter's control. Traditionally this has been attributed solely to the difficulty in characterising the friction interface.

Issues of uncertainty and sensitivity are active areas of research within a much wider context of dynamical systems. In the control literature, techniques have been developed over the last few decades to describe uncertainty and design control systems that are robust, both in terms of stability and performance: see for example the textbook by Zhou et al. (1996). Within room acoustics, and somewhat more recently in structural vibration theory, statistical methods have been developed to analyse the ensemble mean response of uncertain systems (e.g., Cremer and Muller, 1982; Lyon and de Jong, 1995). In recent years these techniques have been extended to allow ensemble variance of complex coupled systems to be predicted as well (Langley and Cotoni, 2004). Hybrid models combining these statistical methods consistently with deterministic approaches such as Finite Element methods have been developed (Langley and Bremner, 1999). Much of this dynamics literature draws on mathematical results from quantum physics, developed to analyse nuclear energy levels or the behaviour of electrons in solids (e.g., Mehta, 1991). A variety of other techniques to take uncertainty and sensitivity into account within the context of structural uncertainty are undergoing continuing development (e.g., Manson and Worden, 2007).

Study of the sensitivity of friction-induced vibration to parameter uncertainty has lagged behind developments in these other contexts, but it has begun to receive increasing attention. Lee et al. (2001) developed a model of a drum brake: of most interest here was the recognition of the sensitive dependence of the predictions to small parameter changes. A small design modification was proposed that was theoretically shown to stabilise the system. An experimental investigation was carried out to validate this and the results supported a greatly reduced occurrence of squeal. The study highlighted, theoretically and experimentally, that small changes to parameters can cause large changes in coupled behaviour. Huang et al. (2006) carried out a sensitivity analysis of some of the model parameters such as the brake lining stiffness in order to suggest squeal reduction techniques.

Huang et al. (2007) in a subsequent paper described a method for estimating the critical value for the coefficient of friction using a sensitivity analysis of a local approximation of the characteristic equation. Their assumption was that ‘mode coupling’ is the cause of squeal: the boundary between stable and unstable systems occurs when two coupled poles combine and bifurcate. The quadratic equation that governs this merging was factorised out of the whole characteristic equation and an expression was derived for the sensitivity of the solutions with respect to the coefficient of friction: the critical coefficient of friction occurs when this gradient is infinite. The method was then applied to a multiple mode system and showed good local agreement when pairs of modes were close. Two features of this work were particularly interesting in the context of the present paper: the local approximation of a large-scale system using just a quadratic, and the sensitivity analysis of these modes with respect to the coefficient of friction.

Guan et al. (2006) carried out a sensitivity analysis of a finite element model of a disk brake. Sensitivity was measured for each coupled pole as the proportional rate of change of the real part (in the complex *s*-domain) with respect to the modal parameters of the subsystems. The dominant modal parameter for the unstable modes was then identified and an optimisation problem set up to change the relevant parameter so that the system is stabilised within a given frequency range. Kessler et al. (2007) recognised that the eigenvalues of a sliding-coupled system were sensitive to perturbations and that these perturbations conformed to a structure determined by the parameters that characterise the system dynamics. They defined a new quantity referred to as ‘structured pseudospectra’ which gave a measure of the effect of the structured uncertainty on state-space systems in general.

The focus of these previous sensitivity studies has been either on design optimisation to stabilise unstable poles or on the presentation of new methods for determining sensitivity, rather than on looking at the conditions under which high sensitivity occurs for sliding contact systems. This paper focuses on understanding sensitivity as an intrinsic part of friction-induced vibration. A computational and experimental study will be described which aims to document in a rather complete manner the instability and sensitivity behaviour of a particular class of systems. This class has been chosen as the logical starting point for a systematic investigation which will gradually expand to include the full range of phenomena and systems. Two assumptions define this class: only linear instability will be studied, and the sliding frictional contact is limited to a single point. The linear vibration characteristics of the systems on either side of this contact point are described by transfer functions, so there is no restriction on what could be included: for example, the ‘mode coupling’ bifurcation route to instability is a special case.

It is envisaged that in future studies the model could be progressively extended to include two or more discrete contact points, line contact parallel or perpendicular to the sliding direction, and finally area contact as in a vehicle disk brake (though it is recognised that extensions to continuous contact regions would require a more substantially adapted framework). Each of these enhancements might be expected to introduce new physical mechanisms of instability and sensitivity. But it will be seen that the single-point system is already sufficiently complicated to contain a rich range of phenomena and to raise several important general issues. Three main topics will be addressed in this paper. First, a computationally efficient way to analyse sensitivity is presented, and validated against Monte Carlo simulations. Second, the issue of low-order models is discussed: how sensitive are the predictions of a low-order model to the other, neglected, modes of the system and how well can one expect to approximate a real system by a low-order model? Third, an experimental study will be presented in which enough data was collected to obtain direct information about sensitivity, and thus to compare with the predictions of a theoretical model which also takes sensitivity into account. This paper will give an overview of the work to highlight the main results. More details can be found in Butlin (2007).

2 Modelling

2.1 Friction modelling

One of the key difficulties in modelling friction-induced vibration lies in characterising the frictional interface between the two subsystems. This has been an active area of research for many decades, with a wide range of approaches taken to modelling it. A brief review of some of the primary models used in the context of friction-induced vibration is given here: for a fuller discussion see Ibrahim (1994a, 1994b) or Sheng (2008).

The simplest model is the familiar Coulomb law: the frictional force resisting sliding, F, is taken to be proportional to the normal reaction force, N: so $F = \mu N$, where μ is known as the coefficient of friction. Empirically, μ is found to be independent of the apparent area of contact and to depend only on the two materials, their surface finish, and any lubricant or contaminant layer. This behaviour is fairly well understood in terms of the contact mechanics of rough surfaces (e.g., Johnson, 1985; Sheng, 2008). At the next level of approximation a distinction is made between static and sliding contact: often a higher coefficient of friction is measured when there is no relative sliding between the two bodies. This has been proposed as another mechanism that can lead to squeal (e.g., Blok, 1940), but it is not relevant to the kind of linear instability being studied here because the distinction only matters when nonlinear stick-slip motion occurs.

The usual next stage of modelling is driven by empirical measurements. The most common method of testing the friction (and wear) properties of two contacting materials is to load a hemispherical pin of the softer material against a rotating disk of the harder material and measure the friction force, see for example Williams (1994). This kind of pin-on-disk test rig, and related rigs which impose a state of steady sliding, allows the coefficient of friction to be measured at different sliding speeds. In many cases some variation is observed, leading to the family of models, such as the 'Stribeck model', in which μ is a function of sliding speed. There is no general consensus on the reasons for this velocity dependence nor a generally accepted standard for its measurement though it is often thought to be a key factor in the generation of brake noise. It is common to assume a form for the functional dependence (e.g., Bengisu and Akay, 1994) that can fit observations of friction over a range of steady sliding speeds. A large volume of experimental data has been analysed in this way, see for example Kragelskii (1965) and Rabinowicz (1965). These authors also proposed empirical formulae for the observed velocity dependence. If μ is plotted against sliding speed then both positive and negative gradients are observed and Kragelskii (1965) reported that there is usually a velocity at which μ is a maximum. Rabinowicz (1965) noted that for many purposes μ can be taken as being independent of velocity over a limited velocity range as the relative change in μ is small. The question of whether this velocity dependence is important within the context of friction-induced vibration continues to be debated.

If such a dependence is taken into account, there is no reason to expect that velocity and normal force should be the only parameters upon which the frictional force depends, especially at higher frequencies: brake squeal often happens in the kilohertz range. Numerous laws from the microscopic to the macroscopic scale have been developed that propose descriptions of its dependence on temperature, relative displacement, asperity deformation history, material damping, strain rates and other phenomenological system states that are not directly related to physical properties of the system (see for example Sheng, 2008).

Within the context of linearised theory a major simplification can be made which unifies all these models within a single framework. Consider how the frictional force varies when a small oscillating tangential velocity is superimposed on a larger steady sliding velocity. Provided the perturbation is small enough that a linear approximation is appropriate, the fluctuations in force will be sinusoidal at the same frequency as the imposed velocity fluctuation. The amplitude and phase of the force fluctuations might vary with frequency, described by a kind of transfer function of sliding friction which can be written in the form:

$$F' \approx \varepsilon N_0 v'. \tag{1}$$

where $F = F_0 + F'e^{i\omega t}$ and $v = v_0 + v'e^{i\omega t}$ describe the sinusoidally fluctuating force and sliding velocity, N_0 is the constant normal pre-load and the complex quantity $\varepsilon(\omega)$ represents the linearised transfer function. Different physical models of friction will produce different functions $\varepsilon(\omega)$.

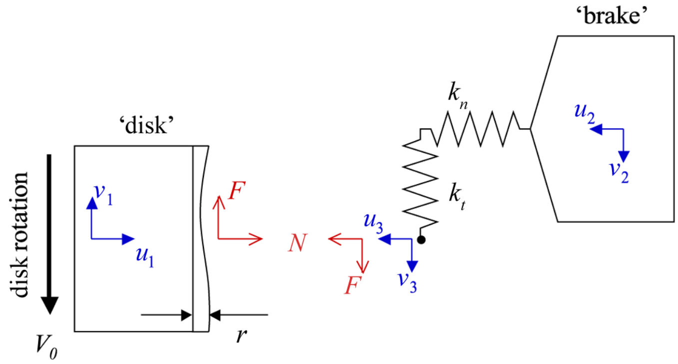
Measurement of the magnitude and phase of ε at a range of frequencies of imposed oscillation about a steady sliding velocity would provide the necessary input data for any prediction model based on linear theory, and in addition it would give clues as to the physical processes governing the variation in frictional force. For example if friction really is a function of relative sliding speed only, then equation (1) would simply represent the linear term in a Taylor expansion of this function, and ε would be real and independent of frequency. In that case the value really could be deduced from the slope of the curve measured from a pin-on-disk test, as is often assumed, but for any other friction law no such deduction can be drawn and ε should be measured for the contacting materials and frequency range of interest.

A test rig for measuring $\varepsilon(\omega)$ has recently been developed and tested by Wang (2008). This rig is essentially a modified pin-on-disk apparatus, with a PZT drive system to impose the required small fluctuations in sliding speed. The design constraints on such a rig are quite challenging, but preliminary results have been obtained and further developments can be hoped for. Prior to this work, there appear to have been no systematic efforts to measure $\varepsilon(\omega)$. Experimental and theoretical work has been done for the quasi-static case where there is no overall sliding and microslip occurs (e.g., Johnson, 1961, 1985). Chen and Zhou (2003) described low frequency tests to explore the velocity dependence of friction but the oscillation was not superposed on a mean sliding velocity so the direction of the frictional force changed for each cycle and the results cannot be interpreted using linear theory. The influence of vibration *normal* to the contact surface on the mean coefficient of friction has been extensively studied (Tolstoi et al., 1973, is an early example), but this is not relevant to the present study.

2.2 Dynamic modelling

The system to be analysed is sketched in Figure 1. A 'disk' is driven at steady velocity, V_0, and a 'brake' is pushed against it with a dynamically varying normal force, N, composed of a steady equilibrium pre-load, N_0, plus a small fluctuating component, N', such that $N = N_0 + N'e^{i\omega t}$. Similarly, the force tangential to the sliding direction due to friction, F, can be expressed as a steady equilibrium force, F_0, plus a fluctuating component, F', such that $F = F_0 + F'e^{i\omega t}$. With a Coulomb friction law the normal and tangential forces are related by $F = \mu_0 N$ where μ_0 is the coefficient of friction. Consequently the sign of μ_0 defines the direction of rotation of the disk: if V_0 is positive then μ_0 is negative and vice versa. The normal and tangential displacements from equilibrium of the disk are denoted u_1 and v_1 respectively, and u_2 and v_2 for the brake. The normal and tangential displacements from equilibrium of the point of contact are denoted u_3 and v_3. The springs of stiffness k_n and k_t represent the linearised contact stiffness in the normal and tangential directions respectively. Any damping that may result from the contact has been ignored.

Figure 1 Two linear subsystems coupled by a single point sliding contact with definition of variables (see online version for colours)



Displacements represented by solid arrows, forces represented by open arrows.

The dynamics of the disk and brake can be described in terms of transfer functions:

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} G_{11}(\omega) & G_{12}(\omega) \\ G_{21}(\omega) & G_{22}(\omega) \end{bmatrix} \begin{bmatrix} N' \\ F' \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix} \begin{bmatrix} N' \\ F' \end{bmatrix} \tag{3}$$

where $G_{ij}(\omega)$ are the transfer functions representing the disk's response and $H_{ij}(\omega)$ represent the equivalent responses for the brake. These transfer functions can be determined using standard vibration measurement techniques. The convention of the vibration literature is followed by using transfer functions defined as the Fourier transform of an impulse response. To convert to the Laplace formalism the complex ω-plane should be rotated anticlockwise by 90° to correspond to the complex s-plane as $s = i\omega$.

Assuming a constant coefficient of friction, this system will be stable if and only if all zeros of the characteristic function $D(\omega)$ lie in the upper half-plane, where:

$$D = G_{11} + \mu_0 G_{12} + H_{11} + \mu_0 H_{12} + 1/k_n \tag{4}$$

as derived by Duffour and Woodhouse (2004a, 2004b). The corresponding condition for stability in the Laplace formalism would require all the zeros to lie in the left half-plane.

If a linearised velocity-dependent coefficient of friction is included, the relationship between F and N can be written:

$$F \approx [\mu_0 - i\omega\varepsilon\,(v_1 + v_3)]N \tag{5}$$

and only first order terms are kept on expansion. The factor $i\omega$ converts the displacements v_1 and v_3 into velocities and ε is as discussed in the previous section. Considering the transfer functions from any possible input to any output results in two characteristic functions:

$$E_1 = D + i\omega\varepsilon N_0[(G_{11} + H_{11} + 1/k_n)(G_{22} + H_{22} + 1/k_t) - (G_{12} + H_{12})]^2 \quad (6)$$

$$E_2 = D + i\omega\varepsilon N_0(G_{22} + H_{22} + 1/k_t). \quad (7)$$

The system will be unstable if and only if all the zeros of both $E_1(\omega)$ and $E_2(\omega)$ lie in the upper half-plane (Butlin and Woodhouse, 2009).

3 Prediction issues

The framework presented above can be used to describe any two subsystems coupled at a single point by a sliding contact. In order to obtain a prediction, the various subsystem and contact parameters must be established. These measurements are inherently uncertain and this will affect the predictions to an extent determined by both the degree of uncertainty and relative sensitivity of each parameter. The reliability of predictions is also affected by the sensitivity to modelling changes: if the successive inclusion of increasing detail does not lead to converging predictions then little confidence can be placed in them. This raises the question as to the usefulness of very simplistic models, such as low-order lumped-parameter models, in capturing the mechanisms of squeal.

3.1 Method and model system

In order to obtain a prediction, the transfer function matrices of each subsystem must first be determined in a suitable parametric form. If the system is a theoretical one, the modal properties can be found by standard analysis methods. If it is a physical system the transfer function can be measured and then mathematically fitted using pole-residue extraction techniques (e.g., Ewins, 2000) to infer modal properties of the systems, in particular the natural frequencies, damping factors and modal amplitudes at the contact point. These properties can be used to obtain the characteristic functions (equations (6) and (7)). After truncation of the number of modes at a chosen number, and some manipulation, these give polynomial equations which can be solved to obtain the complex roots which approximate the poles of the sliding-coupled system. Each stage of the prediction process introduces uncertainties and it is of interest to understand to what extent the predictions are affected by them.

In order to show an example, a particular system has been selected for study. This consists of a circular disk, and a point-contact ‘brake’ subsystem with carefully designed dynamic characteristics to give only a small number of modes in the frequency range studied (up to 15 kHz). Experimental results for this system will be presented in Section 4; more details are given by Duffour and Woodhouse (2007). The transfer function matrices *G* and *H* for the two subsystems were measured using standard vibration test methods and fitted using modal analysis procedures. The natural frequencies of the dominant modes below 15 kHz of each subsystem are summarised in Table 1.

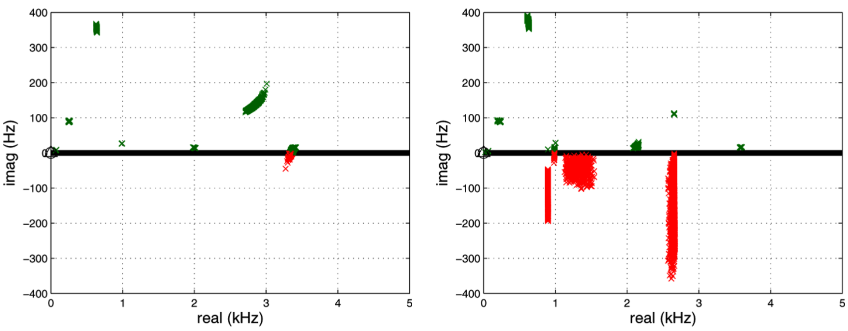
Table 1 Summary of natural frequencies of the pin and disk modes. The notation (m, n) denotes a disk mode with m nodal diameters and n nodal circles

	0–1 kHz				*1–5 kHz*				*5–10 kHz*					*10–15 kHz*		
Natural frequency (Hz)	*20*	*193*	*613*	*896*	*1002*	*2083*	*2656*	*3573*	*5380*	*6327*	*7469*	*8698*	*9806*	*11154*	*11571*	*12365*
Pin mode No.	1			2			3							4		
Disk mode No.		1	2		3	4		5	6	7	8	9	10		11	12
Disk mode type		(1, 0)	(0, 0)		(2, 0)	(3, 0)		(4, 0)	(5, 0)	(2, 1)	(6, 0)	(3, 1)	(7, 0)		(4, 1)	(8, 0)

3.2 *Quantifying sensitivity*

A straightforward way to explore the effect of uncertainties on predictions is to repeatedly solve the characteristic equation, each time using different values for the system parameters within a given range. Experimental work described in more detail in Section 4 has indicated that natural frequencies for this system are uncertain to within 0.05%, damping factors to 2%, modal amplitudes to 10% and the coefficient of friction to 20%. Figure 2(a) shows an example of a 'cloud plot' of the poles obtained by solving the characteristic equation 1000 times using these representative uncertainties for the range of parameters considered, using a model that neglects contact stiffness and any velocity-dependence of the coefficient of friction. It is interesting that some of the clusters of poles are rather robust to these perturbations, while others are much more sensitive. In some cases, the uncertainty is significant enough to affect the stability of the poles. Figure 2(b) shows the equivalent plot but with both contact stiffness and a velocity-dependent coefficient of friction included in the model. For this case it is estimated that the contact stiffnesses and ε are uncertain to within 50%. As might be expected, the uncertainty has a more significant effect, though further tests reveal that the large uncertainty estimates of the contact parameters are not always the most significant contributors to the prediction uncertainties (for more details see Butlin, 2007).

Figure 2 Effect of representative parametric uncertainties on the solutions to the characteristic equation in the complex-ω plane: stable solutions have a positive imaginary part: (a) k_n and ε neglected and (b) k_n and ε included (see online version for colours)



(a) (b)

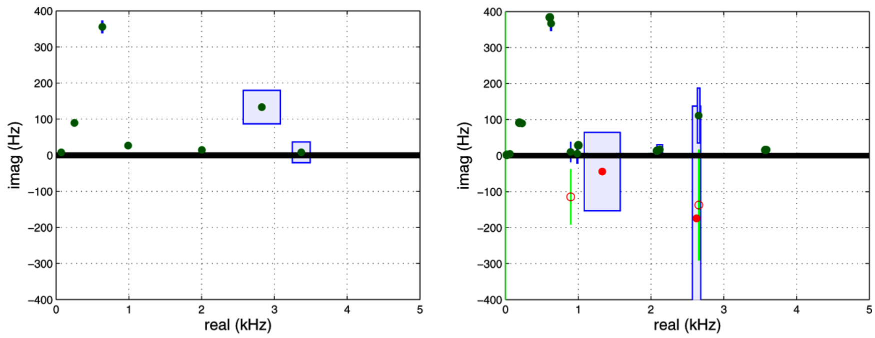
Estimating sensitivity by repeatedly solving the characteristic equation is computationally expensive, motivating a need for a more efficient estimate of the effects of uncertainty.

A first-order perturbation analysis can be carried out, taking into account some of the structure of the uncertainty. Assuming small changes in the coefficients of the characteristic polynomial, an expression for the approximate change in root location can be shown to be:

$$\delta\lambda_i \approx \frac{\lambda_i \sum_{k=0}^{N-1} \delta p_k \lambda_i^k}{\sum_{k=0}^{N-1} (N-k)\, p_k \lambda_i^k} \tag{8}$$

where λ_i is the nominal location of the ith root and each δp_k is the change in the characteristic polynomial coefficient p_k. The changes in coefficients are related to the parametric uncertainties and can be calculated directly, providing a method for estimating the effect of uncertainties on predictions. This can be validated against repeatedly solving the characteristic equation using matching distributions of parametric uncertainty. Figure 3 shows the equivalent estimated prediction uncertainties to the cloud plots of Figure 2: it can be seen that the estimate is good even for the more sensitive roots. The nominal characteristic equation needs to be solved only once, increasing the computational speed of uncertainty estimates by a factor of approximately 40 for this particular Matlab implementation.

Figure 1 First-order estimate of the effect of uncertainties on the solutions to the characteristic equation: nominal poles shown as dots, uncertainty ranges by rectangular boxes: (a) k_n and ε neglected and (b) k_n and ε included (see online version for colours)



(a) (b)

3.3 Low-order models, sensitivity and convergence

A significant proportion of the friction-induced vibration literature focuses on highly simplified models with very few degrees of freedom (e.g., Hoffmann et al., 2002; Kinkaid et al., 2005; von Wagner et al., 2007; Emira, 2007). Their simplicity enables clearer analysis than more realistic models and provides some intuition as to the mechanisms that can lead to squeal. However, the question of their accuracy for approximating more complicated systems is often left untested. A study of very low-order models has demonstrated that predictions can be highly sensitive to system parameters under sometimes surprising conditions (Butlin and Woodhouse, 2009). Related to this issue is the sensitivity of predictions to changes in the complexity of the model, rather than just parameter values. This is important to understand, as even large-scale models that include all identified modes over a wide bandwidth are nevertheless approximations of the true system. If these models are to have any predictive value, their output must converge as the model complexity is increased.

A systematic sequence of tests has been carried out on the system of Section 3.1: again we choose a representative subset that illustrates interesting convergence behaviour. For this sequence, only the number of modes is changed, and we choose the case which neglects both contact stiffness and a velocity-dependent coefficient of friction. It is insufficient simply to compare the nominal predictions of different models: if models are to provide good approximations to the global model, the relative trends as parameters vary should also be similar. To illustrate the idea two parameters are chosen to vary, the natural frequency of the third pin mode by ±10% and the coefficient of friction from 0.4 to 0.6.

Table 2 shows the relevant modes from Table 1, together with the sequence of uncoupled modes included in the models. The dotted line gives an indication of the frequency of an instability predicted by the full model. The first model includes just the third pin mode and fourth disk mode (3 nodal diameters, 0 nodal circles). Figure 4(a) shows a comparison of predictions from this model (crosses) with those from the full 15-mode model (circles). The vertical dotted lines mark the minimum and maximum natural frequencies included in each case, representing the bandwidth over which the low-order model might be expected to approximate the full model. It is very clear that predictions from this two-mode model do not approximate the full model in any useful way: sensitivities and root locations are both very different.

Table 2 Sequence of uncoupled modes included in reduced-order models, with numbered pin and disk modes. Sequence (a)–(d) corresponds to sequence in Figure 4

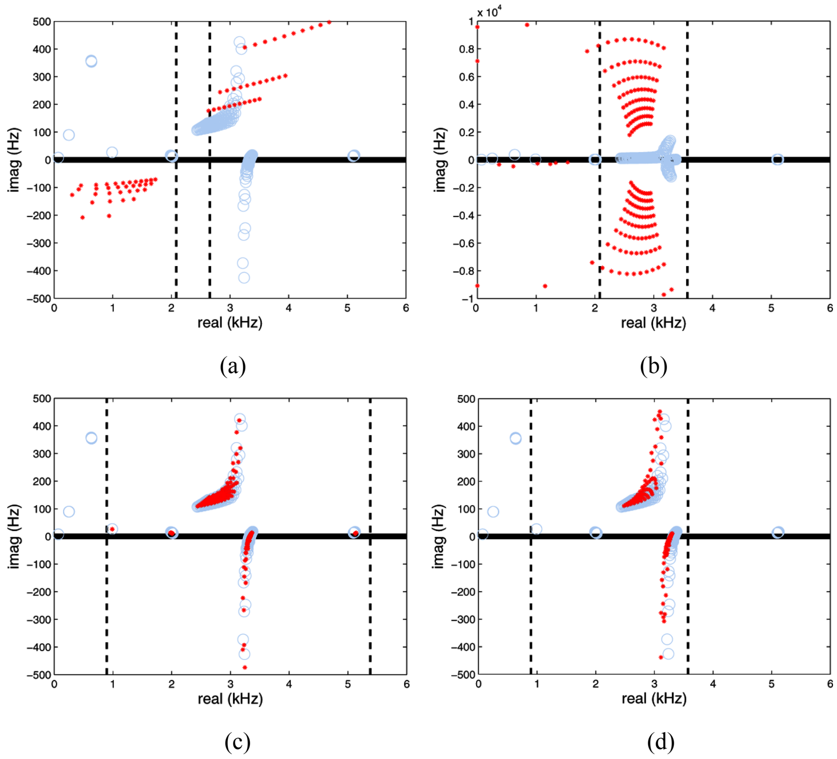
Frequency (Hz)	*896*	*1002*	*2083*	*2656*	*3573*	*5380*
Pin mode No.	2			3		
Disk mode No.		$3^{(2,0)}$	$4^{(3,0)}$		$5^{(4,0)}$	$6^{(5,0)}$
(a)			+	+		
(b)			+	+	+	
(c)	+	+	+	+	+	+
(d)	+			+	+	

The superscript (m, n) denotes a disk mode with m nodal diameters and n nodal circles.

Figure 4(b) shows the next logical step, adding the fifth disk mode (4, 0) to the low-order model. Again the three-mode model is a poor approximation, but it is interesting that the addition of an extra mode also causes a very sensitive change in the predictions such that it bears little resemblance to the two-mode predictions: the vertical scale of the plot has been changed to show this. Sequentially adding modes symmetrically about the third pin mode continues to show little convergence to predictions from the full model until the six-mode model, shown in Figure 4(c), that includes the second and third pin modes and disk modes three to six (2, 0), (3, 0), (4, 0), (5, 0). Convergence occurs suddenly and only upon inclusion of the most remote mode: the second pin mode. In this case predictions from the two models are almost indistinguishable.

The step improvement upon inclusion of the second pin mode suggests a further test. Figure 4(d) shows a comparison of predictions when some of the intermediate disk modes are neglected, leaving three modes: the second and third pin modes and the fifth disk mode (4, 0). It can be seen that this three-mode model provides a rather good approximation to the full model. Further tests showed that usually three modes could be found that provided a good 'local' approximation to any given feature of the full-model prediction, but that their choice is non-obvious as this example has illustrated. Thus low-order models can be valuable, but guaranteeing their validity for describing more general systems is difficult as predictions may be highly sensitive to neglected modes in addition to system parameters. Great care must be taken when generalising conclusions from such models.

Figure 4 Comparisons between predictions from low- and full-order models, with two parameters varying in order to compare the trends of the predictions: (*) low-order model prediction; (O) full-order model prediction: (a) pin 3/disk 4; (b) pin 3/disk 4, 5; (c) pin 2, 3/ disk 3, 4, 5, 6 and (d) pin 2, 3/disk 5 (see online version for colours)



4 Experimental studies

4.1 Measurement methodology

The pin-on-disk test rig used for the experimental work is sketched in Figure 5. A motor drives an aluminium disk of radius 129 mm. A polycarbonate pin is mounted on a dynamometer fitted with strain gauges to measure the normal and tangential forces. The dynamometer acts as a rigid body over the frequency range of interest (0–15 kHz) and is in turn mounted to one of two thin metal strips as illustrated in Figure 6.

The modes of this assembly are responsible for most of the dynamics of the pin subsystem in the frequency range of interest. The two strips allowed two different systems to be tested: asymmetrical and nominally symmetrical. (This comparison is interesting because theoretical predictions for any symmetric assembly are independent of the steady sliding direction.) The strip is mounted to a heavy bracket grounded by leaf springs and having a low natural frequency compared with the rest of the system dynamics. A soft spring and screw provide a means of adjusting the normal pre-load of the pin on the disk. The two subsystems can be separated in order to measure their transfer function matrices. In addition, extra masses (1, 4 and 14 g) can be attached to the dynamometer to perturb the system by varying amounts in order to experimentally explore sensitivity to perturbations. There are two stages to the experimental work: measuring the transfer function matrices and carrying out sliding contact tests to observe the coupled behaviour.

Figure 5 Diagrammatic sketch of the pin-on-disk test rig

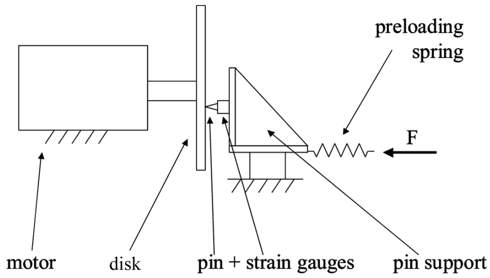
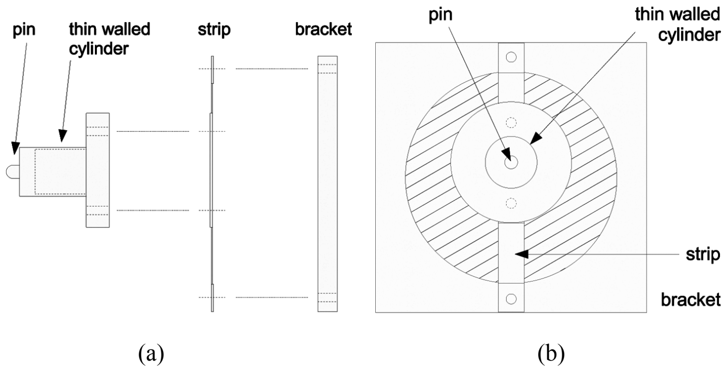
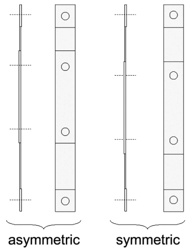


Figure 6 The pin assembly: (a) from side; (b) from disk and (c) asymmetric and symmetric metal strips





(c)

The transfer function matrices of each subsystem were measured and fitted for each pin assembly (symmetric and asymmetric) and each perturbation level. Each measurement consisted of the average of 40 consecutive measurements and the modal parameters for each of the 40 were estimated. This provided a mean value to be used in the characteristic equation and a measure of the scatter, used as an input to the first-order perturbation analysis.

Sliding contact tests were carried out using both symmetric and asymmetric pin assemblies with the disk rotating in either the clockwise or anticlockwise direction. For each direction four different disk speeds were tested. For each pin assembly, different levels of perturbation were applied, overall resulting in 64 tests that were repeated each day for 20 days.

Each test within the sequence occupied a 40 second period. With the disk in motion, the pin was brought into contact with the disk and the normal pre-load increased until squeal was audible. The normal pre-load was then decreased until squeal stopped and the cycle repeated in order to obtain as many initiations as possible. This varied from 1–40 depending on the test. The polycarbonate pin was replaced every 32 tests and worn in to a set procedure. This prevented the contact area of the hemispherical pin tip from varying excessively due to wear. For further discussion regarding the repeatability of results see Butlin (2007).

Predictions from linear models can only describe the onset of instability, not the subsequent non-linear limit cycle reached. Comparisons of fully developed squeal frequencies with those predicted by linear models may therefore be misleading: although they may sometimes be correlated there is evidence to suggest that this is not always so (e.g., Chen and Zhou, 2007; Duffour, 2002). In addition, squeal occurrences could be the result of an intrinsically non-linear route to instability: in this case a linear model would not be expected to predict it.

A data-processing method was therefore developed to extract the initial growth of instabilities and assess their consistency with a linear time-invariant model. The process is summarised in Figure 7. Firstly initiations are extracted from the time series data. A sonogram (time-frequency analysis) of this section of data is then calculated and the dominant frequency estimated. The amplitude of this frequency can then be found as a function of time. If the event is to be consistent with the assumed model then exponential growth is expected. A least squares line of best fit can be used to find the rate of growth. A normalised measure of the least squares error quantifies the degree of confidence that can be placed in the measurement. In addition, measurements for which the normal pre-load varied by more than 10 % during the initiation were discarded as being inconsistent with a time-invariant model.

4.2 Comparisons with predictions

Over the course of the experiment a total of 5731 squeal initiations were identified as ‘acceptable’. This large volume of data can be analysed in many ways (see Butlin, 2007). This paper describes only a minimal, but representative, comparison with theoretical predictions. For the following discussion the system chosen is the symmetrical pin assembly.

Figure 7 Example initiation to illustrate stages of data processing: (a) normal velocity and envelope; (b) envelope and identified initiation; (c) sonogram of initiation (grey scale used is logarithmic, with a displayed range of 20 dB) and (d) fitting exponential growth (see online version for colours)

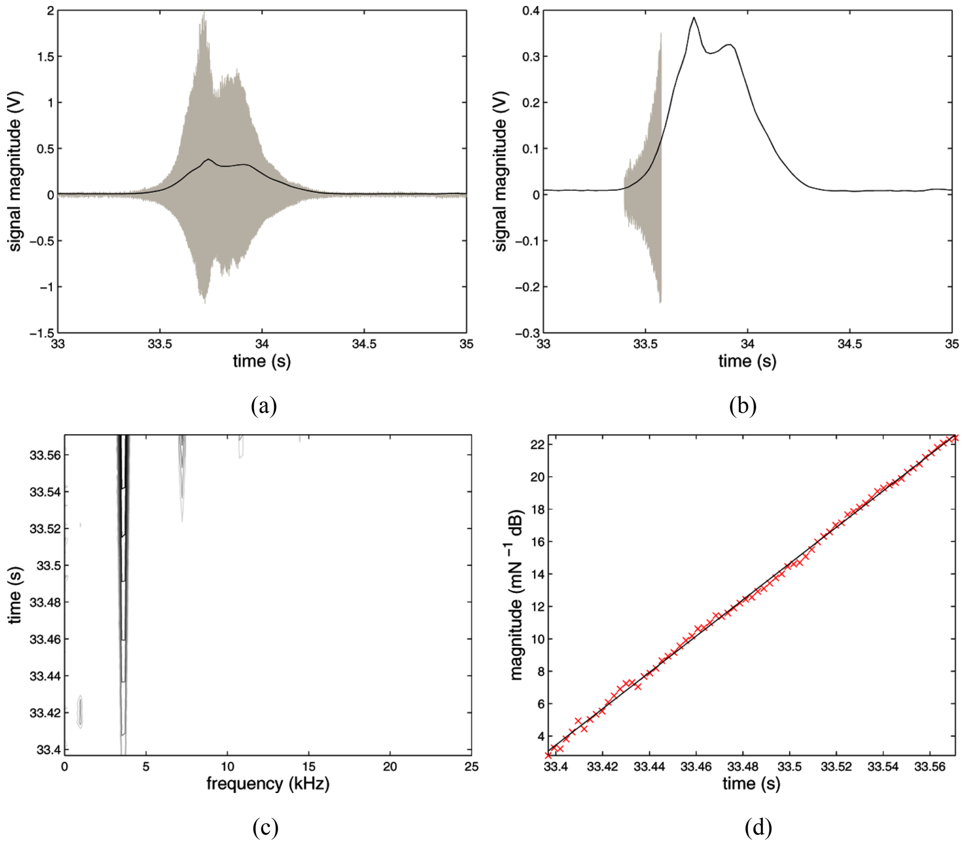
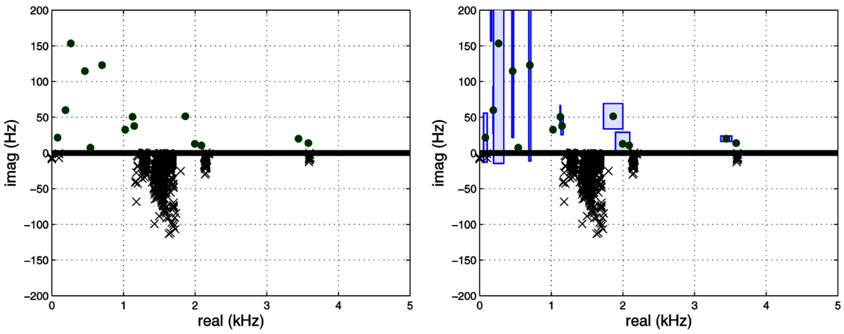


Figure 8(a) shows a comparison between the nominal predictions (dots) and the poles inferred from the measured initiations (crosses) using the simplest model that neglects both contact stiffness and any velocity dependence of friction. The coefficient of friction throughout this section is taken to be 0.5 and only uncoupled modes with natural frequencies less than 5 kHz have been included in the model. The restricted bandwidth avoids numerical sensitivity when a velocity-dependent coefficient of friction is included. It would seem that none of the instabilities are predicted, particularly the large cluster around 1.5 kHz. Figure 8(b) shows the same comparison but with shaded rectangles representing first-order uncertainty estimates using the method described in Section 3.2 together with the measured uncertainties described in Section 4.1. This confirms that the largest cluster is unaccounted for and while the feasible frequency ranges of the 2.1 kHz and 3.5 kHz clusters may just encompass the measured frequencies, instability is still not predicted.

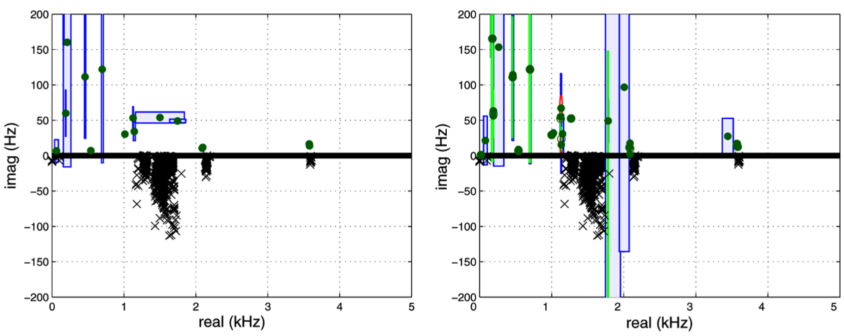
Figure 8 Comparison of predicted poles (•) with measured coupled poles (×). Model does not include either contact stiffness or a velocity-dependent coefficient of friction. Shaded rectangles in (b) represent uncertainty estimates: (a) nominal prediction and (b) prediction with estimated uncertainty (see online version for colours)



(a) (b)

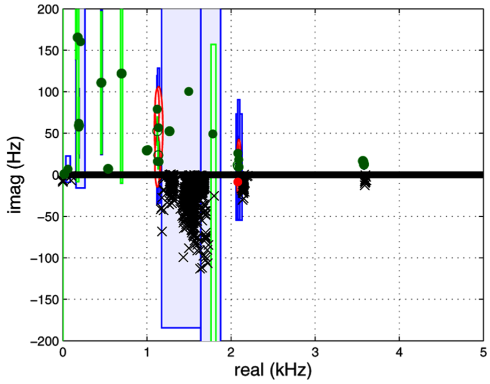
Figure 9(a) shows the case when only contact stiffness is included in the model ($k_n = 2 \times 10^6$ Nm^{-1} ± 50%) and Figure 9(b) shows the case when only a velocity dependent coefficient of friction is included ($\varepsilon N_0 = 20$ Nsm^{-1} ± 50%): in both cases there is still little agreement between predictions and measurements. Only when both contact stiffness and a velocity-dependent coefficient of friction are included in the model is reasonable agreement obtained. Figure 10 shows this case, using independently measured values for these parameters: $k_n = k_t = 2 \times 10^6$ Nm^{-1} ± 50%, $\varepsilon N_0 = 20$ Nsm^{-1} ± 50%. The predictions now feasibly account for almost all of the observed data points. The large cluster around 1.5 kHz is overlaid by the large box corresponding to a nominally stable but highly uncertain pole of the same frequency. The observed poles near 2.1 kHz are on the borderline of the estimated error bounds. The instability near 3.5 kHz remains unaccounted for. A likely explanation is that all of the data for the clockwise symmetric case has been included in these comparisons and the particular cluster near 3.5 kHz is made up of initiations that only occurred when the largest perturbation of 14 g was added: for more detail see Butlin (2007).

Figure 9 Comparison of predicted poles (•) with measured coupled poles (×). Model includes (a) only contact stiffness and (b) only a velocity-dependent coefficient of friction. Shaded rectangles represent uncertainty estimates: (a) only k_n included in model and (b) only ε included in model (see online version for colours)



(a) (b)

Figure 10 Comparison of predicted poles (•) with measured coupled poles (×). Model includes both contact stiffness and a velocity-dependent coefficient of friction (see online version for colours)



The level of agreement between predictions and experimental results is now moderately encouraging. This would not have been possible to see without taking account of uncertainty and sensitivity, both in the predictions and the measurements. Perhaps the greatest value of the data and uncertainty estimates is to inform the process of developing the model: the above analysis suggests that in order to predict the observed instabilities within the linearised framework, a model that includes both contact stiffness and a velocity-dependent coefficient of friction is needed.

5 Conclusions

Friction-induced vibration is well known to be a twitchy phenomenon: nominally identical experiments often produce different behaviour. This paper has opened several avenues of fruitful investigation by regarding twitchiness as important part of the phenomenon, rather than as simply an experimental difficulty to overcome. The model chosen for study is based on a linear transfer function analysis of two systems coupled by a single point sliding contact: simple enough to be mathematically tractable but general enough to be amenable to experimental testing. Linear instability is arguably the simplest route to squeal and has the largest representation within the literature, making it a logical and uncontroversial starting point for a systematic study. Future models could be extended to include more complicated contact geometries, but the single-point system already captures a rich range of behaviour and raises some important issues.

The twitchiness of squeal has often been attributed to the frictional interface. The most common models include a coefficient of friction that varies with velocity. This paper brings to attention a generalisation of this concept: within the context of linear models, the relationship between the frictional force and change in sliding velocity can be written as a transfer function. If velocity is the only dependent parameter, then this will just be a constant value. However, it is highly likely that the coefficient of friction depends on a variety of other parameters, which will lead to a frequency-dependent, complex relationship. No change is needed in the methodology of squeal prediction to account for any such relationship, but at present there is almost no measured data for friction materials of interest.

Stability predictions were obtained using transfer function measurements of a test pin-on-disk system. Using representative parametric uncertainties as an input to Monte Carlo simulations and a first-order perturbation analysis, the effect of realistic uncertainty and sensitivity were shown to be significant enough to affect the stability of some predicted poles.

The convergence behaviour of low-order models was demonstrated to be non-trivial. As the model complexity was increased by progressively including nearby modes, convergence to the full model sometimes occurred suddenly upon the inclusion of a particular mode, sometimes quite remote in frequency from the squeal instability. While usually three uncoupled modes can be identified that sufficiently approximate the complete model, choosing which three modes are required to describe a given bandwidth is very difficult. This highlights that predictions can be sensitive to modelling changes in addition to parameter changes, and raises a warning as to the accuracy of low-order models in locally approximating more general systems. This contrasts with the situation in normal vibration theory, where such local approximations are a powerful tool (e.g., Skudrzyk, 1968).

A large-scale experiment was carried out with the aim of generating a large number of initiations of squeal. A data-processing strategy estimated frequencies and growth rates from squeal initiations, allowing direct comparison with predictions including explicit allowance for sensitivity. Combined with first-order estimates of the prediction uncertainties, this enabled logical development of the model details. For the test case presented it was seen that both contact stiffness and a velocity-dependent coefficient of friction were needed within the model to predict the observed instabilities.

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