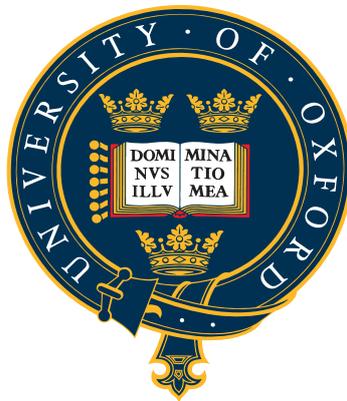


**Computing Laboratory**

THREE QUBIT ENTANGLEMENT  
IN GRAPHICAL Z/X-CALCULUS

Bob Coecke and Bill Edwards

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Oxford University Computing Laboratory  
Wolfson Building, Parks Road, Oxford, OX1 3QD

## Abstract

The compositional techniques of categorical quantum mechanics are applied to analyse 3-qubit quantum entanglement. In particular the graphical calculus of complementary observables and corresponding phases due to Duncan and one of the authors is used to construct representative members of the two genuinely tripartite SLOCC classes of 3-qubit entangled states, GHZ and W. This nicely illustrates the respectively pairwise and global tripartite entanglement found in the W- and GHZ-class states. A new concept of supplementarity allows us to characterise inhabitants of the W class within the abstract diagrammatic calculus; these method extends to more general multipartite qubit states.

## 1 Introduction

The structure of multipartite entanglement has been a subject of much research in recent years. Much work has been done on trying to classify the entanglement in many body states (for example [11, 19, 16]), and investigating the properties and uses of particular multipartite entangled states, for example graph states in measurement based quantum computing [12]. Three qubit states with genuine tripartite entanglement fall into two SLOCC<sup>1</sup>-classes: one inhabited by the GHZ-state, which is a graph states, and one inhabited by the W-state [11]. But beyond these states, there is little structural understanding of general multipartite entangled states; even their classification remains mysterious. Such states also remain virtually untapped as resources for quantum information processing protocols.

Most of the work described above has so far employed rather technical arguments using linear algebra. This paper describes some tentative steps towards a different, *compositional* approach, where we view entangled states as being built up from simpler components. This approach springs from the programme, initiated by Abramsky and one of the authors [1], to analyse quantum mechanics in terms of symmetric monoidal categories. We assume that the reader is familiar with the basics of this approach; for an introductory account see [3]. One feature of this programme is that it allows us to use a very intuitive graphical language to describe quantum states and processes, and this is utilised in much of what follows.

Here we show how to build up examples from the W SLOCC class using simple graphical building blocks; the GHZ-state is itself one of the building blocks. These building blocks are categorical structures called *basis structures* [7]; in the categorical quantum mechanics programme these provide an abstract counterpart to orthonormal bases. In particular we will use a pair of *complementary* basis structures - this notion was introduced by Duncan and one of the authors [4] to model mutually unbiased bases. Their *graphical Z/X-calculus* employed a vivid graphical convention where two complementary basis structures were depicted using green and red dots respectively.

To understand the difference between the GHZ class states and W class states within the context of abstract categorical quantum mechanics, we introduce a concept of the *supplementarity* of certain elements relative to complementary basis structures; W states then arise in situations of supplementarity. We sketch how this concept leads to distinct subclasses of more general multipartite qubit states.

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<sup>1</sup>SLOCC: Stochastic Local Operations and Classical Communication.

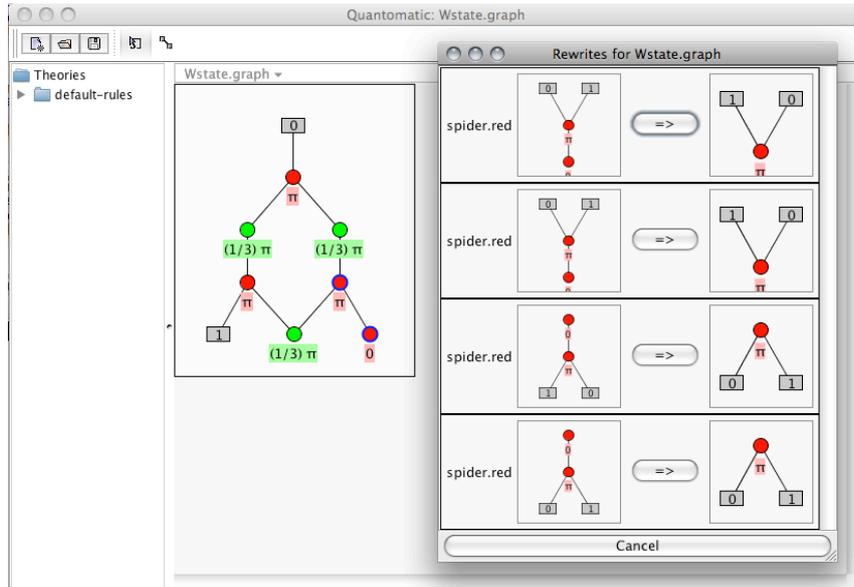


Figure 1: Screenshot of some suggested rewrites in `quantomatic`.

One important advantage of the graphical presentation of entangled states is that it is subject to automated reasoning, by means of the `quantomatic` software [10], developed by Dixon, Duncan, Kissinger and Merry. Meanwhile, the results in this paper have also led to a new graphical calculus, which takes both the GHZ and W state as its primitives [5]. Rather than being mediated by the laws of complementarity, this graphical GHZ/W-calculus is mediated by the laws of basic arithmetic [6].

This paper is structured as follows: in section 2 we give a brief introduction to some key aspects of three-qubit entanglement, and section 3 reviews some essential properties of complementary basis structures. Next, in section 4 we show (via concrete linear algebra calculations) how certain states from the GHZ and W classes can be built up from the morphisms of a pair of complementary basis structures. In section 5 we show how one can identify W class states within the graphical calculus by means of the notion of supplementarity. We conclude with some speculations on whether our methods can be used for the study of general multipartite entanglement.

## 2 Background: 3-qubit entanglement

Entangled states are classified primarily according to the following criterion: If state  $|\psi\rangle$  can be transformed into state  $|\phi\rangle$  via local operations on the components of the system (we also allow classical communication between the agents acting on the components, so that they can condition their operations on the outcomes of measurements performed by other agents, for example) then  $|\psi\rangle$  is more entangled than, or as entangled as  $|\phi\rangle$ .

In the case of bipartite systems (with components whose state spaces have arbitrary dimen-

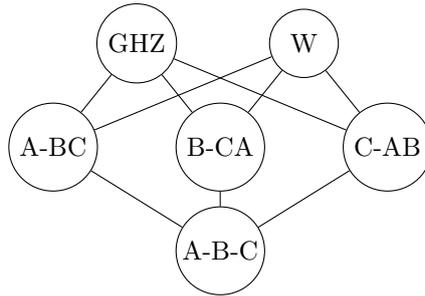
sion) the entanglement of quantum states has been completely classified, in that there exists a well-defined mathematical criterion for determining whether one state can be transformed into another via LOCC (local operations and classical communication). This leads to the well-known *majorization order* [17].

For systems with more than two subsystems (henceforth we will refer to this as *multipartite* entanglement) no such classification has been achieved. However in certain cases a weaker classification has been achieved. This is based on whether one state can be converted into another via LOCC *with some non-zero probability*. In this case we say that the two states are related via SLOCC (*stochastic* local operations and classical communication).

Mathematically SLOCC translates into a very simple condition [11]. For the  $n$ -partite state  $|\psi\rangle$  to be transformable into  $|\phi\rangle$  via SLOCC there must exist local linear operations  $A_1, A_2, \dots, A_n$  such that:

$$|\phi\rangle = A_1 \otimes A_2 \otimes \dots \otimes A_n |\psi\rangle \quad (2.1)$$

In practice determining whether this condition is satisfied for two states is not straightforward. However in the case of three qubits the states have been completely classified under SLOCC [11]. There are six classes of states arranged in a hierarchy (in fact a partial order). The states of a given class are all interconvertible under SLOCC. States from higher classes can be converted via SLOCC into states from lower classes.



The bottom class contains the completely separable states. The middle three classes consist of states where one system is unentangled, while the other two are entangled with each other. There are two classes of true tripartite entanglement, not interconvertible, each named after a particular member state:

$$|\Psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \quad (2.2)$$

$$|\Psi_W\rangle = \frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle) \quad (2.3)$$

Is there any intuitive difference between the GHZ-class and W-class states? One difference is mostly clearly seen using a measure of entanglement called the *tangle*, introduced in [9]. Label the three qubits  $A$ ,  $B$  and  $C$ . Strictly the tangle is defined on mixed states of two qubits. However it is possible to sensibly extend this definition and calculate the tangle,  $\tau_{A(BC)}$  for the entanglement between qubit  $A$  and qubits  $B$  and  $C$  viewed as a single 4-dimensional system. The tangle for the entanglement between  $A$  and  $B$ ,  $\tau_{AB}$  and between  $A$  and  $C$ ,  $\tau_{AC}$ , can also be calculated. The following inequality then always holds:

$$\tau_{A(BC)} \geq \tau_{AB} + \tau_{AC} \quad (2.4)$$

and likewise for permutations of  $A$ ,  $B$  and  $C$ . Interestingly the quantities:

$$\tau_{A(BC)} - \tau_{AB} - \tau_{AC} \tag{2.5}$$

$$\tau_{B(AC)} - \tau_{AB} - \tau_{BC} \tag{2.6}$$

$$\tau_{C(AB)} - \tau_{BC} - \tau_{AC} \tag{2.7}$$

are all equal. This quantity is termed the *3-tangle* and is denoted  $\tau_{ABC}$ . We can now write, for example:

$$\tau_{A(BC)} = \tau_{AB} + \tau_{AC} + \tau_{ABC} \tag{2.8}$$

This seems to say that the entanglement between  $A$  and the combined system of  $B$  and  $C$  consists of the pairwise entanglement of  $A$  with  $B$  and  $A$  with  $C$ , plus some kind of global tripartite entanglement, quantified by the 3-tangle.

It can be shown [11] that all GHZ-class states have non-zero 3-tangle, while all W states have zero 3-tangle. Thus it would seem that in some sense, the entanglement in W-class states is all pairwise, or local, while in the case of GHZ-class states, at least some of the entanglement is genuinely global, shared between all three qubits.

### 3 Background: Red-Green calculus

We consider as given a dagger symmetric monoidal category ( $\dagger$ -SMC) [1], that is, a symmetric monoidal category with an identity-on-objects involutive contravariant functor  $f \mapsto f^\dagger$ . We will work within its corresponding graphical calculus [13].

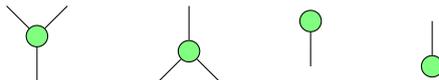
In a  $\dagger$ -SMC a *basis structure* is a dagger special commutative Frobenius algebra [7] - i.e. a refinement of Carboni and Walters' Frobenius algebras [2]. It consists of an internal commutative monoid

$$(A, \delta : A \rightarrow A \otimes A, \epsilon : A \rightarrow I)$$

for which  $\delta$  and  $\delta^\dagger$  satisfy the specialness and Frobenius equations, that is:



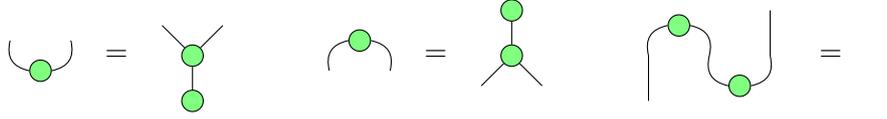
where we represented  $\delta$ ,  $\delta^\dagger$ ,  $\epsilon$  and  $\epsilon^\dagger$  graphically as:



A key result holding for any basis structure is the *spider theorem* - a high-level abstract account of which is due to Lack [15]. This spider theorem essentially allows us, when working with the graphical language, to fuse together any directly connected dots representing  $\delta$ ,  $\delta^\dagger$ ,  $\epsilon$  and  $\epsilon^\dagger$  from the same basis structure: the theorem guarantees that all morphisms which *look* the same graphically after such a fusing procedure *are* indeed equal. In fact, this property

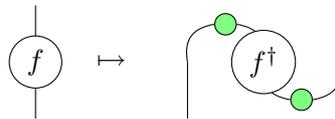
provides an equivalent definition of basis structures, since all the defining axioms of a basis structure are implied by it [3, 4].

Among many other things, the spider theorem implies that we have a compact structure [14]:



Such a compact structure allows one to exchange the roles of inputs to outputs, and vice versa, so we can essentially ignore those roles. When writing equations it suffices to identify open-ended wires in the LHS with those in the RHS.

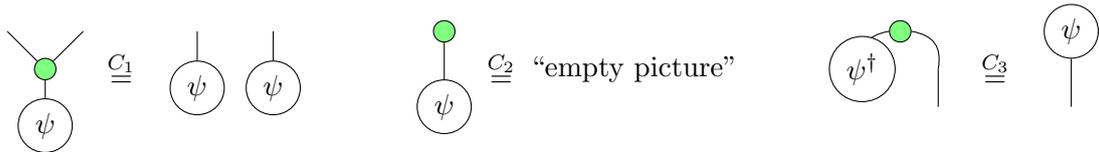
Compact structure induces a covariant involutive *conjugation* functor:



In the category **FdHilb** of finite dimensional Hilbert spaces, linear maps, tensor products and adjoints, basis structures are in bijective correspondence with orthonormal bases [8]. Explicitly, for a Hilbert space  $\mathcal{H}$ ,  $\delta$  ‘copies’ basis vectors and  $\epsilon$  ‘uniformly erases’ them:

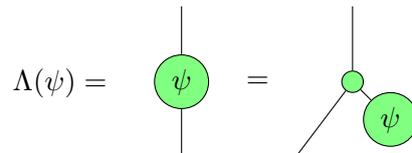
$$\delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H} :: |i\rangle \mapsto |ii\rangle \quad \text{and} \quad \epsilon : \mathcal{H} \rightarrow \mathbb{C} :: |i\rangle \mapsto 1. \quad (3.9)$$

Basis structures act as the abstract counterparts to orthonormal bases, and in analogy to the concrete case, we define the *basis elements* of general basis structures as the elements  $\psi : I \rightarrow A$  that are self-conjugate comonoid homomorphisms [4], that is, explicitly:



In the concrete basis structure of equation 3.9), conjugation boils down to conjugating matrix entries when matrices are expressed in the corresponding orthonormal basis.

In a  $\dagger$ -SMC  $\mathcal{C}$ , any basis structure induces a bijection  $\Lambda : \mathcal{C}(I, A) \rightarrow \mathcal{C}(A, A)$  between states and endomorphisms of  $A$ , depicted graphically as:



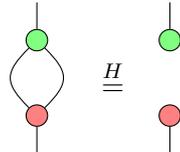
In **FdHilb**  $\Lambda(\psi)$  is unitary iff  $|\psi\rangle$  is unbiased with respect to the basis  $\{|i\rangle\}$  copied by  $\delta$ . Inspired by this, in the general setting a state  $\psi$  is defined to be *unbiased with respect to a basis structure*  $(A, \delta, \epsilon)$  iff  $\Lambda(\psi)$  is *unitary* i.e. the dagger and the inverse coincide.

States which are unbiased w.r.t. a particular basis structure are filled with the same colour as the basis structure. In the explicit case of qubits (i.e. the object  $\mathbb{C}^2$  in **FdHilb**) the unbiased states are those of the form  $|e_1\rangle + e^{i\alpha}|e_2\rangle$  where  $\{|e_i\rangle\}$  is the copied basis, and we label these states with the phase,  $\alpha$ . For example, if the green basis structure corresponds to  $\{|0\rangle, |1\rangle\}$  we respectively depict  $|0\rangle + e^{i\alpha}|1\rangle$  and  $\Lambda(|0\rangle + e^{i\alpha}|1\rangle)$  by:

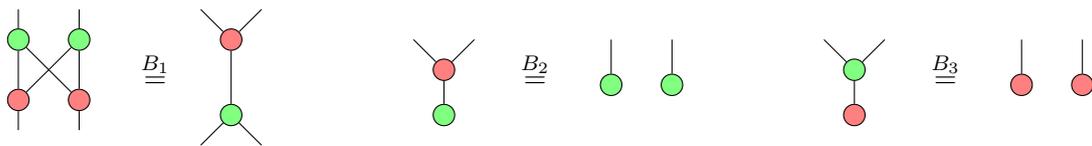


It's easy to show that the unitary  $\Lambda(\psi)$ s form a group. In the case of any basis structure on the object  $\mathbb{C}^2$  in **FdHilb** (i.e. the qubit case), these unitaries are exactly the phase rotations w.r.t. the copied basis; for this reason the group formed by the  $\Lambda(\psi)$  unitaries is termed the *phase group* [4].

In quantum mechanics the relationship between *different* orthonormal bases is clearly of crucial importance – they represent incompatible observables. Work has been done on abstractly characterising *mutually unbiased* basis structures [4] – those corresponding to bases which are unbiased w.r.t. one another. It was shown that the basis elements of one basis structure are unbiased w.r.t. the other basis structure -i.e. imitating the concrete definition of unbiased bases- if and only if [4]:



where conventionally we denote one such basis structure with green dots and the other with red dots. As also shown in [4], certain important pairs of mutually unbiased bases in quantum theory (e.g. the Z and X spin observables) obey a strictly stronger set of equations,<sup>2</sup> namely those that define up-to-scalar-multiples a so-called bialgebra [18]:



below we will omit specifying ‘up-to-scalar-multiples’ when referring to these laws.

The key feature of the graphical calculus is that particular behaviors correspond to radical changes of the topology of the picture e.g. being ‘an eigenstate’ or being ‘mutually unbiased’ both correspond to pictures decomposing into disconnected components - cf. equations ( $C_1$ ) and ( $H$ ) respectively. This is the very heart of categorical quantum mechanics: essential concepts are expressed in a language for which all can be reduced to tensor product structure, ‘disconnecting’ then standing for ‘disentanglement’. It goes without saying that topological distinctions come with clear behavioral differences. In Section 5 we will classify tripartite entanglement also according to this paradigm.

<sup>2</sup>That is, strictly stronger provided that we are considering basis structures rather than a monoid-comonoid pair.

## 4 GHZ and W states represented graphically

We now move to consider the concrete case of qubits. We will be using two mutually unbiased basis structures,  $\Delta_Z = (\mathbb{C}^2, \delta_Z, \epsilon_Z)$  which corresponds to the  $|0\rangle, |1\rangle$  basis and which will be represented graphically by green dots, and  $\Delta_X = (\mathbb{C}^2, \delta_X, \epsilon_X)$  which corresponds to the  $|+\rangle, |-\rangle$  basis and which will be represented graphically by red dots. Viewed as a tripartite state  $\mathbb{C} \rightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$  the green  $\delta_Z$  is in fact the GHZ state. Thus the GHZ state can be depicted graphically as:



The same diagram but with a red dot is also a GHZ class state: it's the state  $|+++ \rangle + |-- \rangle$  which is clearly obtained from the standard GHZ state via local basis transformations. Since any state in the GHZ class is related to the GHZ state via local linear operations (recall equation 2.1), such states can be depicted as:



where  $A_1, A_2$  and  $A_3$  are linear maps. Now consider the following diagram:



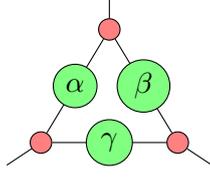
When evaluated in **FdHilb** (i.e. concretely composing the linear maps represented by the graphical components), and ignoring global phase and normalisation, we get the W state.

Comparing the two diagrams 4.10 and 4.12 it is striking to see how they seem to embody the global/local entanglement distinction discussed in section 2. In the GHZ state, 4.10, the three systems are all connected together via the green dot, mirroring the genuine global tripartite entanglement which they share in this state. In contrast, in the W state, 4.12, the systems are connected in a pairwise fashion, mirroring the pairwise entanglement in this state.

If we change the phases, from  $\pi/3$  to 0 then we would wind up with a GHZ-class state:



So depending on the choice of phases we may end up with a W-class or a GHZ-class state. For the general case:



$$(4.14)$$

we will now verify for which values of  $\alpha$ ,  $\beta$  and  $\gamma$  this state is a GHZ-class state, and for which it is a W-class state. Concretely composing linear maps gives:

$$|\psi\rangle = (1 + e^{i(\alpha+\beta+\gamma)})|000\rangle + (e^{i\alpha} + e^{i(\beta+\gamma)})|011\rangle + (e^{i\beta} + e^{i(\alpha+\gamma)})|101\rangle + (e^{i\gamma} + e^{i(\alpha+\beta)})|110\rangle \quad (4.15)$$

for which the 3-tangle is equal to:

$$\tau_{ABC} = 16|a||b||c||d| \quad (4.16)$$

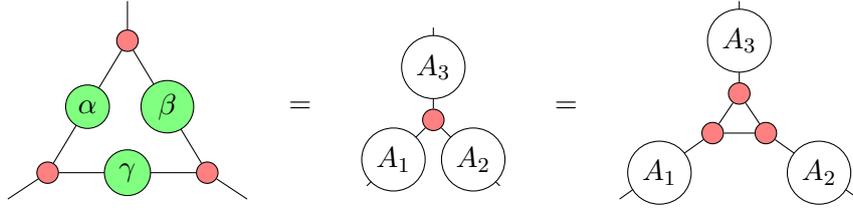
where

$$a = 1 + e^{i(\alpha+\beta+\gamma)} \quad b = e^{i\gamma} + e^{i(\alpha+\beta)} \quad c = e^{i\alpha} + e^{i(\beta+\gamma)} \quad d = e^{i\beta} + e^{i(\alpha+\gamma)} \quad (4.17)$$

Thus, the tangle is zero (and  $|\psi\rangle$  not a GHZ-class state) when one of the following conditions holds:

$$\begin{aligned} \alpha + \beta + \gamma &= \pi \\ \gamma - \alpha - \beta &= \pi \\ \alpha - \beta - \gamma &= \pi \\ \beta - \alpha - \gamma &= \pi \end{aligned} \quad (4.18)$$

So unless one of these conditions holds, we must be able to find linear maps  $A_1$ ,  $A_2$ ,  $A_3$ , such that:



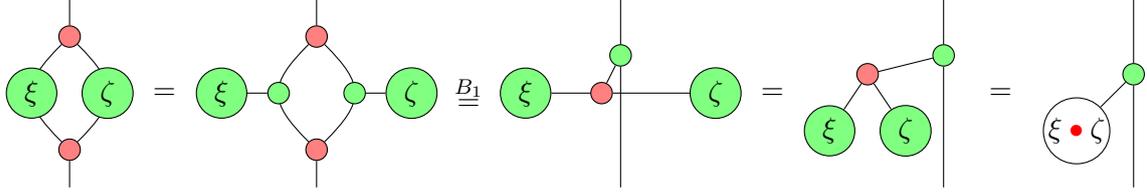
$$(4.19)$$

i.e. we should be able to ‘pull out’ the three phases, in the process transforming them into linear maps, leaving a free central triangle, which can then be closed down to a single red dot using the spider law. In the case where one of the four conditions does hold, something stops us from pulling the phases out, and they cause a ‘log jam’, preventing us from closing down the triangle to a dot, and dooming the state to be without genuine global tripartite entanglement.

## 5 GHZ and W states analyzed graphically

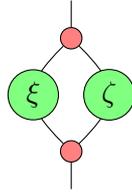
In this section we wish to produce the constraints (4.18) not by direct computation but within the diagrammatic language. We will show how these four constraints can be classified.

First observe that:



where we set  $\xi \bullet \zeta := \delta_X \circ (\xi \otimes \zeta)$ . It then follows by equation  $C_1$  that:

**Proposition 5.1** *If basis structures  $(\text{red circle}, \text{red circle})$  and  $(\text{green circle}, \text{green circle})$  form a bialgebra then the endomorphism:*



*is disconnected whenever  $\xi \bullet \zeta$  is a basis element of  $(\text{green circle}, \text{green circle})$ . Explicitly, up-to-a-scalar we obtain:*



**Definition 5.2** For basis structures  $(\text{red circle}, \text{red circle})$  and  $(\text{green circle}, \text{green circle})$  on  $A$  which form a bialgebra we call a pair of  $(\text{green circle}, \text{green circle})$ -phases  $\xi, \zeta : I \rightarrow A$  *supplementary* when  $i := \xi \bullet \zeta : I \rightarrow A$  is (up-to-a-scalar) a basis element of  $(\text{red circle}, \text{red circle})$ . More specifically, we say that  $\xi$  and  $\zeta$  are *i-supplementary*.

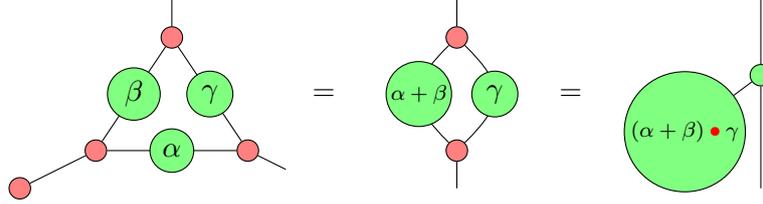
In the case of the concrete basis structures  $\Delta_Z$  and  $\Delta_X$ , since  $\Delta_Z$  has two basis elements  $|0\rangle$  and  $|1\rangle$ , there will be two kinds of supplementary. We have

$$\begin{aligned} \xi \bullet \zeta &= \delta_Z \left( (|0\rangle + e^{i\xi}|1\rangle) \otimes (|0\rangle + e^{i\zeta}|1\rangle) \right) \\ &= \delta_Z \left( (|00\rangle + e^{i(\xi+\zeta)}|11\rangle) + (e^{i\xi}|01\rangle + e^{i\zeta}|10\rangle) \right) \\ &= (1 + e^{i(\xi+\zeta)})|0\rangle + (e^{i\xi} + e^{i\zeta})|1\rangle \end{aligned}$$

Hence  $\xi \bullet \zeta$  is either proportional to  $|0\rangle$  and  $|1\rangle$  respectively when:

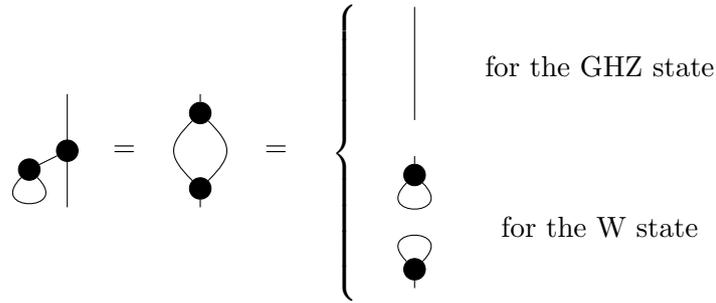
$$\xi + \zeta = \pi \Rightarrow |1\rangle\text{-supplementary} \quad \text{or} \quad \xi + \pi = \zeta \Rightarrow |0\rangle\text{-supplementary}$$

We can now reproduce equations (4.18) in terms of ‘disconnectedness of a picture’. We have:

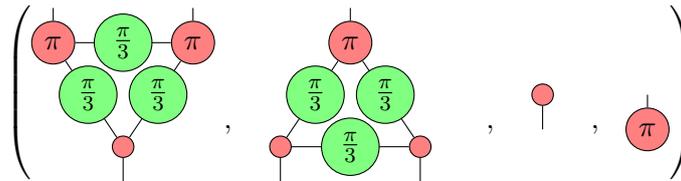


so we obtain the two first of equations (4.18); the remaining two equations arise when ‘plugging’  $\bullet$  in the other corners of the W-state.

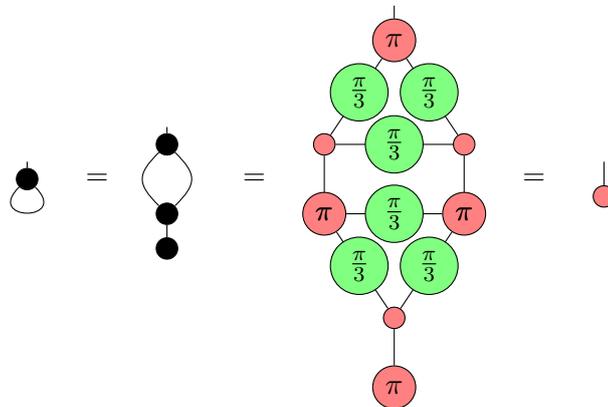
Now, since by plugging  $\bullet$  we obtain equations that are characteristic for the W class states, one expects that one can find a derivation of  $\bullet$  from such a state. This is indeed the case. As shown in [5] also the W state can be endowed with a commutative Frobenius algebra structure, although this structure is not dagger, and more importantly, not special. We still have some kind of a spider theorem, but now it also accounts for the number of loops in a picture, which has to be preserved. It is indeed ‘the value of the loop’ that is the characteristic difference between GHZ class states and W class states; we have:



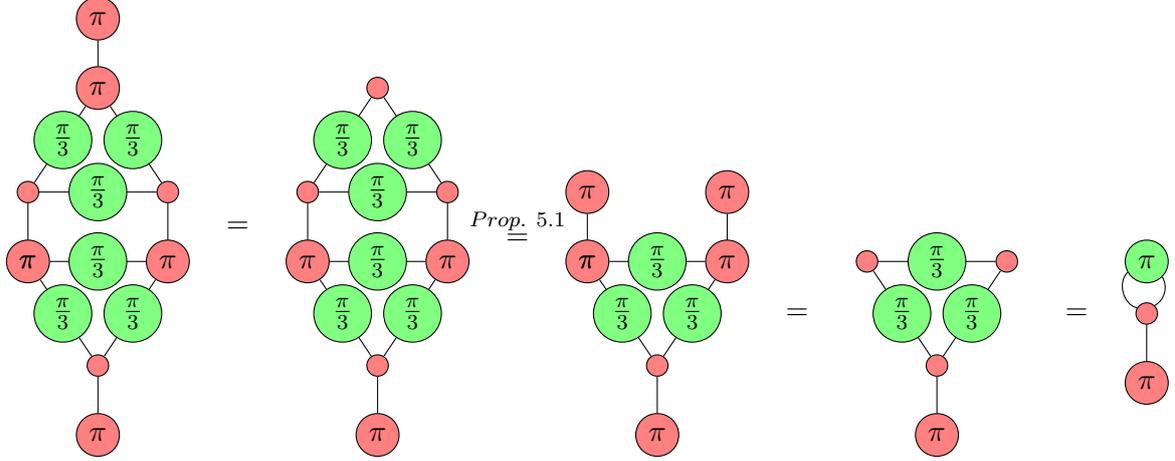
The commutative Frobenius algebra structure on W is given by:



and one can compute that:



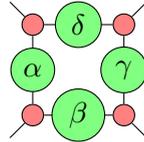
We can establish the last equality within graphical calculus by showing that, like RHS, also LHS is orthogonal to  $\pi$ . Indeed:



and a  $\pi$ -gate of one color within a loop-point of the other color is always 0.

## 6 Outlook

We showed that W class states occur when a singularity takes place in the graphical calculus: plugging with a certain point ‘special’ structural value leads to disconnectedness. The methods in this paper for analyzing the tripartite qubit states admit extrapolation to multiple qubits; for example, for:



by plugging two corners, analogous singularities occur when either  $(\alpha+\beta)\bullet(\gamma+\delta)$  or  $\alpha\bullet(\beta+\gamma+\delta)$  are basis elements for  $(\bullet, \bullet)$ . In this case the status of  $\bullet$  isn’t as clear as in the previous section where we could rely on the results of [5], and hence requires further investigation. On the other hand, while the results in [5] are restricted to the highly symmetrical so-called Frobenius states, the methods presented here don’t have such a constraint, and may lead to an algebraic account on more general states than tripartite ones.

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