THE THURSTON NORM, FIBERED MANIFOLDS AND TWISTED ALEXANDER POLYNOMIALS

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ABSTRACT. Every element in the first cohomology group of a 3–manifold is dual to embedded surfaces. The Thurston norm measures the minimal 'complexity' of such surfaces. For instance the Thurston norm of a knot complement determines the genus of the knot in the 3–sphere. We show that the degrees of twisted Alexander polynomials give lower bounds on the Thurston norm, generalizing work of McMullen and Turaev. Our bounds attain their most elegant form when interpreted as the degrees of the Reidemeister torsion of a certain twisted chain complex. The bounds are very powerful and can be easily implemented with a computer. We show that these lower bounds determine the genus of all knots with 12 crossings or less, including the Conway knot and the Kinoshita–Terasaka knot which have trivial Alexander polynomial. We also give many examples of closed manifolds and link complements where twisted Alexander polynomials detect the correct Thurston norm.

We also give obstructions to fibering 3–manifolds using twisted Alexander polynomials and detect all knots with 12 crossings or less that are not fibered. For some of these it was unknown whether or not they are fibered. Our obstructions also extend work of Cha to the case of closed manifolds.

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1. Introduction

1.1. **Definitions and history.** Let M be a 3–manifold. Throughout the paper we will assume that all 3–manifolds are compact, orientable and connected. Let $\phi \in H^1(M)$ (integral coefficients are understood). The *Thurston norm* of ϕ is defined as

$$||\phi||_T := \min\{\textstyle\sum_{i=1}^k \max\{-\chi(S_i),0\} \mid \begin{array}{l} S_1 \cup \dots \cup S_k \subset M \text{ properly embedded,} \\ \text{dual to } \phi, S_i \text{ connected for } i=1,\dots,k\}. \end{array}$$

Thurston [Th86] showed that this defines a seminorm on $H^1(M)$ which can be extended to a seminorm on $H^1(M;\mathbb{R})$. As an example consider $X(K) := S^3 \setminus \nu K$, where $K \subset S^3$ is a knot and νK denotes an open tubular neighborhood of K in S^3. Let $\phi \in H^1(X(K))$ be a generator, then $||\phi||_T = 2\,\text{genus}(K) - 1$ (cf. Lemma 2.2).

It is an important problem to find methods for computing the Thurston norm. Such methods have many applications even outside of topology. For example using work of Freedman and He [FH91] bounds on the Thurston norm translate into lower bounds for the ropelength [CKS02] (cf. Section 9.5). Furthermore bounds on the Thurston norm also have applications in electrodynamics [CK02, Ko04].

Kronheimer and Mrowka [KM97] showed that Seiberg–Witten monopole homology determines the Thurston norm. Similarly Oszváth and Szabó [OS04a] proved that the Thurston norm is determined by Heegaard Floer homology, at least in the case that M is closed. But both homologies are non–combinatorial and therefore impractical to compute in most cases. We refer to [Kr98, Kr99] for more on the connection between the Thurston norm, Seiberg–Witten theory and 4–dimensional geometry.

Methods from algebraic topology can also be used to give lower bounds on the Thurston norm. For example it is a classical result of Alexander that

$$2\text{genus}(K) \geq \deg(\Delta_K(t)),$$

where $\Delta_K(t)$ denotes the Alexander polynomial of K. In recent years this was greatly generalized. Let M be a 3–manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M) \cong \text{Hom}(H_1(M),\mathbb{Z})$ be primitive, i.e., the corresponding homomorphism $\phi : H_1(M) \to \mathbb{Z}$ is surjective. Then McMullen [Mc02] showed that if the Alexander polynomial $\Delta_1(t) \in \mathbb{Q}[t^{\pm 1}]$ of (M,ϕ) is non–zero, then

$$||\phi||_T \geq \deg(\Delta_1(t)) - (1 + b_3(M)).$$

This result has been reproved for closed manifolds by Vidussi [Vi99, Vi03] using Seiberg–Witten theory.

Cochran [Co04] in the knot complement case and Harvey [Ha05] and Turaev [Tu02a, Tu02b] in the general case generalized McMullen's inequality. They studied maps $\mathbb{Z}[\pi_1(M)] \to \mathbb{K}[t^{\pm 1}]$ where $\mathbb{K}$ is a skew field and $\mathbb{K}[t^{\pm 1}]$ is a skew Laurent polynomial ring. The resulting lower bounds are very powerful and in general much stronger than McMullen's lower bounds. Unfortunately the algebra is difficult and the lower bounds are hard to compute in a given specific situation.

We will show how the degrees of twisted Alexander polynomials give lower bounds on the Thurston norm. These bounds are easy to compute and remarkably strong.

1.2. **Twisted Alexander polynomials and Reidemeister torsion.** In the following let $\mathbb{F}$ be a commutative field. Let $\phi \in H^1(M)$ and $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ a representation. Then $\alpha \otimes \phi$ induces an action of $\pi_1(M)$ on $\mathbb{F}^k \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] =: \mathbb{F}^k[t^{\pm 1}]$ and we can therefore consider the twisted homology $\mathbb{F}[t^{\pm 1}]$–module $H_i^\alpha(M; \mathbb{F}^k[t^{\pm 1}])$. We define $\Delta_i^\alpha(t) \in \mathbb{F}[t^{\pm 1}]$ to its order; it is called *the i-th twisted Alexander polynomial of* (M, ϕ, α) and well–defined up to multiplication by a unit in $\mathbb{F}[t^{\pm 1}]$. We point out that $\Delta_i^\alpha(t) \neq 0$ if and only if $H_i^\alpha(M; \mathbb{F}^k[t^{\pm 1}])$ is $\mathbb{F}[t^{\pm 1}]$–torsion, and in that case $\deg\left(\Delta_i^\alpha(t)\right) = \dim_{\mathbb{F}}\left(H_i^\alpha(M; \mathbb{F}^k[t^{\pm 1}])\right)$. We refer to Section 2.2 for more details. If K is a knot in S^3, ϕ is a generator of $H^1(X(K))$ and $\alpha : \pi_1(X(K)) \to \mathrm{GL}(\mathbb{Q}, 1)$ is the trivial representation, then $\Delta_1^\alpha(t) \in \mathbb{Q}[t^{\pm 1}]$ equals the usual Alexander polynomial $\Delta_K(t)$ of K.

The twisted Alexander polynomial of a knot was introduced by Lin [Lin01] in 1990 who used it to distinguish knots with the same Alexander polynomial. Twisted Alexander polynomials have been successfully used in many situations to provide more information than can be extracted from the untwisted Alexander polynomial [JW93, Wa94, Kit96, KL99a, KL99b, Ch03, HLN04]. In particular we note that Kirk and Livingston [KL99a] first introduced the above homological definition of twisted Alexander polynomials for a finite complex. We refer to [KL99a, Section 4] for the relationship between our definition and the other definitions of twisted Alexander polynomials.

If ∂M is empty or consists of tori and if $\Delta_1^\alpha(t) \neq 0$, then $\Delta_i^\alpha(t) \neq 0$ for all i and $\Delta_3^\alpha(t) = 1$ (see Corollary 4.3). This implies that $H_i^\alpha(M; \mathbb{F}^k[t^{\pm 1}] \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t)) = 0$ for all i . Therefore the Reidemeister torsion $\tau(M, \phi, \alpha) \in \mathbb{F}(t)$ is defined (cf. [Tu01] for a definition) and (cf. [Tu01, p. 20])

$$\tau(M, \phi, \alpha) = \prod_{i=0}^{2} \Delta_i^\alpha(t)^{(-1)^{i+1}} \in \mathbb{F}(t).$$

The equality holds up to multiplication by a unit in $\mathbb{F}[t^{\pm 1}]$. We will use this equality as a definition for $\tau(M, \phi, \alpha)$ and will not make use of the fact that $\tau(M, \phi, \alpha)$ has in general a smaller indeterminacy. For $f(t)/g(t) \in \mathbb{F}(t)$ we define $\deg(f(t)/g(t)) := \deg(f(t)) - \deg(g(t))$ for $f(t), g(t) \in \mathbb{F}[t^{\pm 1}]$. This allows us to consider $\deg(\tau(M, \phi, \alpha))$.

1.3. **Lower bounds on the Thurston norm.** The following is one of our main results.

Theorem 3.1 (Main Theorem 1). *Let M be a 3–manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M)$ be non–trivial and $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ a representation such that $\Delta_1^\alpha(t) \neq 0$. Then*

$$||\phi||_T \geq \frac{1}{k} \deg(\tau(M, \phi, \alpha)).$$

Equivalently,

$$||\phi||_T \geq \frac{1}{k}\big(\deg\left(\Delta_1^\alpha(t)\right) - \deg\left(\Delta_0^\alpha(t)\right) - \deg\left(\Delta_2^\alpha(t)\right)\big).$$

The proof of Theorem 3.1 is partly based on ideas of McMullen [Mc02] and Turaev [Tu02b]. In Section 3 we will show that Theorem 3.1 generalizes McMullen's theorem [Mc02] and Turaev's abelian invariants in [Tu02a]. In Section 8 we will see that $\Delta_1^\alpha(t)$ and $\Delta_0^\alpha(t)$ can easily be computed given a presentation of $\pi_1(M)$. Furthermore by duality $\Delta_2^\alpha(t)$ equals $\Delta_0^\beta(t)$ for a certain representation β, and hence can be computed the same way as $\Delta_0^\alpha(t)$ (cf. Proposition 3.2 for details).

In Theorem 5.1 we show that the condition $\Delta_1^\alpha(t) \neq 0$ can sometimes be dropped and in Section 7 we state a version of Theorem 3.1 over skew fields, which combines our lower bounds from Theorem 3.1 with the lower bounds of Cochran, Harvey and Turaev [Co04, Ha05, Tu02b]. We concentrate on proving Theorem 3.1, i.e., the case for the modules over a commutative ring, and we only point out the changes to the proof of Theorem 3.1 which have to be made to prove the non–commutative generalization.

An important source of representations is given by homomorphisms $\alpha : \pi_1(M) \to G$, G a finite group. This induces a representation $\alpha : \pi_1(M) \to G \to \mathrm{GL}(\mathbb{F}, |G|)$ where the map $G \to \mathrm{GL}(\mathbb{F}, |G|)$ is the regular representation of G. (Note that $\mathrm{GL}(\mathbb{F}, |G|)$ is isomorphic to $\mathrm{GL}(\mathbb{F}[G])$.) In Section 3.4 we give an elegant short proof of Theorem 3.1 in the case of a representation $\pi_1(M) \to G \to \mathrm{GL}(\mathbb{F}, |G|)$, using only McMullen's theorem and well–known properties of finite covers.

1.4. **Fibered manifolds.** Let $\phi \in H^1(M)$ be non–trivial. We say (M, ϕ) *fibers over* S^1 if the homotopy class of maps $M \to S^1$ induced by $\phi : \pi_1(M) \to H_1(M;\mathbb{Z}) \to \mathbb{Z}$ contains a representative that is a fiber bundle over S^1. If K is a fibered knot, i.e., if $X(K)$ fibers, then it is a classical result that $2\mathrm{genus}(K) = \deg(\Delta_K(t))$ and that $\Delta_K(t) \in \mathbb{Z}[t^{\pm1}]$ is monic, i.e., its top coefficient is $+1$ or -1.

Theorem 6.1 (**Main Theorem 2**). *Assume that* (M, ϕ) *fibers over* S^1 *and that* $M \neq S^1 \times D^2, M \neq S^1 \times S^2$. *Let* $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ *be a representation. Then* $\Delta_1^\alpha(t) \neq 0$ *and*

$$||\phi||_T = \frac{1}{k}\deg(\tau(M, \phi, \alpha)).$$

This theorem has been known for a long time for the untwisted Alexander polynomial of fibered knots. McMullen, Cochran, Harvey and Turaev prove corresponding theorems in their respective papers [Mc02, Co04, Ha05, Tu02b]. This result clearly generalizes the first classical condition on fibered knots.

Let R be a Noetherian unique factorization domain (henceforth UFD), for example $R = \mathbb{Z}$ or a field. Given a representation $\pi_1(M) \to \mathrm{GL}(R, k)$ Cha [Ch03] defined a twisted Alexander polynomial $\Delta_1^\alpha(t) \in R[t^{\pm1}]$, which is well–defined up to multiplication by a unit in $R[t^{\pm1}]$. This is a generalization of the Alexander polynomial

$\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ and coincides with the first twisted Alexander polynomial defined in Section 2.2 in the case that R is a field. We say a polynomial $\Delta_1^\alpha(t) \in R[t^{\pm 1}]$ is *monic*, if its top coefficient is a unit in R. Using Theorem 6.1 we get the following theorem.

Theorem 6.4. *Let M be a 3–manifold. Let $\phi \in H^1(M)$ be non–trivial such that (M, ϕ) fibers over S^1 and such that $M \neq S^1 \times D^2, M \neq S^1 \times S^2$. Let R be a Noetherian UFD and let $\alpha : \pi_1(M) \to GL(R, k)$ be a representation. Then $\Delta_1^\alpha(t) \in R[t^{\pm 1}]$ is monic and*

$$||\phi||_T = \frac{1}{k} \deg(\tau(M, \phi, \alpha)).$$

In fact in Proposition 6.3 we show that if the fibering obstruction of Theorem 6.1 vanishes, then the conclusion of Theorem 6.4 holds. In particular this shows that our fibering obstruction of Theorem 6.1 also contains the second classical condition on a fibered knot K that $\Delta_K(t)$ is monic. Theorem 6.4 also shows that the fibering obstruction of Theorem 6.1 contains Cha's [Ch03] obstruction for fibered knots. Goda, Kitano and Morifuji [GKM05] use the Reidemeister torsion corresponding to representations $\pi_1(X(K)) \to \mathrm{SL}(\mathbb{F}, k)$, $\mathbb{F}$ a field, to give fibering obstructions for knots. The precise relationship to our obstructions is not known.

We point out that the before mentioned fibering obstructions of McMullen, Cochran, Harvey and Turaev do not detect this monicness, and therefore they are only of limited use as fiberedness obstructions. The idea of using monicness of the twisted Alexander polynomial originated from Cha [Ch03] and our work generalizes Cha's work to the case of closed 3–manifolds.

The significance of Theorem 6.1 lies in the fact that it gives a fibering obstruction for a wide class of representations and for any 3–manifold M, whereas the arguments of [Ch03] and [GKM05] relied on the fact that $\partial M \neq \emptyset$. Note that our obstructions only require the computation of twisted Alexander polynomials over principal ideal domains $\mathbb{F}[t^{\pm 1}]$ (see Corollary 6.2). This can be done efficiently, whereas the computation of determinants over rings like $\mathbb{Z}[t^{\pm 1}]$ as in [Ch03] can be time consuming since the size of the integers during the computation can become very large.

1.5. **Examples.** Consider the Conway knot $K = 11_{401}$ (*knotscape* notation, cf. [HT]). The diagram is given in Figure 1. For this knot $\Delta_K(t) = 1$, therefore the genus bounds of McMullen, Turaev, Cochran and Harvey vanish. We found a representation $\alpha : \pi_1(X(K)) \to \mathrm{GL}(\mathbb{F}_{13}, 4)$ such that $\deg(\tau(M, \phi, \alpha)) = 14$. These computations and all the following computations were done using the program *KnotTwister* [F05]. It follows from Theorem 3.1 that

$$2\,\mathrm{genus}(K) - 1 = ||\phi||_T \geq \frac{14}{4}.$$

Hence $\mathrm{genus}(K) \geq \frac{18}{8} = 2.25$. Since $\mathrm{genus}(K)$ is an integer we get $\mathrm{genus}(K) \geq 3$. Since there exists a Seifert surface of genus 3 for K (cf. Figure 1) it follows that the genus of the Conway knot is 3. Gabai [Ga84] proved the same result using geometric methods.

FIGURE 1. The Conway knot 11_{401} and a Seifert surface of genus 3 (from [Ga84]).

We went over all knots with up to 12 crossings such that $2\,\text{genus}(K) \neq \deg(\Delta_K(t))$. In all cases we found representations $\alpha : \pi_1(X(K)) \to \mathrm{GL}(\mathbb{F}_{13}, k)$ which give the right genus bounds. Using *KnotTwister* this process just takes a few seconds. We also investigated the closed manifolds which are the result of 0–framed surgery along these knots. Again in all cases we found representations such that twisted Alexander polynomials give the right bound on the Thurston norm. In fact experience suggests that if $b_1(M) = 1$ then in most cases taking only a few non–trivial representations will give the correct bound on the Thurston norm, regardless of whether M is closed or not.

The situation for links is more complex. On the one hand in many interesting cases twisted Alexander polynomials give the correct bound. For example in Section 9.5 we reprove results of Harvey on the ropelength of a certain link [Ha05]. In Section 9.6 we also give further evidence that McMullen's Alexander norm [Mc02] and the Thurston norm agree for all links with up to 9 crossings.

On the other hand boundary links have vanishing twisted Alexander polynomials and therefore our lower bounds vanish. In Section 5 we show that in some cases we can still extract lower bounds from the degrees of twisted Alexander polynomials corresponding to the $\mathbb{F}[t^{\pm 1}]$–torsion submodule of $H_1^\alpha(X(L); \mathbb{F}^k[t^{\pm 1}])$ where $X(L)$ is the link complement in the 3–sphere (cf. Theorem 5.1).

In [FK05] we will show how twisted *multivariable* Alexander polynomials can be used in many cases to completely determine the Thurston norm ball of link complements, generalizing results of McMullen and Turaev.

It is known that a knot K with 11 or fewer crossings is fibered if and only if K satisfies

$$\Delta_K(t) \text{ is monic and } \deg(\Delta_K(t)) = 2\,\text{genus}(K). \tag{1}$$

Hirasawa and Stoimenow had started a program to find all non–fibered 12–crossing knots. Using methods of Gabai they showed that except for thirteen knots a 12–crossing knot is fibered if and only if it satisfies condition (1). Furthermore they showed that among these 13 knots the knots 12_{1498}, 12_{1502}, 12_{1546} and 12_{1752} are not fibered even though they satisfy condition (1).

Using Theorem 6.1 we confirmed the non–fiberedness of these 4 knots and we showed that the remaining 9 knots are not fibered either. These 9 knots are:

$$12_{1345}, 12_{1567}, 12_{1670}, 12_{1682}, 12_{1771}, 12_{1823}, 12_{1938}, 12_{2089}, 12_{2103}.$$

This result completes the classification of all fibered 12–crossing knots. Jacob Rasmussen confirmed our results using knot Floer homology which gives a fibering obstruction as well (cf. [OS02, Section 3]).

As we pointed out our fibering obstructions work for closed manifolds as well. If K is one of the 13 12–crossing knots in the previous paragraph, then we can easily show using Theorem 6.1 and *KnotTwister* that the zero surgery on K in S^3 is not fibered. (See Section 9.2.)

1.6. **Conjectures and symplectic manifolds.** We propose the following conjecture.

Conjecture 10.1. *Let M be a 3–manifold and $\phi \in H^1(M)$ non–trivial. Then (M, ϕ) fibers over S^1 if and only if for all epimorphisms $\alpha : \pi_1(M) \to G$, G a finite group, the twisted Alexander polynomial $\Delta_1^\alpha(t) \in \mathbb{Z}[t^{\pm 1}]$ is monic and*

$$||\phi||_T = \frac{1}{|G|} \deg(\tau(M, \phi, \alpha)).$$

We discuss this conjecture in detail in Section 10.6. We show that it holds in an important case of satellite knots. We also show that it would follow from the geometrization conjecture and the following conjecture.

Conjecture 10.5. *Let S be an incompressible surface in M and let N be M cut along S. Let $i : S \to N$ be one of the two canonical inclusions of S into N. If $i_* : H_1^\alpha(S; \mathbb{Z}[G]) \to H_1^\alpha(N; \mathbb{Z}[G])$ is surjective for every homomorphism $\pi_1(M) \to G$, G a finite group, then $i_* : \pi_1(S) \to \pi_1(N)$ is surjective.*

Note that the inclusion induced homomorphisms $\pi_1(S) \to \pi_1(M)$ and $\pi_1(N) \to \pi_1(M)$ are clearly injections. Therefore Conjecture 10.5 becomes a conjecture in the theory of 3–manifold groups.

Let M be a closed 3–manifold. Taubes conjectured that if $S^1 \times M$ is symplectic then (M, ϕ) fibers over S^1 for some $\phi \in H^1(M)$. Vidussi [Vi99, Vi03] and Fintushel and Stern [FS98, p. 398] showed using work of Taubes [Ta94, Ta95] that if $S^1 \times M$ is symplectic then there exists $\phi \in H^1(M)$ such that all abelian invariants of (M, ϕ) look like invariants of fibered manifolds. In [FV05] the first author and Stefano Vidussi will show that if $S^1 \times M$ is symplectic then there exists $\phi \in H^1(M)$ such that (M, ϕ) satisfies the assumptions of Conjecture 10.1. This gives very strong evidence for Taubes' conjecture and shows that Conjecture 10.1 implies Taubes' conjecture.

In Conjecture 10.6 we conjecture that twisted Alexander polynomials detect the genus of hyperbolic knots. We show that this can also be reduced to a group theoretic problem as in Conjecture 10.5.

1.7. **Outline of the paper.** In Section 2 we state some properties of the Thurston norm and give a definition of twisted Alexander polynomials. In Section 3 we state Theorem 3.1 (Main Theorem 1) and discuss related theorems. We give a proof of Theorem 3.1 in Section 4. In Section 5 we show how in many important cases we can drop the assumption that $\Delta_1^\alpha(t) \neq 0$ in Theorem 3.1 and still get lower bounds on the Thurston norm. In Section 6 we consider fibered manifolds and give a proof of Theorems 6.1 (Main Theorem 2) and 6.4. We formulate non–commutative versions of Theorem 3.1 and Theorem 6.1 in Section 7. After showing in Section 8 how Fox calculus can be used to efficiently compute twisted Alexander polynomials we discuss a wealth of examples in Section 9. Finally in Section 10 we discuss and give further evidence for Conjectures 10.1 and related conjectures.

Notations and conventions: We assume that all 3–manifolds are compact, oriented and connected. All homology groups and all cohomology groups are with respect to $\mathbb{Z}$–coefficients, unless it specifically says otherwise. For a knot K in S^3, we denote the result of zero framed surgery along K by M_K. For a link L in S^3, $X(L)$ denotes the exterior of L in S^3. (That is, $X(L) = S^3 \setminus \nu L$ where νL is an open tubular neighborhood of L in S^3). An arbitrary (commutative) field is denoted by $\mathbb{F}$. We identify the group ring $\mathbb{F}[\mathbb{Z}]$ with $\mathbb{F}[t^{\pm 1}]$. We denote the permutation group of order k by S_k. For a 3–manifold M we use the canonical isomorphisms to identify $H^1(M) = \mathrm{Hom}(H_1(M), \mathbb{Z}) = \mathrm{Hom}(\pi_1(M), \mathbb{Z})$. Hence sometimes $\phi \in H^1(M)$ is regarded as a homomorphism $\phi : \pi_1(M) \to \mathbb{Z}$ (or $\phi : H_1(M) \to \mathbb{Z}$) depending on the context.

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2. The Thurston norm and twisted Alexander polynomials

2.1. **The Thurston norm.** For a connected CW complex X denote by $\chi(X)$ the Euler characteristic of X. We define $\chi_-(X) := \max\{-\chi(X), 0\}$. For a non-connected CW complex X define $\chi_-(X) = \sum \chi_-(X_i)$ where we sum over the connected components of X. This is called the *complexity* of X. Now let M be a 3–manifold and

$\phi \in H^1(M)$. Then we define

$$||\phi||_{T,M} := \min\{\chi_-(S)\},$$

where we take the minimum with respect to all properly embedded surfaces S which are dual to ϕ. Note that we take the minimum over a non–empty set since any ϕ is dual to a properly embedded surface (cf. [Th86]). If the manifold M is clear we will just write $||\phi||_T$.
Thurston [Th86] introduced $||-||_{T,M}$ in a preprint in 1976 and proved the following theorem (cf. [Oe86] [Kr98]) which justifies the name Thurston *norm*.

Theorem 2.1. (1) *For $\phi, \phi_1, \phi_2 \in H^1(M)$ and $k \in \mathbb{N}$ we have the following:*

$$\begin{aligned} ||k\phi||_T &= k||\phi||_T, \\ ||\phi_1+\phi_2||_T &\le ||\phi_1||_T + ||\phi_2||_T. \end{aligned}$$

(2) *There exists a seminorm $||-||_T$ on $H^1(M;\mathbb{R})$ which equals $||-||_T$ on the integral lattice $H^1(M)$.*
(3) *If no element in $H^1(M)$ is dual to an embedded surface of non–negative Euler characteristic, then $||-||_T$ is in fact a norm. In particular if M is hyperbolic, then $||-||_T$ is a norm.*
(4) *The unit ball of the Thurston norm is a finite, convex, possibly non–compact polyhedron.*

Proof. (1) We refer to [Th86].
(2) It is now easy to see that the convex function $||-||_T$ on $H^1(M)$ can be extended to a seminorm $||-||_T$ on $H^1(M;\mathbb{R})$ (cf. [Th86, p. 104] for details).
(3) This follows immediately from the definition and from Thurston's hyperbolicity theorem (cf. [Th82]).
(4) We refer to [Th86, Theorem 2].

□

As an illustration we outline the proof of the following lemma. At several later occasions we will make use of the arguments in this proof.

Lemma 2.2. *Let $K \subset S^3$ be a non–trivial knot and $\phi \in H^1(X(K))$ a generator. Then $||\phi||_T = 2\,genus(K) - 1$.*

Proof. Clearly a Seifert surface for K is dual to ϕ, hence $||\phi||_T \le 2\,\mathrm{genus}(K) - 1$. Denote the longitude of K by λ. If S is dual to ϕ with minimal complexity, then $\partial S = [\lambda] \in H_1(K \times S^1)$ (we identify $\partial X(K)$ with $K \times S^1$). Note that every boundary component of S is essential in $K \times S^1$. Otherwise we can find an innermost circle which bounds a disk, which can then be attached to find a dual surface of lower complexity.

Therefore each component of ∂S is non–trivial in $H_1(K \times S^1)$. Since at least one component of ∂S represents $[\lambda] \in H_1(K \times S^1)$ it follows that in fact each component of ∂S represents $\pm[\lambda]$ since the components are disjoint. In particular it follows that ∂S has an odd number of components. If ∂S has $2k+1$ components, then to two adjacent components of ∂S with opposite orientations we can attach an annulus and push this part off the boundary of $X(K)$. Note that adding an annulus does not change the complexity. Repeating this process gives a possibly disconnected surface. One component is a Seifert surface F for K, and the remaining components are closed. Since we can throw away the closed components it now follows that $||\phi||_T = b_1(F) - 1 \geq 2\,\mathrm{genus}(K) - 1$. □

Strictly speaking the Thurston norm is not a norm, but only a seminorm. For example if $M = S^1 \times S^1 \times S^1$, then clearly every element in $H_2(M)$ is represented by a disjoint union of tori, hence the Thurston norm vanishes completely on $H^1(M)$.

2.2. **Alexander polynomials.** Let M be a 3–manifold and $\phi \in H^1(M)$. Let $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ be a representation. We can now define a left $\mathbb{Z}[\pi_1(M)]$–module structure on $\mathbb{F}^k \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] =: \mathbb{F}^k[t^{\pm 1}]$ via $\alpha \otimes \phi$ as follows:

$$g \cdot (v \otimes p) := (\alpha(g) \cdot v) \otimes (\phi(g) \cdot p) = (\alpha(g) \cdot v) \otimes (t^{\phi(g)} p)$$

where $g \in \pi_1(M), v \otimes p \in \mathbb{F}^k \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] = \mathbb{F}^k[t^{\pm 1}]$.

Denote by $\tilde{M}$ the universal cover of M. Then the chain groups $C_*(\tilde{M})$ are in a natural way right $\mathbb{Z}[\pi_1(M)]$–modules. Therefore we can form the tensor product $C_*(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{F}^k[t^{\pm 1}]$. Now we define *the i–th twisted Alexander module of* (M, ϕ, α) to be

$$H_*^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}]) := H_*(C_*(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{F}^k[t^{\pm 1}]).$$

Usually we drop the notation ϕ and write $H_*^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}])$. Note that $H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}])$ is a finitely generated module over the PID $\mathbb{F}[t^{\pm 1}]$. Therefore there exists an isomorphism

$$H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \cong \mathbb{F}[t^{\pm 1}]^f \oplus \bigoplus_{i=1}^{k} \mathbb{F}[t^{\pm 1}]/(p_i(t))$$

for $p_1(t), \ldots, p_k(t) \in \mathbb{F}[t^{\pm 1}]$. We define

$$\Delta^{\alpha}_{M,\phi,i} := \begin{cases} \prod_{i=1}^{k} p_i(t), & \text{if } f = 0 \\ 0, & \text{if } f > 0. \end{cases}$$

This is called the *i–th twisted Alexander polynomial* of (M, ϕ, α). We furthermore define $\tilde{\Delta}^{\alpha}_{M,\phi,i} := \prod_{i=1}^{k} p_i(t)$ regardless of f. In most cases we drop the notations M and ϕ and write $\Delta_i^{\alpha}(t)$ and $\tilde{\Delta}_i^{\alpha}(t)$. It follows from the structure theorem of finitely generated modules over a PID that these polynomials are well–defined up to multiplication by a unit in $\mathbb{F}[t^{\pm 1}]$. In Section 8 we will see that $\Delta_i^{\alpha}(t)$ and $\tilde{\Delta}_i^{\alpha}(t)$ can be computed easily for $i = 0, 1$ given a presentation of $\pi_1(M)$.

Remark. The first twisted Alexander polynomial for a knot was originally defined by Lin in 1990 using a presentation of the fundamental group [Lin01]. This was generalized by Jiang and Wang [JW93] and the multivariable twisted Alexander polynomial was first introduced by Wada [Wa94] given only a presentation of a group and a representation to $\mathrm{GL}(R,k)$ where R is a UFD. Wada's definition differs slightly from our definition even in the case that it is associated to a representation to $\mathbb{Z}$. Our homological definition of twisted Alexander polynomials in the above was originally introduced by Kirk and Livingston in [KL99a].

For an oriented knot K we always assume that ϕ denotes the generator of $H^1(X(K))$ given by the orientation. If $\alpha : \pi_1(X(K)) \to \mathrm{GL}(\mathbb{Q},1)$ is the trivial representation then the Alexander polynomial $\Delta_1^\alpha(t)$ equals the classical Alexander polynomial $\Delta_K(t) \in \mathbb{Q}[t^\pm]$ of the knot K.

Remark. When K is a knot, then the untwisted homology $H_1(X(K);\mathbb{F}[t^{\pm1}])$ is $\mathbb{F}[t^{\pm1}]$–torsion. But even for a knot complement it can happen that in the twisted case $H_1^\alpha(X(K);\mathbb{F}^k[t^{\pm1}])$ is not $\mathbb{F}[t^{\pm1}]$–torsion (cf. e.g. [KL99a]).

If $f=0$ then we write $\deg(f)=\infty$, otherwise, for $f=\sum_{i=m}^n a_it^i \in \mathbb{F}[t^{\pm1}]$ with $a_m\neq 0, a_n\neq 0$ we define $\deg(f)=n-m$. Note that $\deg\left(\Delta_i^\alpha(t)\right)$ is well–defined. The following observation follows immediately from the classification theorem of finitely generated modules over a PID.

Lemma 2.3. *$H_i^\alpha(M;\mathbb{F}^k[t^{\pm1}])$ is a finite–dimensional $\mathbb{F}$–vector space if and only if $\Delta_i^\alpha(t)\neq 0$. If $\Delta_i^\alpha(t)\neq 0$, then*

$$\deg\left(\Delta_i^\alpha(t)\right) = \dim_{\mathbb{F}}\left(H_i^\alpha(M;\mathbb{F}^k[t^{\pm1}])\right).$$

Furthermore $\deg(\tilde{\Delta}_i^\alpha(t)) = \dim_{\mathbb{F}}\left(Tor_{\mathbb{F}[t^{\pm1}]}(H_i^\alpha(M;\mathbb{F}^k[t^{\pm1}]))\right)$.

If ∂M is empty or consists of tori and if $\Delta_1^\alpha(t)\neq 0$, then $\Delta_i^\alpha(t)\neq 0$ for all i and hence $H_i^\alpha(M;\mathbb{F}^k[t^{\pm1}]\otimes_{\mathbb{F}[t^{\pm1}]}\mathbb{F}(t))=0$ for all i (see Corollary 4.3). Therefore the Reidemeister torsion $\tau(M,\phi,\alpha)\in\mathbb{F}(t)$ is defined. We refer to [Tu01] for an excellent introduction into the theory of Reidemeister torsion. $\tau(M,\phi,\alpha)\in\mathbb{F}(t)$ is well–defined up to multiplication by an element of the form $rt^k, r\in \mathrm{Im}\{\pi_1(M)\xrightarrow{\alpha}\mathrm{GL}(\mathbb{F},k)\xrightarrow{det}\mathbb{F}\}$. We will not make use of this, and mostly use $\tau(M,\phi,\alpha)$ as a convenient way to store information.

Lemma 2.4. *(cf.* [Tu01, p. 20]*) If $\Delta_1^\alpha(t)\neq 0$, then $\tau(M,\phi,\alpha)$ is defined and*

$$\tau(M,\phi,\alpha)=\prod_{i=0}^{3}\Delta_i^\alpha(t)^{(-1)^{i+1}}\in\mathbb{F}(t).$$

3. Main Theorem 1: Lower bounds on the Thurston norm

3.1. **Statement of Main Theorem 1.** Our main theorem gives a lower bound for the Thurston norm of a non–trivial element $\phi\in H^1(M)$.

Theorem 3.1 (Main Theorem 1). *Let M be a 3–manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M)$ be non–trivial and let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a representation such that $\Delta_1^\alpha(t) \neq 0$. Then*

$$\begin{aligned} ||\phi||_T &\geq \tfrac{1}{k} \deg(\tau(M, \phi, \alpha)) \\ &= \tfrac{1}{k}\big(\deg(\Delta_1^\alpha(t)) - \deg(\Delta_0^\alpha(t)) - \deg(\Delta_2^\alpha(t))\big). \end{aligned}$$

The proof of the above theorem is given in Section 4. In Section 8 we will show how to compute $\Delta_1^\alpha(t)$ and $\Delta_0^\alpha(t)$ using a presentation of $\pi_1(M)$. In Section 4.4 we will also give a proof of the following proposition which allows us to compute $\Delta_2^\alpha(t)$ using the algorithm for computing the 0-th twisted Alexander polynomial.

Assume that $\mathbb{F}$ has a (possibly trivial) involution. Let $\langle\, ,\, \rangle$ be the canonical hermitian inner product on $\mathbb{F}^k$. Then there exists a unique representation $\overline{\alpha} : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ such that

$$\langle \alpha(g^{-1})v, w\rangle = \langle v, \overline{\alpha}(g)w\rangle$$

for all $g \in \pi_1(M)$ and $v, w \in \mathbb{F}^k$.

Proposition 3.2. *Let M be a 3–manifold whose boundary is empty or consists of tori and let $\phi \in H^1(M)$ be non–trivial. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a representation such that $\Delta_1^\alpha(t) \neq 0$.*

(1) *If M is closed, then*

$$\Delta_2^\alpha(t) = \Delta_0^{\overline{\alpha}}(t^{-1}).$$

(2) *If M has non–empty boundary, then $\Delta_2^\alpha(t) = 1$.*

In particular

$$\deg(\Delta_2^\alpha(t)) = b_3(M) \deg\big(\Delta_0^{\overline{\alpha}}(t)\big).$$

For a unitary representation we have $\overline{\alpha} = \alpha$, therefore we get the following important special case of Theorem 3.1:

Theorem 3.3. *Let M be a 3–manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M)$ be non–trivial and let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a unitary representation such that $\Delta_1^\alpha(t) \neq 0$. Then*

$$||\phi||_T \geq \frac{1}{k}\big(\deg(\Delta_1^\alpha(t)) - \big(1 + b_3(M)\big) \deg(\Delta_0^\alpha(t))\big).$$

Remark. (1) Our restriction to closed manifolds or manifolds whose boundary consists of tori is not a significant restriction. Indeed, if ∂M has a spherical boundary component, then gluing in a 3–ball does not change the Thurston norm. Furthermore manifolds with a boundary component of genus greater than 1 have in most cases vanishing twisted Alexander polynomials.

(2) We point out that a slightly different proof of Theorem 3.1 shows that in fact Theorem 3.3 holds for all 3–manifolds if we replace $b_3(M)$ by $\tilde{b}_3(M)$ where we define $\tilde{b}_3(M) = 1$ if M is closed or if the boundary of M consists only of spheres, and $\tilde{b}_3(M) = 0$ otherwise.

The following lemma shows that in most cases we can determine for a given $\phi \in H^1(M)$ whether $||\phi||_T$ is even or odd. This means that we can 'round up' the lower bounds from Theorem 3.1 to an even or odd number, depending on the parity of $||\phi||_T$. Recall that for a non–trivial $\phi \in H^1(M)$ the *divisibility* of ϕ equals the maximum natural number n such that $\frac{1}{n}\phi \in H^1(M)$.

Lemma 3.4. *Let $\phi \in H^1(M)$ be primitive. If M is closed, then $||\phi||_T$ is even. Assume that ∂M consists of a non–empty collection of tori $N_1 \cup \cdots \cup N_s$. If $\phi|_{H_1(N_i)} = 0$ then let $n_i := 0$, otherwise define n_i to be the divisibility of $\phi|_{H_1(N_i)}$. Then*

$$||\phi||_T \equiv \left(\sum_{i=1}^{s} n_i \right) \mathit{mod}\ 2.$$

Proof. Let S be a properly embedded surface dual to ϕ with minimal complexity. If M is closed then S is closed, hence $\chi(S)$ is even. Now assume that ∂M is a collection of tori. Then

$$\chi_-(S) \equiv b_0(\partial S) \mod 2.$$

This follows from the observation that adding a 2–disk to each component of ∂S gives a closed surface, which has even Euler characteristic. Now consider N_i. Clearly $S \cap N_i$ is Poincaré dual to $\phi|_{H_1(N_i)}$. It follows from an argument as in Lemma 2.2 that, modulo 2, $\partial S \cap N_i$ has n_i components. $\square$

In Section 9.2 we will see that Theorem 3.1 can be very successfully used to determine the genus of knots and the Thurston norm of closed manifolds. In particular we give many examples of triples $(M, \phi \in H^1(M), \alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k))$ for which

$$\begin{array}{rll} & b(M,\phi,\alpha) & := \frac{1}{k}\big(\deg\left(\Delta_1^\alpha(t)\right) - \deg\left(\Delta_0^\alpha(t)\right) - \deg\left(\Delta_2^\alpha(t)\right)\big) \\ > & b(M,\phi) & := \deg(\Delta_1(t)) - \deg(\Delta_0(t)) - \deg(\Delta_2(t)), \end{array}$$

i.e., we have many examples where the degrees of twisted Alexander polynomials give better bounds on the Thurston norm than the degree of the untwisted Alexander polynomial. In most cases we have $b(M, \phi, \alpha) \geq b(M, \phi)$. But if we take K to be the knot 9_4, ϕ a generator of $H^1(X(K))$, then there exists a map $\alpha : \pi_1(X(K)) \twoheadrightarrow S_3 \to \mathrm{GL}(\mathbb{Q}, |S_3|)$ with $\deg\left(\Delta_0^\alpha(t)\right) = 2$, and such that

$$\Delta_1^\alpha(t) = 9 - 13t^2 - 8t^4 + 24t^6 - 8t^8 - 13t^{10} + 9t^{12} \in \mathbb{Q}[t^{\pm 1}].$$

Since $\Delta_1(t) = 3 - 5t + 5t^2 - 5t^3 + 3t^4$ it follows that

$$b(M, \phi, \alpha) = \frac{5}{3} < 3 = b(M, \phi).$$

This example shows that twisted Alexander polynomials give at times bounds on the Thurston norm that are worse than the bounds from the ordinary Alexander polynomials. This should be compared to the situation of [Co04] (cf. also [Ha06]) : Cochran's sequence of higher order Alexander polynomials gives a never decreasing sequence of lower bounds on the genus of a knot.

In the following sections we discuss several interesting types of representations which allow us to recover work of McMullen [Mc02] and Turaev [Tu02b].

3.2. **The trivial representation: McMullen's theorem.** If we let $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, 1)$ be the trivial representation then given any primitive $\phi \in H^1(M)$ we have

$$H_0^\alpha(M; \mathbb{F}[t^{\pm 1}]) \cong \mathbb{F}[t^{\pm 1}]/\{t^i f - f | i \in \mathbb{Z},\, f \in \mathbb{F}[t^{\pm 1}]\} \cong \mathbb{F}[t^{\pm 1}]/(t-1).$$

Therefore $\Delta_0(t) = t - 1$. Since the trivial representation is unitary we immediately get McMullen's theorem from Theorem 3.3:

Theorem 3.5. [Mc02, Proposition 6.1] *Let M be a 3–manifold whose boundary is empty or consists of tori and $\phi \in H^1(M)$ primitive. If $\Delta_1(t) \neq 0$, then for any field $\mathbb{F}$*

$$||\phi||_T \geq \deg(\Delta_1(t)) - \big(1 + b_3(M)\big).$$

In fact McMullen showed more: he introduced a norm $|| - ||_A$ on $H^1(M; \mathbb{R})$, called the Alexander norm, and showed that if $b_1(M) > 1$ then

$$||\phi||_T \geq ||\phi||_A$$

for all $\phi \in H^1(M; \mathbb{R})$. Furthermore $||\phi||_A \geq \deg(\Delta_1(t)) - 1 - b_3(M)$ for all $\phi \in H^1(M)$, and equality holds for almost all $\phi \in H^1(M)$. In [FK05] the authors will introduce *twisted Alexander norms* which give lower bounds on the Thurston norm, extending the work of McMullen and Turaev [Tu02a].

3.3. **Abelian representations: Turaev's theorem.** The following was first shown by Turaev [Tu02a].

Theorem 3.6. *Let M be a 3–manifold whose boundary is empty or consists of tori, $\phi \in H^1(M)$ primitive, and $\alpha : \pi_1(M) \to H_1(M) \to GL(\mathbb{F}, 1)$ a one–dimensional representation which is non–trivial on Ker(ϕ). If $\Delta_1^\alpha(t) \neq 0$, then*

$$||\phi||_T \geq \deg\left(\Delta_1^\alpha(t)\right).$$

Proof. Let $\beta : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, 1)$ be a representation. Using that $\phi : \pi_1(M) \to \mathbb{Z}$ is surjective one can easily show that

$$H_0^\beta(M; \mathbb{F}[t^{\pm 1}]) \cong \mathbb{F}[t^{\pm 1}]/\{\beta(g) t^{\phi(g)} f - f \,|\, g \in \pi_1(M),\, f \in \mathbb{F}[t^{\pm 1}]\} \cong \mathbb{F}$$

if β is trivial on $\mathrm{Ker}(\phi)$, and $H_0^\beta(M; \mathbb{F}[t^{\pm 1}]) = 0$ otherwise. We apply this to $H_0^\alpha(M; \mathbb{F}[t^{\pm 1}])$ and $H_0^{\overline{\alpha}}(M; \mathbb{F}[t^{\pm 1}])$. Turaev's theorem now follows from Theorem 3.1 and Proposition 3.2. $\square$

This simple abelian version of Theorem 3.1 can already be very useful. Using results of [FK05] one can show that for primitive $\phi \in H^1(M)$

$$\begin{aligned} ||\phi||_A = \max\{\deg(\Delta_1^\alpha(t))\,|\alpha : \pi_1(M) \to H_1(M)/\mathrm{Tor}(H_1(M)) \to \mathrm{GL}(\mathbb{C},1) \\ \text{non–trivial on } \mathrm{Ker}(\phi)\}, \end{aligned}$$

where $||\phi||_A$ denotes McMullen's Alexander norm. Harvey [Ha05, Proposition 3.12] showed that the invariant $\bar{\delta}_0(\phi)$ in [Ha05] equals $||\phi||_A$. This shows that Alexander polynomials corresponding to one–dimensional representations contain all known lower bounds on the Thurston norm coming from abelian covers.

3.4. **Finite covers.** Let M be a 3–manifold, $\phi \in H^1(M)$ non–trivial and $\alpha : \pi_1(M) \to G$ a homomorphism to a finite group. Then we get a representation $\alpha : \pi_1(M) \to G \to \mathrm{GL}(\mathbb{F}, |G|)$ via the regular representation of G. In Section 9.1 we will see that this gives an abundant supply of representations. We denote the resulting Alexander polynomials by $\Delta_i^G(t)$, suppressing the homomorphism α in the notation.

Theorem 3.7. *Let M be a 3–manifold whose boundary is empty or consists of tori, $\phi \in H^1(M)$ primitive, and $\alpha : \pi_1(M) \to G$ an epimorphism to a finite group. If $\Delta_1^G(t) \neq 0$ then*

$$||\phi||_T \geq \frac{1}{|G|}\Big(\deg\big(\Delta_1^G(t)\big) - \big(1 + b_3(M)\big)\deg\big(\Delta_0^G(t)\big)\Big).$$

Since the representation $\alpha : \pi_1(M) \to G \to \mathrm{GL}(\mathbb{F}, |G|)$ is unitary, Theorem 3.7 follows from Theorem 3.3. We outline an illuminating alternative proof of Theorem 3.7, using only McMullen's Theorem 3.5. Let M be a 3–manifold and $\alpha : \pi_1(M) \to G$ a homomorphism to a finite group G. We denote the induced G–cover of M by M_G. For $\phi : H_1(M) \to \mathbb{Z}$ we denote the induced map $H_1(M_G) \to H_1(M) \to \mathbb{Z}$ by ϕ_G. Note that if $\phi : H_1(M) \to \mathbb{Z}$ is non–trivial, then ϕ_G is non–trivial as well.

Lemma 3.8.

$$\begin{array}{rcl} |G| \cdot ||\phi||_{T,M} & = & ||\phi_G||_{T,M_G} \\ \Delta^G_{M,\phi,i}(t) & = & \Delta_{M_G,\phi_G,i}(t), \text{ for all } i \end{array}.$$

For the second part we refer to [FV05]. The first part was shown by Gabai [Ga83]. In fact we will only need the inequality $|G| \cdot ||\phi||_{T,M} \geq ||\phi_G||_{T,M_G}$ which can easily be seen directly using the fact that if S_G is the G–cover of a surface S, then $\chi(S_G) = |G|\chi(S)$.

Alternative proof of Theorem 3.7. It is easy to see that $\phi(\mathrm{Ker}(\alpha)) \neq 0$. Therefore we can define $n \in \mathbb{N}$ to be the divisibility of ϕ restricted to $\mathrm{Ker}(\alpha)$. Recall that ϕ_G is given by $H_1(M_G) \cong H_1(\mathrm{Ker}(\alpha)) \to H_1(M) \xrightarrow{\phi} \mathbb{Z}$. It follows that the element $\frac{1}{n}\phi_G \in H^1(M_G)$ is defined and primitive. Since α is an epimorphism, M_G is connected and we can apply Theorem 3.5 to conclude that

$$\begin{array}{rcl} \big|\big|\frac{1}{n}\phi_G\big|\big|_{T,M_G} & \geq & \deg\big(\Delta_{M_G,\frac{1}{n}\phi_G,1}(t)\big) - \big(1 + b_3(M)\big) \\ & = & \deg\big(\Delta_{M_G,\frac{1}{n}\phi_G,1}(t)\big) - \big(1 + b_3(M)\big)\deg\big(\Delta_{M_G,\frac{1}{n}\phi_G,0}(t)\big). \end{array}$$

The second equality follows from the observation that $\Delta_{M_G,\frac{1}{n}\phi_G,0}(t) = t-1$. By Lemma 3.8 and the homogeneity of the Thurston norm we get

$$\begin{array}{rcl} |G|\cdot ||\phi||_{T,M} &=& ||\phi_G||_{T,M_G} \\ &=& n\big|\big|\frac{1}{n}\phi_G\big|\big|_{T,M_G} \\ &\geq& n\Big(\deg\big(\Delta_{M_G,\frac{1}{n}\phi_G,1}(t)\big) - \big(1+b_3(M)\big)\deg\big(\Delta_{M_G,\frac{1}{n}\phi_G,0}(t)\big)\Big). \end{array}$$

Clearly $\Delta_{M_G,\phi_G,i}(t) = \Delta_{M_G,\frac{1}{n}\phi_G,i}(t^n)$. Therefore $n\deg\big(\Delta_{M_G,\frac{1}{n}\phi_G,i}(t)\big) = \deg\big(\Delta_{M_G,\phi_G,i}(t)\big)$. The theorem now follows immediately from Lemma 3.8. and the observation that $b_3(M_G) = b_3(M)$. □

4. Proof of Main Theorem 1

4.1. Twisted Alexander polynomials of (M,ϕ).

Lemma 4.1. *Let $\phi \in H^1(M)$ be non–trivial and $\alpha : \pi_1(M) \to GL(\mathbb{F},k)$ a representation. Then $H_3^\alpha(M;\mathbb{F}^k[t^{\pm 1}]) = 0$ and $H_0^\alpha(M;\mathbb{F}^k[t^{\pm 1}])$ is finite dimensional as a $\mathbb{F}$–vector space.*

Proof. First assume that M is closed. We make use of an argument in the proof of [Mc02, Theorem 5.1]. Choose a triangulation τ of M. Let T be a maximal tree in the 1-skeleton of τ and let T' be a maximal tree in the dual 1-skeleton. We collapse T to form a single 0-cell and join the 3-simplices of T' to form a single 3-cell. Denote the number of 1-cells by n. It follows from M closed that $\chi(M)=0$, hence there are n 2-cells.

Write $\pi := \pi_1(M)$. From the CW structure we obtain a chain complex $C_* := C_*(\tilde{M})$

$$0 \to C_3^1 \xrightarrow{\partial_3} C_2^n \xrightarrow{\partial_2} C_1^n \xrightarrow{\partial_1} C_0^1 \to 0$$

where the C_i are free $\mathbb{Z}[\pi]$-right modules. In fact $C_i^k \cong \mathbb{Z}[\pi]^k$. Let A_i, $i=0,\dots,3$, be the matrices with entries in $\mathbb{Z}[\pi]$ corresponding to the boundary maps $\partial_i : C_i \to C_{i-1}$ with respect to the bases given by the lifts of the cells of M to $\tilde{M}$. Then we can arrange the lifts such that

$$\begin{array}{rcl} A_3 &=& (1-g_1, 1-g_2, \dots, 1-g_n)^t, \\ A_1 &=& (1-h_1, 1-h_2, \dots, 1-h_n), \end{array}$$

where $\{g_1,\dots,g_n\}$ and $\{h_1,\dots,h_n\}$ are generating sets for $\pi_1(M)$. Consider the chain complex $C_* \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[t^{\pm 1}]$

$$0 \to C_3^1 \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[t^{\pm 1}] \xrightarrow{\partial_3\otimes id} C_2^n \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[t^{\pm 1}] \xrightarrow{\partial_2\otimes id} C_1^n \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[t^{\pm 1}] \xrightarrow{\partial_1\otimes id} C_0^1 \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[t^{\pm 1}] \to 0.$$

Let $B=(b_{rs})$ be a $p\times q$ matrix with entries in $\mathbb{Z}[\pi]$. We write $b_{rs} = \sum b_{rs}^g g$ for $b_{rs}^g \in \mathbb{Z}, g \in \pi$. We define $(\alpha\otimes\phi)(B)$ to be the $p \times q$ matrix with entries $\sum b_{rs}^g \alpha(g) t^{\phi(g)}$. Since each $\sum b_{rs}^g \alpha(g)t^{\phi(g)}$ is a $k\times k$ matrix with entries in $\mathbb{F}[t^{\pm 1}]$ we can think of $(\alpha\otimes\phi)(B)$ as a $pk \times qk$ matrix with entries in $\mathbb{F}[t^{\pm 1}]$.

Note that $\partial_i \otimes \mathrm{id}$ is represented by $(\alpha \otimes \phi)(A_i)$. Since ϕ is non-trivial there exist k, l such that $\phi(g_k) \neq 0$ and $\phi(h_l) \neq 0$. It follows that $(\alpha \otimes \phi)(A_1)$ and $(\alpha \otimes \phi)(A_3)$ have full rank over $\mathbb{F}[t^{\pm 1}]$. The lemma now follows immediately in the case that M is closed.

If M has boundary, then clearly $H_3^\alpha(M; \mathbb{F}^k[t^{\pm 1}]) = 0$. The argument that $H_0^\alpha(M; \mathbb{F}^k[t^{\pm 1}])$ is finite dimensional is exactly the same as in the closed case. □

We note that the fact that $H_0^\alpha(M; \mathbb{F}^k[t^{\pm 1}])$ is finite–dimensional was first proved by Kirk and Livingston in [KL99a, Proposition 3.5].

Lemma 4.2. *Assume that ∂M is empty or consists of tori and $\phi \in H^1(M)$ is non–trivial. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a representation. If $\Delta_1^\alpha(t) \neq 0$, then $H_2^\alpha(M; \mathbb{F}^k[t^{\pm 1}])$ is $\mathbb{F}[t^{\pm 1}]$–torsion. In particular $\Delta_2^\alpha(t) \neq 0$.*

Proof. We know that $\Delta_i^\alpha(t) \neq 0$ for $i = 0, 1, 3$ by assumption and by Lemma 4.1. It follows from the long exact homology sequence for $(M, \partial M)$ and from duality that $\chi(M) = \frac{1}{2}\chi(\partial M)$. Hence $\chi(M) = 0$ in our case. It follows from Lemma 4.14 (applied to the field $\mathbb{F}(t)$) that

$$\sum_{i=0}^{3} (-1)^i \dim_{\mathbb{F}(t)} \left(H_i^\alpha(M; \mathbb{F}^k[t^{\pm 1}] \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t)) \right) = k \cdot \chi(M) = 0.$$

Note that $H_i^\alpha(M; \mathbb{F}^k[t^{\pm 1}] \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t)) = H_i^\alpha(M; \mathbb{F}^k[t^{\pm 1}]) \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t)$ since $\mathbb{F}(t)$ is flat over $\mathbb{F}[t^{\pm 1}]$. By assumption $H_i^\alpha(M; \mathbb{F}^k[t^{\pm 1}]) \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t) = 0$ for $i \neq 2$, hence $H_2^\alpha(M; \mathbb{F}^k[t^{\pm 1}]) \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t) = 0$ by Lemma 4.14. □

We get the following corollary immediately from Lemmas 4.1 and 4.2.

Corollary 4.3. *Let M be a 3–manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M)$ be non–trivial and $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ a representation. If $\Delta_1^\alpha(t) \neq 0$ then $\Delta_i^\alpha(t) \neq 0$ for all i, and $\Delta_3^\alpha(t) = 1$.*

4.2. **Main argument.** In this section we prove Theorem 3.1. Before beginning the proof we give relevant propositions and lemmas. In an attempt to make the proof easier to read we prove several technical lemmas separately in Section 4.3. We also need one delicate duality argument which we explain in detail in Section 4.4

Let M be a 3–manifold and $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ a representation. We will endow any subset $X \subset M$ with the representation given by $\pi_1(X) \to \pi_1(M) \xrightarrow{\alpha} \mathrm{GL}(\mathbb{F}, k)$. Note that because of base point issues this induced homomorphism is only defined up to conjugacy. But the homology groups $H_*^\alpha(X; \mathbb{F}^k)$ are isomorphic, and their dimensions over $\mathbb{F}$ are well-defined. We will therefore suppress base points and the choice of paths connecting base points in our notation. Let $b_n^\alpha(X) := \dim_{\mathbb{F}}(H_n^\alpha(X; \mathbb{F}^k))$ for $n \geq 0$.

Proposition 4.4. *Let $\phi \in H^1(M)$ and S a properly embedded surface dual to ϕ. Then*

$$b_1^\alpha(S) \geq \dim_{\mathbb{F}}\left(Tor_{\mathbb{F}[t^{\pm 1}]}\left(H_1^\alpha(M;\mathbb{F}^k[t^{\pm 1}])\right)\right).$$

In particular if $\Delta_1^\alpha(t) \neq 0$, then $b_1^\alpha(S) \geq \deg(\Delta_1^\alpha(t))$.

Proof. Denote the components of S by $S_1,\dots,S_l$. Denote by N the result of cutting M along S. Denote by i_+ and i_- the two inclusions of S into ∂N induced by taking the positive and the negative inclusions of S into N. We use the same notations i_+ and i_- for the induced homomorphisms on homology groups. Note that ϕ vanishes on $\pi_1(N)$ and on every $\pi_1(S_i)$. Indeed, every curve in S_i can be pushed off into N, where ϕ vanishes. It follows that $H_i^\alpha(N;\mathbb{F}^k[t^{\pm 1}]) \cong H_i^\alpha(N;\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm 1}]$ and $H_i^\alpha(S;\mathbb{F}^k[t^{\pm 1}]) \cong H_i^\alpha(S;\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm 1}]$. Therefore we have a commutative diagram if exact sequences

(2)

$$\begin{array}{cccccccc}
\to & H_i^\alpha(S;\mathbb{F}^k[t^{\pm 1}]) & \xrightarrow{ti_- - i_+} & H_i^\alpha(N;\mathbb{F}^k[t^{\pm 1}]) & \to & H_i^\alpha(M;\mathbb{F}^k[t^{\pm 1}]) & \to \\
 & \downarrow\cong & & \downarrow\cong & & \downarrow = & \to \\
\to & H_i^\alpha(S;\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm 1}] & \xrightarrow{ti_- - i_+} & H_i^\alpha(N;\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm 1}] & \to & H_i^\alpha(M;\mathbb{F}^k[t^{\pm 1}]). & \to
\end{array}$$

Note that $\mathrm{Ker}\{H_0^\alpha(S;\mathbb{F}^k[t^{\pm 1}]) \to H_0^\alpha(N;\mathbb{F}^k[t^{\pm 1}])\} \subset H_0^\alpha(S;\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm 1}]$ is a (possibly trivial) free $\mathbb{F}[t^{\pm 1}]$–module F. Therefore we get an exact sequence

$$H_1^\alpha(S;\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm 1}] \xrightarrow{ti_- - i_+} H_1^\alpha(N;\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm 1}] \to H_1^\alpha(M;\mathbb{F}^k[t^{\pm 1}]) \xrightarrow{\partial} F \to 0.$$

Since $\mathbb{F}[t^{\pm 1}]$ is a PID the map ∂ splits, i.e., $H_1^\alpha(M;\mathbb{F}^k[t^{\pm 1}]) \cong \mathrm{Ker}(\partial)\oplus F$. Using appropriate bases the map $ti_- - i_+$, which represents the module $\mathrm{Ker}(\partial)$, is presented by a matrix of size $\dim_{\mathbb{F}}\left(H_1^\alpha(N;\mathbb{F}^k)\right) \times \dim_{\mathbb{F}}\left(H_1^\alpha(S;\mathbb{F}^k)\right)$ of the form $At+B$, A,B matrices over $\mathbb{F}$. It follows from Lemma 4.10 that

$$b_1^\alpha(S) = \dim_{\mathbb{F}}\left(H_1^\alpha(S;\mathbb{F}^k)\right) \geq \dim_{\mathbb{F}}\left(\mathrm{Tor}_{\mathbb{F}[t^{\pm 1}]}(H_1^\alpha(M;\mathbb{F}^k[t^{\pm 1}]))\right).$$

The last part of this proposition is obvious by Lemma 2.3. □

A *weighted surface* $\tilde{S}$ in M is a collection of pairs $(S_i,w_i), i=1,\dots,k$ where $S_i \subset M$ are properly embedded, oriented, disjoint surfaces in M and w_i are non–negative integers. We denote the union $\bigcup_{i,w_i\neq 0} S_i \subset M$ by $|\tilde{S}|$.

Every weighted surface $\tilde{S}$ defines an element $\phi_{\tilde{S}} := \sum_{i=1}^k w_i\cdot PD([S_i]) \in H^1(M)$ where $PD(f)\in H^1(M)$ denotes the Poincaré dual of an element $f\in H_2(M,\partial M)$. By taking w_i parallel copies of S_i we get an (unweighted) properly embedded oriented surface $\tilde{S}^\#$ such that $\phi_{\tilde{S}} = PD(\tilde{S}^\#)$. An example of a the surface $\tilde{S}^\#$ for a weighted surface $\tilde{S}$ is given in Figure 2.

We need the following proposition proved by Turaev in [Tu02b].

Proposition 4.5. *Let $\phi \in H^1(M)$. Then there exists a weighted surface $\tilde{S}$ with*

(1) $\phi_{\tilde{S}} = \phi$,

FIGURE 2. Weighted surface in a handlebody.

(2) $\chi_-(\tilde{S}^\#) = ||\phi||_T$,
(3) $M \setminus |\tilde{S}|$ *connected,*
(4) $w_i \neq 0$ *for all* i.

We give a quick outline of the proof.

Proof. Since ϕ is dual to an embedded surface, which we can view as a weighted surface by giving weight 1 to each component, there exist weighted surfaces $\tilde{S}$ such that $\tilde{S}$ satisfies properties (1), (2) and (4).

Now among these let $\tilde{S} = (S_i, w_i), i \in I$, be a weighted surface with minimal $b_0(M \setminus |\tilde{S}|)$. We have to show that $b_0(M \setminus |\tilde{S}|) = 1$. Assume the contrary. Let N be a component of $M \setminus |\tilde{S}|$. Denote by $\bar{N}$ the closure of N in M. Let I_+ (respectively I_-) be the set of all $i \in I$ such that $S_i \subset \partial\bar{N}$ and the orientation of $\partial\bar{N}$ induced by the one of M induces the given orientation of S_i (respectively the opposite orientation of S_i). Note that $I_+ \cup I_-$ is non–empty since we assume that $N \neq M \setminus |\tilde{S}|$. We can write

$$\partial\bar{N} = \bigcup_{i \in I_+} S_i \cup \bigcup_{i \in I_-} -S_i \cup S'$$

for some surface S'. Clearly $\sum\limits_{i \in I_+} S_i$ and $\sum\limits_{i \in I_-} S_i$ are homologous in $C_2(M, \partial M)$. Without loss of generality we can assume that $\sum\limits_{i \in I_+} \chi_-(S_i) \geq \sum\limits_{i \in I_-} \chi_-(S_i)$. Let $l := \min\{w_i | i \in I_+\}$. Now let $\tilde{T}$ be the weighted surface which is the result of reducing the weights of $S_i, i \in I_+$, by l and increasing the weights of $S_i, i \in I_-$, by l.

Clearly $\phi_{\tilde{T}} = \phi_{\tilde{S}}$, $\chi_-(\tilde{T}^\#) \leq \chi_-(\tilde{S}^\#)$. Furthermore $b_0(M \setminus |\tilde{T}|) < b_0(M \setminus |\tilde{S}|)$ since at least one weight of a component S_i, $i \in I_+ \cup I_-$, became zero. This contradicts the minimality of $\tilde{S}$. $\square$

Lemma 4.6. *Let* $\phi \in H^1(M)$ *be primitive. Let* $\tilde{S}$ *denote the weighted surface as in Proposition 4.5. Assume* $\Delta_1^\alpha(t) \neq 0$. *Then* $S := \tilde{S}^\#$ *is either connected or* $b_0^\alpha(S_i) = 0$ *for any component* S_i *of* S.

Proof. Denote by N the result of cutting M along S. Consider the Mayer–Vietoris sequence (2) in Proposition 4.4

$$\begin{array}{rl} & \to \; H_1^\alpha(M;\mathbb{F}^k[t^{\pm 1}]) \\ \to H_0^\alpha(S;\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm1}] \xrightarrow{ti_- - i_+} H_0^\alpha(N;\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm1}] & \to \; H_0^\alpha(M;\mathbb{F}^k[t^{\pm1}]) \to 0. \end{array}$$

From $\Delta_1^\alpha(t)\neq 0$ it follows that $H_1^\alpha(M;\mathbb{F}^k[t^{\pm1}])$ is $\mathbb{F}[t^{\pm1}]$–torsion. By Lemma 4.1 $H_0^\alpha(M;\mathbb{F}^k[t^{\pm1}])$ is a finite–dimensional $\mathbb{F}$–vector space, hence $\mathbb{F}[t^{\pm1}]$–torsion. If we now consider the above exact sequence with $\mathbb{F}(t)$–coefficients it follows that

$$H_0(S;\mathbb{F}^k)\cong H_0(N;\mathbb{F}^k). \tag{3}$$

Since we can arrange w_i parallel copies of S_i inside $\nu(S_i)$ in M, we see that $N\cong (M\setminus \nu|\tilde{S}|)\cup \bigcup_{i=1}^{l}\bigcup_{j=1}^{w_i-1} S_i\times[-1,1]$. Therefore we have the following isomorphisms

$$\begin{array}{rcl} H_0^\alpha(S;\mathbb{F}^k) & \cong & \bigoplus_{i=1}^{l} H_0^\alpha(S_i;\mathbb{F}^k) \quad\oplus\quad \bigoplus_{i=1}^{l} H_0^\alpha(S_i;\mathbb{F}^k)^{w_i-1} \\ H_0^\alpha(N;\mathbb{F}^k) & \cong & H_0^\alpha(M\setminus\nu|\tilde{S}|;\mathbb{F}^k) \quad\oplus\quad \bigoplus_{i=1}^{l} H_0^\alpha(S_i;\mathbb{F}^k)^{w_i-1} \end{array} \tag{4}$$

where $H_0^\alpha(S_i;\mathbb{F}^k)^{w_i-1} := \bigoplus^{w_i-1} H_0^\alpha(S_i;\mathbb{F}^k)$. Note that the maps $i_+,i_- : \pi_1(S_i)\to \pi_1(M)\xrightarrow{\alpha}\mathrm{GL}(\mathbb{F},k)$ factor through $\pi_1(M\setminus\nu|\tilde{S}|)$. Therefore

$$b_0^\alpha(S_i)\geq b_0^\alpha(M\setminus\nu|\tilde{S}|), i=1,\dots,l \tag{5}$$

by Lemma 4.13.

First consider the case $b_0^\alpha(M\setminus\nu|\tilde{S}|)=0$. In that case it follows from the isomorphisms in (3) and (4) that $\bigoplus_{i=1}^{l} H_0(S_i;\mathbb{F}^k)=0$, hence $b_0^\alpha(S_i)=0$ for all $i=1,\dots,l$.

Now assume that $b_0^\alpha(M\setminus\nu|\tilde{S}|)>0$. It follows immediately from the isomorphisms in (3) and (4) and from the inequality (5) that $l=1$. But since ϕ is primitive it also follows that $w_1=1$, i.e., S is connected. $\square$

Lemma 4.7. *Let $\phi\in H^1(M)$ be primitive and $\Delta_1^\alpha(t)\neq 0$. Let $S:=\tilde{S}^\#$ denote the same surface as in Proposition 4.5. Then*

$$b_0^\alpha(S)=\deg\left(\Delta_0^\alpha(t)\right).$$

Proof. Let N be M cut along S. Since $\Delta_1^\alpha(t)\neq 0$, we have $H_0^\alpha(S;\mathbb{F}^k)\cong H_0^\alpha(N;\mathbb{F}^k)$ as $\mathbb{F}$–vector spaces (see (2) in the proof of Lemma 4.6). First assume that $b_0^\alpha(S_i)=0$ for every component S_i of S. Then $H_0^\alpha(S;\mathbb{F}^k)=H_0^\alpha(N;\mathbb{F}^k)=0$. This implies that $H_0^\alpha(M;\mathbb{F}^k[t^{\pm1}])=0$ from the exact sequence (2) in the proof of Proposition 4.4, hence $\Delta_0^\alpha(t)=1$.

Now assume that $b_0^\alpha(S_i)\neq 0$ for some i. By Lemma 4.6 S is connected. Hence N is connected. It follows from Lemma 4.13 that the maps $i_+,i_- : H_0^\alpha(S;\mathbb{F}^k)\to$

$H_0^\alpha(N;\mathbb{F}^k)$ are surjective. Since $H_0^\alpha(S;\mathbb{F}^k)\cong H_0^\alpha(N;\mathbb{F}^k)$ it follows that i_+ and i_- induce isomorphisms on $H_0^\alpha(S;\mathbb{F}^k)$. Note that this argument uses that S is connected.

Let $b:=b_0^\alpha(S)=b_0^\alpha(N)$. Picking appropriate bases for $H_0^\alpha(S;\mathbb{F}^k)$ and $H_0^\alpha(N;\mathbb{F}^k)$ the sequence (2) becomes

$$\mathbb{F}^b\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm1}]\xrightarrow{t\cdot Id-J}\mathbb{F}^b\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm1}]\to H_0^\alpha(M;\mathbb{F}^k[t^{\pm1}])\to 0,$$

where $J:\mathbb{F}^b\to\mathbb{F}^b$ is an isomorphism. It follows that $H_0^\alpha(M;\mathbb{F}^k[t^{\pm1}])\cong\mathbb{F}^b\cong H_0^\alpha(S;\mathbb{F}^k)$. The lemma now follows from Lemma 2.3. $\square$

We note that from Lemmas 4.6 and 4.7 we immediately get the following useful corollary:

Corollary 4.8. *If $\Delta_0^\alpha(t)\neq 1$ and $\Delta_1^\alpha(t)\neq 0$, then there exists a surface of minimal complexity dual to ϕ which is connected.*

Lemma 4.9. *Assume that ∂M is empty or consists of tori. Let $\phi\in H^1(M)$ be primitive and $\Delta_1^\alpha(t)\neq 0$. Let $S:=\tilde{S}^\#$ denote the same surface as in Proposition 4.5. Then*

$$b_2^\alpha(S)=\deg\left(\Delta_2^\alpha(t)\right).$$

Proof. Let $\tilde{S}=(S_i,w_i)_{i=1,\dots,l}$ be the weighted surface with $w_i\neq 0$ for all i from Proposition 4.5. Let N be M cut along $S=\tilde{S}^\#$. Let $I':=\{i\in\{1,\dots,l\}|S_i \text{ closed}\}$ and $I'':=\{i\in\{1,\dots,l\}|S_i \text{ has non–empty boundary}\}$. Denote the union of w_i parallel copies of $S_i, i\in I'$, by $S'\subset S$. Clearly $b_2^\alpha(S)=b_2^\alpha(S')$.

Note that we can write $\partial N=S'_+\cup S'_-\cup W$ for some surface W where S'_- and S'_+ are the images of the two canonical inclusion maps of $S'\to N$. It follows from Lemmas 4.1 and 4.2 that the long exact sequence (2) becomes

$$0\to H_2^\alpha(S';\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm1}]\xrightarrow{ti_--i_+}H_2^\alpha(N;\mathbb{F}^k)\otimes_{\mathbb{F}}\mathbb{F}[t^{\pm1}]\to H_2^\alpha(M;\mathbb{F}^k[t^{\pm1}])\to 0.$$

Clearly we are done once we show that $i_-,i_+:H_2^\alpha(S';\mathbb{F}^k)\to H_2^\alpha(N;\mathbb{F}^k)$ are isomorphisms. Considering the sequence with $\mathbb{F}(t)$–coefficients it follows that $H_2^\alpha(S';\mathbb{F}^k)$ and $H_2^\alpha(N;\mathbb{F}^k)$ have the same dimension as $\mathbb{F}$-vector spaces. It is therefore enough to show that i_- and i_+ are injections, or equivalently that the maps $H_2^\alpha(S'_\pm;\mathbb{F}^k)\to H_2^\alpha(N;\mathbb{F}^k)$ are injections.

Consider the short exact sequence

$$H_3^\alpha(N,S_+;\mathbb{F}^k)\to H_2^\alpha(S'_+;\mathbb{F}^k)\to H_2^\alpha(N;\mathbb{F}^k).$$

By Poincaré duality and by Lemma 4.15 we have

$$H_3^\alpha(N,S'_+;\mathbb{F}^k)\cong H^0_\alpha(N,S'_-\cup W;\mathbb{F}^k)\cong \operatorname{Hom}_{\mathbb{F}}(H_0^{\overline{\alpha}}(N,S'_-\cup W;\mathbb{F}^k),\mathbb{F}).$$

Claim.

$$H_0^{\overline{\alpha}}(N,S'_-\cup W;\mathbb{F}^k)=0.$$

Recall that

$$N \cong M \setminus \nu|\tilde{S}| \cup \bigcup_{i\in I'} \bigcup_{j=1}^{w_i-1} S_i \times [0,1] \cup \bigcup_{i\in I''} \bigcup_{j=1}^{w_i-1} S_i \times [0,1]$$

which equals the decomposition of N into connected components. Clearly there exists a surjective map

$$\varphi : \{\text{components of } S'_- \cup W\} \to \{\text{components of } N\},$$

such that $S_0 \subset \partial(\varphi(S_0))$ for every component S_0 of $S'_- \cup W$. Therefore it follows from Lemma 4.13 that $H_0^{\overline{\alpha}}(S'_- \cup W; \mathbb{F}^k[t^{\pm 1}]) \to H_0^{\overline{\alpha}}(N; \mathbb{F}^k[t^{\pm 1}])$ is surjective. The claim now follows from the long exact homology sequence. □

Now we can conclude the proof of Theorem 3.1.

Proof of Theorem 3.1. Without loss of generality we can assume that ϕ is primitive since the Thurston norm and the degrees of twisted Alexander polynomials are homogeneous. Let $\tilde{S}$ be the weighted surface from Proposition 4.5. Let $S := \tilde{S}^{\#}$. By Lemma 4.14 we have

$$\begin{array}{rcl} ||\phi||_T &=& \max\{0, b_1(S) - (b_0(S) + b_2(S))\} \\ &\geq& b_1(S) - (b_0(S) + b_2(S)) \\ &=& \frac{1}{k}\left(b_1^\alpha(S) - (b_0^\alpha(S) + b_2^\alpha(S))\right). \end{array}$$

The Theorem now follows immediately from Proposition 4.4 and Lemmas 4.1, 4.7, 4.9 and 2.4.

□

4.3. Lemmas used in the proof of Main Theorem 1.

Lemma 4.10. *Let A, B be $p \times q$–matrices over a field $\mathbb{F}$. Let H be a $\mathbb{F}[t^{\pm 1}]$–module with the presentation matrix $At + B$. Then $\dim_{\mathbb{F}}(Tor_{\mathbb{F}[t^{\pm 1}]}(H)) \leq \min(p, q)$.*

Proof. This lemma is well–known, and a proof in the much harder non–commutative case is given by Harvey [Ha05, Proposition 9.1]. Therefore we give just an outline for the proof. Let $C = At + B$. Using row and column operations over the PID $\mathbb{F}[t^{\pm 1}]$ we can transform C into a matrix of the form

$$\begin{pmatrix} f_1(t) & 0 & \dots & 0 & 0 \\ 0 & f_2(t) & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & f_l(t) & 0 \\ 0 & 0 & \dots & 0 & (0)_{p-l\times q-l} \end{pmatrix}$$

for some $f_i(t) \in \mathbb{F}[t^{\pm 1}] \setminus \{0\}$. Clearly $\dim_{\mathbb{F}}(\mathrm{Tor}_{\mathbb{F}[t^{\pm 1}]}(H)) = \sum_{i=1}^{l} \deg(f_i(t))$. Since row and column operations do not change the ideals of $\mathbb{F}[t^{\pm 1}]$ generated by minors (cf. [CF77, p. 101]), and since any $k \times k$ minor of $At + B$ has degree at most k, it follows that $\sum_{i=1}^{l} \deg(f_i(t)) \leq \min(p, q)$. □

Lemma 4.11. *Let X be a CW–complex and let A be a left $\mathbb{Z}[\pi_1(X)]$–module. Then $H_i(X;A) \cong H_i(\pi_1(X);A)$ for $i = 0,1$.*

The above lemma is immediate from the observation that adding cells of dimension greater than or equal to 3 to X gives the Eilenberg–Maclane space $K(\pi_1(X),1)$, but does not change the two lowest homology groups.
The following lemma was proved by Kirk and Livingston in [KL99a, Theorem 2.1].

Lemma 4.12. *Let $\phi \in H^1(M)$ be primitive. Denote the cover of M corresponding to ϕ by M_ϕ, i.e., $\pi_1(M_\phi) = Ker(\phi)$. Then*

$$H_i^\alpha(M;\mathbb{F}^k[t^{\pm 1}]) \cong H_i^\alpha(M_\phi;\mathbb{F}^k),$$

for all i and

$$H_i^\alpha(M;\mathbb{F}^k[t^{\pm 1}]) \cong H_i(Ker(\phi);\mathbb{F}^k)$$

for $i = 0,1$.

Proof. Let $\mu \in \pi_1(M)$ such that $\phi(\mu) = 1$. Then the chain map

$$\begin{array}{rcl} C_*(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{F}^k[t^{\pm 1}] & \to & C_*(\tilde{M}) \otimes_{\mathbb{Z}[ker(\phi)]} \mathbb{F}^k \\ s \otimes (v \otimes t^k) & \mapsto & s\mu^k \otimes \alpha(\mu)^{-k} v \end{array}$$

defines a chain homotopy which shows that $H_*^\alpha(M;\mathbb{F}^k[t^{\pm 1}]) \cong H_*^\alpha(M_\phi;\mathbb{F}^k)$. Furthermore $H_i^\alpha(M_\phi;\mathbb{F}^k[t^{\pm 1}]) \cong H_i^\alpha(\mathrm{Ker}(\phi);\mathbb{F}^k[t^{\pm 1}])$ for $i = 0,1$ by Lemma 4.11. □

Lemma 4.13. *Let V be an $\mathbb{F}$–vector space. Let A be a group and $\alpha : A \to GL(V)$ a representation. If $\varphi : B \to A$ is a homomorphism, then $H_0^{\alpha\circ\varphi}(B;V) \to H_0^\alpha(A;V)$ is surjective.*

Proof. The lemma follows immediately from the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \{\alpha(\varphi(b))v - v | b \in B, v \in V\} & \to & V & \to & H_0(B;V) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \{\alpha(a)v - v | a \in A, v \in V\} & \to & V & \to & H_0(A;V) & \to & 0 \end{array}$$

and the observation that the vertical map on the left is injective. □

Lemma 4.14. *Let X be an n–manifold, $\mathbb{K}$ a field, and $\alpha : \pi_1(X) \to GL(\mathbb{K},k)$ a representation. Then*

$$\sum_{i=0}^{n} (-1)^n dim_{\mathbb{K}}(H_*^\alpha(X;\mathbb{K}^k)) = k\chi(X).$$

Proof. Write $\pi := \pi_1(X)$ and denote the universal cover of X by $\tilde{X}$. Pick a finite cell decomposition of X and pick an π–equivariant lifting of the cell decomposition to a cell decomposition of $\tilde{X}$. Then we get the following equalities for Euler characteristics of $\mathbb{K}$–complexes:

$$\begin{array}{rcl}\sum_{i=0}^{n}(-1)^n\dim_{\mathbb{K}}(H_*^{\alpha}(X;\mathbb{K}^k)) & = & \chi(H_*^{\alpha}(X;\mathbb{K}^k)) \\ & = & \chi(H_*(C_*(\tilde{X};\mathbb{Z})\otimes_{\mathbb{Z}[\pi]}\mathbb{K}^k)) \\ & = & \chi(C_*(\tilde{X};\mathbb{Z})\otimes_{\mathbb{Z}[\pi]}\mathbb{K}^k) \\ & = & \chi(C_*(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[\pi]\otimes_{\mathbb{Z}[\pi]}\mathbb{K}^k) \\ & = & \chi(C_*(X;\mathbb{K})\otimes_{\mathbb{K}}\mathbb{K}^k) \\ & = & k\chi(C_*(X;\mathbb{K})) \\ & = & k\chi(H_*(X;\mathbb{K})) \\ & = & k\chi(X).\end{array}$$

□

4.4. **Duality arguments.** In this section we clarify a delicate duality argument. Since this is perhaps of independent interest, and since we need it in [F05b] we will explain this in the non–commutative setting.

In this section let R be a (possibly non–commutative) ring with involution $r \mapsto \overline{r}$ such that $\overline{ab} = \overline{b}\cdot\overline{a}$. Let V be a left R–module together with a map $\beta : \pi_1(M) \to \mathrm{GL}(V,R)$. This representation β can be used to define a left $\mathbb{Z}[\pi_1(M)]$–module structure on V which commutes with the R–module structure. Pick a non–singular R–sesquilinear inner product $\langle\, , \,\rangle : V\times V \to R$. This means that for all $v,w\in V$ and $r\in R$ we have

$$\langle vr, w\rangle = \langle v,w\rangle r, \quad \langle v, wr\rangle = \overline{r}\langle v,w\rangle$$

and $\langle\, , \,\rangle$ induces via $v\mapsto (w\mapsto \langle v,w\rangle)$ an R–module isomorphism $V\cong \mathrm{Hom}_R(V,R)$. Here we view $\mathrm{Hom}_R(V,R)$ as right R–module homomorphisms where R gets the right R–module structure given by involuted left multiplication. Furthermore consider $\mathrm{Hom}_R(V,R)$ as a right R–module via right multiplication in the target R.

There exists a unique representation $\bar{\beta} : \pi_1(M) \to \mathrm{GL}(V,R)$ such that

$$\langle \beta(g^{-1})v, w\rangle = \langle v, \bar{\beta}(g)w\rangle$$

for all $v,w\in V, g\in\pi_1(M)$. Note that $\bar{\beta}$ induces a left $\mathbb{Z}[\pi_1(M)]$–module structure on V (which is possibly different from that induced from β) which commutes with the R–module structure. To clarify which $\mathbb{Z}[\pi_1(M)]$–module structure we use, we sometimes denote V with the $\mathbb{Z}[\pi_1(M)]$–module structure induced from β (respectively $\bar{\beta}$) by $V(\beta)$ (respectively $V(\bar{\beta})$). Note that they are the same viewed as R–modules.

Lemma 4.15. [KL99a, p. 638] *Let X be an n–manifold, V an R–module and $\beta : \pi_1(X) \to GL(V)$ a representation. Let $\langle\, , \,\rangle : V\times V\to R$ be a non–singular sesquilinear inner product as above. If R is a PID then*

$$H_{n-i}^{\beta}(X;V(\beta)) \cong Hom_R(H_i^{\overline{\beta}}(X,\partial X;V(\bar{\beta})),R)\oplus Ext_R(H_{i-1}^{\overline{\beta}}(X,\partial X;V(\bar{\beta})),R)$$

as R–modules.

Here we equip $H_*(-,V), H^*(-,V)$ with the right R–module structures given on V. Also for a right R–module H we view $\mathrm{Hom}_R(H,R)$ as a right R–module homomorphisms where R gets the right R–module structure given by involuted left multiplication. We consider $\mathrm{Hom}_R(H,R)$ as a right R–module via right multiplication in the target R.

Proof. Let $\pi := \pi_1(X)$. Let $V(\beta)' = V(\beta)$ as R–modules equipped with the right $\mathbb{Z}[\pi_1(M)]$–module structure given by $v \cdot g := \beta(g^{-1})v$ for $v \in V(\beta)$ and $g \in \pi$. By Poincaré duality

$$H^{\beta}_{n-i}(X;V(\beta)) \cong H^i(X,\partial X;V(\beta)') := H_i\big(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{X},\partial\tilde{X}),V(\beta)')\big),$$

where $\tilde{X}$ denotes the universal cover of X. Using the inner product we get a map

$$\begin{array}{rcl}\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{X},\partial\tilde{X}),V(\beta)') & \to & \mathrm{Hom}_R\big(C_*(\tilde{X},\partial\tilde{X})\otimes_{\mathbb{Z}[\pi]} V(\bar{\beta}),R\big)\\ f & \mapsto & ((c\otimes w)\mapsto \langle f(c),w\rangle).\end{array}$$

Note that this map is well–defined since $\langle \beta(g^{-1})v,w\rangle = \langle v,\bar{\beta}(g)w\rangle$. It is now easy to see that this defines in fact an isomorphism of right R–module chain complexes.

Now we can apply the universal coefficient theorem for chain complexes over the PID R to $C_*(\tilde{X},\partial\tilde{X})\otimes_{\mathbb{Z}[\pi]} V(\bar{\beta})$. The lemma is now immediate. □

Now assume that the field $\mathbb{F}$ has a (possibly trivial) involution. We equip $\mathbb{F}^k$ with a hermitian inner product, denoted by $\langle\, ,\, \rangle$.

Proof of Proposition 3.2. We extend the involution on $\mathbb{F}$ to $\mathbb{F}[t^{\pm 1}]$ by taking $t \mapsto t^{-1}$. Now equip $\mathbb{F}^k[t^{\pm1}]$ with the hermitian inner product defined by $\langle vt^i, wt^j\rangle := \langle v,w\rangle t^i t^{-j}$ for all $v,w\in\mathbb{F}^k$. To simplify the notation we denote $\mathbb{F}^k[t^{\pm1}](\alpha\otimes\phi)$ and $\mathbb{F}^k[t^{\pm1}](\overline{\alpha\otimes\phi})$ just by $\mathbb{F}^k[t^{\pm1}]$. The $\mathbb{Z}[\pi_1(M)]$–module structure on $\mathbb{F}^k[t^{\pm1}]$ will always be clear from the context.

Note that $\mathbb{F}[t^{\pm1}]$ is a PID. We apply Lemma 4.15 with $R=\mathbb{F}[t^{\pm1}]$, $V=\mathbb{F}^k[t^{\pm1}]$ and $\beta=\alpha\otimes\phi$, and get

$$\begin{array}{rcl}H_2^{\alpha\otimes\phi}(M;\mathbb{F}^k[t^{\pm1}]) & \cong & \mathrm{Hom}_{\mathbb{F}[t^{\pm1}]}\Big(H_1^{\overline{\alpha\otimes\phi}}(M,\partial M;\mathbb{F}^k[t^{\pm1}]),\mathbb{F}[t^{\pm1}]\Big)\\ & \oplus & \mathrm{Ext}_{\mathbb{F}[t^{\pm1}]}\big(H_0^{\overline{\alpha\otimes\phi}}(M,\partial M;\mathbb{F}^k[t^{\pm1}]),\mathbb{F}[t^{\pm1}]\big)\end{array}$$

as $\mathbb{F}[t^{\pm1}]$–modules. Since $\Delta_1^{\alpha}(t)\neq 0$, $H_2^{\alpha\otimes\phi}(M;\mathbb{F}^k[t^{\pm1}])$ is $\mathbb{F}[t^{\pm1}]$–torsion by Lemma 4.2. Hence the first summand on the right hand side is zero.

By Lemma 4.1 $H_0^{\overline{\alpha\otimes\phi}}(M;\mathbb{F}^k[t^{\pm1}])$ is $\mathbb{F}[t^{\pm1}]$–torsion. From the long exact homology sequence of the pair $(M,\partial M)$ it follows that $H_0^{\overline{\alpha\otimes\phi}}(M,\partial M;\mathbb{F}^k[t^{\pm1}])$ is also $\mathbb{F}[t^{\pm1}]$–torsion. Since $H_0^{\overline{\alpha\otimes\phi}}(M,\partial M;\mathbb{F}^k[t^{\pm1}])$ is a finitely generated $\mathbb{F}[t^{\pm1}]$–torsion module and $\mathbb{F}[t^{\pm1}]$ is a PID, $\mathrm{Ext}_{\mathbb{F}[t^{\pm1}]}(H_0^{\overline{\alpha\otimes\phi}}(M,\partial M;\mathbb{F}^k[t^{\pm1}]),\mathbb{F}[t^{\pm1}])\cong H_0^{\overline{\alpha\otimes\phi}}(M,\partial M;\mathbb{F}^k[t^{\pm1}])$.

If M is closed then we get $H_2^{\alpha}(M;\mathbb{F}^k[t^{\pm1}])\cong H_0^{\overline{\alpha\otimes\phi}}(M;\mathbb{F}^k[t^{\pm1}])$. Note that $\overline{\alpha\otimes\phi} = \overline{\alpha}\otimes(-\phi)$. Therefore we deduce that $\Delta_2^{\alpha}(t)=\Delta_0^{\overline{\alpha}}(t^{-1})$.

If $\partial M \neq 0$, then by Lemma 4.13 the map $H_0^{\overline{\alpha\otimes\phi}}(\partial M;\mathbb{F}^k[t^{\pm 1}]) \to H_0^{\overline{\alpha\otimes\phi}}(M;\mathbb{F}^k[t^{\pm 1}])$ is surjective, hence $H_0^{\overline{\alpha\otimes\phi}}(M,\partial M;\mathbb{F}^k[t^{\pm 1}]) = 0$. This shows that $H_2^\alpha(M;\mathbb{F}^k[t^{\pm 1}]) = 0$ and hence $\Delta_2^\alpha(t) = 1$. $\square$

5. The case of vanishing Alexander polynomials

Let L be a boundary link (for example a split link). It is well–known that the multivariable Alexander polynomial of L has to vanish (cf. [Hi02]). With a little extra care it is not hard to show that the twisted multivariable and twisted one–variable Alexander polynomials vanish as well. (See [FK05] for the definition of twisted multivariable Alexander polynomials.) Therefore Theorem 3.1 can not be applied to get lower bounds on the Thurston norm.

It follows clearly from Proposition 4.4 and Lemma 4.9 that the condition $\Delta_1^\alpha(t) \neq 0$ is only needed to ensure that there exists a surface S dual to ϕ with $b_0^\alpha(S) = \deg(\Delta_0^\alpha(t))$ and $b_2^\alpha(S) = \deg(\Delta_2^\alpha(t))$. Sometimes upper bounds on $b_0^\alpha(S)$ and $b_2^\alpha(S)$ can be obtained using alternative arguments.

Theorem 5.1. *Let M be a 3–manifold such that $H^1(M) \xrightarrow{i^*} H^1(\partial M)$ is an injection where i^* is the inclusion–induced homomorphism. Let N be a torus component of ∂M and $\phi \in H^1(N) \cap Im(i^*)$ primitive, and $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ a representation. Then*

$$||(i^*)^{-1}(\phi)||_{T,M} \geq \frac{1}{k}\deg(\tilde{\Delta}_1^\alpha(t)) - 1.$$

Proof. Let us consider the following commutative diagram

$$\begin{array}{ccc} H^1(M) & \hookrightarrow & H^1(\partial M) \\ \downarrow & & \downarrow \\ H_2(M,\partial M) & \hookrightarrow & H_1(\partial M), \end{array}$$

where the vertical maps are given by Poincaré duality. An embedded surface S in M is dual to $(i^*)^{-1}(\phi)$ if and only if ∂S is dual to ϕ. It follows that

$$||(i^*)^{-1}(\phi)||_{T,M} = \min\{\chi_-(S) | S \text{ properly embedded}, \partial S \text{ Poincaré dual to } \phi \in H^1(\partial M)\}.$$

Let $c \subset N$ be a simple closed curve Poincaré dual to ϕ. Since N is a torus we can use an argument as in the proof of Lemma 2.2 and it follows that

$$||(i^*)^{-1}(\phi)||_{T,M} = \min\{\chi_-(S) | S \text{ properly embedded and } \partial S = c\}.$$

Let S be a (possibly disconnected) surface with minimal complexity such that $\partial S = c$. By throwing away the components of S which do not contain c, we can assume that S is connected. Hence $b_0(S) = 1$ and $b_2(S) = 0$. In particular $b_0^\alpha(S) \leq k$ and $b_2^\alpha(S) = 0$. Therefore using Proposition 4.4 and Lemmas 2.3 and 4.14 we obtain that

$$\begin{array}{rcl} ||\phi||_T & \geq & b_1(S) - b_0(S) - b_2(S) \\ & = & \frac{1}{k}\left(b_1^\alpha(S) - b_0^\alpha(S) - b_2^\alpha(S)\right) \\ & \geq & \frac{1}{k}\deg(\tilde{\Delta}_1^\alpha(t)) - 1. \end{array}$$

□

We will apply this theorem later to the complement of a link $L = L_1 \cup \cdots \cup L_m \subset S^3$. In this case we can take ϕ to be dual to the meridian of the i^{th} component L_i. Then it follows from the proof of Theorem 5.1 and an argument as in the proof of Lemma 2.2 that $||(i^*)^{-1}(\phi)||_T = 2\,\text{genus}(L_i) - 1$, where genus$(L_i)$ denotes the minimal genus of a surface in $X(L)$ bounding a parallel copy of L_i. Similar results were obtained by Turaev [Tu02b, p. 14] and Harvey [Ha05, Corollary 10.4].

The following observation will show that in more complicated cases there is no immediate way to determine $b_0(S)$: if $L = L_1 \cup L_2$ is a split oriented link, and $\phi : H_1(X(L)) \to \mathbb{Z}$ given by sending the meridians to 1, then the surface of minimal complexity dual to ϕ is the disjoint union of the Seifert surfaces of L_1 and L_2. This follows immediately from the proof that the genus is additive, i.e., genus$(L_1 \# L_2) =$ genus(L_1) + genus(L_2) (cf. [Lic97, p. 18]). In particular $b_0(S) = 2$. On the other hand if L_1 and L_2 are parallel copies of a knot with opposite orientations and $\phi : H_1(X(L)) \to \mathbb{Z}$ is again given by sending the meridians to 1, then the annulus S between L_1 and L_2 is dual to ϕ with complexity zero. In particular it is connected, hence $b_0(S) = 1$.

6. Main theorem 2: Obstructions to fiberedness

Let M be a 3–manifold and $\phi \in H^1(M)$. We say (M, ϕ) *fibers over* S^1 if the homotopy class of maps $M \to S^1$ induced by $\phi : \pi_1(M) \to H_1(M) \to \mathbb{Z}$ contains a representative that is a fiber bundle over S^1.

Theorem 6.1 (Main Theorem 2). *Let M be a 3–manifold and $\phi \in H^1(M)$ such that (M, ϕ) fibers over S^1 and such that $M \neq S^1 \times D^2, M \neq S^1 \times S^2$. If $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ is a representation, then $\Delta_1^\alpha(t) \neq 0$ and*

$$\begin{aligned} ||\phi||_T &= \tfrac{1}{k} \deg(\tau(M, \phi, \alpha)) \\ &= \tfrac{1}{k}\big(\deg(\Delta_1^\alpha(t)) - \deg(\Delta_0^\alpha(t)) - \deg(\Delta_2^\alpha(t)) \big). \end{aligned}$$

If α is unitary, then also

$$||\phi||_T = \frac{1}{k}\big(\deg(\Delta_1^\alpha(t)) - \big(1 + b_3(M)\big) \deg(\Delta_0^\alpha(t)) \big).$$

Proof. Let S be a fiber of the fiber bundle $M \to S^1$. Clearly S is dual to $\phi \in H^1(M)$ and it is well–known that S is Thurston norm minimizing. Note that $H_i^\alpha(M; \mathbb{F}^k[t^{\pm 1}]) \cong$

$H_i^\alpha(S;\mathbb{F}^k)$ by Lemma 4.12. By assumption $S \ne D^2$ and $S \ne S^2$. Therefore by Lemmas 4.14, 4.12 and 2.3 we get

$$\begin{array}{rcl} ||\phi||_T &=& \chi_-(S) \\ &=& b_1(S) - b_0(S) - b_2(S) \\ &=& \frac{1}{k}\left(b_1^\alpha(S) - b_0^\alpha(S) - b_2^\alpha(S)\right) \\ &=& \frac{1}{k}\big(\dim_{\mathbb{F}}\big(H_1^\alpha(M;\mathbb{F}^k[t^{\pm1}])\big) - \dim_{\mathbb{F}}\big(H_0^\alpha(M;\mathbb{F}^k[t^{\pm1}])\big) - \dim_{\mathbb{F}}\big(H_2^\alpha(M;\mathbb{F}^k[t^{\pm1}])\big)\big) \\ &=& \frac{1}{k}\big(\deg\left(\Delta_1^\alpha(t)\right) - \deg\left(\Delta_0^\alpha(t)\right) - \deg\left(\Delta_2^\alpha(t)\right)\big) \\ &=& \deg(\tau(M,\phi,\alpha)). \end{array}$$

The unitary case follows now immediately from Proposition 3.2. □

Remark. This theorem has been known for a long time for the untwisted Alexander polynomial of fibered knots. McMullen, Cochran, Harvey and Turaev prove corresponding equalities in their respective papers [Mc02, Co04, Ha05, Tu02b].

Since $||\phi||_T$ might be unknown for a given example the following corollary gives a more practical fibering obstruction.

Corollary 6.2. *Let M be a 3–manifold and $\phi \in H^1(M)$ such that (M,ϕ) fibers over S^1 and such that $M \ne S^1 \times D^2, M \ne S^1 \times S^2$. Let $\mathbb{F}$ and $\mathbb{F}'$ be fields. Consider the untwisted Alexander polynomial $\Delta_1(t) \in \mathbb{F}[t^{\pm1}]$. For any representation $\alpha : \pi_1(M) \to GL(\mathbb{F}',k)$ we have*

$$\deg(\Delta_1(t)) - \big(1 + b_3(M)\big) = \frac{1}{k}\big(\deg\left(\Delta_1^\alpha(t)\right) - \deg\left(\Delta_0^\alpha(t)\right) - \deg\left(\Delta_2^\alpha(t)\right)\big).$$

Proof. The Corollary follows immediately from applying Theorem 6.1 to the trivial representation $\pi_1(M) \to \mathrm{GL}(\mathbb{F},1)$ and to the representation α. □

Let $\alpha : \pi_1(M) \to \mathrm{GL}(R,k)$ be a representation where R is a Noetherian UFD, for example $R = \mathbb{Z}$ or a field $\mathbb{F}$. Then Cha [Ch03] defined the twisted Alexander polynomial $\Delta_1^\alpha(t) \in R[t^{\pm1}]$ which is well–defined up to multiplication by a unit in $R[t^{\pm1}]$. Cha's definition of twisted Alexander polynomials generalizes our definition. Given a prime ideal $\mathfrak{p} \subset R$ we denote the quotient field of $R/\mathfrak{p}$ by $\mathbb{F}_\mathfrak{p}$. Furthermore we denote by $\alpha_\mathfrak{p}$ the representation $\pi_1(M) \xrightarrow{\alpha} \mathrm{GL}(R,k) \to \mathrm{GL}(\mathbb{F}_\mathfrak{p},k)$ where $\mathrm{GL}(R,k) \to \mathrm{GL}(\mathbb{F}_\mathfrak{p},k)$ is induced from the canonical map $\pi_\mathfrak{p} : R \to R/\mathfrak{p} \to \mathbb{F}_\mathfrak{p}$.

Proposition 6.3. *Let M be a 3–manifold whose boundary is empty or consists of tori and let R be a Noetherian UFD. Let $\phi \in H^1(M)$ be non–trivial and $\alpha : \pi_1(M) \to GL(R,k)$ a representation. Then $\Delta_1^{\alpha_\mathfrak{p}}(t)$ is non–trivial and*

$$||\phi||_T = \frac{1}{k}\deg(\tau(M,\phi,\alpha_\mathfrak{p}))$$

for all prime ideals $\mathfrak{p}$ if and only if $\Delta_1^\alpha(t) \in R[t^{\pm1}]$ is monic and

$$||\phi||_T = \frac{1}{k}\deg(\tau(M,\phi,\alpha))$$

We will prove Proposition 6.3 at the end of this section. By combining Theorem 6.1 and Proposition 6.3 we immediately get the following theorem.

Theorem 6.4. *Let M be a 3–manifold. Let $\phi \in H^1(M)$ be non–trivial such that (M, ϕ) fibers over S^1 and such that $M \neq S^1 \times D^2, M \neq S^1 \times S^2$. Let R be a Noetherian UFD and $\alpha : \pi_1(M) \to GL(R, k)$ a representation. Then $\Delta_1^\alpha(t) \in R[t^{\pm 1}]$ is monic and*

$$||\phi||_T = \frac{1}{k} \deg(\tau(M, \phi, \alpha)).$$

Remark.

1. Theorem 6.4 shows that the fibering obstructions from Theorem 6.1 contain Neuwirth's theorem that $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ is monic for a fibered knot.
2. Cha's methods in [Ch03] can be used to show that if (M, ϕ) fibers over S^1, $\partial M \neq \emptyset$ and if $\alpha : \pi_1(M) \to \mathrm{GL}(R, k)$, R a Noetherian UFD, is a representation factoring through a finite group G, then the corresponding Alexander polynomial $\Delta_1^\alpha(t) \in R[t^{\pm 1}]$ is monic. Thus Theorems 6.1 and 6.4 generalize Cha's results.
3. Goda, Kitano and Morifuji [GKM05] use the Reidemeister torsion corresponding to representations $\pi_1(X(K)) \to \mathrm{SL}(\mathbb{F}, k)$, $\mathbb{F}$ a field, to give fibering obstructions for a knot K.
4. As we explain in the introduction the significance of our results lies in the fact that they also give fibering obstructions for *closed* manifolds and that they are very efficient to compute.

Remark. In Section 10 we conjecture that a converse to Theorem 6.4 holds. As we explained in the introduction a proof of this conjecture implies Taubes' conjecture.

Proof of Proposition 6.3. We only prove this proposition in the case that M is closed. The case that ∂M is a non–empty collection of tori is very similar. Note that in either case $\chi(M) = 0$.

As in the proof of Lemma 4.1 we can find a CW–structure for M such that the chain complex of the universal cover $\tilde{M}$ is as follows

$$0 \to C_3(\tilde{M}) \xrightarrow{\partial_3} C_2(\tilde{M}) \xrightarrow{\partial_2} C_1(\tilde{M}) \xrightarrow{\partial_1} C_0(\tilde{M}) \to 0$$

where $C_i(\tilde{M}) \cong \mathbb{Z}[\pi_1(M)]$ for $i = 0, 3$ and $C_i(\tilde{M}) \cong \mathbb{Z}[\pi_1(M)]^n$ for $i = 1, 2$. Let $A_i, i = 0, \dots, 3$ over $\mathbb{Z}[\pi_1(M)]$ be the matrices corresponding to the boundary maps $\partial_i : C_i \to C_{i-1}$ with respect to the bases given by the lifts of the cells of M to $\tilde{M}$. We can arrange the lifts such that

$$\begin{array}{rcl} A_3 &=& (1 - g_1, 1 - g_2, \dots, 1 - g_n)^t, \\ A_1 &=& (1 - h_1, 1 - h_2, \dots, 1 - h_n), \end{array}$$

where $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_n\}$ are generating sets for $\pi_1(M)$. Since ϕ is non–trivial there exist r, s such that $\phi(g_r) \neq 0$ and $\phi(h_s) \neq 0$. Let B_3 be the r-th row of

A_3. Let B_2 be the result of deleting the r-th column and the s–th row from A_2. Let B_1 be the s–th column of A_1. Note that

$$\det((\alpha\otimes\phi)(B_3)) = \det(\mathrm{id}-(\alpha\otimes\phi)(g_r)) = \det(\mathrm{id}-\phi(g_r)\alpha(g_r)) \neq 0$$

since $\phi(g_r)\neq 0$. Similarly $\det((\alpha\otimes\phi)(B_1))\neq 0$ and $\det((\alpha_{\mathfrak{p}}\otimes\phi)(B_i))\neq 0$, $i=1,3$ for any prime ideal $\mathfrak{p}$. We need the following theorem which can be found in [Tu01].

Theorem 6.5. [Tu01, Theorem 2.2, Lemma 2.5, Theorem 4.7] *Let S be a Noetherian UFD. Let $\beta:\pi_1(M)\to GL(S,k)$ be a representation and $\varphi\in H^1(M)$.*

1. *If $\det((\beta\otimes\varphi)(B_i))\neq 0$ for $i=1,2,3$, then $H_i^{\beta}(M;S^k[t^{\pm1}]))$ is $S[t^{\pm1}]$–torsion for all i.*
2. *If $H_i^{\beta}(M;S^k[t^{\pm1}]))$ is $S[t^{\pm1}]$–torsion for all i, and if $\det((\beta\otimes\varphi)(B_i))\neq 0$ for $i=1,3$, then $\det((\beta\otimes\varphi)(B_2))\neq 0$ and*
$$\prod_{i=1}^{3}\det((\beta\otimes\varphi)(B_i))^{(-1)^{i+1}} = \prod_{i=0}^{3}\left(\Delta_i^{\beta}(t)\right)^{(-1)^{i+1}} = \tau(M,\varphi,\beta).$$

First assume that $\Delta_1^{\alpha_{\mathfrak{p}}}(t)\neq 0$ and

$$||\phi||_T = \frac{1}{k}\left(\deg\left(\Delta_1^{\alpha_{\mathfrak{p}}}(t)\right)-\deg\left(\Delta_0^{\alpha_{\mathfrak{p}}}(t)\right)-\deg\left(\Delta_2^{\alpha_{\mathfrak{p}}}(t)\right)\right)$$

for all prime ideals $\mathfrak{p}$. By Corollary 4.3 we get $\Delta_i^{\alpha_{\mathfrak{p}}}(t)\neq 0$ for all i, in particular $H_i^{\alpha_{\mathfrak{p}}}(M;\mathbb{F}_{\mathfrak{p}}^k[t^{\pm1}])$ is $\mathbb{F}_{\mathfrak{p}}[t^{\pm1}]$–torsion for all i and all prime ideals $\mathfrak{p}$. It follows from Theorem 6.5 that $\det((\alpha_{\mathfrak{p}}\otimes\phi)(B_2))\neq 0$. Clearly this also implies that $\det((\alpha\otimes\phi)(B_2))\neq 0$. Since we already know that $\det((\alpha\otimes\phi)(B_i))\neq 0$ for $i=1,3$ it follows from Theorem 6.5 that $H_i^{\alpha}(M;R^k[t^{\pm1}])$ is $R[t^{\pm1}]$–torsion for all i.

It follows from [Tu01, Lemma 4.11] that $\Delta_0^{\alpha}(t)$ divides $\det((\alpha\otimes\phi)(B_1)) = \det(\mathrm{id}-\phi(h_s)\alpha(h_s))$ which is a monic polynomial in $R[t^{\pm1}]$ since $\phi(h_s)\neq 0$ and since $\det(\alpha(g))$ is a unit. But then $\Delta_0^{\alpha}(t)$ is monic as well. The same argument (again using [Tu01, Lemma 4.11]) shows that $\Delta_2^{\alpha}(t)$ is monic. It follows from the proof of Lemma 4.1 that $H_3^{\alpha}(M;R^k[t^{\pm1}])=0$, hence $\Delta_3^{\alpha}(t)=1$.

Denote the map $R\to R/\mathfrak{p}\to\mathbb{F}_{\mathfrak{p}}$ by $\pi_{\mathfrak{p}}$. We also denote the induced map $R[t^{\pm1}]\to\mathbb{F}_{\mathfrak{p}}[t^{\pm1}]$ by $\pi_{\mathfrak{p}}$. It follows from Theorem 6.5 that

$$\begin{aligned}\prod_{i=0}^{3}\pi_{\mathfrak{p}}\left(\Delta_i^{\alpha}(t)^{(-1)^{i+1}}\right) &= \prod_{i=1}^{3}\pi_{\mathfrak{p}}\left(\det((\alpha\otimes\phi)(B_i))\right)^{(-1)^{i+1}}\\ &= \prod_{i=1}^{3}\det((\alpha_{\mathfrak{p}}\otimes\phi)(B_i))^{(-1)^{i+1}}\\ &= \prod_{i=0}^{3}\Delta_i^{\alpha_{\mathfrak{p}}}(t)^{(-1)^{i+1}}\end{aligned}$$

for all prime ideals $\mathfrak{p}$. By assumption we get

$$\frac{1}{k}\sum_{i=0}^{3}(-1)^{i+1}\deg\left(\pi_{\mathfrak{p}}\left(\Delta_i^{\alpha}(t)\right)\right) = \frac{1}{k}\sum_{i=0}^{3}(-1)^{i+1}\deg\left(\Delta_i^{\alpha_{\mathfrak{p}}}(t)\right) = ||\phi||_T$$

for all $\mathfrak{p}$. Since $\Delta_i^\alpha(t)$ is monic for $i=0,2,3$ it follows that

$$\deg\big(\pi_{\mathfrak{p}}(\Delta_1^\alpha(t))\big) = \deg\big(\pi_{\mathfrak{q}}(\Delta_1^\alpha(t))\big)$$

for all prime ideals $\mathfrak{p}$ and $\mathfrak{q}$. Since R is a UFD it follows that $\Delta_1^\alpha(t)$ is monic. Hence $\deg(\pi_{\mathfrak{p}}(\Delta_i^\alpha(t))) = \deg(\Delta_i^\alpha(t))$ for all i and all prime ideals $\mathfrak{p}$ and clearly

$$||\phi||_T = \frac{1}{k}\big(\deg(\Delta_1^\alpha(t)) - \deg(\Delta_0^\alpha(t)) - \deg(\Delta_2^\alpha(t))\big).$$

Now assume that $\Delta_1^\alpha(t) \in R[t^{\pm 1}]$ is monic and

$$||\phi||_T = \frac{1}{k}\big(\deg(\Delta_1^\alpha(t)) - \deg(\Delta_0^\alpha(t)) - \deg(\Delta_2^\alpha(t))\big).$$

The same argument as above shows that $\Delta_i^\alpha(t)$, $i=0,2,3$, are monic as well. Recall that $\det(\alpha\otimes\phi)(B_i)$, $i=1,3$, are monic polynomials. It follows from Theorem 6.5 that

$$\det(\alpha\otimes\phi)(B_2) = \det(\alpha\otimes\phi)(B_1)\,\det(\alpha\otimes\phi)(B_3)\prod_{i=0}^{3}(\Delta_i^\alpha(t))^{(-1)^{i+1}}$$

is a quotient of monic non–zero polynomials. In particular $\det(\alpha_{\mathfrak{p}}\otimes\phi)(B_2) = \pi_{\mathfrak{p}}(\det(\alpha\otimes\phi)(B_2)) \neq 0$. It now follows immediately from Theorem 6.5 that $H_i^{\alpha_{\mathfrak{p}}}(M;\mathbb{F}_{\mathfrak{p}}^k[t^{\pm1}]))$ is $\mathbb{F}_{\mathfrak{p}}[t^{\pm1}]$–torsion for all i. In particular $\Delta_1^{\alpha_{\mathfrak{p}}}(t)\neq 0$. Using arguments as above we now see that

$$\begin{aligned}
\deg(\tau(M,\phi,\alpha_p)) &= \tfrac{1}{k}\left(\deg\left(\Delta_1^{\alpha_{\mathfrak{p}}}(t)\right) - \deg\left(\Delta_0^{\alpha_{\mathfrak{p}}}(t)\right) - \deg\left(\Delta_2^{\alpha_{\mathfrak{p}}}(t)\right)\right)\\
&= \tfrac{1}{k}\sum_{i=0}^{3}(-1)^{i+1}\deg\left(\Delta_i^{\alpha_{\mathfrak{p}}}(t)\right)\\
&= \tfrac{1}{k}\sum_{i=0}^{3}(-1)^{i+1}\deg\left(\pi_{\mathfrak{p}}(\Delta_i^{\alpha}(t))\right)\\
&= \tfrac{1}{k}\sum_{i=0}^{3}(-1)^{i+1}\deg\left(\Delta_i^{\alpha}(t)\right)\\
&= ||\phi||_T.
\end{aligned}$$

□

Remark. Let $\alpha:\pi_1(M)\to \mathrm{GL}(\mathbb{Z},k)$ be a representation. Then it is in general *not* true that $\Delta_1^{\alpha_p}(t) = \pi_p(\Delta_1^\alpha(t)) \in \mathbb{F}_p[t^{\pm1}]$ (we use the notation of Proposition 6.3), not even if (M,ϕ) fibers over S^1. Indeed, let K be the trefoil knot and $\varphi:\pi_1(X(K))\to S_3$ the unique epimorphism. Consider the representation $\alpha(\varphi):\pi_1(X(K))\to\mathrm{GL}(\mathbb{Z},2)$ as in Section 9.1. Then $\deg(\pi_3(\Delta_1^\alpha(t)))=2$, but $\deg(\Delta_1^{\alpha_3}(t))=3$.

7. Non–commutative versions of the Main Theorems

In this section we formulate and outline the proof of non–commutative versions of Theorems 3.1 and 6.1. Let $\mathbb{K}$ be a skew field and $\gamma:\mathbb{K}\to\mathbb{K}$ a ring homomorphism. Then denote by $\mathbb{K}_\gamma[t^{\pm1}]$ the twisted Laurent polynomial ring over $\mathbb{K}$. More precisely the elements in $\mathbb{K}_\gamma[t^{\pm1}]$ are formal sums $\sum_{i=-r}^{s} a_i t^i$ with $a_i\in\mathbb{K}$. Addition is given by addition of the coefficients, and for multiplication one has to apply the rule $t^i a = \gamma(a)t^i$ for any $a \in \mathbb{K}$. We denote $\mathbb{K}^k \otimes_{\mathbb{K}} \mathbb{K}_\gamma[t^{\pm 1}]$ by $\mathbb{K}^k_\gamma[t^{\pm 1}]$.

Let π be a group and $\phi : \pi \to \mathbb{Z}$ a homomorphism. A representation $\alpha : \pi \to \mathrm{GL}(\mathbb{K}_\gamma[t^{\pm 1}], k)$ is called *ϕ–compatible* if for any $g \in \pi$ we have $\alpha(g) = At^{\phi(g)}$ with $A \in \mathrm{GL}(\mathbb{K}, k)$. This generalizes a notion of Turaev [Tu02b]. Note that if $\beta : \pi \to \mathrm{GL}(\mathbb{F}, k)$ is a representation, then $\beta \otimes \phi : \pi \to \mathrm{GL}(\mathbb{F}[t^{\pm 1}], k)$ is ϕ–compatible.

Theorem 7.1. *Let M be a 3–manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M)$ be non–trivial, and $\alpha : \pi_1(M) \to GL(\mathbb{K}_\gamma[t^{\pm 1}], k)$ a ϕ–compatible representation. Then*

$$||\phi||_T \geq \frac{1}{k} \left(dim_{\mathbb{K}} \left(H_1^\alpha(M; \mathbb{K}^k_\gamma[t^{\pm 1}]) \right) - dim_{\mathbb{K}} \left(H_0^\alpha(M; \mathbb{K}^k_\gamma[t^{\pm 1}]) \right) - dim_{\mathbb{K}} \left(H_2^\alpha(M; \mathbb{K}^k_\gamma[t^{\pm 1}]) \right) \right).$$

Furthermore this inequality becomes an equality if (M, ϕ) fibers over S^1 and if $M \neq S^1 \times D^2, M \neq S^1 \times S^2$.

Proof. First note that if ϕ vanishes on $X \subset M$ then α restricted to $\pi_1(X)$ lies in $\mathrm{GL}(\mathbb{K}, k) \subset \mathrm{GL}(\mathbb{K}_\gamma[t^{\pm 1}], k)$ since α is ϕ–compatible. Therefore $H_i^\alpha(X; \mathbb{K}^k_\gamma[t^{\pm 1}]) \cong H_i^\alpha(X; \mathbb{K}^k) \otimes_{\mathbb{K}} \mathbb{K}_\gamma[t^{\pm 1}]$. The proofs of Theorem 3.1 and Theorem 6.1 can now easily be translated to this non–commutative setting. One only has to replace Lemma 4.10 (which uses determinants) by its non–commutative version proved by Harvey [Ha05, Proposition 9.1]. □

Since $\dim_{\mathbb{F}} \left(H_i^\alpha(M; \mathbb{F}^k[t^{\pm 1}]) \right) = \deg \left(\Delta_i^\alpha(t) \right)$ this theorem is a generalization of Theorems 3.1 and 6.1. It also generalizes the results of Cochran [Co04], Harvey [Ha05] and Turaev [Tu02b]. To our knowledge it is the strongest theorem of its kind.

8. Computing twisted Alexander polynomials

In this section we will show how to compute $\Delta_0^\alpha(t)$, $\Delta_1^\alpha(t)$ and $\tilde{\Delta}_1^\alpha(t)$ efficiently. Let M be a 3–manifold and let $\langle g_1, \dots, g_s | r_1, \dots, r_q \rangle$ be a presentation of $\pi_1(M)$. Let $\phi \in H^1(M)$ be non–trivial and $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ a representation.

First, $\Delta_1^\alpha(t)$ and $\tilde{\Delta}_1^\alpha(t)$ can be computed using Fox calculus as follows. By [CF77, p. 98] there exist unique maps $\partial_i : \langle g_1, \dots, g_s \rangle \to \mathbb{Z}\langle g_1, \dots, g_s \rangle$ such that

$$\begin{array}{rcll} \partial_i(g_j) & = & \delta_{ij}, & \text{for any } i, j, \\ \partial_i(uv) & = & \partial_i(u) + u\partial_i(v), & \text{for any } u, v \in \langle g_1, \dots, g_s \rangle. \end{array}$$

This gives indeed well–defined maps. Denote by $f \mapsto \bar{f}$ for $f \in \mathbb{Z}[\pi_1(M)]$ the involution induced by $\bar{g} = g^{-1}$ for any $g \in \pi_1(M)$. Then apply the map

$$\alpha \otimes \phi : \mathbb{Z}[\pi_1(M)] \to M_{k \times k}(\mathbb{F}[t^{\pm 1}])$$

to the entries of the $s \times q$–matrix $(\overline{\partial_i(r_j)})$. We denote the resulting $sk \times qk$–matrix over $\mathbb{F}[t^{\pm 1}]$ by A. Since $\mathbb{F}[t^{\pm 1}]$ is a PID we can do row and column operations to get

A into the following form

$$\begin{pmatrix} p_1(t) & 0 & \dots & 0 & 0 \\ 0 & p_2(t) & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & p_l(t) & 0 \\ 0 & 0 & \dots & 0 & (0)_{ks-l\times kq-l} \end{pmatrix}$$

where $p_i(t) \in \mathbb{F}[t^{\pm 1}] \setminus \{0\}$.

Proposition 8.1. *If $l < k(s-1)$ then $\Delta_1^\alpha(t) = 0$. Otherwise*

$$\Delta_1^\alpha(t) = \prod_{i=1}^{l} p_i(t).$$

Furthermore $\tilde{\Delta}_1^\alpha(t) = \prod_{i=1}^{l} p_i(t)$ regardless of l.

Proof. Write $\pi := \pi_1(M)$ and $K := K(\pi, 1)$. By Lemma 4.11 $H_1^\alpha(M; \mathbb{F}^k[t^{\pm 1}]) \cong H_1^\alpha(K; \mathbb{F}^k[t^{\pm 1}])$. Therefore it suffices to compute the latter homology.

Note that we can assume that K has one 0–cell, s 1–cells corresponding to the generators $g_1, \dots, g_s$ and q 2–cells corresponding to the relations $r_1, \dots, r_q$. Denote the universal cover of K by $\tilde{K}$. Let $p \in K$ be the point corresponding to the 0–cell. Denote the preimage of p under the map $\tilde{K} \to K$ by $\tilde{p}$. Note that $C_i(\tilde{K}, \tilde{p}) = C_i(\tilde{K})$ for $i \geq 1$. Therefore we get an exact sequence

$$C_2(\tilde{K}) \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[t^{\pm 1}] \xrightarrow{d_2 \otimes id} C_1(\tilde{K}) \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[t^{\pm 1}] \to H_1^\alpha(K, p; \mathbb{F}^k[t^{\pm 1}]) \to 0.$$

The equivariant lifts of the cells gives $\mathbb{Z}[\pi]$–bases for $C_2(\tilde{K})$ and $C_1(\tilde{K})$. As Harvey [Ha05, Section 6] pointed out, the $\mathbb{Z}[\pi]$–right module homomorphism $d_2 : C_2(\tilde{K}) \to C_1(\tilde{K})$ with respect to these bases is given by the $s \times q$–matrix $(\overline{\partial_i(r_j)})$. Clearly $A = (\alpha \otimes \phi)(\overline{\partial_i(r_j)})$ now represents $d_2 \otimes \mathrm{id}$. Therefore A is a presentation matrix for $H_1^\alpha(K, p; \mathbb{F}^k[t^{\pm 1}])$. Now consider the following diagram whose rows are exact:

$$\begin{array}{ccccccc} 0 \to H_1^\alpha(K; \mathbb{F}^k[t^{\pm 1}]) \to & H_1^\alpha(K, p; \mathbb{F}^k[t^{\pm 1}]) & \to H_0^\alpha(p; \mathbb{F}^k[t^{\pm 1}]) \to & H_0^\alpha(K; \mathbb{F}^k[t^{\pm 1}]) \\ \| & \downarrow\cong & \downarrow\cong & \| \\ 0 \to H_1^\alpha(K; \mathbb{F}^k[t^{\pm 1}]) \to & \bigoplus_{i=1}^{l} \mathbb{F}[t^{\pm 1}]/(p_i(t)) \oplus \mathbb{F}^{ks-l}[t^{\pm 1}] \to & \mathbb{F}^k[t^{\pm 1}] & \to H_0^\alpha(K; \mathbb{F}^k[t^{\pm 1}]). \end{array}$$

By Lemma 4.1 $H_0^\alpha(K; \mathbb{F}^k[t^{\pm 1}])$ is torsion over $\mathbb{F}[t^{\pm 1}]$. It follows that the kernel of the homomorphism $\mathbb{F}^k[t^{\pm 1}] \to H_0^\alpha(K; \mathbb{F}^k[t^{\pm 1}])$ is isomorphic to $\mathbb{F}^k[t^{\pm 1}]$ again. Putting all these together it follows that if $ks - l > k$ then $\Delta_1^\alpha(t) = 0$, otherwise $\Delta_1^\alpha(t) = \prod_{i=1}^{l} p_i(t)$. Clearly it also follows that $\tilde{\Delta}_1^\alpha(t) = \prod_{i=1}^{l} p_i(t)$. □

Now apply $\alpha \otimes \phi$ to the $1 \times s$–matrix $(1 - g_1^{-1}, \dots, 1 - g_s^{-1})$. Denote the resulting $k \times sk$–matrix by B. Since $\mathbb{F}[t^{\pm 1}]$ is a PID we can do row and column operations to get B into the following form

$$\begin{pmatrix} q_1(t) & \dots & 0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 & \dots & 0 \\ 0 & \dots & q_k(t) & 0 & \dots & 0 \end{pmatrix}$$

where $q_i(t) \in \mathbb{F}[t^{\pm 1}]$.

Proposition 8.2.

$$\Delta_0^\alpha(t) = \prod_{i=1}^{k} q_i(t).$$

Proof. We use the same notation as in the proof of the previous proposition. We have an exact sequence

$$C_1(\tilde{K}) \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[t^{\pm 1}] \xrightarrow{d_1 \otimes id} C_0(\tilde{K}) \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[t^{\pm 1}] \to H_0^\alpha(K; \mathbb{F}^k[t^{\pm 1}]) \to 0.$$

Consider the $\mathbb{Z}[\pi]$–right module homomorphism $d_1 : C_1(\tilde{K}) \to C_0(\tilde{K})$ together with the bases given by cells. Then d_1 is represented by the $1 \times s$–matrix $(1 - g_1^{-1}, \dots, 1 - g_s^{-1})$. Therefore B is a presentation matrix for $H_0^\alpha(K; \mathbb{F}^k[t^{\pm 1}])$. □

We recall that by Proposition 3.2 we can compute $\Delta_2^\alpha(t) = 1$ if M has non–empty boundary, and $\Delta_2^\alpha(t) = \Delta_0^{\overline{\alpha}}(t^{-1})$ if M is closed. Therefore $\Delta_2^\alpha(t)$ can be computed using the above algorithm.

Remark. All the computations can be done over the ring $\mathbb{F}[t^{\pm 1}]$. Therefore we can apply the Euclidean algorithm to quickly find a 'diagonal' form for the matrix A.

9. Examples

9.1. **Representations of 3–manifold groups.** Let M be a 3–manifold. Assume we are given a presentation $\langle g_1, \dots, g_s | r_1, \dots, r_t \rangle$ for $\pi_1(M)$. Then finding a representation to $\mathrm{GL}(\mathbb{F}, k)$ for some k is easy in theory: it is enough to assign arbitrary elements in $\mathrm{GL}(\mathbb{F}, k)$ to $g_1, \dots, g_s$ and check whether these satisfy the relations. Our experience shows that this is not an effective way of finding representations since $\mathrm{GL}(\mathbb{F}_p, k)$ has approximately p^{k^2} elements, and therefore there are $\left(p^{k^2}\right)^s$ possible assignments of elements in $\mathrm{GL}(\mathbb{F}_p, k)$ to s generators.

Therefore in our applications we first find homomorphisms $\pi_1(M) \to G$, G a finite group, and then find a representation of $\mathbb{F}[G]$. In all our examples we take $G = S_k$ for some k. (Metabelian groups can also be useful.) The first choice of a representation for S_k that comes to mind is $S_k \to \mathrm{GL}(\mathbb{F}, k)$ where S_k acts by permuting the coordinates. But S_k leaves the subspace $\{(v, v, \dots, v) | v \in \mathbb{F}\} \subset \mathbb{F}^k$ invariant, hence this representation is 'not completely non–trivial'. To avoid this we prefer to work with a slightly different representation of S_k. If $\varphi : \pi_1(M) \to S_k$ is a homomorphism then we consider

$$\alpha(\varphi) : \pi_1(M) \xrightarrow{\varphi} S_k \to \mathrm{GL}(V_{k-1}(\mathbb{F})),$$

where

$$V_l(\mathbb{F}) := \{(v_1, \dots, v_{l+1}) \in \mathbb{F}^{l+1} | \sum_{i=1}^{l+1} v_i = 0\}.$$

Clearly $\dim_{\mathbb{F}}(V_l(\mathbb{F})) = l$ and S_{l+1} acts on it by permutation. Since $\alpha(\varphi)$ is a subrepresentation of a unitary representation, $\alpha(\varphi)$ is unitary itself. These representations are easy to find and remarkably useful for our purposes.

We quickly explain why this approach is promising. Recall that irreducible manifolds with $b_1(M) \geq 1$ are Haken. Thurston [Th82] (cf. also [He87]) showed that the fundamental group of a Haken manifold is residually finite. Recall that a group G is called *residually finite* if for every $g \neq e \in G$ there exists a homomorphism $\alpha : G \to H$, H a finite group, such that $\alpha(g) \neq e$. Furthermore the free product of residually finite groups is residually finite. This shows that any manifold we are interested in has many homomorphisms to finite groups. In fact the geometrization conjecture implies that all 3–manifold groups are residually finite.

Note that every finite group G is a subgroup of $S_{|G|}$. In particular the homomorphisms to $S_k, k \in \mathbb{N}$, contain all homomorphisms to all finite groups.

9.2. **Knots with up to 12 crossings: genus bounds and fiberedness.** In this section we give sharp bounds on the genus of all knots with 12 crossings or less. Also we detect all non–fibered knots with 12 crossings or less.

I. Knot genus: Given a diagram for a knot one can find a Seifert surface using Seifert's algorithm (cf. [Rol90]). This gives an upper bound on the genus of a knot. It turns out that for all knots with 10 or fewer crossings the (untwisted) Alexander norm determines the knot genus, that is, we have

$$2\,\text{genus}(K) = \deg(\Delta_K(t)).$$

This equality also holds for all alternating knots (cf. [Cr59, Mu58a, Mu58b]). On the other hand it is known that

$$2\,\text{genus}(K) > \deg(\Delta_K(t))$$

for many knots with more than 10 crossings. We will discuss all 11–crossing and all 12–crossing knots with this property in this section.

There are 36 knots with 12 crossings or less for which $\text{genus}(K) > \frac{1}{2}\deg \Delta_K(t)$. The most famous and interesting examples are $K = 11_{401}$ (the Conway knot) and 11_{409} (the Kinoshita–Terasaka knot). Here we use the *knotscape* notation. Using geometric methods Gabai [Ga84] showed that the genus of the Conway knot is three and that the genus of the Kinoshita–Terasaka knot is two. The computation of the genus for all 11–crossing knots was done by Jacob Rasmussen, using a computer assisted computation of the Oszváth–Szabó knot Floer homology (cf. also [OS04a] and [OS04b]).

We first consider the Conway knot $K = 11_{401}$ whose diagram is given in Figure 3. This knot has Alexander polynomial one, i.e., the degree of $\Delta_K(t)$ equals zero. Furthermore this implies that $\pi_1(X(K))^{(1)}$ is perfect, i.e., $\pi_1(X(K))^{(n)} = \pi_1(X(K))^{(1)}$ for any $n > 1$. (For a group G, $G^{(n)}$ is defined inductively as follows; $G^{(0)} := G$ and $G^{(n+1)} := [G^{(n)}, G^{(n)}]$.) Therefore the genus bounds of Cochran [Co04] and Harvey [Ha05] vanish as well.

FIGURE 3. The Conway knot 11_{401} and the Kinoshita–Terasaka knot 11_{409}.

The fundamental group $\pi_1(X(K))$ is generated by the meridians $a, b, \dots, k$ of the segments in the knot diagram of Figure 3. The relations are

$$\begin{array}{llllllllllll} a &=& jbj^{-1}, & b &=& fcf^{-1}, & c &=& g^{-1}dg, & d &=& k^{-1}ek, \\ e &=& h^{-1}fh, & f &=& igi^{-1}, & g &=& e^{-1}he, & h &=& c^{-1}ic, \\ i &=& aja^{-1}, & j &=& iki^{-1}, & k &=& e^{-1}ae. & & & \end{array}$$

Using the program *KnotTwister* [F05] we found the homomorphism $\varphi : \pi_1(X(K)) \to S_5$ given by

$$\begin{array}{llllllllllll} A &=& (142), & B &=& (451), & C &=& (451), & D &=& (453), \\ E &=& (453), & F &=& (351), & G &=& (351), & H &=& (431), \\ I &=& (351), & J &=& (352), & K &=& (321), & & & \end{array}$$

where we use cycle notation. The generators of $\pi_1(X(K))$ are sent to the element in S_5 given by the cycle with the corresponding capital letter. We then consider $\alpha := \alpha(\varphi) : \pi_1(X(K)) \xrightarrow{\varphi} S_5 \to \mathrm{GL}(V_4(\mathbb{F}_{13}))$. Using *KnotTwister* we compute $\deg(\Delta_0^\alpha(t)) = 0$ and we compute the twisted Alexander polynomial to be

$$\Delta_1^\alpha(t) = 1 + 6t + 9t^2 + 12t^3 + t^5 + 3t^6 + t^7 + 3t^8 + t^9 + 12t^{11} + 9t^{12} + 6t^{13} + t^{14} \in \mathbb{F}_{13}[t^{\pm 1}].$$

Note that α is unitary and we can therefore apply Theorem 3.3 which says that if $\Delta_1^\alpha(t) \neq 0$, then

$$\mathrm{genus}(K) \geq \frac{1}{2k}\big(\deg(\Delta_1^\alpha(t)) - \deg(\Delta_0^\alpha(t))\big) + \frac{1}{2}.$$

Therefore in our case we get

$$\text{genus}(K) \geq \frac{1}{8} \cdot 14 + \frac{1}{2} = \frac{18}{8} = 2.25.$$

Since genus(K) is an integer we get genus$(K) \geq 3$. Since there exists a Seifert surface of genus 3 for K (cf. [Ga84] and Figure 1) it follows that the genus of the Conway knot is three.

For the Kinoshita–Terasaka knot K we found a map $\varphi : \pi_1(X(K)) \to S_5$ such that $\Delta_1^{\alpha(\varphi)}(t) \in \mathbb{F}_{13}[t^{\pm 1}]$ has degree 12 and $\deg\big(\Delta_0^{\alpha(\varphi)}(t)\big) = 0$. It follows from Theorem 3.3 that genus$(K) \geq \frac{1}{8} \cdot 12 + \frac{1}{2} = 2$. A Seifert surface of genus two is given in [Ga84]. Note that in this case our inequality is strict, hence 'rounding up' is not necessary. Our table below shows that this is surprisingly often the case.

Table 1 gives all knots with 12 crossings or less for which $\deg(\Delta_K(t)) < 2\,\text{genus}(K)$. We obtained the list of these knots from Alexander Stoimenow's knot page [Sto]. One can also find the genus of all these knots on his knot page. We compute twisted Alexander polynomials using *KnotTwister* and representations $\alpha(\varphi) : \pi_1(X(K)) \xrightarrow{\varphi} S_k \to \mathrm{GL}(V_{k-1}(\mathbb{F}_{13}))$. Our genus bounds from Theorem 3.3 give (by rounding up if necessary) the correct genus in each case. All of the representations which give us the correct genus bounds have the property that $\deg\big(\Delta_0^{\alpha(\varphi)}(t)\big) = 0$.

Using *KnotTwister* it takes only a few seconds to find such representations and to compute the twisted Alexander polynomial.

II. Fiberedness: It is known that a knot with 11 or fewer crossings is fibered if and only if the Alexander polynomial is monic and $\deg(\Delta_K(t)) = 2\,\text{genus}(K)$. Hirasawa showed that the knots 12_{1498}, 12_{1502}, 12_{1546} and 12_{1752} are not fibered even though their Alexander polynomials are monic and $\deg(\Delta_K(t)) = 2\,\text{genus}(K)$.

Now consider the knot $K = 12_{1345}$. Its Alexander polynomial equals $\Delta_K(t) = 1 - 2t + 3t^2 - 2t^3 + t^4$ and its genus equals two. Therefore K has the abelian invariants of a fibered knot, i.e., $\Delta_K(t)$ is monic and $2\,\text{genus}(K) = \deg(\Delta_K(t))$. It follows from Corollary 6.2 that if K were fibered, then for any field $\mathbb{F}$ and any representation $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ the following would hold:

$$\deg(\Delta_K(t)) = \frac{1}{k}\big(\deg\left(\Delta_1^{\alpha}(t)\right) - \deg\left(\Delta_0^{\alpha}(t)\right)\big) + 1.$$

We found a representation $\alpha : \pi_1(X(K)) \to S_4$ such that for the canonical representation $\alpha : \pi_1(X(K)) \to S_4 \to \mathrm{GL}(\mathbb{F}_3, 4)$ given by permuting the coordinates, we get $\deg(\Delta_1^{\alpha}(t)) = 7$ and $\deg\left(\Delta_0^{\alpha}(t)\right) = 1$. We compute

$$\frac{1}{4}\deg\left(\Delta_1^{\alpha}(t)\right) - \frac{1}{4}\deg\left(\Delta_0^{\alpha}(t)\right) + 1 = \frac{10}{4} \neq 4 = \deg(\Delta_K(t)).$$

Hence K is not fibered.

Now consider $\alpha : \pi_1(X(K)) \to S_4 \to \mathrm{GL}(\mathbb{Z}, 4)$, the second map being the canonical representation induced from permutation on the basis elements. Then according to

Knotscape name	11_{401}	11_{409}	11_{412}	11_{434}	11_{440}	11_{464}
genus bound from $\Delta_K(t)$	0	0	2	1	2	1
dimension of $\alpha(\varphi)$	4	4	4	4	2	4
degree of $\Delta_1^{\alpha(\varphi)}(t)$	19	12	20	12	10	12
genus bound from $\Delta_1^{\alpha}(t)$	2.25	2.00	3.00	2.00	3.00	2.00
Knotscape name	11_{519}	12_{1311}	12_{1316}	12_{1319}	12_{1339}	12_{1344}
genus bound from $\Delta_K(t)$	2	1	2	1	1	2
dimension of $\alpha(\varphi)$	4	3	2	2	4	4
degree of $\Delta_1^{\alpha(\varphi)}(t)$	20	9	10	10	12	20
genus bound from $\Delta_1^{\alpha(\varphi)}(t)$	3.00	2.00	2.75	3.00	2.00	3.00
Knotscape name	12_{1351}	12_{1375}	12_{1412}	12_{1417}	12_{1420}	12_{1509}
genus bound from $\Delta_K(t)$	2	2	1	1	2	2
dimension of $\alpha(\varphi)$	2	4	4	4	4	2
degree of $\Delta_1^{\alpha(\varphi)}(t)$	10	20	12	20	20	10
genus bound from $\Delta_1^{\alpha(\varphi)}(t)$	3.00	3.00	2.00	3.00	3.00	3.00
Knotscape name	12_{1519}	12_{1544}	12_{1545}	12_{1552}	12_{1555}	12_{1556}
genus bound from $\Delta_K(t)$	2	2	2	2	2	1
dimension of $\alpha(\varphi)$	4	4	4	4	2	2
degree of $\Delta_1^{\alpha(\varphi)}(t)$	20	20	20	20	10	6
genus bound from $\Delta_1^{\alpha(\varphi)}(t)$	3.00	3.00	3.00	3.00	3.00	2.00
Knotscape name	12_{1581}	12_{1601}	12_{1609}	12_{1699}	12_{1718}	12_{1745}
genus bound from $\Delta_K(t)$	1	0	1	1	0	1
dimension of $\alpha(\varphi)$	4	5	4	4	4	4
degree of $\Delta_1^{\alpha(\varphi)}(t)$	12	13	12	12	12	12
genus bound from $\Delta_1^{\alpha(\varphi)}(t)$	2.00	1.80	2.00	2.00	2.00	2.00
Knotscape name	12_{1807}	12_{1953}	12_{2038}	12_{2096}	12_{2100}	12_{2118}
genus bound from $\Delta_K(t)$	1	2	2	2	2	2
dimension of $\alpha(\varphi)$	4	2	4	4	2	4
degree of $\Delta_1^{\alpha(\varphi)}(t)$	12	10	20	20	10	20
genus bound from $\Delta_1^{\alpha(\varphi)}(t)$	2.00	3.00	3.00	3.00	3.00	3.00

TABLE 1. Computation of degrees of twisted Alexander polynomials.

Proposition 6.3 our computation can also be interpreted as saying that $\Delta_1^\alpha(t) \in \mathbb{Z}[t^{\pm 1}]$ is not monic.

Similarly we found altogether 13 knots which are not fibered; we list them in Table 2. We used Corollary 6.2 as above. That is, we compared the degrees of untwisted Alexander polynomials with the degrees of twisted Alexander polynomials corresponding to some representation $\pi_1(S^3 \setminus K) \to S_k \to \mathrm{GL}(\mathbb{F}_p, k)$ where $S_k \to$

$\mathrm{GL}(\mathbb{F}_p, k)$ is the canonical representation. Stoimenow and Hirasawa then showed that

Knotscape name	12_{1345}	12_{1498}	12_{1502}	12_{1546}	12_{1567}	12_{1670}	12_{1682}
Order of permutation group k	4	5	5	3	5	5	4
Order p of finite field	3	2	11	2	3	2	3
Knotscape name	12_{1752}	12_{1771}	12_{1823}	12_{1938}	12_{2089}	12_{2103}	
Order of permutation group k	3	3	5	5	5	4	
Order p of finite field	2	7	7	11	2	3	

TABLE 2. Alexander polynomials of non–fibered knots

the remaining 12–crossing knots are fibered if and only if the Alexander polynomial is monic and if $\deg(\Delta_K(t)) = 2\,\mathrm{genus}(K)$. So Corollary 6.2 was crucial in finding all non–fibered 12–crossing knots.

Remark. Jacob Rasmussen confirmed our results using knot Floer homology which gives a fibering obstruction as well (cf. [OS02, Section 3]).

9.3. **Closed manifolds.** In this short section we intend to show that twisted Alexander polynomials are also very useful for studying closed manifolds.

Let $K \subset S^3$ be a non–trivial knot and ϕ a generator of $H^1(X(K))$. Since $H^1(X(K)) \cong H^1(M_K)$ we will denote the corresponding generator of $H^1(M_K)$ by ϕ as well. Let S be a minimal Seifert surface for K. Adding a disk to S along the boundary clearly gives a closed surface $\hat{S}$ dual to $\phi \in H^1(M_K)$, hence $||\phi||_{T,M_K} \le ||\phi||_{T,X(K)} - 1$. Gabai [Ga87, Theorem 8.8] showed that $\hat{S}$ is in fact norm minimizing. In particular for a non–trivial knot K

$$||\phi||_{T,M_K} = ||\phi||_{T,X(K)} - 1 = 2\,\mathrm{genus}(K) - 2.$$

If K fibers, then clearly (M_K, ϕ) fibers over S^1 as well. Gabai [Ga87] showed the converse; a knot K is fibered if and only if M_K is fibered.

We will quickly discuss the manifolds M_K with K a knot with 12 crossings or less. Let $K = 11_{409}$, the Kinoshita–Terasaka knot. Previously we computed that $\mathrm{genus}(K) = 3$. By Gabai's theorem above $||\phi||_{T,M_K} = 3 - 1 = 2$. We will confirm this using Theorem 3.3. The fundamental group $\pi_1(X(K))$ is generated by the meridians $a, b, \dots, k$ of the segments in the knot diagram of Figure 3. The relations are

$$\begin{array}{rclrclrclrcl} a &=& hbh^{-1}, & b &=& i^{-1}ci, & c &=& fdf^{-1}, & d &=& k^{-1}ek, \\ e &=& gfg^{-1}, & f &=& dgd^{-1}, & g &=& j^{-1}hj, & h &=& kik^{-1}, \\ i &=& g^{-1}jg, & j &=& b^{-1}kb, & k &=& e^{-1}ae. \end{array}$$

Let $\lambda \in \pi_1(X(K))$ represent the longitude, then $\pi_1(M_K) = \pi_1(X(K))/\langle\lambda\rangle$. Note that

$$\lambda = hi^{-1}fk^{-1}gdj^{-1}kg^{-1}b^{-1}e^{-1}a.$$

This can be seen by following the knot starting at a, and picking up a generator at each undercrossing. The extra term a is needed to get a curve which has linking number zero with the knot K.

Up to conjugation there exists a unique homomorphism $\varphi : \pi_1(X(K)) \to S_5$ and it factors through $\varphi : \pi_1(M_K) \to S_5$. For M_K using *KnotTwister* we compute $\deg\big(\Delta^{\alpha(\varphi)}_{X(K),0}(t)\big) = \deg\big(\Delta^{\alpha(\varphi)}_{M_K,0}(t)\big) = 0$ and $\deg\big(\Delta^{\alpha(\varphi)}_{X(K),1}(t)\big) = 12$ and $\deg\big(\Delta^{\alpha(\varphi)}_{M_K,1}(t)\big) = 8$ where $\alpha(\varphi) : \pi_1(M_K) \to S_5 \to \mathrm{GL}(V_4(\mathbb{F}_{13}))$. From Theorem 3.3 and the computation of twisted Alexander polynomials we get the following bounds on the Thurston norm:

$$\begin{array}{llcccccl} \text{for } X(K), & \deg(\Delta_1^\alpha(t)) & = & 12 & \Rightarrow & ||\phi||_{T,X(K)} & \geq & \frac{12}{4} = 3, \\ \text{for } M_K, & \deg(\Delta_1^\alpha(t)) & = & 8 & \Rightarrow & ||\phi||_{T,M_K} & \geq & \frac{8}{4} = 2. \end{array}$$

Note that the degree of the twisted Alexander polynomial of $X(K)$ 'drops by just the right amount' to give again the correct Thurston norm for M_K. In particular this shows that twisted Alexander polynomials also determine $||\phi||_{T,M_K}$. Our computations show that in fact for all knots with 12 crossings or less twisted Alexander polynomials determine the Thurston norm of M_K.

Let K be one of the 13 non–fibered knots with 12 crossings with monic Alexander polynomial and $\deg(\Delta_K(t)) = 2\,\mathrm{genus}(K)$. Then M_K is not fibered by Gabai [Ga87]. Corollary 6.2 and the computations with *KnotTwister* show that twisted Alexander polynomials also detect that these manifolds are not fibered, confirming Gabai's result.

9.4. **Satellite knots.** We will show how to find lower bounds for the genus of satellite knots. We will see that even though we are interested in the genus of a knot we sometimes have to study the Thurston norm of a link complement.

Let K and C be knots in S^3. Let $A \subset S^3 \setminus K$ be a simple closed curve, unknotted in S^3. Then $X(A)$ is a solid torus. Let $\psi : \partial X(A) \to \partial X(C)$ be a diffeomorphism which sends a meridian of A to a longitude of C, and a longitude of A to a meridian of C. The space

$$X(A) \cup_\psi X(C)$$

is a 3-sphere and the image of K is denoted by $S := S(K, C, A)$. We say S is the *satellite knot* with *companion* C, *orbit* K and *axis* A. Note that we replaced a tubular neighborhood of C by a knot in a solid torus, namely $K \subset X(A)$.

In [FT05] the first author and Peter Teichner study examples where K is the knot 6_1, C is an arbitrary knot, and A is a knot as in Figure 4. In [FT05] they show that all these knots are topologically slice, but it is not known whether they are smoothly slice or not. Chuck Livingston asked what the genus of these satellite knots equals.

Proposition 9.1. *Let $K \subset S^3$ be a non–trivial knot, and $A \subset X(K)$ a simple closed curve such that $[A] = 0 \in H_1(X(K))$, which is unknotted if considered as a knot in*

FIGURE 4. Knot 6_1 with choice of A and considered as a knot in the solid torus $X(A)$.

S^3. Let C be another knot. Now let $S := S(K, C, A)$ be the satellite knot. Then

$$\textit{genus}(S) = \frac{1}{2}(||\phi||_{T,X} + 1),$$

where $X := S^3 \setminus (\nu K \cup \nu A)$ and $\phi : H_1(X) \to \mathbb{Z}$ is given by sending the meridian of K to one, and the meridian of A to zero.

Proof. For convenience let us identify ∂X with $K \times S^1 \cup A \times S^1$. We also identify K with $K \times \{*\} \subset \partial X$. It follows from [Sc53], [BZ03, p. 21] that (F denotes a surface)

$$\text{genus}(S) = \min\{\text{genus}(F) | F \subset X \text{ properly embedded and } \partial F = K\}$$

since the linking number of A and K equals zero. This also implies that $\phi : H_1(A \times S^1) \to H_1(X) \xrightarrow{\phi} \mathbb{Z}$ is the zero map. Similar to the proof of Lemma 2.2 one can now show that

$$\begin{aligned} ||\phi||_T &= \min\{2\,\text{genus}(F) - 1 | F \subset X \text{ properly embedded and } K \subset \partial F, F \text{ dual to } \phi\} \\ &= \min\{2\,\text{genus}(F) - 1 | F \subset X \text{ properly embedded and } \partial F = K\} \\ &= 2\,\text{genus}(S) - 1. \end{aligned}$$

□

Hence in order to determine the genus of $S(K, C, A)$ for any knot C we have to determine the Thurston norm of $||\phi||_{T,X}$. For X, we compute that the untwisted Alexander polynomial $\Delta_1(t) = 0$. Hence we need twisted coefficients to get non–trivial bounds.

Now consider the representation $\alpha : \pi_1(X) \to \text{GL}(\mathbb{F}_{13}, 1)$ given by $\alpha(\mu_K) = 6$ and $\alpha(\mu_A) = 2$, where μ_K (respectively μ_A) denotes the meridian of K (respectively A). For X we compute $\Delta_1^\alpha(t) = 1 + 2t + 2t^2 + 4t^3 \in \mathbb{F}_{13}[t^{\pm 1}]$. It follows from Theorem 3.6 that $||\phi||_{T,X} \geq 3$.

Consider Figure 5. It shows the link $K \cup A$ and a Seifert surface of genus one for K. The knot A intersects this Seifert surface twice. Therefore adding a hollow 1–handle gives a Seifert surface of genus two for K which does not intersect A. Therefore $||\phi||_{T,X} \leq 3$. Hence $||\phi||_{T,X} = 3$ and by Proposition 9.1 we get genus$(S) = 2$.

FIGURE 5. Seifert surface for $K \subset S^3 \setminus K \cup A$.

Using Fox calculus we computed the untwisted multivariable Alexander polynomial

$$\Delta(x,y) = x^3y^3 - x^3y - x^2y^3 + x^2y + xy^2 - x - y^2 + 1 \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}],$$

here $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}] = \mathbb{Z}[H_1(X)]$, where x corresponds to a meridian of K and y corresponds to a meridian of A. It follows that the Alexander norm $||\phi||_A$ equals 3. The Alexander norm gives a better bound than the untwisted Alexander polynomial since ϕ is an extreme point of the Alexander norm ball (cf. [Mc02]). For manifolds the multivariable Alexander polynomial has to be computed by finding the greatest common divisor of the determinants of minors, which is not an easy task in more complex cases. In contrast, the computation of twisted Alexander polynomials is very fast. This shows that even if the Alexander norm gives the correct result for the Thurston norm, it is still useful to study twisted Alexander polynomials.

9.5. **Ropelength.** For a smooth curve K in S^3 we can define its *length* $\mathrm{Len}(K)$, and for a collection of curves L we can define its *thickness* $\tau(L)$. The *ropelength* of a knot K is defined to be the quotient of its length and its thickness. We refer to [CKS02] for more details. Cantarella, Kusner and Sullivan [CKS02, Corollary 22] proved the following theorem.

Theorem 9.2. *If K is a non–trivial curve of unit thickness, then*

$$Len(K) \geq 2\pi \left(2 + \sqrt{2\, genus(K) - 1}\right).$$

Let $L = L_1 \cup \cdots \cup L_m$ be a collection of smooth curves of unit thickness with meridians $\mu_1, \ldots, \mu_m$. If $\phi_1, \ldots, \phi_m \in H^1(X(L))$ is the dual basis, then

$$Len(L_i) \geq 2\pi \left(1 + \sqrt{||\phi_i||_T}\right).$$

Clearly our methods now give lower bounds on the length of a component of a collection of curves of unit thickness. In [CKS02] the authors consider the link in Figure 6. Denote the component represented by a circle by L_1, and the other com-

FIGURE 6

ponent by L_2. Denote the meridian of L_i by μ_i for $i = 1, 2$. Attach a half–sphere to L_1 'below the paper plane'. Removing three disks and adding star shaped 'pants' gives a Seifert surface for L_1 of genus two which does not intersect L_2. In [CKS02] the authors conjectured that the above surface is minimal, i.e., that $||\phi_1||_T = 3$ for ϕ_1 the homomorphism with $\phi_1(\mu_1) = 1$, $\phi_1(\mu_2) = 0$. Harvey [Ha05, Section 8] used higher–order Alexander polynomials to prove the conjecture in the positive. We will reprove this using twisted Alexander polynomials.

Consider the representation $\alpha : \pi_1(X(L)) \to \mathrm{GL}(\mathbb{F}_{13}, 1)$ given by $\alpha(\mu_1) = 10$ and $\alpha(\mu_2) = 7$. Then $\tilde{\Delta}_1^\alpha(t) = 1 + 4t^2 + 2t^3 \in \mathbb{F}_{13}[t^{\pm 1}]$. It follows from Theorem 5.1 that $||\phi||_T \geq \frac{3}{1} - 1 = 2$. By Lemma 3.4 $||\phi||_T \equiv 10 + 7 \mod 2$, hence it is odd. Therefore $||\phi||_T \geq 3$, which reproves the conjecture of [CKS02]. In particular it follows that the ropelength of L_1 is at least $2\pi(1 + \sqrt{3})$. Using similar arguments, one can show that the ropelength of L_2 is also at least $2\pi(1 + \sqrt{3})$.

9.6. **9–crossing links.** McMullen [Mc02] determined the Thurston norm for all links with 9 or fewer crossings and with three or fewer components, except for $9^3_{21}, 9^2_{41}, 9^2_{50}$ and 9^3_{15} (here we use Rolfsen's [Rol90] notation). For the link 9^3_{21} the multivariable Alexander polynomial vanishes, and also all twisted Alexander polynomials we computed vanish.

In the case of the other three links McMullen computed the Alexander norm, but could not show that it agrees with the Thurston norm for all ϕ. We computed twisted Alexander polynomials for the extreme points of the Alexander norm ball for the links 9^2_{41} and 9^2_{50} and our computations strongly suggest that the Alexander norm agrees with the Thurston norm in these two cases.

9.7. **Dunfield's link.** We will show that our invariants also detect subtle examples of pairs $(X(L), \phi)$ where L is a link in S^3 and $\phi \in H^1(X(L))$, which do not fiber over S^1. Consider the link L in Figure 7 from [Du01]. Denote the knotted component by L_1 and the unknotted component by L_2. Let $x, y \in H_1(X(L))$ be the elements

Figure 7. Dunfield's link.

represented by a meridian of L_1 respectively L_2. Then the multivariable Alexander polynomial equals

$$\Delta_{X(L)} = xy - x - y + 1 \in \mathbb{Z}[H_1(X(L))] = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}].$$

The Alexander norm ball (cf. [Mc02] for a definition) and the Thurston norm ball (which is determined in [FK05]) are given in Figure 8. Dunfield [Du01] showed that

Figure 8. Alexander norm ball and Thurston norm ball for Dunfield's link.

$(X(L), \phi)$ fibers over S^1 for all $\phi \in H^1(M)$ in the cone on the two open faces with vertices $(-\frac{1}{2}, \frac{1}{2}), (0,1)$ respectively $(0,-1), (\frac{1}{2}, -\frac{1}{2})$. He also showed that $(X(L), \phi)$ does not fiber over S^1 for any $\phi \in H^1(X(L))$ lying outside the cone. Later in [FK05] the authors completely determined the Thurston norm of $X(L)$.

Now let ϕ be the homomorphism given by $\phi(x) = 1$, $\phi(y) = -1$. In that case ϕ is inside the cone on an open face of the Alexander norm ball and $\Delta_1(t) = 1 - 3t + 3t^2 - t^3 \in \mathbb{Z}[t^{\pm 1}]$ is monic. Hence the abelian invariants are inconclusive whether $(X(L), \phi)$ is fibered or not. On the other hand we found a representation $\pi_1(X(L)) \to S_3 \to \mathrm{GL}(\mathbb{F}_2, 3)$ such that $\Delta_1^\alpha(t) = 0 \in \mathbb{F}_2[t^{\pm 1}]$. Therefore $(X(L), \phi)$ does not fiber over S^1 by Theorem 6.1. Note that the fact that $(X(L), \phi)$ does not fiber over S^1 also follows from the fact that ϕ is not in the cone on a top–dimensional open face of the Thurston norm ball (cf. [Th86] and [Oe86]).

10. Conjectures

10.1. **Fiberedness conjecture.** In Section 9.2 we saw that twisted Alexander polynomials very successfully detect non–fibered knots and non–fibered manifolds. In fact, if K is not fibered, then in most cases $\Delta_K(t)$ is not monic. If $\Delta_K(t)$ is not monic, then the computations suggest that most non–trivial representations will give a twisted Alexander polynomial which is not monic. Loosely speaking, it seems that if K is not fibered, then a twisted Alexander polynomial is monic only by chance. In fact we think that the representations from homomorphisms to finite groups suffice to detect non–fibered knots. More precisely, we propose the following conjecture.

Conjecture 10.1. *Let M be a 3–manifold and $\phi \in H^1(M)$ non–trivial. Then (M, ϕ) fibers over S^1 if and only if for all epimorphisms $\alpha : \pi_1(M) \to G$, G a finite group, the twisted Alexander polynomial $\Delta_1^G(t) \in \mathbb{Z}[t^{\pm1}]$ is monic and*

$$||\phi||_T = \frac{1}{|G|}\deg(\tau(M, \phi, \alpha)).$$

We give some further supporting evidence. It is well–known that fibered manifolds are prime. We have the following result.

Lemma 10.2. *Let M be a 3–manifold which is not prime and let $\phi \in H^1(M)$. If the geometrization conjecture holds, then there exists an epimorphism $\pi_1(M) \to G$, G a finite group, such that $\Delta_1^G(t) = 0 \in \mathbb{Z}[t^{\pm1}]$.*

Proof. First assume that $M = M_1 \# M_2$ with $b_1(M_i) > 0, i = 1, 2$. If $\phi : H_1(M_i) \to H_1(M) \to \mathbb{Z}$, $i = 1, 2$ is non–trivial, then $H_0(M_i; \mathbb{Z}[t^{\pm1}])$ is $\mathbb{Z}[t^{\pm1}]$–torsion for $i = 1, 2$, but $H_0(S^2; \mathbb{Z}[t^{\pm1}])$ is not $\mathbb{Z}[t^{\pm1}]$ torsion. It follows from the Mayer–Vietoris exact sequence that $H_1(M; \mathbb{Z}[t^{\pm1}])$ is not $\mathbb{Z}[t^{\pm1}]$–torsion.

Now assume that $\phi : H_1(M_1) \to H_1(M) \to \mathbb{Z}$ is trivial. Then $H_0(M_1\mathbb{Z}[t^{\pm1}])$ is not $\mathbb{Z}[t^{\pm1}]$–torsion, hence by Lemma 4.14 (applied to the field $\mathbb{Q}(t)$) together with Proposition 3.2 it follows that $H_1(M_1; \mathbb{Z}[t^{\pm1}])$ is not $\mathbb{Z}[t^{\pm1}]$–torsion. Since $H_1(S^2; \mathbb{Z}[t^{\pm1}]) = 0$ it follows from the Mayer–Vietoris exact sequence that $H_1(M; \mathbb{Z}[t^{\pm1}])$ is not $\mathbb{Z}[t^{\pm1}]$–torsion.

If $M = M_1 \# M_2$ with $b_1(M_i) > 0, i = 1, 2$, then an easy Mayer–Vietoris argument for the homology of $M = M_1 \cup_{S^2} M_2$ with $\mathbb{Z}[t^{\pm1}]$–coefficients shows that we have $\Delta_1^\alpha(t) = 0$ for all $\alpha : \pi_1(M) \to G$, G a finite group. In the proof we need the fact which follows from Lemma 4.14 (applied to the field $\mathbb{Q}(t)$) that if ϕ vanishes on M_i, then $H_1^\alpha(M_i; \mathbb{Z}[t^{\pm1}])$ is not $\mathbb{Z}[t^{\pm1}]$–torsion.

Now assume that $M = M_1 \# M_2$ with $b_1(M_1) > 0$ and $b_1(M_2) = 0$, $M_2 \neq S^3$. By the Poincaré conjecture (a consequence of the geometrization conjecture), $\pi_1(M_2) \neq 0$, and by [He87] and the geometrization conjecture there exists a non–trivial epimorphism $\alpha : \pi_1(M_2) \to G$, G a finite group. Denote the homomorphism $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2) \to \pi_1(M_2) \to G$ by α as well. Then

$$\Delta_1^G(t) = \Delta_{M_G, \phi_G, 1}(t)$$

by Lemma 3.8 where M_G is the cover of M corresponding to M. But the prime decomposition of M_G has $|G|$ copies of M_1, in particular by the above observation that $\Delta_{M,\phi,1}(t) = \Delta_{M_G,\phi_G,1}(t) = 0$. $\square$

Proposition 10.3. *Let $S = S(K, C, A)$ be a satellite knot with $A \in \pi_1(S^3 \setminus K)^{(1)}$ and $\Delta_C(t) \neq 1$. Then there exists a homomorphism $\alpha : \pi_1(S) \to G$, G a finite group, such that $\Delta^G_{X(S),1}(t)$ is not monic.*

Note that combining this proposition with Theorem 6.1 we come close to reproving the more general fact that if K and C are any knots and $A \in \pi_1(S^3 \setminus K)^{(1)}$ then S is not fibered, which follows from studying $\pi_1(S^3 \setminus S)^{(1)}$ (cf. [BZ03, p. 64]).

Proof. First let $\alpha : \pi_1(X(K)) \to G$ be any homomorphism to a finite group. Denote the order of $\alpha(A) \in G$ by r and write $L_{C,r}$ for the r–fold cover of S^3 branched along C. By [Ch03, Section 4] there exists a surjective map $\pi_1(X(S)) \to \pi_1(X(K))$ such that $\Delta^G_{X(S),1}(t) \in \mathbb{Z}[t^{\pm 1}]$ is an annihilator for $H_1(L_{C,r}) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]$.

Clearly we are done, once we can arrange α such that $H_1(L_{C,r})$ is either infinite or non–trivial torsion. By Riley [Ri90] the finite values $|H_1(L_{C,r})|$ grow exponentially in r, in particular there exists $R \in \mathbb{N}$ such that $H_1(L_{C,r})$ is either infinite or non–trivial torsion for all $r \geq R$.

Since $\pi_1(X(K))$ is residually finite (cf. [Th82, He87]) an easy argument shows that we can find an epimorphism $\alpha : \pi_1(X(K)) \to G$, G finite, such that $\alpha(A^k) \neq e$ for any $k < R$. In particular the order of $\alpha(A)$ in G is bigger than or equal to R. $\square$

We explain a possible approach to Conjecture 10.1. Let M be a 3–manifold whose boundary is empty or consists of tori and $\phi \in H^1(M)$ non–trivial. Assume that for every representation $\alpha : \pi_1(M) \to G$, G a finite group, the twisted Alexander polynomial $\Delta^G_1(t)$ is monic and

$$||\phi||_T = \frac{1}{|G|} \deg(\tau(M, \phi, \alpha)).$$

Assuming the geometrization conjecture we get from Lemma 10.2 that M is prime.

Let S be a surface dual to ϕ with minimal complexity. Clearly $H^\alpha_0(M; \mathbb{Z}[G][t^{\pm 1}]) \neq 0$, hence $\Delta^G_0(t) \neq 1$. Combining this observation with the fact that $\Delta^G_1(t) \neq 0$ (which is obvious from the assumption), we can assume by Corollary 4.8 that S is connected. Then let N be the result of cutting M along S. Denote the positive and the negative inclusions of S into N by i_+ and i_-. Since S is minimal, $i_+ : \pi_1(S) \to \pi_1(N)$ is injective by Dehn's Lemma. Since M is prime it follows easily from Stallings' theorem [St62] that (M, ϕ) fibers over S^1 if and only if $i_+ : \pi_1(S) \to \pi_1(N)$ is surjective (cf. also [Ka96, p. 84] in the knot complement case).

Proposition 10.4. *$H^\alpha_1(S; \mathbb{Z}[G])$ and $H^\alpha_1(N; \mathbb{Z}[G])$ are free abelian groups of the same rank.*

Proof. First note that $H_1^\alpha(S;\mathbb{Z}[G])$ is the first homology of the cover of S corresponding to $\pi_1(S)\to G$, hence $H_1^\alpha(S;\mathbb{Z}[G])$ is free abelian.

Let p be a prime. It follows from the proof of Proposition 6.3 that $\Delta_1^{\alpha_p}(t)\neq 0$ for the representation $\alpha_p:\pi_1(M)\to G\to \mathrm{GL}(\mathbb{F}_p,|G|)$. In particular . By Lemma 4.2 $H_2^\alpha(M;\mathbb{F}_p[G][t^{\pm1}])$ is $\mathbb{F}_p[t^{\pm1}]$–torsion. Therefore we have a short exact sequence

$$0\to H_1^{\alpha_p}(S;\mathbb{F}_p[G])\otimes_{\mathbb{F}_p}\mathbb{F}_p[t^{\pm1}]\to H_1^{\alpha_p}(N;\mathbb{F}_p[G])\otimes_{\mathbb{F}_p}\mathbb{F}_p[t^{\pm1}]\to H_1^{\alpha_p}(M;\mathbb{F}_p[G][t^{\pm1}])\to 0.$$

In particular $H_1^{\alpha_p}(S;\mathbb{F}_p[G])\cong H_1^{\alpha_p}(N;\mathbb{F}_p[G])$ for every prime p as $\mathbb{F}_p$–vector spaces. Note that $H_0^\alpha(S;\mathbb{Z}[G])\cong \mathbb{Z}[G/\mathrm{Im}\{\pi_1(S)\to G\}]$ and $H_0^\alpha(N;\mathbb{Z}[G])\cong \mathbb{Z}[G/\mathrm{Im}\{\pi_1(N)\to G\}]$ in particular both are $\mathbb{Z}$–torsion free. It follows from the universal coefficient theorem applied to the $\mathbb{Z}$–module complex $C_*(\tilde{S})\otimes_{\mathbb{Z}[\pi_1(S)]}\mathbb{Z}[G]$ that

$$H_1^\alpha(S;\mathbb{Z}[G])\otimes_{\mathbb{Z}}\mathbb{F}_p\cong H_1^{\alpha_p}(S;\mathbb{F}_p[G])$$

for every prime p. The same statement holds for N. Combining our results we see that $H_1^\alpha(S;\mathbb{Z}[G])\otimes_{\mathbb{Z}}\mathbb{F}_p\cong H_1^\alpha(N;\mathbb{Z}[G])\otimes_{\mathbb{Z}}\mathbb{F}_p$ for any prime p. It follows that $H_1^\alpha(S;\mathbb{Z}[G])\cong H_1^\alpha(N;\mathbb{Z}[G])$.

□

Now consider the exact sequence

$$H_1^\alpha(S;\mathbb{Z}[G])\otimes\mathbb{Z}[t^{\pm1}]\xrightarrow{ti_+-i_-}H_1^\alpha(N;\mathbb{Z}[G])\otimes\mathbb{Z}[t^{\pm1}]\to H_1^\alpha(M;\mathbb{Z}[G][t^{\pm1}])\to 0.$$

Since $H_1^\alpha(S;\mathbb{Z}[G])$ and $H_1^\alpha(N;\mathbb{Z}[G])$ are free abelian groups of the same rank it follows that $\Delta_1^G(t)=\det(ti_+-i_-)$ (we refer to [Ch03] for the definition of $\Delta_1^G(t)\in\mathbb{Z}[t^{\pm1}]$). Using Lemmas 4.7, 4.9 and 4.14 we see that the assumption

$$||\phi||_T=\frac{1}{|G|}\big(\deg\big(\Delta_1^G(t)\big)-\deg\big(\Delta_0^G(t)\big)-\deg\big(\Delta_2^G(t)\big)\big)$$

implies that $\deg(\det(ti_+-i_-))=\deg\big(\Delta_1^G(t)\big)=\mathrm{rank}(H_1^\alpha(S;\mathbb{Z}[G]))$. In particular $\det(i_+)\neq 0$ and $\det(i_-)\neq 0$. The assumption that the twisted Alexander polynomial $\Delta_1^G(t)$ is monic implies that in fact $\det(i_+)=\pm1$ and $\det(i_-)=\pm1$. Therefore the question is whether $i_+:\pi_1(S)\to\pi_1(N)$ is surjective if $i_+:H_1^\alpha(S;\mathbb{Z}[G])\to H_1^\alpha(N;\mathbb{Z}[G])$ is surjective for every representation $\pi_1(M)\to G$, G a finite group.

This discussion shows that Conjecture 10.1 follows from the geometrization conjecture and the following group–theoretic question.

Conjecture 10.5. *Let S be an incompressible surface in a 3–manifold M and let N be M cut along S. Let $i:S\to N$ be one of the positive and the negative inclusions of S into N. If $i:H_1^\alpha(S;\mathbb{Z}[G])\to H_1^\alpha(N;\mathbb{Z}[G])$ is surjective for every homomorphism $\pi_1(M)\to G$, G a finite group, then $i:\pi_1(S)\to\pi_1(N)$ is surjective.*

Note that it is well–known that the inclusion induced homomorphisms $\pi_1(S)\to\pi_1(M)$ and $\pi_1(N)\to\pi_1(M)$ are injections. We think that an affirmative answer to the above conjecture will need a strong condition on $\pi_1(M)$ (of which $\pi_1(N)$ and $\pi_1(S)$ are subgroups) such as *subgroup separability*, which is conjectured to hold for fundamental groups of hyperbolic manifolds (cf. [Th82]) but does not hold for fundamental groups in general (cf. [NW01]). On the other hand, knots which are not hyperbolic are either torus knots (and therefore fibered), or satellite knots, which can perhaps be dealt with differently, as in Proposition 10.3.

10.2. **Knot genus conjecture.** As we pointed out in Section 5 there exist many manifolds such that $\Delta_1^\alpha(t) = 0$ for all $\phi \in H^1(M)$ and all representations α. Even though Theorem 5.1 can still give some partial information, it is clear that in the case $b_1(M) > 1$ in general we can not expect to get the Thurston norm from twisted Alexander polynomials. As we pointed out in Section 5, we can not even determine whether a surface with minimal complexity dual to some $\phi \in H^1(M)$ is connected or not.

It follows immediately from the definition of twisted Alexander polynomials that $\Delta_1^\alpha(t)$ is determined by the fundamental group of M. Therefore it is a natural question whether the Thurston norm is determined by the fundamental group. Note that the Thurston norm of a manifold is determined by its prime components. If M is a prime manifold with $H^1(M) \neq 0$, then M is Haken and by Waldhausen's work it is determined by its peripheral system. In particular if M is closed and Haken, then it is determined by its fundamental group.

In the case of a knot complement $X(K)$ Feustel and Whitten [FW78, Corollary 3] showed that the genus of a knot K is determined by $\pi_1(X(K))$. We do not know whether $\pi_1(M)$ determines the Thurston norm for every manifolds with boundary.

We conjecture that twisted Alexander polynomials determine the genus of hyperbolic knots. More precisely we formulate the following conjecture.

Conjecture 10.6. *Let $K \subset S^3$ be a hyperbolic knot and $\phi \in H^1(X(K))$ a generator. Then there exists an epimorphism $\alpha : \pi_1(M) \to G$ such that*

$$\frac{1}{|G|}\big(\deg\big(\Delta_1^G(t)\big) - \deg\big(\Delta_0^G(t)\big)\big) > ||\phi||_T - 1.$$

It turns out that this question is similar in nature to Conjecture 10.1. Indeed, let S be a minimal Seifert surface for K. Then the maps

$$i_+, i_- : \pi_1(S) \to \pi_1(S^3 \setminus \nu S)$$

are injective. The goal now is to show that there exists a map $\pi_1(X(K)) \to G$ such that the induced maps on homology

$$i_+, i_- : H_1^\alpha(S; \mathbb{Z}[G]) \to H_1^\alpha(S^3 \setminus \nu S; \mathbb{Z}[G])$$

are 'almost' injective. For example if $H_1^\alpha(S; \mathbb{Z}[G])$ and $H_1^\alpha(S^3 \setminus \nu S; \mathbb{Z}[G])$ are isomorphic as $\mathbb{Z}$–modules and i_+, i_- are isomorphisms, then $\Delta_1^G(t)$ has degree $\mathrm{rank}_{\mathbb{Z}}(H_1^\alpha(S; \mathbb{Z}[G]))$ and therefore

$$||\phi||_T = \frac{1}{|G|}\big(\deg\big(\Delta_1^G(t)\big) - \deg\big(\Delta_0^G(t)\big)\big).$$

Hence again we have to detect a property of the map on the fundamental group level in a map of appropriate homology groups.

Remark. Let K_1 and K_2 be knots and assume there exists an epimorphism $\varphi : \pi_1(X(K_1)) \to \pi_1(X(K_2))$. Simon asked (cf. question 1.12 (b) on Kirby's problem list [Kir97]) whether this implies that genus$(K_1) \geq$ genus(K_2). Let $\alpha : \pi_1(X(K_2)) \to G$ be an epimorphism to a finite group. By [KSW04] $\Delta^{\alpha}_{K_2}(t)$ divides $\Delta^{\alpha\circ\varphi}_{K_1}(t)$. Thus our results strongly suggest an affirmative answer to Simon's question, and clearly a positive solution to our conjecture would answer Simon's question.

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