PROF . DR. ING. VASILE SZOLGA

THEORETICAL MECHANICS

LECTURE NOTES AND SAMPLE PROBLEMS

PART ONE

- ***STATICS OF THE PARTICLE, OF THE RIGID BODY AND OF THE SYSTEMS OF BODIES***
- ***KINEMATICS OF THE PARTICLE***

2010

Contents

Chapter 6. Systems of rigid bodies………. 125

Chapter 7. Trusses………………………….... 165

KINEMATICS

Chapter 8. Kinematics of the particle…… 188

Chapter 1. Introduction

1.1. The object of the course

Mechanics may be defined as that science that describe and develop the conditions of equilibrium or of the motion of the material bodies under the action of the forces. Mechanics can be divided in three large parts, function of the studied object: mechanics of the no deformable bodies (mechanics of the rigid bodies), mechanics of the deformable bodies (strength of the materials, elasticity, building analysis) and fluid mechanics.

Mechanics of the no deformable bodies, or theoretical mechanics, may be divided in other three parts: ***statics, kinematics*** *and* ***dynamics.*** *Statics is that part of the theoretical mechanics which studies the transformation of the systems of forces in other simpler systems and of the conditions of equilibrium of the bodies. Kinematics is the part of the theoretical mechanics that deals with the motions of the bodies without to consider their masses and the forces that acts about them, so kinematics studies the motion from geometrical point of view, namely the pure motion. Dynamics is the part of the theoretical mechanics which deals with the study of the motion of the bodies considering the masses of them and the forces that acts about them. In all these definitions the bodies are considered rigid bodies that are the no deformable bodies. It is known that the real bodies are deformable under the action of the forces. But these deformations are generally very small and they produce small effects about the conditions of equilibrium and of the motion.*

Mechanics is a science of the nature because it deals with the study of the natural phenomenon. Many consider mechanics as a science joined to the mathematics because it develops its theory based on mathematical proofs.

At the other hands, mechanics is not an abstract science or a pure one, it is an applied science.

Theoretical mechanics studies the simplest form of the motion of the material bodies, namely the ***mechanical motion.*** *The mechanical motion is defined as that phenomenon in which a body or a part from a body modifies its position with respect to an other body considered as reference system.*

1.2. Fundamental notions in theoretical mechanics

Theoretical mechanics or Newtonian mechanics uses three fundamental notions: ***space, time*** *and* ***mass.*** *These three notions are considered independent one with respect to other two. They are named fundamental notions because they may be not expressed using other simpler notions and they will form the reference frame for to study the theoretical mechanics.*

The notion ***space*** *is associated with the notion of position. For example the position of a point P may be defined with three lengths measured on three given directions, with respect to a reference point. These three lengths are known under the name of the coordinates of the point P. The notion of space is associated also with the notion of largest of the bodies and the area of them.* ***The space*** *in theoretical mechanics is considered to be the real space in which are produced the natural phenomenon and it is considered with the next proprieties:* ***infinity large, three dimensional, continuous, homogeneous and isotropic.*** *The space defined in this way is the Euclidian space with three dimensions that allows to build the like shapes and to obtain the differential computation.*

In the definition of a mechanical phenomenon, generally, is not enough to use only the notion of space, namely is not enough to define only the position and the largest of the bodies. Mechanical phenomena have durations and they are produced in any succession. Joined to these notions: duration and succession, theoretical mechanics considers as fundamental notion ***the time*** *having the following proprieties:* ***infinity large, one-dimensional, continuous, homogeneous and irreversible.*** *The time between two events is named interval of time and the limit among two intervals of time is named instant.*

The notion of ***mass*** *is used for to characterize and compare the bodies in the time of the mechanical events. The* ***mass*** *in theoretical mechanics is the measure of the inertia of bodies in translation motion and will represent the quantity of the substance from the body, constant in the time of the studied phenomenon.*

Besides of these fundamental notions, theoretical mechanics uses other characteristic notions, generally used in each part of the mechanics. These notions will be named as basic notions and they will be defined for each part of mechanics. In Statics we shall use three notions: ***the force****,* ***the moment of the force about a point*** *and* ***the moment of the force about an axis.*** *In Kinematics the basic notions will be:* ***the velocity*** *and* ***the acceleration*** *and in Dynamics we shall use :* ***The linear momentum, the angular momentum, the kinetic energy, the work, the potential energy*** *and* ***the mechanical energy.***

1.3. Fundamental principles of theoretical mechanics

At the base of the theoretical mechanics stay a few fundamental principles (laws or axioms) that cannot be proved theoretical bat they are checked in practice. These principles were formulated by Sir Isaac Newton in the year 1687 in its work named " Philosophiae naturalis principia mathematica". With a few small explanations these principles are used under the same shape also today, in some cases are added a few principles for to explain the behavior of a non deformable body. In this course we shall present five principles from which three are the three laws of Newton.

1) Principle of inertia (Lex prima). *This principle says that:* ***a body keeps its state of rest or of rectilinear and uniformly motion if does not act a force (or more forces) to change this state****. We make the remark that, Newton understands through a body in fact a particle (a small body without dimensions). The statement of this principle may be kept if we say that the motion is a rectilinear uniformly* ***translation*** *motion. This principle does not leave out the possibility of the action of forces about the body, but the forces have to be in equilibrium. About these things we shall talk in a future chapter.*

2) Principle of the independent action of the force (Lex secunda) *has the following statement:* ***if about a body acts a force, this produces an acceleration proportional with them, having the same direction***

and sense as the force, independently by the action of other forces. *Newton has state from this principle the fundamental law of the mechanics:*

$$\boxed{\overline{F} = m\,\overline{a}}$$

3) Principle of the parallelogram *has been stated by Newton as the first "addendum" to the previous principle. This principle has the statement:* ***if about a body act two forces, the effect of these forces may be replaced with a single force having as magnitude, direction and sense of the diagonal of the parallelogram having as sides the two forces.*** *This principle postulates, in fact, the principle of the superposition of the effects. This principle is used under the name of the* ***parallelogram rule****.*

4) Principle of the action and the reaction (Lex tertia) *is the Newton's third law, and says:* ***for each action corresponds a reaction having the same magnitude, direction and opposite sense****, or:* ***the mutual actions of two bodies are equal, with the same directions and opposite senses.***

We make the remark that, in each statement through the notion "body" we shall understand the notion of "particle".

5) Principle of transmissibility *is that principle that defines the non-deformable body and has the statement:* ***the state of a body (non-deformable) does not change if the force acting in a point of the body is replaced with another force having the same magnitude, direction and sense but with the point of application in another point on the support line of the force*** *(Fig.1.). The two forces will have the same effect about the body and we say that they are equivalent forces.*

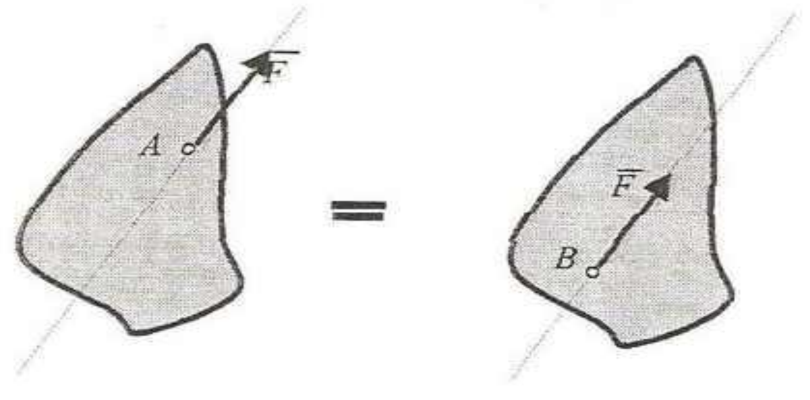


Fig.1.

1.4. Theoretical models and schemes in mechanics

Through a model or a scheme we shall understand a representation of the body or a real phenomenon with a certain degree of approximation. But the approximation may be made so that the body or the phenomenon to keeps the principal proprieties of them.

For to simplify the study of the theoretical mechanics, the material bodies are considered under the form of two models coming from the general model of the ***material continuum****:* ***the rigid body*** *and* ***the particle****.*

The rigid body*, by definition, is the non-deformable material body. This body has the propriety that: the distance among two any points of the body does not change indifferent to the actions of the forces or other bodies about it. This model is accepted in theoretical mechanics because, generally, the deformations of the bodies are very small and they may be neglected without to introduce, in the computations or in the final solutions of the studied problems, substantial errors.*

In the case when the body is very small or the dimensions are not interesting in the studied problem, the used model is ***the particle (the material point).*** *The particle is in fact a geometrical point at which is attached the mass of the body from which is coming the particle.*

The rigid bodies may have different schemes function of the rate of the dimensions. We shall have the next three schemes: material lines (bars), material surfaces (plates) and material volumes (blocks).

Material lines *or* ***bars*** *are rigid bodies at which one dimension (the length) is larger than the other two (width and thickness). These kinds of bodies are reduced to a line representing the locus of the centroids of the cross sections.*

Material surfaces *or* ***plates*** *are bodies at which two dimensions are bigger than the third (the thickness). In this case the body is reduced to a surface representing the median surface of the plate.*

Material volumes *or* ***blocks*** *are bodies at which the three dimensions are comparables.*

Finally, another classification of the bodies is made function the distribution of the mass in the inside of the body. We shall have two kinds of *bodies:* ***homogeneous bodies*** *for which the mass is uniformly distributed in the entire volume of the bodies, and* ***non-homogeneous bodies*** *at which the mass is non-uniformly distributed inside of the bodies.*

STATICS

Chapter 2. Systems of forces

2.1. Introduction

In this chapter we shall study the systems of forces and the way in which they are transformed in other simpler systems. We shall begin with the systems of concurrent forces and after we shall pass to the other systems of forces, like systems of the coplanar forces, parallel forces and arbitrary forces. Also we shall study first the systems in the space with three dimensions, and after the particular case of the systems in the space with two dimensions (the plane problem).

First of all we make a few remarks. If two systems of forces have the same effect about a body we shall say that the two systems are ***equivalent systems of forces****. The reciprocal is also true, namely if two systems are equivalent than they will produce the same effect about same body. Generally we shall look for the simplest equivalent system of forces for the given system.*

2.2. The force

The force *is defined as the action of a body about another body and it is a vector quantity. The vector quantity, the force, has four characteristic:* ***magnitude, direction, sense*** *and* ***point of application****. Being a vector, the force may be represented as in the figure 2, where are represented the four characteristics.*

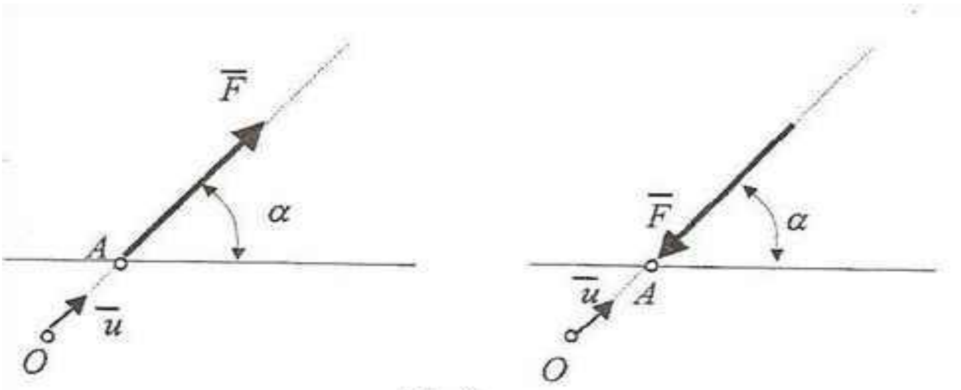


Fig.2.

The magnitude of the force is defined, using the units of measure of the force, by a scalar quantity. The magnitude is represented using an any scale (the correspondence between the units of the force and the unites of the length) through a segment of line.

Direction of the force is defined with ***the support line*** *that is the straight line on which is laying the force. The direction of the support line with respect to an any straight line (or an axis) with known direction is given by the angle (α) between them.*

The sense of the force is represented with an arrowhead in an end of the force. The point of application is, generally, indicated through a letter and may be situated in the same or in the opposite end as the arrowhead.

As it is known, a straight line becomes an axis if on that line is taken a point as the origin of the axis (point O in fig. 2.) and a positive sense. The direction and the positive sense of the axis may be considered also with a unit vector (u in the fig. 2.). With this unit vector we may write:

$$\overline{F} = \pm F.\overline{u}$$

where we have shown three characteristics of the force: the magnitude marked with F, the direction with the unit vector u, and the sense with the sign in front of the magnitude which if it is (+) shows that the force and the unit vector have the same sense, and if it is (-) they are with opposite senses.

We can see easy that in this relation is not represented the position of the point of application of the force and obviously the position of the support line. This means that the position of the force in space have to be expressed with another notion in another future section of this chapter.

2.3. Projection of the force on an axis. Component of the force on the direction of an axis.

Let consider an any force, represented in the figure 3. among the points A and B and an arbitrary axis (Δ) defined with the unit vector $\overline{u}_{\Delta}$. Through the two points we shall consider two parallel planes (P_1) and (P_2) perpendicular on the axis (Δ). These two planes will be intersected by the axis (Δ) in the points A_1 and B_1.

The segment of line A_1B_1 measured at the scale of the force is named ***projection of the force*** *on the axis (Δ) and is marked:*

$$A_1B_1 = pr_{(\Delta)}\overline{F} = F_{\Delta}$$

and as we can see is a scalar quantity.

If through the point A is taken a straight line parallel with the axis (Δ) then this line will intersect the plane (P_2) in the point B and we shall have:*

$$A_1B_1 = AB^*$$

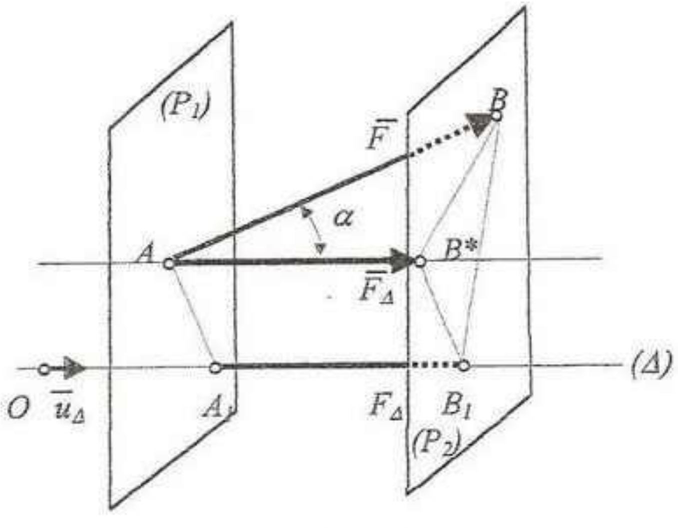


Fig.3.

But this segment of line is the side of the right angle triangle ABB and it may be calculated resulting the expression of the projection of a force on the direction of an axis:*

$$F_{\Delta} = F \cdot \cos\alpha$$

We remark that, this projection may be expressed as a scalar product:

$$F_{\Delta} = \overline{F} \bullet \overline{u}_{\Delta}$$

If through the extremity B we shall consider an arrowhead then AB* becomes a vector quantity that at the scale of the force has the magnitude equal with the projection of the force on the direction of the axis (Δ). This vector is called* ***component of the force $\overline{F}$*** *on the direction of the axis (Δ) and can be expressed:*

$$\overline{F}_{\Delta} = F_{\Delta} \cdot \overline{u}_{\Delta}$$

We remark that the projection of a force on an axis is a scalar quantity and it may be obtained on the axis or on any parallel axis with the given axis and the component of the force on the direction of an axis is a vector quantity, has the magnitude equal with the projection of the force on the axis and has the same point of application as the given force.

2.4. Addition of two concurrent forces

Generally, when we have a system of forces, the main problem is that to transform the system in other simpler one. This is made when we can replace the system with another simpler as the first one but with the same effect about the bodies upon which are they acting. We shall start from the simplest system of forces, the system made from two concurrent forces.

Suppose two forces: $\overline{P}$ and $\overline{Q}$ having the same point of application.

Using the parallelogram's principle these two forces may be replaced with one single force having the same effect. This force marked $\overline{R}$ is called ***resultant force****, or shortly* ***resultant****.*

From mathematical point of view this resultant force is the vector sum of the two forces and we can write:

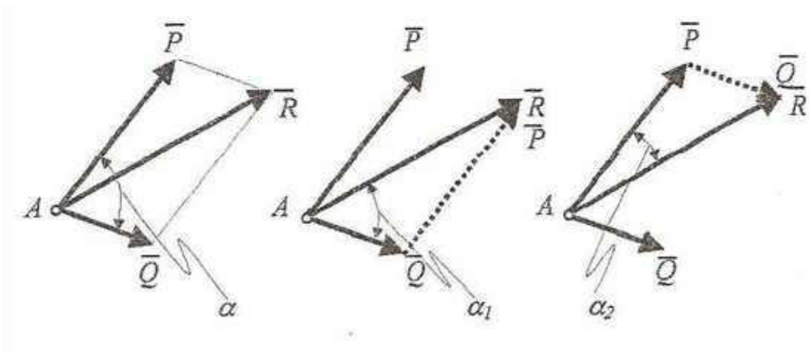


Fig.4.

$$\overline{R} = \overline{P} + \overline{Q}$$

The magnitude of the resultant force is obtained from the cosinus theorem in one of the two triangles that are formed in the parallelogram:

$$R = \sqrt{P^2 + Q^2 + 2PQ.\cos\alpha}$$

The direction may be obtained computing the angle between the resultant force and the direction of one force from the two using the sinus theorem in one of the two triangles made by the resultant with the two forces:

$$\sin\alpha_1 = \frac{P}{R}\sin\alpha\ ;\ \ \sin\alpha_2 = \frac{Q}{R}\sin\alpha$$

From the parallelogram rule result another rule called ***triangle rule****. In this rule the resultant force of the two given forces is obtained in the following way: one force, from the two, brings with its point of application in the top of the other, the resultant force resulting uniting the common point of application of the two forces with the top of the second force.*

In the particular case of two collinear forces with the same sense, this last rule shows that the resultant force is obtained summing in scalar way the magnitudes of the two forces:

$$R = P + Q$$

If the two forces are collinear but with opposite senses, from the same rule, results that the resultant force has the magnitude:

$$R = P - Q$$

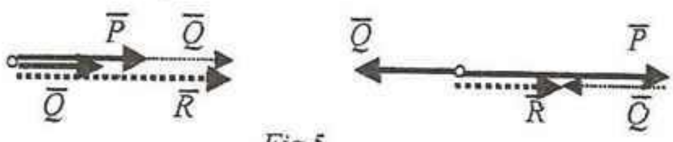


Fig.5.

2.5. The force in Cartesian system of reference

We shall consider now a force with its point of application in the origin of a Cartesian three-orthogonal, right hand reference system. This system of reference has the axes Ox, Oy and Oz (from this reason the name of this system is Cartesian, because Descartes was the first who used this system of notation and the Latin name of him was Cartesius) perpendicular two by two and located so that, the observer looking from the first frame of the system sees the notations of the axes x,y,z in trigonometrically sense (counterclockwise sense). The axes can be defined also (as directions and positive senses) using the unit vectors of the three axes: $\overline{i}$, $\overline{j}$ *and* $\overline{k}$.

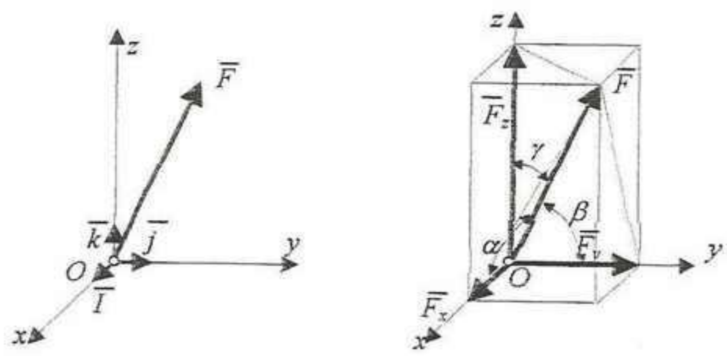


Fig.6.

For to find the expression of the force in Cartesian system of reference we shall make in the next way: first we shall define the projections of *the force on the three axes considering parallel and perpendicular planes on each axis. Marking the three angles with respect to the three axes with: α,β,γ, results the three projections:*

$$F_x = F.\cos\alpha \; ; \; F_y = F.\cos\beta \; ; \; F_z = F.\cos\gamma$$

The three components of the force on the directions of the three axes will be:

$$\overline{F_x} = F_x .\overline{i} \; ; \; \overline{F_y} = F_y .\overline{j} \; ; \; \overline{F_z} = F_z .\overline{k}$$

Adding the three components (first two components and after the resultant with the third) results the relation:

$$\boxed{\overline{F} = \overline{F_x} + \overline{F_y} + \overline{F_z}}$$

or:

$$\boxed{\overline{F} = F_x .\overline{i} + F_y .\overline{j} + F_z .\overline{k}}$$

If we mark, for to simplify, the three projections:

$$F_x = X \; ; \; F_y = Y \; ; \; F_z = Z$$

and the components:

$$\overline{F_x} = \overline{X} = X .\overline{i} \; ; \; \overline{F_y} = \overline{Y} = Y .\overline{j} \; ; \; \overline{F_z} = \overline{Z} = Z .\overline{k}$$

then we shall have the expression of the force in Cartesian system of reference:

$$\boxed{\overline{F} = X .\overline{i} + Y .\overline{j} + Z .\overline{k}}$$

Supposing that we know the projections of the force on the three axes of the Cartesian reference system, the magnitude and the direction of the force results:

$$\boxed{F = \sqrt{X^2 + Y^2 + Z^2}}$$

$$\boxed{\cos\alpha = X/F \; ; \; \cos\beta = Y/F \; ; \; \cos\gamma = Z/F}$$

2.6. The resultant force of a system of concurrent forces. Theorem of projections

Through system of concurrent forces we understand the system of forces in which all forces have the same point of application. If this kind of system of forces is acting about a rigid body it is enough as the support lines of the forces to be concurrent in the same point. Suppose that a system of concurrent forces. For to transform the system of forces in the simplest equivalent system (that has the same effect) we may use one of the two rules used in the case of two forces (the rule of parallelogram or of the triangle). It is easier to use the second rule (of the triangle) obtaining first the resultant of the first two forces that is added with the third force and so on. Acting in this way is obtained a now rule: ***the rule of the polygonal line.*** *This rule say that: for to find the resultant force of a system of concurrent forces it is enough to place, in an any order, the forces of the system so in the top of the previous force to be the point of application of the next force. In this way we shall obtain a polygonal line made from the forces of the system of forces. The resultant force will be obtained uniting the point of application of the first force, from the polygonal line, with the top of the last force from this polygonal line.*

$$\overline{R} = \sum \overline{F_i}$$

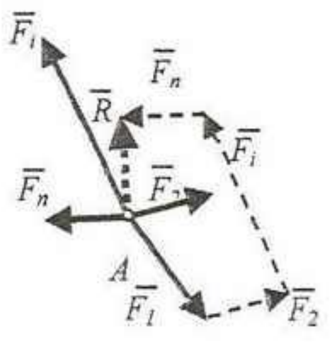


Fig.7.

But this geometrical method to find the resultant force is difficult to apply, especially in space (in three dimensions) and for the systems with more forces. Because this reason, generally is used an analytical method based on the ***theorem of projections****:* ***the projection, on an any axis, of the***

resultant force of a system of concurrent forces is equal to the sum of all projections of the forces from the system on the same axis*. For to prove this theorem we shall compute the scalar product of the previous relation with the unit vector of an any axis:*

$$\overline{R} \bullet \overline{u_\Delta} = \sum \overline{F_i} \bullet \overline{u_\Delta}$$

Knowing that the scalar product among the force and the unit vector of an axis, by definition, is the projection of the force on that axis:

$$\boxed{R_\Delta = \sum F_{i\Delta}}$$

In the case of a Cartesian system of reference we may write:

$$X = \sum X_i \; ; \; Y = \sum Y_i \; ; \; Z = \sum Z_i$$

where we marked X, Y and Z the projections of the resultant force on the three axes and X_i, Y_i and Z_i the projections of the forces from the system of forces on the same axes.

Finally, the resultant force will have the expression as vector, magnitude and direction:

$$\overline{R} = (\Sigma X_i) \cdot \overline{i} + (\Sigma Y_i) \cdot \overline{j} + (\Sigma Z_i) \cdot \overline{k};$$
$$R = \sqrt{(\Sigma X_i)^2 + (\Sigma Y_i)^2 + (\Sigma Z_i)^2}\;;$$
$$\cos\alpha = \frac{\Sigma X_i}{R}\;;\; \cos\beta = \frac{\Sigma Y_i}{R}\;;\; \cos\gamma = \frac{\Sigma Z_i}{R}$$

2.7. Sample problems

Problem 1. *Is given a system of forces as in the figure 8.a. in which the magnitudes of the forces are: F_1 = 10 N; F_2 = 20 N; F_3 = 17,3 N. Calculate and represent the resultant force of the system of forces.*

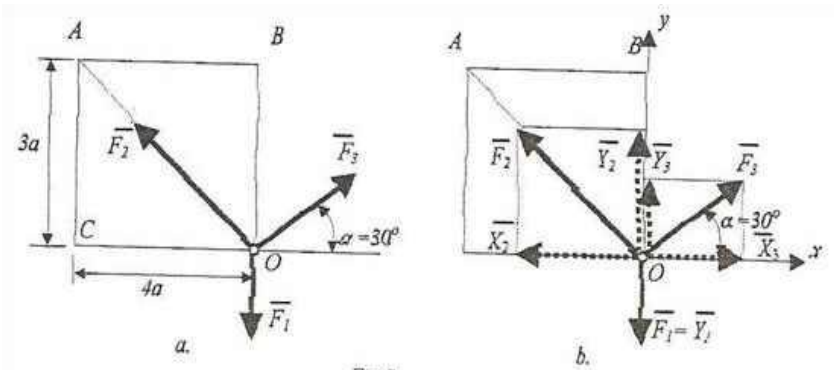


Fig.8.

Solution. Step 1. *First we shall remark that the system of forces is also a coplanar one, namely the forces of the system are located in the same plane. Consequently, we shall choose as reference system the plane system Oxy with the origin in the common point of application of the forces.*

As we may see, there are three kinds of forces in plane (in two dimensions): $\overline{F_1}$ *is a force on the directions of a reference axis (here on the direction of Oy),* $\overline{F_2}$ *is a force al which the direction is given passing through two given points (here the points O and A) and* $\overline{F_3}$ *is a force at which the direction is given through an angle made with respect to a given direction (here the horizontal direction, namely the Ox axis).*

We shall choose as way of computation the analytical way using the theorem of projections. For to determine the projections of the forces we shall use the method of resolution of the forces in components , knowing that the magnitude of the component is equal to the magnitude of the projection and the sign of the projection may be obtained comparing the sense of the component with the positive sense of the corresponding axis. If the component has the sense of the positive axis then the projection will be positive.

The force $\overline{F_1}$ *being on the direction of an axis it have not decompose in components, it is in the same time the component on the direction of that axis:*

$$F_1 = Y_1$$

For the force $\overline{F_2}$ *the resolution in components will give the two components* $\overline{X_2}$ *and* $\overline{Y_2}$*. The magnitudes of them will be obtained knowing that the angle made by the force with the axis is the same with the angle made by the diagonal of the formed rectangle (always the segment OA may be considered as the diagonal of a rectangle). In this way the cosines of the angle among the force and axis may be calculated as the rate of the sides of the formed right angle triangle. results the magnitudes of the components:*

$$X_2 = F_2 \cdot \frac{l_x}{d} = F_2 \cdot \frac{OC}{OA} = 20 \cdot \frac{4a}{5a} = 16N;$$

$$Y_2 = F_2 \cdot \frac{l_y}{d} = F_2 \cdot \frac{OB}{OA} = 20 \cdot \frac{3a}{5a} = 12N.$$

where l_x *,* l_y *and d are the magnitudes of the sides of the rectangle on the directions of the two axes Ox and Oy and the length of the diagonal of the rectangle.*

The force $\overline{F_3}$ *will be resolve in two components also with the magnitudes:*

$$X_3 = F_3 \cdot \cos\alpha = 17,3 \cdot \frac{\sqrt{3}}{2} = 15\ N;$$

$$Y_3 = F_3 \cdot \sin\alpha = 17,3 \cdot \frac{1}{2} = 8,65\ N.$$

Step 2. Calculation of the resultant force. *Knowing the magnitudes of the components (or of the projections) of the forces we shall use the theorem of projections for to determine the projections of the resultant force:*

$$X = \Sigma X_i = -X_2 + X_3 = -16 + 15 = -1\ N;$$
$$Y = \Sigma Y_i = -F_1 + Y_2 + Y_3 = -10 + 12 + 8{,}65 = 10{,}65\ N.$$

The resultant force with respect to the given system of reference will be:

$$\overline{R} = -\overline{i} + 10{,}65\,\overline{j}$$

with the magnitude:

$$R = \sqrt{X^2 + Y^2} = \sqrt{1^2 + 10{,}65^2} = 10{,}75\ N;$$

and the direction defined by the angle α_R:

$$tg\alpha_R = \frac{Y}{X} = \frac{10{,}65}{-1} = +10{,}65 \qquad \alpha_R = 96^\circ.$$

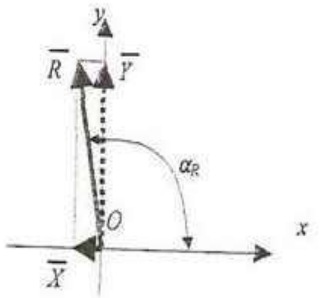


Fig.9.

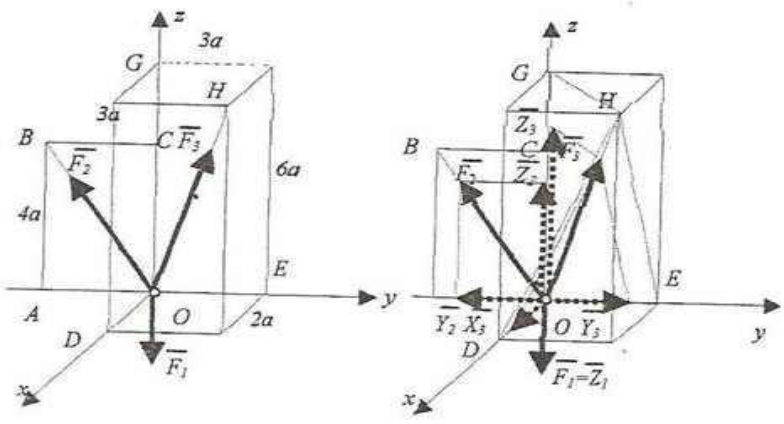
Problem 2. *Are given the forces from the figure 10.a. representing a system of concurrent forces in space (in three dimensions). Knowing the magnitudes of the forces: $F_1 = 5F$, $F_2 = 10F$, $F_3 = 14\ F$ and their directions through the geometrical constructions from the picture, determine and represent the resultant force of the system.*

Solution. Step 1. Calculation of the projections of the forces. *We may see that in space are three kinds of forces: forces parallel with one axis of the reference system (here the force $\overline{F_1}$), forces laying in a reference plane of the system of reference (here the force F_2), and any forces with respect to the axes or the planes of the reference system (here the force F_3). For the forces from the first two categories the rules of resolution and calculation of the projections are the same as in the plane problem (the previous problem), namely the force F_1 has one single component:*

$$F_1 = Z_1 = 5F$$

The force $\overline{F_2}$ has two components on the directions of the axes of the reference plane in which it is located (here the plane yOx), the resolution making with the rule of the parallelogram. The magnitudes of the components will be:

$$Y_2 = F_2 \cdot \frac{l_y}{d} = F_2 \cdot \frac{OA}{OB \cdot} = 10F \cdot \frac{3a}{5a} = 6F;$$
$$Z_2 = F_2 \cdot \frac{l_z}{d} = F_2 \cdot \frac{OC}{OB} = 10F \cdot \frac{4a}{5a} = 8F;$$



a. b.

Fig.10.

where l_y, l_z and $d = \sqrt{l_y^2 + l_z^2}$ are the corresponding sides and the diagonal of the rectangle on which is laying the force $\overline{F_2}$.

The force $\overline{F_3}$ will have three components, and for the calculation of their magnitudes we can use the same rules as for the components of the force $\overline{F_2}$, with the difference that the diagonal:

$$d = \sqrt{l_x^2 + l_y^2 + l_z^2}$$

is the diagonal of the parallelepiped on which is laying the force $\overline{F_3}$. Results the magnitudes:

$$X_3 = F_3 \cdot \frac{l_x}{d} = F_3 \cdot \frac{OD}{OH} = 14F \cdot \frac{2a}{7a} = 4F;$$
$$Y_3 = F_3 \cdot \frac{l_y}{d} = F_3 \cdot \frac{OE}{OH} = 14F \cdot \frac{3a}{5a} = 6F;$$
$$Z_3 = F_3 \cdot \frac{l_z}{d} = F_3 \cdot \frac{OG}{OH} = 14F \cdot \frac{6a}{7a} = 12F.$$

Step 3. Calculation of the resultant force. *Having, now, the magnitudes of the components and their senses with respect to the positive axes of reference we may determine the projections of them on the reference axes. Using the theorem of the projections are obtained the projections of the resultant force:*

$$X = \Sigma X_i = X_3 = 4F;$$
$$Y = \Sigma Y_i = -Y_2 + Y_3 = -6F + 6F = 0;$$
$$Z = \Sigma Z_i = -Z_1 + Z_2 + Z_3 = -5F + 8F + 12F = 15F.$$

that has the expression:

$$\overline{R} = 4F. \overline{i} + 15F . \overline{k}$$

and the magnitude:

$$R = \sqrt{X^2 + Y^2 + Z^2} = F\ \sqrt{4^4 + 15^2} = 15{,}5F$$

The direction of the resultant force will be defined by the angles:

$$\cos\alpha = \frac{X}{R} = \frac{4F}{15{,}5F} = 0{,}15 \longrightarrow \alpha = 75°;$$

$$\cos\beta = \frac{Y}{R} = 0 \longrightarrow \beta = 90°;$$

$$\cos\gamma = \frac{Z}{R} = \frac{15F}{15{,}5F} = 0{,}9 \longrightarrow \gamma = 15°$$

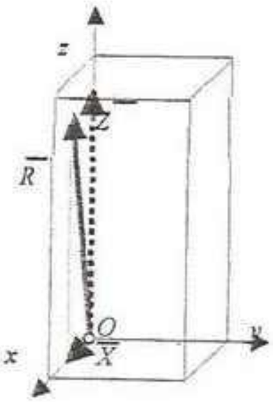


Fig.11.

Problem 3. *Calculate and represent the resultant force of the system of concurrent forces from the figure 12. knowing that:* $F_1 = 3F$, $F_2 = 5F$, $F_3 = 5F$, $F_4 = 5\sqrt{2}\ F$.

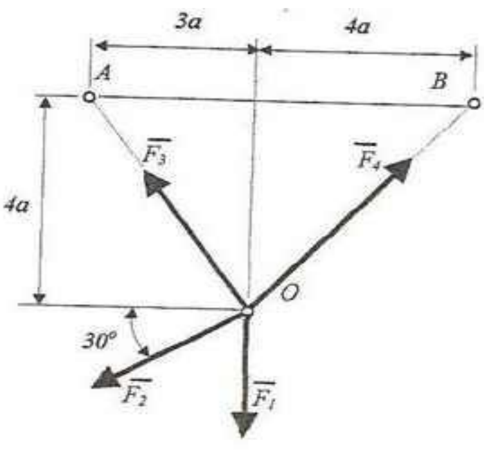


Fig.12.

Problem 4. *Calculate and represent the resultant force of the following system of concurrent forces knowing the magnitudes :* $F_1 = 5F$, $F_2 = 3\sqrt{41}\ F$, $F_3 = 4\sqrt{13}\ F$, $F_4 = 3\sqrt{14}\ F$.

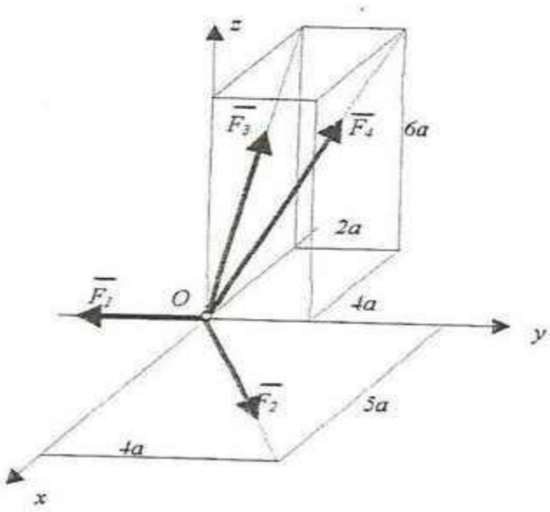


Fig.13.

2.8. Moment of a force about a given point

We have seen that when a force acts about a rigid body, it may be considered as a gliding vector, namely its point of application can be an arbitrary point from its support line, the force keeping the effect about the body. At the other hand, the found expressions of the forces contain only three from the four characteristics that are: the magnitude, the direction and the sense, without to make any reference about the position of the force with respect to the body about it acts or with respect to a system of reference.

If we consider a body (for example a plane body) with a fixed point, is obviously that if the force acts in different positions about the body it will produce different effects. For example if the force acts in the left side of the fixed point it will produce a clockwise rotation (Fig. 14.a.), and if the force acts in the right side of the fixed point it will produce a counterclockwise rotation (Fig. 14.b.), and finally if the force acts so that its support line is passing through the fixed point does not produce any rotation of the body (Fig.14.c.). These effects are obtained keeping, each time, the magnitude, direction and sense of the force.

These facts make that to need to introduce a new notion for to define the position of the force with respect to any systems of reference. This new notion is called ***moment of the force about a given point****.*

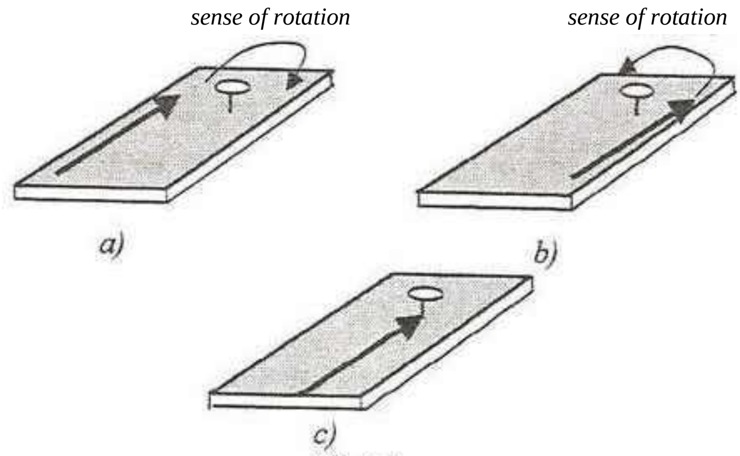


Fig.14.

Suppose a force $\overline{F}$ acting in the point A and another point O (Fig.15.). The position of the point A with respect to the point O may be defined with a vector called ***position vector of the point A with respect to the point O****. We shall mark this vector:*

$$\overline{OA} = \overline{r}$$

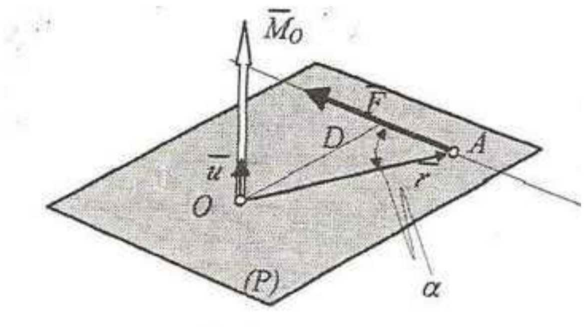


Fig.15.

By definition, is called ***moment of the force $\overline{F}$ with respect to the point O*** *the vector quantity, marked $\overline{M}_O$ and equal to the vector product between the position vector of the point of application of the force with respect to the given point O and the force $\overline{F}$:*

$$\overline{M}_O = \overline{r} \times \overline{F}$$

From the definition of the vector product results the following characteristics of the moment of the force about a point:

- ***The magnitude*** *is:*

$M_O = r \ . F \ . \sin\alpha = F \ . d$

where the distance d is measured from the point O to the support line of the force $\overline{F}$.

- ***The direction*** *is perpendicular on the plane formed by the support line of the force and the point O.*
- ***The sense*** *of the vector moment is so that the three vectors:* $\overline{M_O}$, $\overline{r}$ *and* $\overline{F}$ *make a right hand system, or the sense of the moment is the same as the sense of the advance of the right hand screw rotated by the force* $\overline{F}$ *(the rule of the right hand).*
- ***The point of application*** *is the point O.*

In the definition of the moment is said that the position vector is of the point of application of the force, but the force acting about a rigid body is a gliding vector, so it can change its point of application on its support line. We shall show that the moment of a forcer consider the force as a gliding vector. For that lets consider, on the support line of the force another point B as point of application of the force. We shall mark the new position vector, of this point B, with $\overline{r_1}$*, so the moment of the force about the point O will be:*

$$\overline{M_O}' = \overline{r_1} \times \overline{F}$$

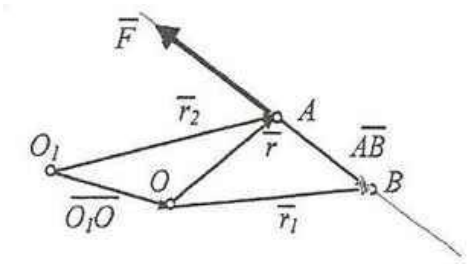


Fig.16.

where the position vector $\overline{r_1}$ *may be expressed as a sum of two vectors:*

$$\overline{r_1} = \overline{r} + \overline{AB}$$

Replacing in the previous relation and knowing the properties of the vector product with respect to the vector sum and that the vector product of two collinear vectors is equal to zero, we have:

$$\overline{M_O}' = (\overline{r} + \overline{AB}) \times \overline{F} = \overline{r} \times \overline{F} + \overline{AB} \times \overline{F} = \overline{r} \times \overline{F} = \overline{M_O}$$

namely we obtain the same result indifferent where is located the point of application of the force on its support line.

But if we change the point about we compute the moment then the moment is modified. For example if the moment is calculated about the point O_1 we have:

$$\overline{M_{O1}} = \overline{r_2} \times \overline{F}$$

Expressing the position vector $\overline{r_2}$ function of the position vector about the point O we may write:

$$\overline{r_2} = \overline{r} + \overline{O_1O}$$

relation that is replaced in the previous relation we shall obtain:

$$\overline{M_{O1}} = (\overline{r} + \overline{O_1O}) \times \overline{F} = \overline{r} \times \overline{F} + \overline{O_1O} \times \overline{F} = \overline{M_O} + \overline{O_1O} \times \overline{F}$$

namely if we change the point about which is calculated the moment, then the moment is changing. The relation express the variation of the moment at the change of the point about is calculated.

2.9. The moment of a force about the origin of the system of reference

Suppose that the force $\overline{F}$ is expressed with respect to a Cartesian system of reference and the point O is the origin of this system. We have, obviously:

$$\overline{F} = X.\overline{i} + Y.\overline{j} + Z.\overline{k}$$
$$\overline{r} = x.\overline{i} + y.\overline{j} + z.\overline{k}$$

where x, y and z are the coordinates of the point A from the support line of the force, and the position vector of this point with respect to the origin of the system of reference has the projections on the axis these coordinates.

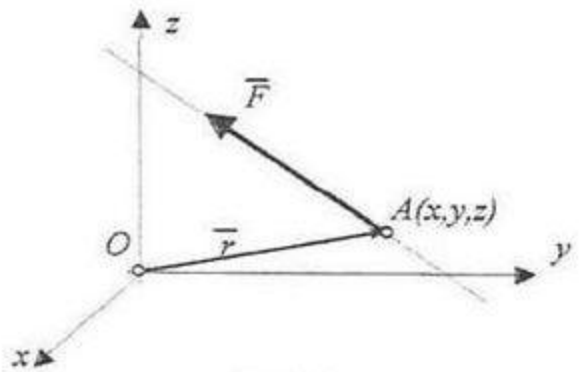


Fig.17.

Knowing that the vector products of the unit vectors of the axes are:

$$\overline{i}\times\overline{i}=0;\ \overline{j}\times\overline{j}=0;\ \overline{k}\times\overline{k}=0;\ \overline{i}\times\overline{j}=-(\overline{j}\times\overline{i})=\overline{k};$$
$$\overline{j}\times\overline{k}=-(\overline{k}\times\overline{j})=\overline{i};\ \overline{k}\times\overline{i}=-(\overline{i}\times\overline{k})=\overline{j};$$

and solving the vector product from the definition of the moment we remark that the moment of a force about the origin of the reference system may be expressed with a determinant:

$$\overline{M}_O=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ x & y & z\\ X & Y & Z\end{vmatrix}=M_x\cdot\overline{i}+M_y\cdot\overline{j}+M_z\cdot\overline{k}$$

where M_x, M_y and M_z are the projections, on the three axes, of the moment of the force about the origin of the reference system.

*Finally we make the remark that through **resultant moment** about a point we shall understand the sum of the moments of a system of forces computed all with respect the same point. Because the System of moments in a point is a concurrent system of vectors the sum is calculated using the same rules as for the concurrent systems of forces.*

2.10. Sample problems

Problem 5. *Is given a system of force as in the figure 18. Is asked to calculate the resultant moment of this system about the point O (the origin of the reference system) knowing the magnitudes of the forces:* $F_1 = 2F$, $F_2 = 4F$, $F_3 = F$ *and* $F_4 = 3F$.

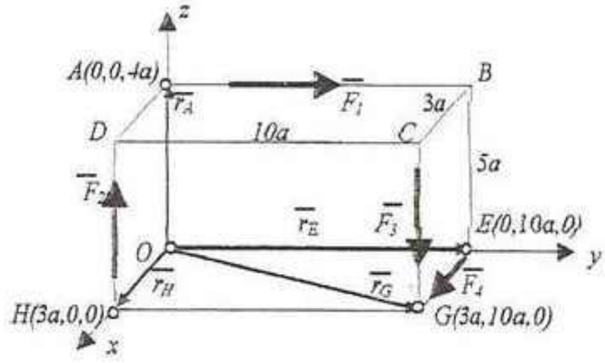


Fig.18.

Solution. *The resultant moment of the given system of forces will be:*

$$\overline{M_O} = \sum \overline{M_{Oi}}$$

For to calculate the resultant moment we shall use the shape of the vector product as a determinant. For this, we shall choose the points of application of the forces as arbitrary points from the support lines of the forces. Obviously in the computation these points will be taken so that to have the maximum simplifications. This will be if one or two coordinates of these points are zero. For the force F_1 *we shall consider the point A(0,0,4a), for the force* F_2 *the point H(3a,0,0), for the force* F_3 *we shall take the point G(3a,10a,0) and for the force* F_4 *the point E(0,10a,0). With these points we have the resultant moment:*

$$\overline{M_O} = \overline{r_A} \times \overline{F_1} + \overline{r_H} \times \overline{F_2} + \overline{r_G} \times \overline{F_3} + \overline{r_E} \times \overline{F_4} =$$

$$= \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 0 & 0 & 4a \\ 0 & 2F & 0 \end{vmatrix} + \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 3a & 0 & 0 \\ 0 & 0 & 4F \end{vmatrix} + \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 3a & 10a & 0 \\ 0 & 0 & -F \end{vmatrix} + \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 0 & 10a & 0 \\ 3F & 0 & 0 \end{vmatrix} =$$

$$= -18Fa \, . \, \overline{i} - 9Fa \, . \, \overline{j} - 30Fa \, . \, \overline{k}$$

Problem 6. *For the system of forces from the figure 19. calculate the resultant moment about the origin of the reference system. Are known:* $F_1 = 5F$, $F_2 = 3F$, $F_3 = 6F$, $F_4 = F$, $F_5 = 2F$.

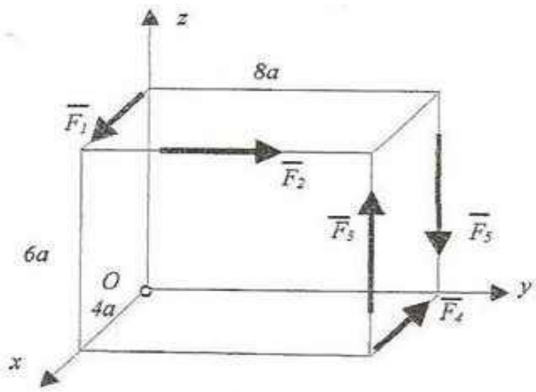


Fig.19.

2.11. Moment of the force about a given axis

Through definition the projection on an axis of the moment of a force about a point from that axis is called ***moment of the force about that axis:***

$$M_\Delta = \overline{M_O} \bullet \overline{u_\Delta} = (\overline{r} \times \overline{F}) \bullet \overline{u_\Delta}$$

relation in which we have marked $\overline{u_\Delta}$ *the unit vector of the axis on which we project the moment.*

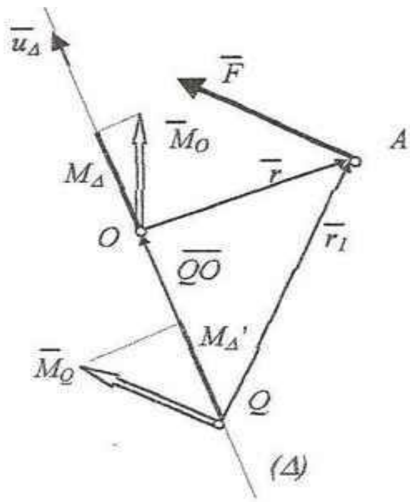


Fig.20.

For to show that we may take any point from the axis about which is calculated the moment and the moment about the axis does not modify. We shall consider a point Q, from the axis, about which we have:

$$M_\Delta' = \overline{M}_Q \cdot \overline{u}_\Delta = (\overline{r_1} \times \overline{F}) \cdot \overline{u}_\Delta$$

where we may remark that the position vector can be expressed as:

$$\overline{r_1} = \overline{r} + \overline{QO}$$

Replacing in the previous relation we obtain finally:

$$M_\Delta' = [(\overline{r} + \overline{QO}) \times \overline{F}] \cdot \overline{u}_\Delta = (\overline{r} \times \overline{F}) \cdot \overline{u}_\Delta + (\overline{QO} \times \overline{F}) \cdot \overline{u}_\Delta = \\ = M_\Delta$$

because the second term is equal to zero being o mixed product with two collinear vectors. Results that: about any point from that axis the moment about the axis is the same.

We shall show a method of calculation of the moment of a force about a given axis. To suppose the same force and axis but also a plane (P) perpendicular on the axis and that pass through the point of application of the force (point A). This plane intersect the axis in point O. It is obviously that the point O can be any point from the axis. We shal resolve the force $\overline{F}$ in two components: one component parallel with the given axis (marked $\overline{F}_\Delta$) and the second located in the plane (P) and marked $\overline{F}_P$. We may write:

$$\overline{F} = \overline{F}_\Delta + \overline{F}_P$$

Now we shall use the definition relation of the moment of a force about an axis but in which we replace this last relation. We obtain:

$$M_\Delta = \overline{M}_O \cdot \overline{u}_\Delta = (\overline{r} \times \overline{F}) \cdot \overline{u}_\Delta = [\overline{r} \times (\overline{F}_\Delta + \overline{F}_P)] \cdot \overline{u}_\Delta = \\ = (\overline{r} \times \overline{F}_\Delta) \cdot \overline{u}_\Delta + (\overline{r} \times \overline{F}_P) \cdot u_\Delta = \overline{M}_O(\overline{F}_P) \cdot \overline{u}_\Delta = \\ = \pm \left| \overline{M}_O(\overline{F}_P) \right|$$

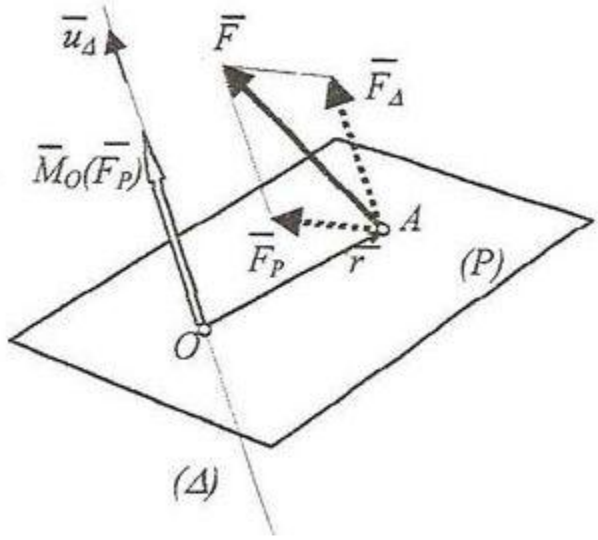


Fig. 21.

The last result shows as that the moment of a force about an axis has the magnitude equal to the magnitude of the moment of the force component from a perpendicular plane on that axis, moment calculated with respect to the intersection point between the plane and the axis. For to find the sign of this moment we shall use the same rules as for the moment of a force about a point but we shall consider the rotation sense around the axis. For example using the right hand rule we consider the palm of the right hand with the fingers in the sense of action of the force and with the palm looking towards the axis, the thumb shows as the sense of the moment about that axis. In the figure 22 the moment is negative because the thumb shows in opposite sense as the positive axis.

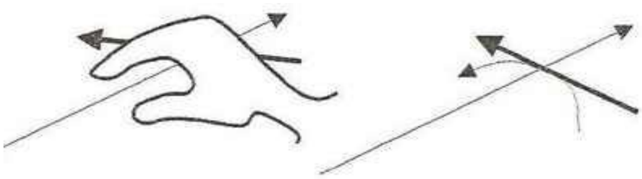


Fig.22.

Another way to determine the sign of the moment of a force about an axis is that to define the sense of rotation produced by the force around the axis, looking from the positive sense of the axis. If the sense of rotation is trigonometrically (counterclockwise sense), looking from the positive sense of the axis then the moment is positive.

Results also that if the force is parallel to the axis or intersects the axis then the moment of that force is equal to zero.

2.12. Sample problems

Problem 7. *For the system of forces from the figure 23. that acts on the sides of a parallelipipedium by known sides calculate the resultant moment with respect to the origin of the reference system. Are known:* $F_1 = 2F$, $F_2 = F$, $F_3 = 3F$, $F_4 = 2F$, $F_5 = 4F$.

Solution*. The system of forces is made from forces having the directions parallel with the directions of the axes. Being parallel with the axes the forces are located in the perpendicular planes on the other axes, so they are ready for to calculate the moments about the axes.*

The moment about the axis Ox is calculated eliminating first the forces without moments about this axis, namely the forces that intersect the axis and the forces parallel with Ox. These kind of forces are $\overline{F_2}$ *and* $\overline{F_5}$ *(forces intersecting the axis Ox) and also the forces* $\overline{F_3}$ *and* $\overline{F_4}$ *(forces parallel with Ox). In this way only the force* $\overline{F_1}$ *will have moment about the axis Ox. Results:*

$$M_x = -Y_1 \,.\, l_z = -2F \,.\, 4a = -8Fa$$

The sign (-) results from the right hand rule or using the following rule: we suppose that the side of the parallelipipedium laying on the axis Ox is fixed, so the parallelipipedium can rotate around this axis because the forces acting about it. If we look about the body (the parallelipipedium) from positive sense of the axis and the body rotates in clockwise sense then the moment is negative.

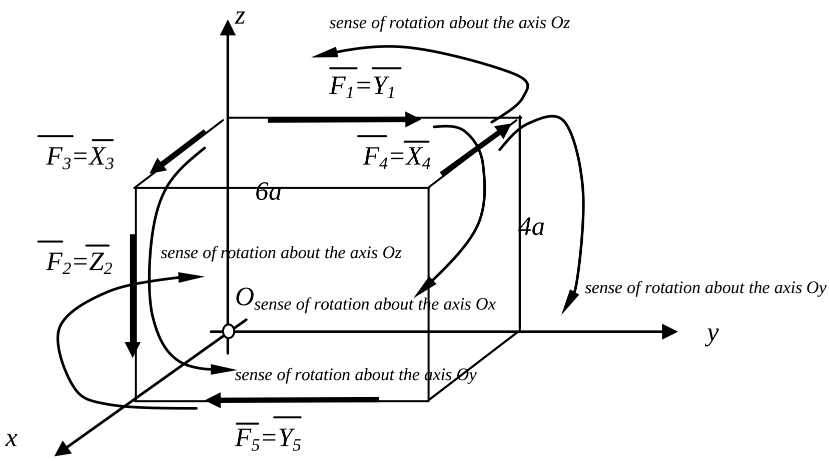


Fig.23.

The distance from the axis Ox to the support line of the force is perpendicular on the both directions (of the axis and of the support line of the force). If the body is a parallelipipedium, like in this problem, then the distance is the side perpendicular on the both directions, here the side l_z.

For the moment about the axis Oy we shall make in the same way, namely firs we shall eliminate the forces without moment about this axis (here the forces $\overline{F_1}$ *and* $\overline{F_5}$ *parallel with the axisOy) remaining the forces* $\overline{F_2}$ *,* $\overline{F_3}$ *and* $\overline{F_4}$ *having moments about the axis Oy. Is obtained:*

$$M_y = Z_2 \,.\, l_x + X_3 \,.\, l_z - X_4 \,.\, l_z = F \,.\, 3a + 3F \,.\, 4a - 2F \,.\, 3a = 9Fa$$

The forces $\overline{F_2}$ and $\overline{F_3}$ have positive moments because they produce counterclockwise rotations of the paralelipipedum about its side from the Oy axis if we look about the body from positive sense of the axis (from the right part). For to determine the distances, the simplest way is that to mark the forces with the name of the corresponding component and so that the perpendicular on $\overline{Z_2}$ and Oy is the side l_x, the perpendicular on $\overline{X_3}$ and Oy is the side l_z and, finally, on the $\overline{X_4}$ and Oy is the side l_x.

The calculation about the axis Oz starts with the elimination of the forces without moment about this axis: $\overline{F_1}$ and $\overline{F_3}$ intersect the axis and $\overline{F_2}$ is parallel with the axis. In this way only the forces $\overline{F_4}$ and $\overline{F_5}$ produce moment about the axis Oz. We shall have:

$$M_z = +X_4 \cdot l_y - Y_5 \cdot l_x = 2F \cdot 6a - 4F \cdot 3a = 0$$

Finally we have the resultant moment about the point O:

$$\overline{M_O} = -8Fa\,\overline{i} + 9Fa\,\overline{j}$$

Problem 8. *Is given the system of forces from the figure 24. acting about the paralelipipedium. Calculate the resultant moment of the system with respect to the point O. Are known: $F_1 = 9F$, $F_2 = 5F$, $F_3 = 10\sqrt{2}\,F$.*

Solution. *Because the forces have any directions we shall solve the problem in two steps: first we shall decompose the forces in components parallel with the reference axes and after we shall calculate the moment about the origin of the system of reference as in the previous problem.*

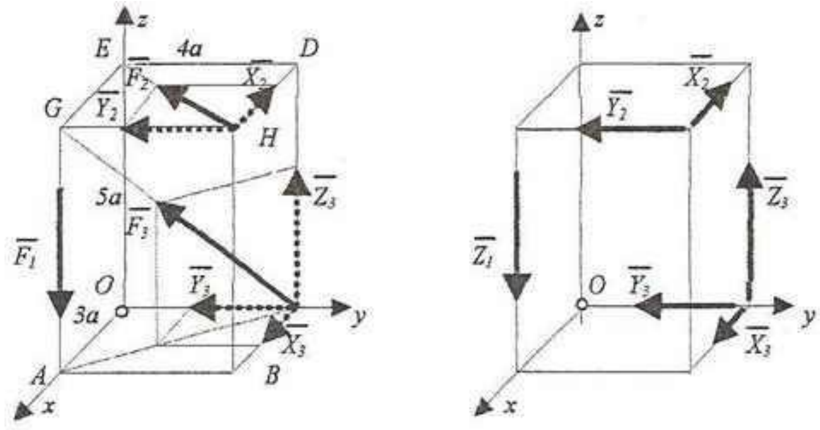


Fig.24.

Step 1. *We shall decompose the force $\overline{F_2}$ in two components, and the force $\overline{F_3}$ in three components having the magnitudes:*

$$X_2 = F_2 \cdot \frac{HD}{HE} = 5F \cdot \frac{3a}{5a} = 3F;\; Y_2 = F_2 \cdot \frac{HG}{HE} = 5F \cdot \frac{4a}{5a} = 4F;$$

$$X_3 = F_3 \cdot \frac{CB}{CG} = 10\sqrt{2}\,F \cdot \frac{3a}{5\sqrt{2}a} = 6F;\; Y_3 = F_3 \cdot \frac{CO}{CG} =$$

$$= 10\sqrt{2}\,F \cdot \frac{4a}{5\sqrt{2}a} = 8F;\; Z_3 = F_3 \cdot \frac{CD}{CG} = 10\sqrt{2}\,F \cdot \frac{5a}{5\sqrt{2}a} = 10F.$$

Step 2. *Fellows the calculation of the resultant moments about the three reference axes. Results:*

$$M_x = Y_2 \,.\, l_z + Z_3 \,.\, l_y = 4F \,.\, 5a + 10F \,.\, 4a = 60Fa;$$
$$M_y = Z_1 \,.\, l_x - X_2 \,.\, l_z = 9F \,.\, 3a - 3F \,.\, 5a = -12Fa;$$
$$M_z = -X_3 \,.\, l_y = -6F \,.\, 4a = -24Fa.$$

In the case of the moment about the axis Oz in place of the moments of the two components of the force $\overline{F_2}$ we have considered the moment of the entire force that intersect the axis with respect to which is calculated the moment. The same rule may be used for the force $\overline{F_3}$ with respect to the axis Oy.

Problem 9. *Is given a system of forces as in the figure 25. Calculate the resultant moment about the origin of the reference system. The magnitudes of the forces are given in parenthesis on the picture.*

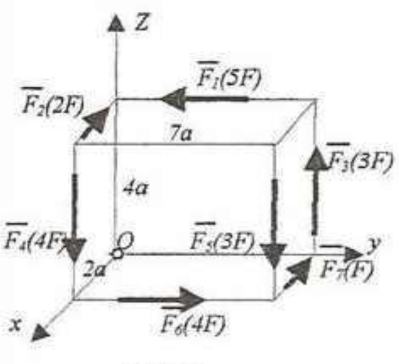


Fig.25.

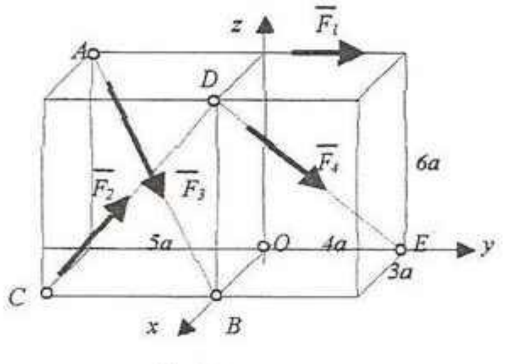


Fig.26.

Problem 10. *Calculate the resultant moment about the point O for the system of forces from the figure 26. Are known: $F_1 = 4F$, $F_2 = 2\sqrt{61}\,F$, $F_3 = 3\sqrt{70}\,F$, $F_4 = 3\sqrt{61}\,F$.*

2.13. Couple

*We shall define a particular system of forces. This system is called **couple** and by definition is the system made from two parallel forces with the same magnitudes and opposite senses.*

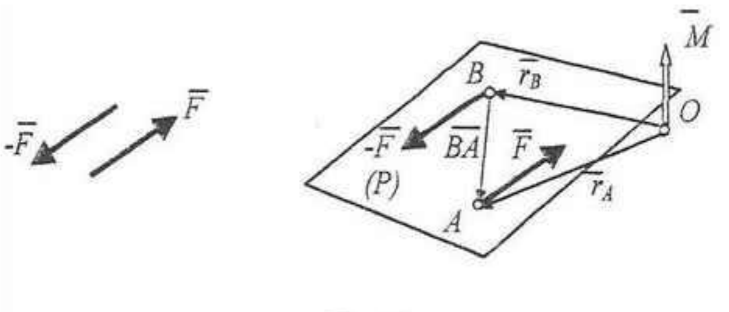


Fig.27.

This system has a particular behavior because has zero projections on the direction of any axis, namely the force effect of them is zero. So we can say that:

$\overline{R} = 0$

At the other hand if we calculate the resultant moment of this system with respect to any point O it is obtained:

$$\overline{M} = \overline{M}_O(\overline{F}) + \overline{M}_O(-\overline{F}) = \overline{r}_A \times \overline{F} + \overline{r}_B \times (-\overline{F}) =$$
$$= \overline{r}_A \times \overline{F} - \overline{r}_B \times \overline{F} = (\overline{r}_A - \overline{r}_B) \times \overline{F} = \overline{BA} \times \overline{F} = \overline{AB} \times (-\overline{F})$$

This relations says:

- *The resultant moment of a couple does not depend by the point about that is calculated. This propriety results from the fact that the final result does not refers about the point O. Consequently the moment of a couple is a **free vector**.*
- *The moment of the couple may be calculated as the moment of one force (from two) about a point from the other force.*

*The moment of the couple being a vector quantity is defined by the following characteristics: **the magnitude** is equal to the magnitude of the force multiplied with the distance between the support lines of the two parallel forces:*

$M = F \cdot d$

***the direction** of the moment of the couple is perpendicular on the plane containing the two forces; **the sense** is determined with the right hand rule or the right screw rule; **the point of application** can be any point from space because the moment of the couple is a free vector.*

2.14. Varignon's theorem

We shall state a theorem for the systems of concurrent forces, but later we will state also for the systems of any forces.

*This theorem says: **for a system of concurrent forces the resultant moment of the system, calculated about a point, is equal to the moment of the resultant force of the system about the same point:***

$$\Sigma \overline{M}_O(\overline{F}_i) = \overline{M}_O(\overline{R})$$

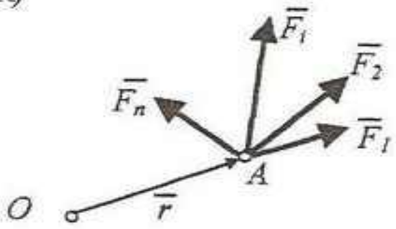


Fig.28.

For to prove we have:

$$\overline{M_O}(\overline{F_i}) = \Sigma\, \overline{r} \times \overline{F_i} = \overline{r} \times \Sigma \overline{F_i} = \overline{r} \times \overline{R} = \overline{M_O}(\overline{R})$$

2.15. Reduction of a force in a given point

When a force acts about a body it produces a mechanical effect (usual it produces a motion that generally is a combination of a translation and a rotation). The determination of the mechanical effect of a force in a given point of the body about that the force is acting is called ***reduction of the force in that point.***

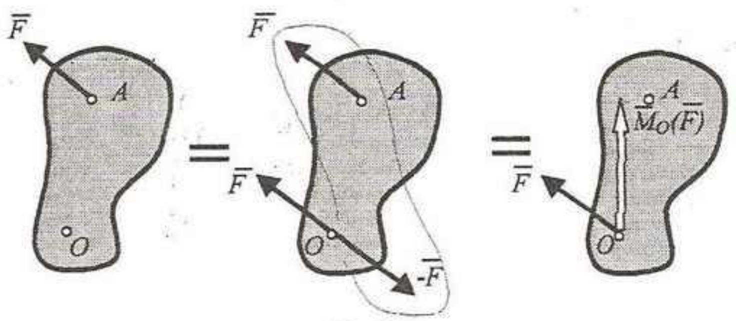


Fig.29.

Suppose that in the point A of a body acts a force $\overline{F}$ and we want to determine its mechanical effect in the point O. For this we shall make an artifice but without to modify the mechanical effect of the force about the body. In point O we shall introduce a system of forces equivalent with zero (without effect about the body) made from two forces having the same magnitudes and opposite senses. This system does not modify the mechanical state of the body. Now if we consider together the force $\overline{F}$ from the point A and the force $-\overline{F}$ from the point O, then these two forces make a couple, that being a free vector may be *considered acting in point O. The moment of this couple is, as we have seen, equal to the moment of one force, from the two, about a point from the support line of the other force, here we shall consider the point O. In this way we may consider that we have transformed the body acted by the force $\overline{F}$ from A in the body acted by the force $\overline{F}$ acting in O and a moment equal to the moment of the force $\overline{F}$ from A about the point O, without to modify the mechanical state of the body.*

The ensemble made from the two vectors: $\overline{F}$ in O and the moment of the force about the point O is called ***force-couple system*** *of the force $\overline{F}$ in point O:*

$$\tau_O(\overline{F}) = \begin{cases} \overline{F} \\ \overline{M}_O(\overline{F}) \end{cases}$$

We can state that the force $\overline{F}$ and its force-couple system in a point are equivalent (they produce the same effect about the body).

2.16. Reduction of a system of forces in a given point

To consider now a system of any forces acting about a rigid body. We shall determine the mechanical effect of this system in point O. For this we shall replace each force with the corresponding force-couple system in point O.

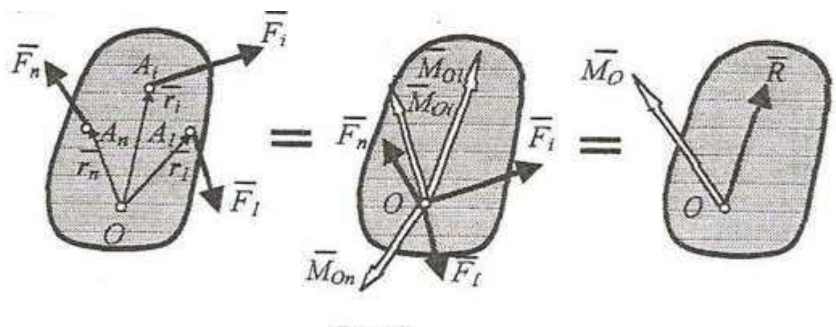


Fig.30.

In this way we have transformed, without to modify the mechanical state of the body, the given system of forces in two systems of *concurrent vectors: the first is the system of the given forces bat acting all in point O and the second the system of the moments of the given forces calculated about the point O. The simplest equivalent systems of these two systems of vectors are two vectors: the resultant force of the forces from the given system of forces (considered acting all in the same point) and the resultant moment of the forces from the given system of forces calculated about the point O. The ensemble of these two vectors form the* **force-couple system** *in point O of the given system of forces and it has the expression:*

$$\tau_O(\overline{F}_i) = \begin{cases} \overline{R} = \Sigma \overline{F}_i \\ \overline{M}_O = \Sigma \overline{M}_O(\overline{F}_i) \end{cases}$$

We may state that: a system of forces is equivalent to its force-couple system in a point, and also that two systems of forces with the same force-couple system in the same point are equivalent systems of forces.

If we change the point of reduction, (for example the point A) and performing the same steps is obtained the force-couple system in point A:

$$\tau_O(\overline{F}_i) = \begin{cases} \overline{R} = \Sigma \overline{F}_i \\ \overline{M}_O = \Sigma \overline{M}_O(\overline{F}_i) = \Sigma [\overline{M}_O(\overline{F}_i) + \overline{AO} \times \overline{F}_i] = \\ \quad = \overline{M}_O + \overline{AO} \times \overline{R} \end{cases}$$

2.17. Sample problems

Problem 11. *About the paralelipipedium from the figure 31. acts a system of forces as in the picture. Knowing the magnitudes of the forces: $F_1 = 3F$, $F_2 = F$, $F_3 = 5F$, $F_4 = 3F$, $F_5 = 4F$, $F_6 = 2F$, calculate and represent the force-couple system in point O and after in point A.*

Solution. *Because the forces are collinear with the sides of the paralelipipedium this problem will be solved in two steps: in the first we determine the resultant force (like the system of forces is a system of concurrent forces) and in the second we shall determine the resultant moment in the given point.*

We begin with the force-couple system in point O.

The projections of the resultant force on the three reference axes are:

$$X = \Sigma X_i = F_2 - F_5 = -3F;$$
$$Y = \Sigma Y_i = F_1 + F_4 - F_6 = 4F;$$
$$Z = \Sigma Z_i = F_3 = 5F;$$

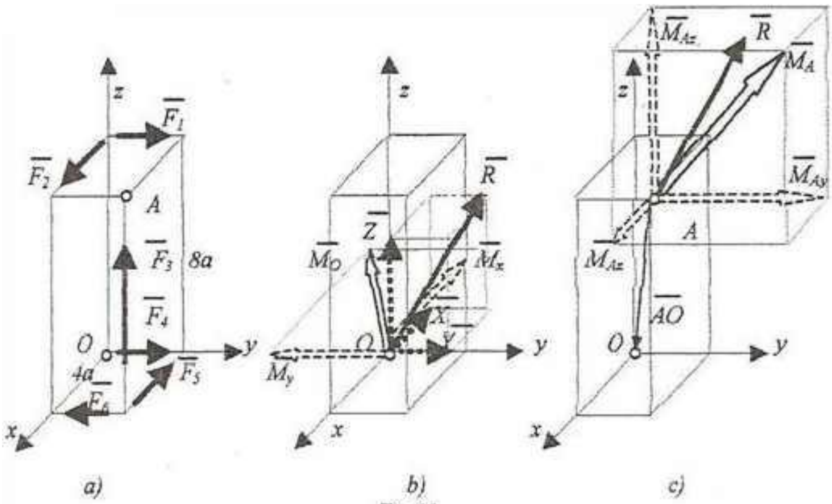


Fig.31.

and the resultant force is (with its magnitude):

$$\overline{R} = -3F\,\overline{i} + 4F\,\overline{j} + 5F\,\overline{k};\quad R = F\sqrt{3^2 + 4^2 + 5^2} = 7{,}07F.$$

Now we shall calculate the moments of the forces about the three axes:

$$M_x = \Sigma M_{xi} = -F_1 \cdot 8a + F_3 \cdot 2a = -14Fa;$$
$$M_y = \Sigma M_{yi} = F_2 \cdot 8a - F_3 \cdot 4a = -12Fa;$$
$$M_z = \Sigma M_{zi} = F_5 \cdot 2a - F_6 \cdot 4a = 0$$

Results the moment of the system about the point O:

$$\overline{M_O} = -14Fa\,\overline{i} - 12Fa\,\overline{j}$$

The representation of this force- couple system is made in the figure 31.b.

For to calculate the force-couple system in point A the resultant force is the same and the resultant moment may be computed with the following formula:

$$\overline{M_A} = \overline{M_O} + \overline{AO} \times \overline{R}$$

Where the position vector $\overline{AO}$ is, with opposite sign, the position vector of the point A with respect to the origin of the reference system O, namely the projections of this vector on the axes are the coordinates of the point A with respect to the reference system taken with opposite signs. The calculation of the vector product is made with the determinant:

$$\overline{M_A} = -14Fa\,\overline{i} - 12Fa\,\overline{j} + \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ -4a & -2a & -8a \\ -3F & 4F & 5F \end{vmatrix} = $$
$$= 6Fa\,\overline{i} + 32Fa\,\overline{j} + 22Fa\,\overline{k}$$

The representation of this force-couple system is made in the figure 31.c.

Problem 12. *For the given system of forces from the figure 32. acting about the paralelipipedium, calculate and represent the force-couple system in point O and in point A. Are known:* $F_1 = 4F$, $F_2 = 3F$, $F_3 = 5F$, $F_4 = F$.

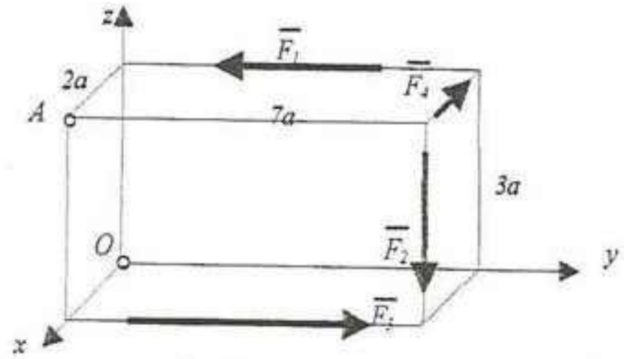


Fig.32.

2.18. Invariants of the systems of forces. Minimum moment, central axis

We have seen in the previously section that if we change the point of reduction the resultant force of the system of forces remains unchangeable. We may say that the resultant force of a system of forces is the ***vector invariant*** *of that system, or the first invariant.*

The second term of the force-couple system, the resultant moment, is changeable, modifying if is modified the point of reduction.

If the relation of variation of the moment when we change the point of reduction is multiplied (scalar product) with the vector invariant (namely the resultant force) is obtained:

$$\overline{M_A} = \overline{M_O} + \overline{AO} \times \overline{R} \,\Big|\, \cdot \overline{R}$$

or:

$$\overline{M_A} \cdot \overline{R} = \overline{M_O} \cdot \overline{R} + (\overline{AO} \times \overline{R}) \cdot \overline{R}$$

But because the last term is a mixed product with two collinear vectors it will be equal to zero and finally we shall have:

$$\overline{M_A} \cdot \overline{R} = \overline{M_O} \cdot \overline{R}$$

namely the scalar product of the two vectors of the force – couple system is also an invariant of the systems of forces. This invariant is called ***the scalar invariant*** *or the second invariant of the systems of forces.*

Suppose now the force – couple system of a system of forces in any point (Fig.33.). Because the resultant force is an invariant we shall decompose the resultant moment (that is not an invariant) in two components: one on the direction of the resultant force and the second on perpendicular direction to the direction of the resultant force.

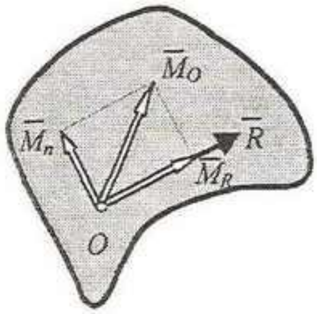


Fig.33.

We shall mark these two components : $\overline{M_R}$ *and* $\overline{M_n}$ *and we have:*

$$\overline{M_O} = \overline{M_R} + \overline{M_n}$$

The magnitude of the component on the direction of the resultant force is determined as the projection of the moment on that direction:

$$M_R = pr_R \overline{M}_O = \overline{M}_O \cdot \overline{u}_R = \overline{M}_O \cdot \frac{\overline{R}}{R} = \frac{\overline{M}_O \cdot \overline{R}}{R}$$

namely this component has the magnitude equal to the rate of the two invariants, consequently it is also an invariant. We remark from this that from the two components of the resultant moment only the normal component changes when we change the point of reduction. Corresponding to the minimum value of this component we shall find the minimum moment, and this is obtained when the normal component is zero as value. But in this case the resultant moment is in fact the component having the direction of the resultant force, so we may write:

$$\overline{M}_{min} = \overline{M}_R$$

Now remains to determine the points from space in which is obtained this minimum moment. For this to suppose that we have the components of the force-couple system in the origin of the reference system:

$$\overline{R} = X\overline{i} + Y\overline{j} + Z\overline{k}$$
$$\overline{M}_O = M_x\overline{i} + M_y\overline{j} + M_z\overline{k}$$

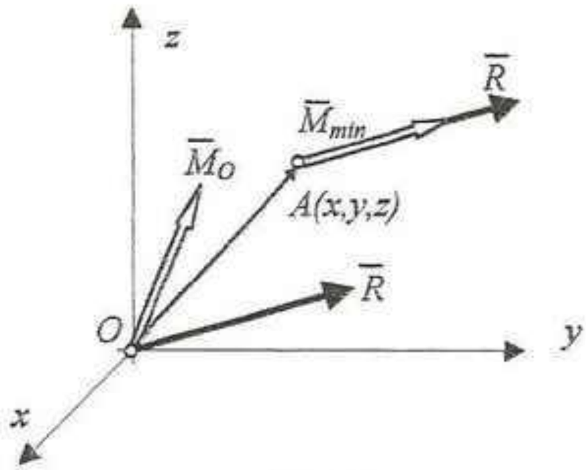


Fig.34.

Suppose that in point A(x,y,z) the resultant moment is minimum, namely it is collinear with the resultant force. In the same time the resultant moment from this point may be expressed function of the components of the force –couple system in point O:

$$\overline{M}_A = \overline{M}_O + \overline{AO} \times \overline{R}$$

Replacing in this relation and taking into account that the vector $\overline{AO}$ is the position vector $\overline{OA}$ with opposite sign we shall obtain:

$$\overline{M}_A = M_x\overline{i} + M_y\overline{j} + M_z\overline{k} + \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ -x & -y & -z \\ X & Y & Z \end{vmatrix} =$$
$$= (M_x - yZ + zY)\,\overline{i} + (M_y - zX + xZ)\,\overline{j} + (M_z - xY + yX)\,\overline{k}$$

The colinearity condition between the resultant force and the resultant moment may be expressed as:

$$\overline{M_A} = \lambda \cdot \overline{R}$$

or in scalar form:

$$\frac{M_{Ax}}{X} = \frac{M_{Ay}}{Y} = \frac{M_{Az}}{Z}$$

Replacing, finally we have:

$$\frac{M_x - yZ + zY}{X} = \frac{M_y - zX + xZ}{Y} = \frac{M_z - xY + yX}{Z}$$

that represent the equation of a straight line. This line is called ***central axis*** *that has the definition: it is the locus of the points in which the resultant moment is co linear with the resultant force or the resultant moment is minimum.*

2.19. Cases of reduction

Function of the particular values of the two components of the force-couple system, in an any point, the systems of forces may be classified in four cases of reduction:

- *1)* $\overline{R} = 0$; $\overline{M_O} = 0$ *. In this case the system of forces equivalent with zero and consequently it has not any effect about the body on which is acting. We say that the system of forces is in* ***equilibrium*** *or it is a* ***balanced*** *system of forces. The two previous relations may be considered as the two vector conditions of equilibrium. These conditions are:*

$$\Sigma \overline{F_i} = 0 \; ; \; \Sigma \overline{M}_{Oi} = 0$$

and in scalar form these conditions are:

$$\Sigma X_i = 0; \; \Sigma Y_i = 0; \; \Sigma Z_i = 0; \; \Sigma M_{xi} = 0; \; \Sigma M_{yi} = 0; \; \Sigma M_{zi} = 0$$

- *2)* $\overline{R} = 0$ *;* $\overline{M_O} \neq 0$. *If we calculate the moment in another point (for example in point A) results:*

$$\overline{M_A} = \overline{M_O}$$

namely anywhere we compute the resultant moment we shall obtain the same force-couple system. This behavior corresponds to a couple. We shall say that the system of forces is reduced to a ***couple****.*

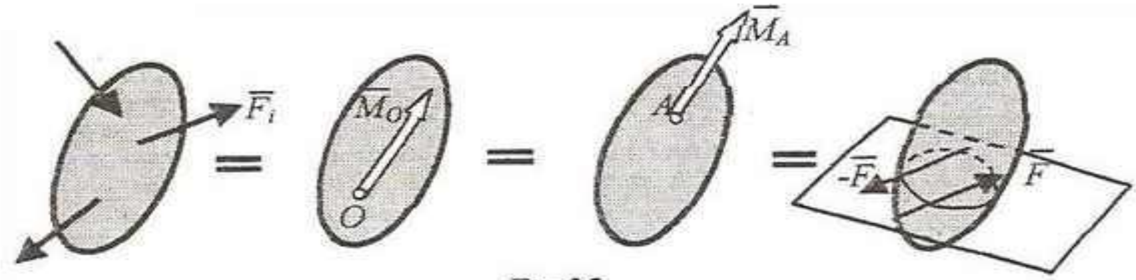


Fig.35.

- *3)* $\overline{R} \neq 0$ *;* $\overline{M_O} = 0$. *The force-couple system in point O has a single component namely the resultant force. In the same time the force couple system is an equivalent system for the given system of forces, so the resultant force is the simplest equivalent system. We shall say that the system of forces is reduced to* ***a unique resultant force that pass through the point O****.*

In this case we have obtained the simplest equivalent system and the calculation of the problem is finished with this result.

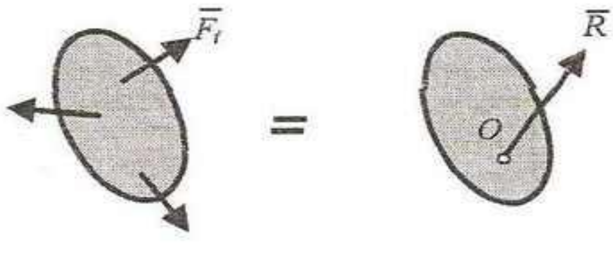


Fig.36.

- *4)* $\overline{R} \neq 0$ *;* $\overline{M_O} \neq 0$. *In this case we shall calculate also the scalar invariant of the system (in the other cases is not necessary to compute this invariant because it is equal to zero) and we obtain two sub cases:*

a) $\overline{R} \cdot \overline{M_O} = 0$, *namely the two vectors of the force-couple system are perpendicular.*

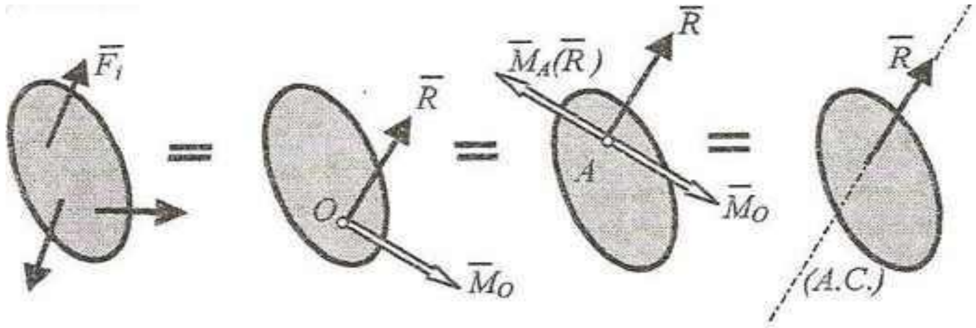


Fig.37.

Obviously we should perform the reduction in another point, for example in point A. If the reduction begin from the force-couple system in point O, because this system is equivalent to the initial system of forces then we shall make in the next way: the resultant moment $\overline{M_O}$ *may be considered as the moment of a couple, then it can be placed in point A without to modify the mechanical state of the body, and the force* $\overline{R}$*, from the point O is reduced in point A finding a force couple system made from the force* $\overline{R}$*, the resultant force of the initial system, and the moment of the force* $\overline{R}$*, from the point O, about the point A,* $\overline{M_A}(\overline{R})$ *moment that is also perpendicular on the resultant force* $\overline{R}$*. But the point A may be chosen so that this moment,* $\overline{M_A}(\overline{R})$ *to have the same magnitude as the resultant moment* $\overline{M_O}$ *but with opposite sense. In this case the two co linear moments gives zero resultant in point A and the force couple system in this point will be made from the resultant force only. We shall say that in this the system is reduced to* ***a unique resultant force that does not pass through the point O****. In this case it is obviously that we have to determine the position of this unique resultant force, representing the simplest equivalent system. Because in point A the resultant moment is zero this value is the minimum value of the resultant moment, so the unique resultant force is located on the central axis.*

b) $\overline{R} \cdot \overline{M_O} \neq 0$ *namely the two vectors of the force couple system are not perpendicular. Consequently the resultant moment has component on the direction of the resultant force, component that is the minimum moment. Together with the resultant force they make the* ***minimum force couple system*** *located on the central axis and called also* ***wrench****.*

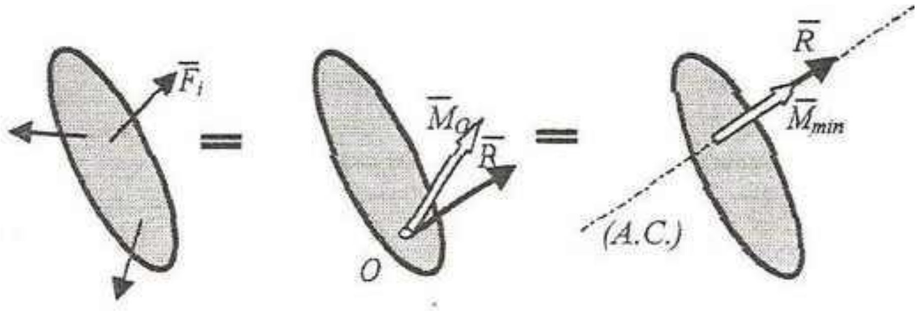


Fig.38.

2.20. Sample problems

Problem 13. *The system of forces from the figure 39. acts about the cube by the side* ***l****. Knowing the magnitudes of the forces: $F_1 = F_2 = F$, $F_3 = F_4 = F\sqrt{2}$ calculate and represent the minimum force-couple system.*

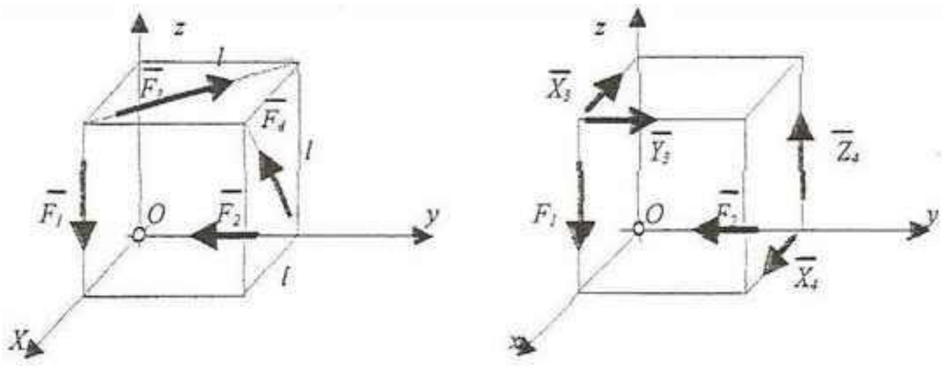


Fig.39.

Solution. *in the first step we shall decompose the forces of the system in components parallel with the reference axes. In this way the forces $\overline{F_1}$ and $\overline{F_2}$ are own components, the force $\overline{F_3}$ has two components: $\overline{X_3}$ and $\overline{Y_3}$ and the force $\overline{F_4}$ will have also two components $\overline{X_4}$ and $\overline{Z_4}$. These components will have the following magnitudes:*

$$X_3 = F_3 \cdot \frac{l}{l\sqrt{2}} = F\,;\; Y_3 = F_3 \cdot \frac{l}{l\sqrt{2}} = F\,;\; X_4 = F_4 \cdot \frac{l}{l\sqrt{2}} = F = Z_4$$

In the second step we shall calculate the force-couple system in point O. The projections of the resultant force will be:

$$X = \Sigma X_i = -X_3 + X_4 = 0;$$
$$Y = \Sigma Y_i = -F_2 + Y_3 = 0;$$
$$Z = \Sigma Z_i = -F_1 + Z_4 = 0.$$

namely the resultant force is equal to zero:

$$\overline{R} = 0$$

The resultant moments about the reference axes are:

$$M_x = \Sigma M_{xi} = -Y_3 \,.\, l + Z_4 \,.\, l = 0;$$
$$M_y = \Sigma M_{yi} = F_1 \,.\, l - X_3 \,.\, l = 0;$$
$$M_z = \Sigma M_{zi} = Y_3 \,.\, l - X_4 \,.\, l = 0.$$

namely also the resultant moment is equal to zero :

$$\overline{M_O} = 0$$

The two vectors of the force-couple system being zero the system of forces is in equilibrium.

Problem 14. *Calculate and represent the force-couple system of the system of forces from the figure 40. Are known:* $F_1 = F_2 = 2F$, $F_3 = 2\sqrt{2}\, F$.

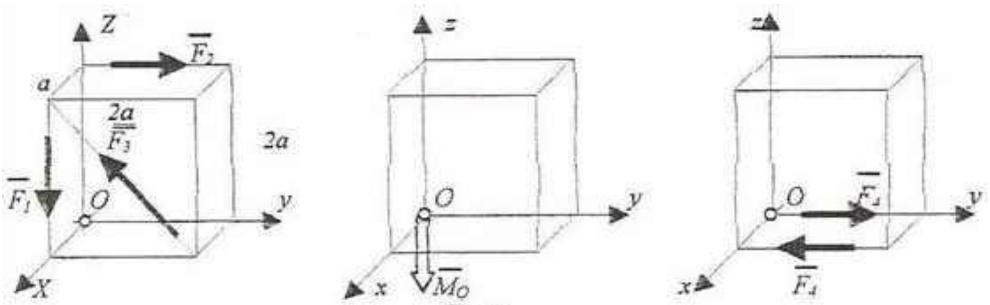


Fig.40.

Solution. *As in the previous problem first we shall decompose the forces in components parallel with the reference axes, in this way the force* F_3 *will have two components:*

$$Y_3 = F_3 \frac{2a}{2a\sqrt{2}} = 2F\,;\; Z_3 = F_3 \frac{2a}{2a\sqrt{2}} = 2F\,;$$

We shall calculate the force-couple system in the point O and we find:

$$X = \Sigma X_i = 0\,;\; Y = \Sigma Y_i = F_2 - Y_3 = 0\,;\; Z = \Sigma Z_i = -F_1 + Z_3 = 0\,;$$

so the resultant force of the system is:

$$\overline{R} = 0$$

The resultant moments about the three axes are:

$$M_x = \Sigma M_{xi} = -F_2 \,.\, 2a + Z_3 \,.\, 2a = 0;\; M_y = \Sigma M_{yi} = F_1 \,.\, a - Z_3 \,.\, a = 0\,;$$
$$M_z = \Sigma M_{zi} = -Y_3 \,.\, a = -2Fa$$

namely the resultant moment is:

$$\overline{M_O} = -2Fa\,\overline{k}$$

These results shows that the system is reduced to a couple having the moment $\overline{M_O}$, couple that may be represented as a vector moment or as a system of two equal and parallel forces with opposite senses located in a perpendicular plane on the vector moment $\overline{M_O}$ and having the magnitudes so that to results the resultant moment of the system. This couple is the simplest equivalent system of the given system of forces.

Problem 15. *Calculate and represent the minimum force-couple system of the following system of forces acting as in the figure 41. Are known: $F_1 = 6F$, $F_2 = 2F$, $F_3 = 3F$, $F_4 = 4F$.*

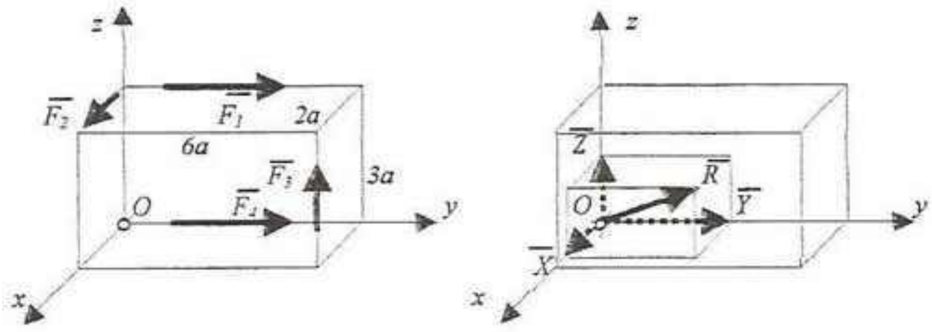


Fig.41.

Solution. *Because the forces of the system are parallel to the reference axes we shall calculate the force-couple system in point O. Are obtained the projections of the resultant force:*

$$X = \Sigma X_i = F_2 = 2F;\ Y = \Sigma Y_i = F_1 + F_4 = 10F;\ Z = \Sigma Z_i = F_3 = 3F$$

namely the resultant force is:

$$\overline{R} = 2F\,\overline{i} + 10F\,\overline{j} + 3F\,\overline{k}$$

with the magnitude:

$$R = \sqrt{X^2 + Y^2 + Z^2} = 10{,}63F$$

For the resultant moment we shall calculate the moments about the three reference axes:

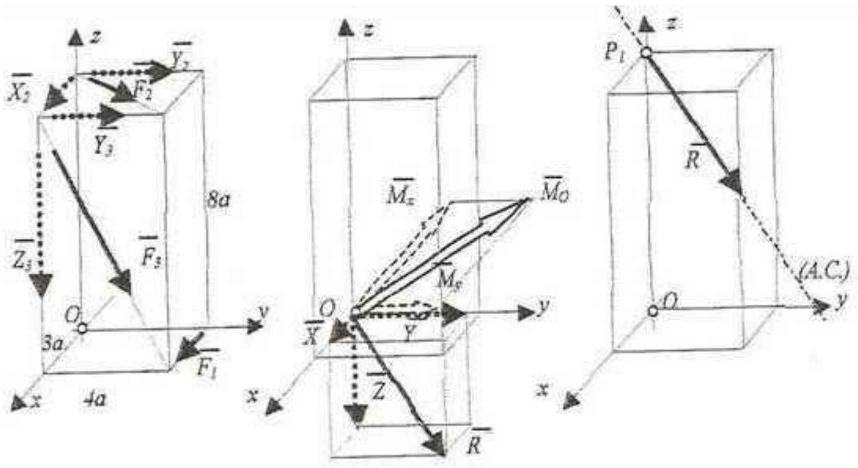
$$M_x = \Sigma M_{xi} = -F_1 \cdot 3a + F_3 \cdot 6a = 0\ ;$$
$$M_y = \Sigma M_{yi} = F_2 \cdot 3a - F_3 \cdot 2a = 0\ ;$$
$$M_z = \Sigma M_{zi} = 0\ ;$$

so the resultant moment of the system is:

$$\overline{M_O} = 0$$

In this case the system of forces is reduced to a unique resultant force that pass through the reduction point O. This resultant force in the point O is the simplest equivalent system of the given system of forces.

Problem 16. *perform the reduction in point O for the system of forces represented in the figure 42.a. and represent the minimum force-couple system. Are given: $F_1 = 3F$, $F_2 = 5F$, $F_3 = 4\sqrt{5}\,F$.*



a) *b)* *c)*

Fig.42.

Solution. *First we shall decompose the forces $\overline{F_2}$ and $\overline{F_3}$ in two components having the magnitudes:*

$$X_2 = F_2 \frac{3a}{5a} = 3F;\ Y_2 = F_2 \frac{4a}{5a} = 4F;$$
$$Y_3 = F_3 \frac{4a}{4a\sqrt{5}} = 4F;\ Z_3 = F_3 \frac{8a}{4a\sqrt{5}} = 8F;$$

and also having the directions and the senses from the figure 42.a.

The projections of the resultant force are:

$$X = \Sigma X_i = F_1 + X_2 = 6F;\ Y = \Sigma Y_i = Y_2 + Y_3 = 8F;\ Z = \Sigma Z_i = -Z_3 = -8F;$$

namely the resultant force is :

$$\overline{R} = 6F\,\overline{i} + 8F\,\overline{j} - 8F\,\overline{k}$$

with the magnitude:

$$R = 12{,}8F$$

The moments about the three reference axes are:

$$M_x = \Sigma M_{xi} = -Y_2 \cdot 8a - Y_3 \cdot 8a = -64Fa;$$
$$M_y = \Sigma M_{yi} = X_2 \cdot 8a + Z_3 \cdot 3a = 48Fa;$$
$$M_z = \Sigma M_{zi} = -F_1 \cdot 4a + Y_3 \cdot 3a = 0;$$

namely the moment about the point O is:

$$\overline{M_O} = -64Fa\,\overline{i} + 48Fa\,\overline{j}$$

The resultant force and the resultant moment of the system are different to zero so this is the fourth case of reduction. We have to calculate in this case the scalar product:

$$\overline{R}\cdot\overline{M_O} = XM_x + YM_y + ZM_z = 6F\cdot(-64Fa) + 8F\cdot 48Fa + (-8F)\cdot 0 = 0$$

This result shows that the system of forces reduces to a unique resultant force that does not pass through the point O. Consequently we have to determine the position of the support line of this unique resultant force. The equation of the support line is obtained with the equation of the central axis:

$$\frac{M_x - yZ + zY}{X} = \frac{M_y - zX + xZ}{Y} = \frac{M_z - xY + yX}{Z}$$

where replacing we find:

$$\frac{-64Fa - y(-8F) + z8F}{6F} = \frac{48Fa - z6F + x(-8F)}{8F} = \frac{-x8F + y6F}{-8F}$$

Considering the rates two by two is obtained the equations of the support line of the unique resultant force:

$$\begin{cases} -800a + 48x + 64y + 100z = 0 \\ 48a - 16y - 6z = 0 \end{cases}$$

For to represent this straight line (that is parallel with the resultant force from the point O because the resultant force is the invariant of the system of forces) it is enough to represent one single point of them. Generally this point is the intersection point of the central axis with one reference plane, for example the plane yOz. For this we shall make zero the term x in the two equations and is obtained one system of two equations with two unknowns: y and z, that solved give us the coordinates of the point in the reference plane yOz:

$$x_1 = 0; \qquad \begin{cases} -800a + 64y + 100z = 0 \\ 48a + 6y - 6z = 0 \end{cases} \longrightarrow y_1 = 0;\ z_1 = 8a.$$

Problem 17. *Calculate and represent the minimum force-couple system of the given system of forces represented in the figure 43. Are known the magnitudes of the forces: $F_1 = 2F$, $F_2 = \sqrt{13}\ F$, $F_3 = 2\sqrt{14}\ F$.*

Solution. *We shall decompose the forces in components parallel with the reference axes and we obtain:*

$$Y_2 = F_2\frac{2a}{a\sqrt{13}} = 2F;\ Z_2 = F_2\frac{3a}{a\sqrt{13}} = 3F;$$
$$X_3 = F_3\frac{a}{a\sqrt{14}} = 2F;\ Y_3 = F_3\frac{2a}{a\sqrt{14}} = 4F;\ Z_3 = F_3\frac{3a}{a\sqrt{14}} = 6F;$$

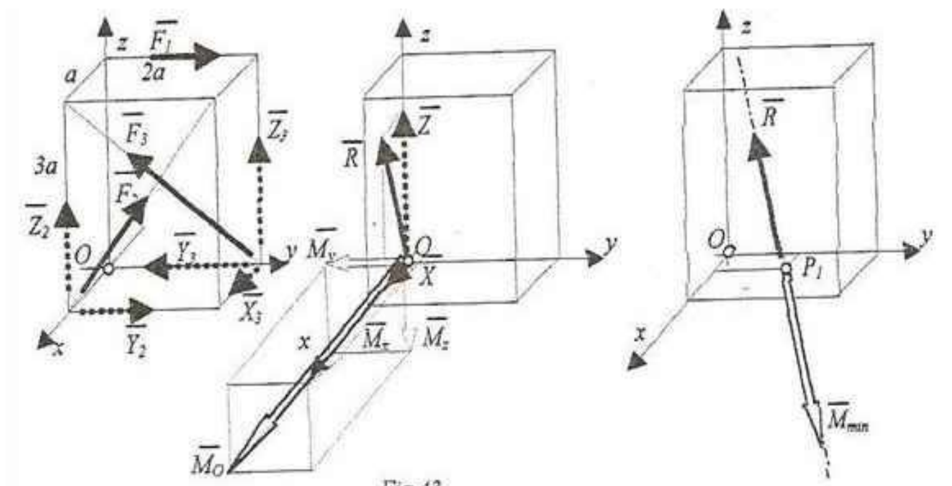


Fig.43.

With these components we shall calculate the projections of the resultant force on the three axes:

$$X = \Sigma X_i = X_3 = 2F\ ;\ Y = \Sigma Y_i = F_1 + Y_2 - Y_3 = 0\ ;\ Z = \Sigma Z_i = Z_2 + Z_3 = 9F$$

The resultant force is:

$\overline{R} = 2F\,\overline{i} + 9F\,\overline{k}$

having the magnitude:

$R = 9{,}22F.$

The resultant moments about the reference axes are:

$$M_x = \Sigma M_{xi} = -F_1 \,.\, 3a + Z_3 \,.\, 2a = 6Fa\ ;$$
$$M_y = \Sigma M_{yi} = -Z_2 \,.\, a\ = -3Fa\ ;$$
$$M_z = \Sigma M_{zi} = Y_2 \,.\, a - X_3 \,.\, 2a = -3Fa\ ;$$

and the resultant moment in point O will be:

$\overline{M_O} = 6Fa\,\overline{i} - 3Fa\,\overline{j} - 3Fa\,\overline{k}$

Also this is the fourth case of reduction because the resultant force and moment are different to zero, that is we have to calculate the scalar product of the two vectors:

$$\overline{R}\,.\,\overline{M_O} = XM_x + YM_y + ZM_z = 2F\,.\,6Fa + 9F\,.\,(-3Fa) = -15F^2a$$

This scalar product being different to zero the system of forces is reduced to a minimum force-couple system made from the resultant force of the system and the minimum moment collinear with the resultant force. The minimum moment has the magnitude:

$$M_{min} = \frac{\overline{R}.\overline{M}_O}{R} = \frac{-15F^2a}{9{,}22F} = -1{,}62Fa$$

Together with the resultant force this minimum moment forms the wrench located on the central axis having the equation:

$$\frac{M_x - yZ + zY}{x} = \frac{M_y - zX + xZ}{Y} = \frac{M_z - xY + yX}{Z}$$

that becomes:

$$\frac{6Fa - y9F + z0}{2F} = \frac{-3Fa - z2F + x9F}{0} = \frac{-3Fa - x0 + y2F}{9F}$$

The first equation is find making zero the denominator of the second rate, and the second equation results from the equality of the first rate with the third:

$$\begin{cases} -3a - 2z + 9x = 0 \\ 60a - 85y = 0 \end{cases}$$

For to represent the force-couple system it is enough to determine one point from the central axis, and this point may be the intersection point with one reference plane, for example the xOy plane:

$z_1 = 0 \longrightarrow x_1 = 0{,}33a;\ y_1 = 0{,}7a.$

Because the minimum moment has resulted with minus sign it will be with opposite sense as the resultant force.

Problem 18. *Calculate and represent the minimum force-couple system of the system of forces from the figure 44. The magnitudes of the forces are:* $F_1 = 7F$, $F_2 = 3\sqrt{13}F$, $F_3 = 10F$, $F_4 = \sqrt{61}F$.

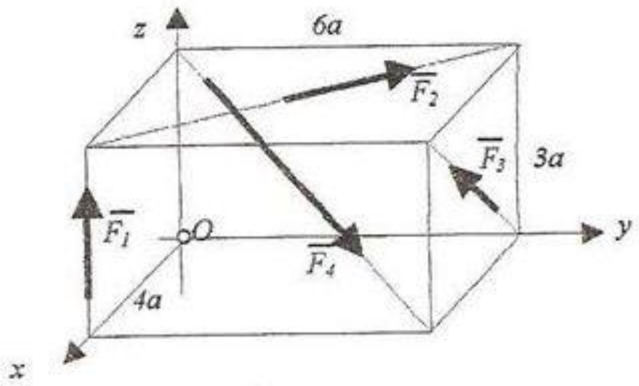


Fig.44.

2.21. Varignon's theorem

We shall give again the Varignon's theorem, enounced for a system of concurrent forces, for an arbitrary system of forces. For these kind of *systems of forces the theorem will have the next statement:* ***for an arbitrary system of forces that has as simplest equivalent system an unique resultant force, the resultant moment of the system about an any point is equal to the moment of the unique resultant force about the same point****.*

For to prove we shall suppose that the system of forces may be reduced to an unique resultant force and this resultant force passes through the point A. The resultant moment of the system about this point may be computed using the resultant moment about an other point, for example the origin of the reference system O:

$$\overline{M}_A = \overline{M}_O + \overline{AO} \times \overline{R}$$

But because the unique resultant force of the system is passing through the point A, the moment of this force about the point A is zero:

$$\overline{M}_A = 0 \longrightarrow \overline{M}_O + \overline{AO} \times \overline{R} = 0$$

But the first term in this relation is the resultant moment in point O and the second if it is passed in the right part then it is the moment of the resultant force about the point O:

$$\Sigma\overline{M}_{Oi} = \overline{OA} \times \overline{R}$$

2.22. Systems of coplanar forces

We shall analyze same particular systems of forces. One of this kind of system is the system of coplanar forces, that is defined as ***the system of forces made from forces lying in the same plane.***

We shall suppose that the plane in which is located the system is the xOy plane.

An any force from the system has the expression in this reference system:

$$\overline{F}_i = X_i\overline{i} + Y_i\overline{j}$$

and the moment of this force about the origin of the reference system will be:

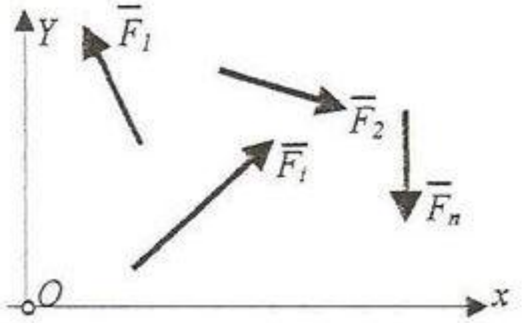


Fig.45.

$$\overline{M}_O(\overline{F}_i) = M_{Oi}\,\overline{k}$$

With these the force-couple system in point O has the components:

$$\overline{R} = \Sigma\overline{F}_i = X\overline{i} + Y\overline{j} = \Sigma X_i\,\overline{i} + \Sigma Y_i\,\overline{j}$$
$$\overline{M}_O = \Sigma\overline{M}_O(\overline{F}_i) = (\Sigma M_{Oi})\,\overline{k}$$

We can see that the two vectors are perpendicular and the scalar product of them, the second invariant of the systems of forces, is equal to zero:

$$\overline{R}\cdot\overline{M}_O = 0$$

The cases of reduction of these kinds of systems of forces are:

- *1) $\overline{R} = 0$; $\overline{M}_O = 0$. The system of forces is in* ***equilibrium****. The two vector equations will give us three scalar conditions of equilibrium:*

$$\Sigma X_i = 0\ ;\ \Sigma Y_i = 0\ ;\ \Sigma M_{Oi} = 0\ ;$$

- 2) $\overline{R} = 0$; $\overline{M_O} \neq 0$. *The system of forces reduces to a* ***couple.*** *This couple may be represented or as a system made from two parallel and opposite forces having the same magnitude, or as the moment of the couple defined by the sense and magnitude.*

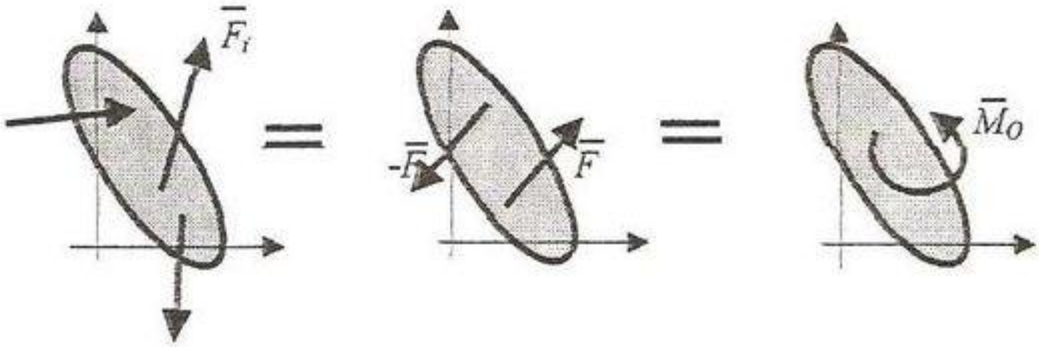


Fig.46.

- 3) $\overline{R} \neq 0$; $\overline{M_O} = 0$. *In this case the simplest equivalent system is* ***the unique resultant force passing through the point O.***

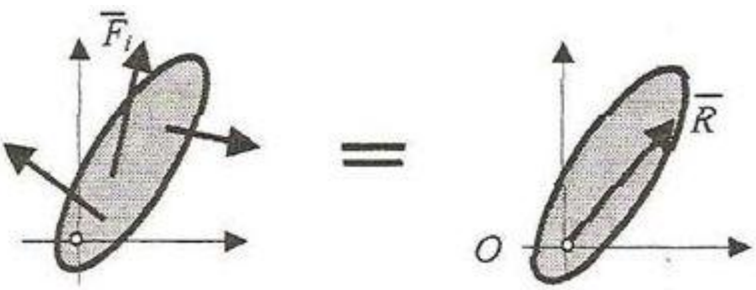


Fig.47.

- 4) $\overline{R} \neq 0$; $\overline{M_O} \neq 0$ *and* $\overline{R} \cdot \overline{M_O} = 0$ *namely the system of forces reduces to an* ***unique resultant force that does not pass through the point O****. In this case we have to determine the position of the unique resultant force that is the simplest equivalent system. The particular shape of the equation of the central axis in two dimensions (in plane) is:*

$$M_O - xY + yX = 0$$

that is the equation of the support line of the resultant force.

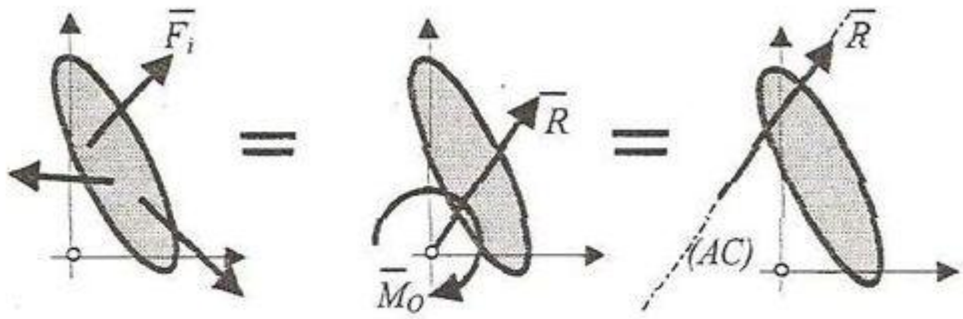


Fig.48.

2.23. Sample problems

Problem 19. *Represent the simplest equivalent system for the given system of coplanar forces from the picture 49. The magnitudes of the forces are:* $F_1 = 3F$, $F_2 = 2F$, $F_3 = 2\sqrt{2}\ F$, $F_4 = \sqrt{13}\ F$.

Solution. *In the first step we shall decompose the forces in components parallel with the axes of the reference system, that is taken arbitrary. We shall obtain:*

$$X_3 = F_3 \frac{2a}{2a\sqrt{2}} = 2F;\ Y_3 = F_3 \frac{2a}{2a\sqrt{2}} = 2F;\ X_4 = F_4 \frac{3a}{a\sqrt{13}} = 3F;$$

$$Y_4 = F_4 \frac{2a}{a\sqrt{13}} = 2F;$$

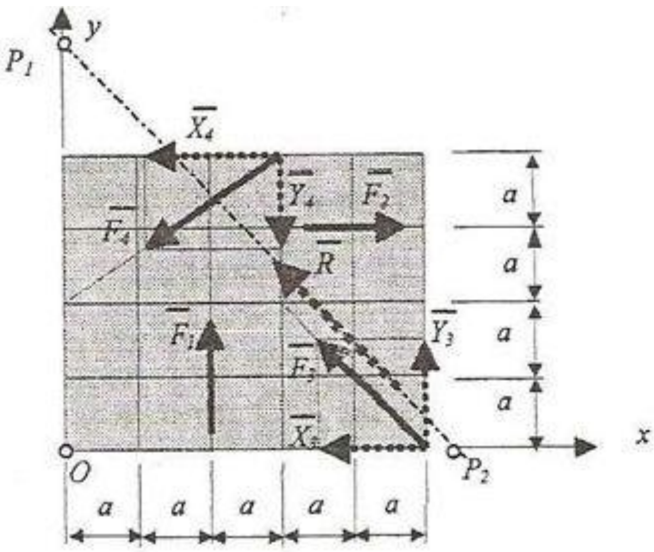


Fig.49.

The resultant force of the system of forces will have the projections on the two axes:

$$X = \Sigma X_i = F_2 - X_3 - X_4 = -3F \; ; \; Y = \Sigma Y_i = F_1 + Y_3 - Y_4 = 3F \; ;$$

with the vector expression:

$$\overline{R} = -3F\overline{i} + 3F\overline{j}$$

and the magnitude:

$$R = \sqrt{X^2 + Y^2} = 4{,}24F.$$

Now, we shall calculate the resultant moment about the point O, the origin of the reference system:

$$M_O = \Sigma M_{Oi} = F_1 \,.\, 2a - F_2 \,.\, 3a + Y_3 \,.\, 5a + X_4 \,.\, 4a - Y_4 \,.\, 3a = 16Fa.$$

As we see, this is the forth case of reduction, namely the system of forces is reduced to an unique resultant force that passes through the point O. We shall find the support line of this resultant force:

$$M_O - xY + y\,X = 0$$

or replacing and simplifying with F we have:

$$16a - x\,3 + y(-3) = 0$$

We shall represent this line using the intersection points with the reference axes:

$$x_1 = 0 \longrightarrow y_1 = 5{,}33a; \quad y_2 = 0 \longrightarrow x_2 = 5{,}33a.$$

We make the remark that in the picture 49 are represented three equivalent systems: the initial system of forces made from four forces, the system of the components of the forces on parallel directions with the axes and finely the unique resultant force. All the three equivalent systems will have the same effect about the body.

Problem 20. *Calculate and represent the simplest equivalent system for the coplanar system of forces from the figure 50 made from the following forces:* $F_1 = 6F$, $F_2 = 2\sqrt{13}\,F$, $F_3 = 3\sqrt{10}\,F$, $F_4 = 4\sqrt{5}\,F$.

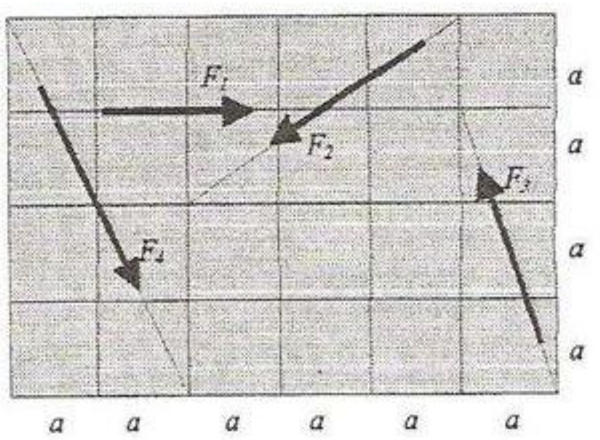


Fig.50.

2.24. Systems of parallel forces. Center of the parallel forces.

*An other particular system of forces is the system of parallel forces, that is **made from forces with parallel support lines**.*

Suppose that the support lines of the forces are parallel with the axis Oz (we may take arbitrary reference system, but for simplifying the calculation we choose the axis Oz parallel with the common direction of the forces). One force from the system will have expression with respect to this reference system:

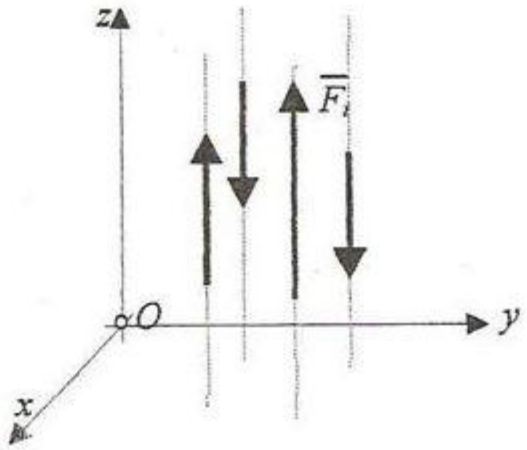


Fig.51.

$$\overline{F_i} = F_i\,\overline{k}$$

and the moment of this force about the origin of the reference system O will be:

$$\overline{M}_O(\overline{F_i}) = M_{xi}\,\overline{i} + M_{yi}\,\overline{j}$$

Making the reduction of the system of forces in point O we shall obtain:

$$\overline{R} = \Sigma\overline{F_i} = (\Sigma F_i)\,\overline{k}$$
$$\overline{M}_O = \Sigma\overline{M}_O(\overline{F_i}) = M_x\,\overline{i} + M_y\,\overline{j}$$

We remark that, the two vectors are perpendicular, so we have:

$$\overline{R} \cdot \overline{M}_O = 0$$

We have the following cases of reduction:

- *1)* $\overline{R} = 0$, $\overline{M}_O = 0$. *The system of forces is in equilibrium. The scalar conditions of equilibrium are:*

$$\Sigma F_i = 0 \, , \, \Sigma M_{xi} = 0 \, , \, \Sigma M_{yi} = 0.$$

- *2)* $\overline{R} = 0$, $\overline{M}_O \neq 0$. *The system of forces is reduced to a couple. Interesting in this case is that the system of parallel forces may be replaced with an other system of parallel forces but having other direction as the initial system. This fact is because the couple may be rotated in its plane without to change the effect of it about the body.*

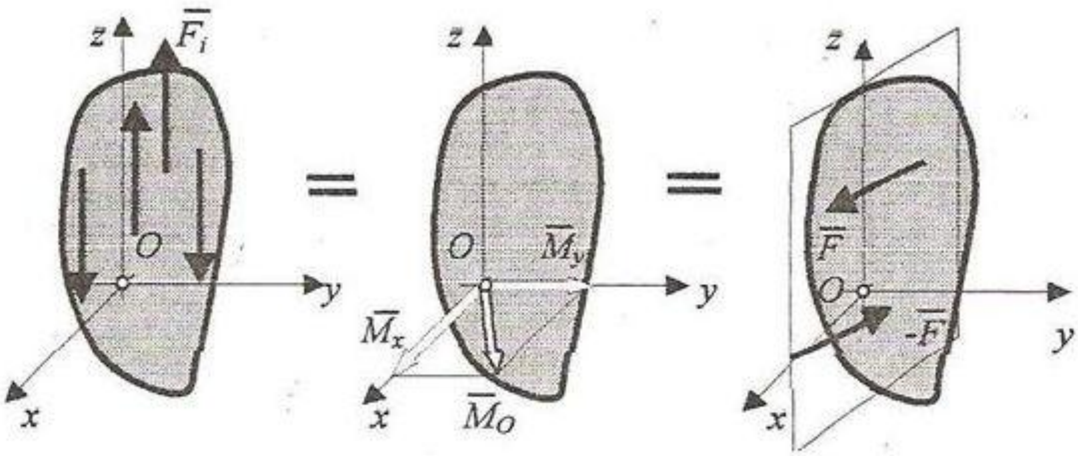


Fig.52.

- *3)* $\overline{R} \neq 0$, $\overline{M}_O = 0$. *The system of forces is reduced to an unique resultant force that passes through the point O. This resultant force is the simplest equivalent system for the given system of forces.*

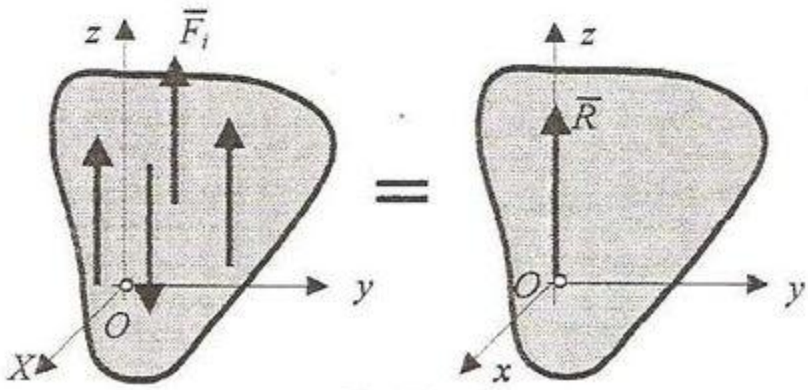


Fig.43.

- *4) $\overline{R} \neq 0$, $\overline{M}_O \neq 0$, and $\overline{R} \cdot \overline{M}_O = 0$. In this case the system is reduced to an unique resultant force that does not passé through the point O. Consequently we have to determine the position of this force using the equation of the central axis. This equation becomes for the parallel forces with the Oz axis:*

$$x = -\frac{\Sigma M_{yi}}{\Sigma F_i}; \quad y = \frac{\Sigma M_{xi}}{\Sigma F_i};$$

We may remark that these two equations define, with z=0, the coordinates of the intersection point between the support line of the resultant force and the xOy plane (the point A).

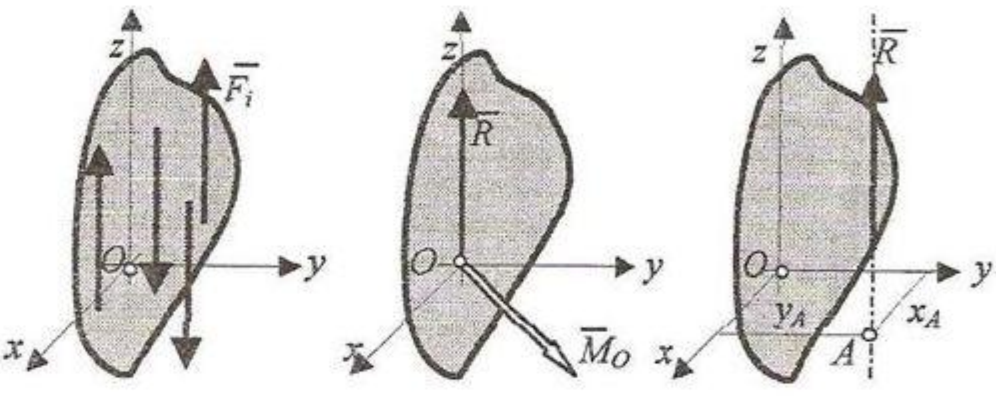


Fig.54.

An important particular case of parallel forces is that when the forces are fixed vectors. This means that the forces of the system have the defined points of application.

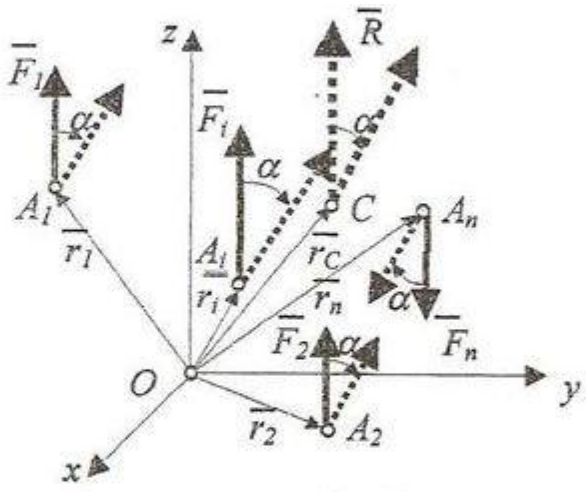


Fig.55.

At this kind of systems we may state two important proprieties:

- *1) If the forces of the system of parallel forces are fixed vectors then the unique resultant force of the system (if it exists) is fixed vector too. The point of application of this resultant force is called* ***center of parallel forces.***
- *2) If the forces of the system of parallel forces rotate with the same angle about their points of application (the system remaining with parallel forces) then the unique resultant force (if it exists) rotates with the same angle about the center of the parallel forces.*

We shall use these two proprieties for to find the center of the parallel forces.

Let be a system of parallel forces, fixed vectors, which is reduced to an unique resultant force. For this kind of system we may write the Varignon's theorem, namely:

$$\Sigma \overline{M}_O(\overline{F}_i) = \overline{M}_O(\overline{R})$$

or expressing the moments:

$$\Sigma \overline{r}_i \times \overline{F}_i = \overline{r}_C \times \overline{R}$$

Considering the forces parallel with Oz axis and bringing the both terms of this relation in the left part, results:

$$\Sigma \overline{r_i} \times F_i \overline{k} - \overline{r_C} \times R \overline{k} = 0$$

We know that the scalar quantities F_i and R may multiply any vector of the vector product, and if we replace the magnitude of the resultant force with the sum of the magnitudes of the forces, the relation is transformed in:

$$\Sigma \overline{r_i} F_i \times \overline{k} - \overline{r_C} (\Sigma F_i) \times \overline{k} = 0$$

Now, we shall take out the common vector $\overline{k}$ from the parenthesis and it is obtained:

$$[\Sigma \overline{r_i} F_i - \overline{r_C} (\Sigma F_i)] \times \overline{k} = 0$$

This vector product may be zero in three cases: if one of the two vectors is equal to zero, or if the two vectors are collinear. But the second vector is an unit vector so it is different as zero. At the other hand if the two vectors (the parenthesis and k) are collinear, using the second property of these kind of forces , the system may be rotated with an any angle and they will be not collinear, so in this case they does not annul the relation. Consequently the single case when this relation will be zero is then the parenthesis is equal to zero. Results :

$$\overline{r_C} = \frac{\Sigma \overline{r_i} F_i}{\Sigma F_i}$$

And finally, projecting on the axes of the reference system, result the coordinates of the center of parallel forces:

$$x_C = \frac{\Sigma x_i F_i}{\Sigma F_i}; \; y_C = \frac{\Sigma y_i F_i}{\Sigma F_i}; \; z_C = \frac{\Sigma z_i F_i}{\Sigma F_i}.$$

Chapter 3. Centers of gravity.

3.1. Introduction.

A particular case of parallel forces, fixed vectors ids the system of the gravitational forces.

If the bodies, or the systems of particles, have small dimensions with respect to the radius of the Earth and they are located in the neighborhood of the Earth's surface then they can be considered acted by the parallel forces, these being the attraction forces exerted by the Earth about them and called weight. These forces may be expressed function the masses of the bodies:

$$\overline{G_i} = m_i \cdot \overline{g}$$

where $\overline{g}$ *is the gravitational acceleration vector that can be considered constant in magnitude, direction and sense. The magnitude of this acceleration will be taken in the usual problems:*

$$g = 9{,}81 m/s^2$$

3.2. Centers of gravity

To suppose a system of particles A_i *by the masses* m_i *and a system of reference Oxyz. The weights of these particles considered as parallel forces, acting all in the same sense will have a resultant force, the total weight of the system of particles, as fixed vector with the point of application (the center of the parallel forces) called* ***center of gravity.***

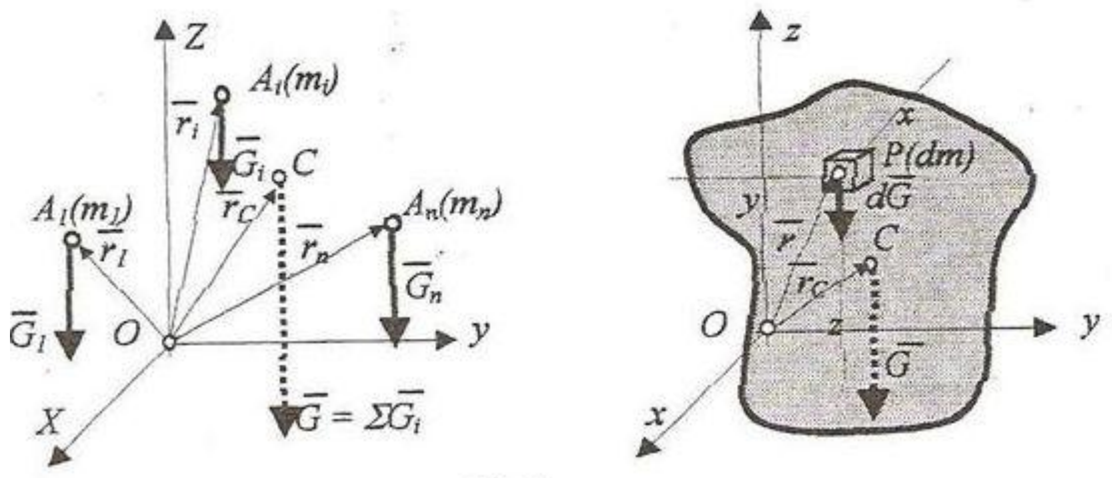


Fig.1.

In accordance with the previous chapter, the position of the center of gravity will be obtained with the relation:

$$\overline{r}_C = \frac{\Sigma\overline{r}_i G_i}{\Sigma G} = \frac{\Sigma\overline{r}_i m_i g}{\Sigma m_i g} = \frac{\Sigma\overline{r}_i m_i}{\Sigma m_i}$$

As we can see, the center of gravity is located in the same point with the ***center of mass****, for the systems satisfying the previous conditions.*

To consider now a rigid body. A rigid body may be considered a continuous system of particles namely a system of infinitesimal masses ***dm*** *called* ***elementary masses*** *tending to zero as magnitude. The number of these elementary masses tends to infinity. Each such element will be acted be an elementary weight:*

$$d\overline{G} = \overline{g} \,.\, dm$$

Using the same relations bat in which we replace the finite sum with the infinite sum, namely the integral, we find the position of the center of gravity for the rigid body:

$$\overline{r}_C = \frac{\int_V \overline{r} dG}{\int_V dG} = \frac{\int_V \overline{r} g dm}{\int_V g dm} = \frac{\int_V \overline{r} dm}{\int_V dm}$$

The coordinates of this center, for the system of particles and for the rigid body are:

$$x_C = \frac{\Sigma x_i m_i}{\Sigma m_i};\ y_C = \frac{\Sigma y_i m_i}{\Sigma m_i};\ z_C = \frac{\Sigma z_i m_i}{\Sigma m_i};$$
$$x_C = \frac{\int_V x dm}{\int_V dm};\ y_C = \frac{\int_V y dm}{\int_V dm};\ z_C = \frac{\int_V z dm}{\int_V dm}.$$

3.3. Statically moments

The fraction's numerators that define the coordinates of the center of gravity are called ***statically moments****. They are calculated with respect to a reference plane, or a reference axis, or a reference point. By definition the statically moments are* ***scalar quantities equal to the sum (or the integral) of the products between the masses and the coordinates with respect to the considered reference system****. In this way, with respect to the reference planes, the statically moments will be:*

$$S_{xOy} = \Sigma z_i \cdot m_i;\ S_{yOz} = \Sigma x_i \cdot m_i;\ S_{xOz} = \Sigma y_i \cdot m_i;$$
$$S_{xOy} = \int_V z.dm;\ S_{yOz} = \int_V x.dm;\ S_{xOz} = \int_V y.dm.$$

We shall state a theorem called ***the statically moment's theorem: the statically moment of a system of particles or of a rigid body may be calculated as the product between the total mass, of the system or of the body, and the corresponding coordinate of the mass center (or of the gravity center):***

$$S_{xOy} = M \cdot z_C;\ S_{yOz} = M \cdot x_C;\ S_{xOz} = M \cdot y_C$$

where we shall marked:

$$M = \Sigma m_i;\ M = \int_V dm$$

To prove this we can see that the fraction's numerators that define the position of the gravity center represent the statically moments and the denominators represent the total mass of the system or of the rigid body.

Besides the importance of this theorem in the calculation of the statically moments the theorem is useful by its consequence that may be stated in the next way: ***a body or a system of particles that accepts a symmetry plane, or a symmetry axis, or a symmetry point has the center of gravity situated in that plane, or on that axis, or in that point of symmetry.***

To prove this statement we shall consider that the system is made from two particles and the symmetry plane is xOz. The distances from the particles to the symmetry plane are marked **d** *and the coordinates with respect to this plane are:*

$$y_{Cs} = -y_{Cd} = d$$

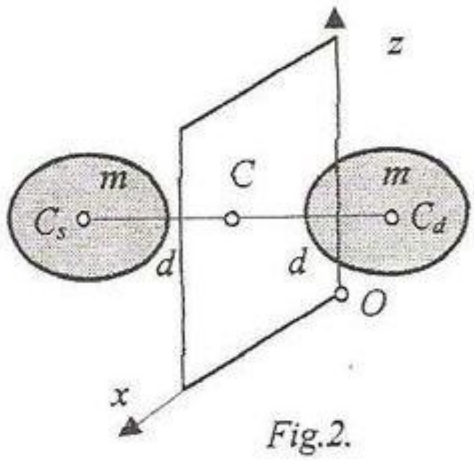


Fig.2.

The masses of the two halves of the system (or of the body) are the same. Consequently we can calculate the statically moment about the symmetry plane:

$$S_{xOz} = m \cdot d + m \cdot (-d) = 0$$

At the same time the statically moment may be calculate using the statically moment's theorem:

$$S_{xOz} = 2m \cdot y_C$$

from which results:

$$y_C = 0$$

namely the center of gravity of the whole system or body is located in the symmetry plane.

In the same way we can prove that the center of gravity is located on the symmetry axis or in the symmetry point.

3.4. Centers of gravity for homogeneous bodies, centroides.

For the homogeneous bodies (bodies having uniformly distribution of the mass) we shall give the relations of calculation of the position of the gravity center considering the three schemes of the bodies, namely for the bars, plates and volumes.

- ***Homogeneous bars.*** *Supposing a homogeneous bar we can express the homogeneity propriety of it in the next way:*

Fig.3.

$$\mu_L = \frac{M}{L} = \frac{m_i}{l_i} = \frac{dm}{dl}$$

relation that represents the ***specific mass*** *of the bar, namely that the rate between the mass and the length is the same indifferent the part considered from the bar. From this relation results:*

$$dm = \mu_L \cdot dl = \frac{M}{L} \cdot dl$$

Replacing this in the coordinates of the center of gravity and simplifying the specific mass results:

$$x_C = \frac{\int_L x dl}{\int_L dl} \; ; y_C = \frac{\int_L y dl}{\int_L dl} \; ; z_C = \frac{\int_L z dl}{\int_L dl}.$$

- ***Homogeneous plates.*** *For a homogeneous plate we can write:*

$$\mu_S = \frac{M}{A} = \frac{m_i}{A_i} = \frac{dm}{dA}$$

representing the specific mass of the plate. Expressing the elementary mass:

$$dm = \mu_A \cdot dA = \frac{M}{A} \cdot dA$$

finally we obtain the coordinates of the center of gravity for the homogeneous plate:

$$x_C = \frac{\int_A x dA}{\int_A dA} \; ; y_C = \frac{\int_A y dA}{\int_A dA} \; ; z_C = \frac{\int_A z dA}{\int_A dA}.$$

- ***Homogeneous volume.*** *The specific mass of the volume, also called density has the expression:*

$$\mu = \frac{M}{V} = \frac{m_i}{V_i} = \frac{dm}{dV}$$

from which results the elementary mass:

$$dm = \mu \cdot dV = \frac{M}{V} \cdot dV$$

and finally the coordinates of the center of gravity:

$$x_C = \frac{\int_V x dV}{\int_V dV} \; ; y_C = \frac{\int_V y dV}{\int_V dV} \; ; z_C = \frac{\int_V z dV}{\int_V dV}.$$

We can remark that for the homogeneous bodies the center of gravity is the geometrical center of gravity and it is called ***centroid.***

3.5. Centers of gravity for composed bodies.

In the case when the body can be considered to be obtained by summing other bodies, for which are known the masses or all geometrical elements and the positions of the centers of gravity, this fact can be used for to determine the position of the center of gravity of the entire body. For simplicity we shall show how is determined the position of the centroid for a homogeneous plate with the total area A.

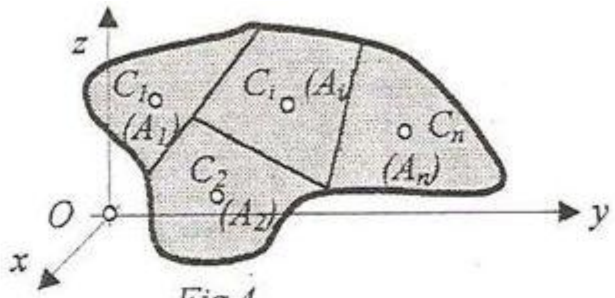


Fig.4.

We shall consider the plate divided into simple plates for which we know the areas and the positions of the centroides. The x_C coordinate of the centroid of the whole plate is calculated:

$$x_C = \frac{\int_A x dA}{\int_A dA} = \frac{\int_{A_1} x dA + \ldots + \int_{A_i} x dA + \ldots + \int_{A_n} x dA}{\int_{A_1} dA + \ldots + \int_{A_i} dA + \ldots + \int_{A_n} dA} = \frac{\Sigma x_i A_i}{\Sigma A_i}$$

This relation is obtained knowing that if the domain of integration can be considered as a sum of domains then the integral can be calculated as a sum of integrals on each separated domain of integration. The terms from the numerators are the statically moments of the component simple plates (or bodies), that can be calculated as the products between the areas and the corresponding coordinates of their centroides. At the denominators we have *the areas of the simple plates. In the same way we shall obtain the other two coordinates:*

$$y_C = \frac{\Sigma y_i A_i}{\Sigma A_i}; \; z_C = \frac{\Sigma z_i A_i}{\Sigma A_i}$$

Obviously, in the same way we have the position of the centroids for the bars and the volumes.

Consequently for to determine the position of the centroid for a composed plane plate we shall pass the following steps:

1) The plate is divided in simple plates for which are known the areas and the positions of the centroides. The division is not unique having more ways to find the same plate from simple plates. For example let be the homogeneous plate from the figure 5. We can see that this plate may be obtained adding two rectangles as in the figure 5.b. and 5.c. or as a difference of two rectangles.

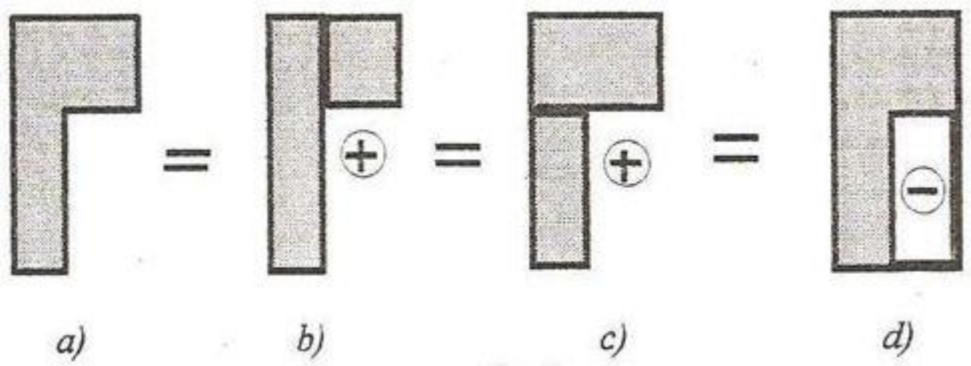


Fig.5.

2) Is chosen a convenient reference system. Through convenient reference system we shall understand that system that makes the biggest simplifications in the computation. Thus if we want to have positive coordinates the reference system will be with the two axes tangent to the plate situated in the first frame. The system can be taken also with the axes passing through the centroides of the simple bodies because in this way a part of the coordinates of the centroides will be zero. In the case when the plate has a symmetry axis then one axis of the reference system is taken this symmetry axis because in this way one coordinate of the centroid of the plate will be zero, te centroid being located on the symmetry axis.

3) With respect to this reference system are calculated the coordinates of the centroides of the simple plates. The coordinates are in fact distances from the centroides to the axes of the system with the sign of the corresponding frame. The distances may be calculated as the sum or the difference of other distances. We make the remark that: for the subtracted plates the coordinates are not considered with minus sign.

4) Are calculated the areas and the statically moments for each simple plate and the area and the statically moment of the entire plate. The calculation is made in a table as the next:

Fig.	A_i	X_i	Y_i	$A_i \cdot x_i$	$A_i \cdot y_i$
1					
...					
Total	ΣA_i			$\Sigma A_i \cdot x_i$	$\Sigma A_i \cdot y_i$

5) Are calculated the coordinates of the centroid of the given plate using the corresponding relations.

6) Is checked the plausibility of the find position of the centroid. The exactly check of it may be obtained remaking calculation, eventually about another reference system. The first check is that the centroid have to be located in the inside of the shape obtained drawing all the tangents to the plate (Fig.6.a.). If the plate is a sum of other simple plates then the centroid is located in the inside of the polygonal line obtained uniting the centroides af the simple plates (Fig.6.b.). Finally, if the body has a hole, then the centroid "runs away" from the hole (Fig.6.c.).

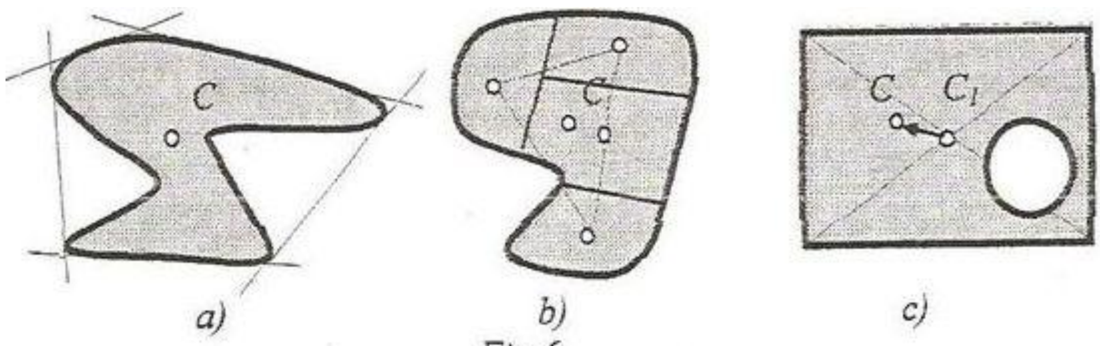


Fig.6.

As we can see, for to solve these kinds of problems we have to know the positions of the centroides of a few simple usual homogeneous bodies.

3.6. Centroides for simple usual homogeneous bodies.

We shall determine the positions of centroides of a few simple homogeneous bodies used in the problems.

Rectilinear bar. *Let be a rectilinear bar by length L.*

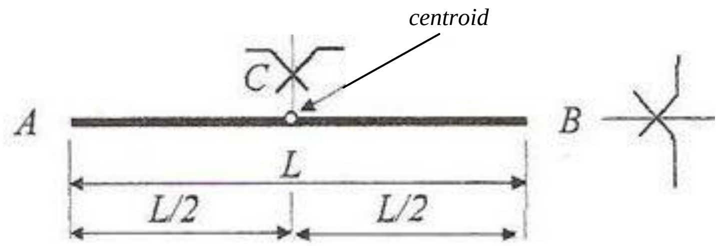


Fig.7.

Because the bar has a symmetry point in the middle of the length of the rods, in that point is located the centroid of it. Namely the centroid of a rectilinear bar is located ***in the middle of the distance between the ends of the bar.***

The circular arch. *Let be a bar having the shape of a circular arch. The circle has the centre in O and the radius R. The length L of the arch can be expressed function of the angle in the centre of the circle:*

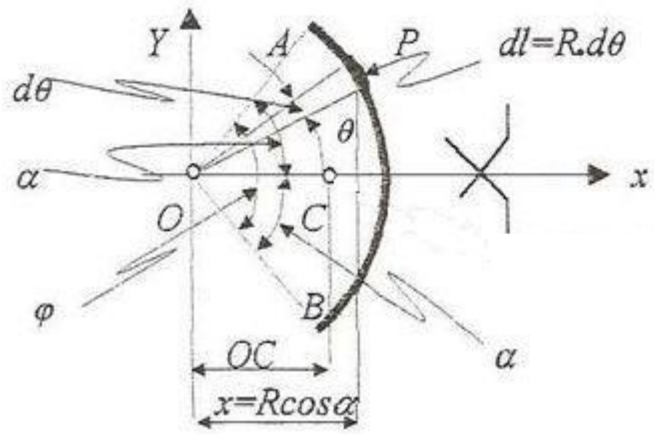


Fig.8.

L = R . $\varphi_{(radians)}$

Because the circular arch is a symmetrical shape having the bisecting line of the angle in the centre as symmetry axis, the centroid of it will be located on this bisecting line somewhere between the arch and the center of the circle. We shall determine the distance OC from the center of the circle to the centroid of the arch.

We shall consider a reference system with the origin O in the centre of the circle and the axis Ox the bisecting line of the angle in the center of the circle. The coordinate x_C *of the centroid will be:*

$$OC = x_C = \frac{\int_L x dl}{\int_L dl}$$

relation in which the denominator is the length of the arch.

For to calculate the numerator, that is the statically moment, we shall choose in the any point P an elementary length **dl**. *To make an simple calculation we shall work in polar coordinates, namely the position of the point P will be expressed function of the radius R and the angle* θ *made by this radius with the axis Ox:*

$$x = R \cdot \cos\theta$$

The elementary length will be:

$$dl = R \cdot d\theta$$

Removing in the previous relation is obtained:

$$x_C = \frac{R^2 \int_{-\alpha}^{\alpha} \cos\theta . d\theta}{R.\varphi} = R\frac{\sin\alpha}{\alpha}$$

Finally we can say: the centroid of a circular arch is located ***on the bisecting line of the angle in the centre at a distance OC = R. sin*** $\boldsymbol{\alpha / \alpha}$

from the centre of the circle, where α is half from the angle in the centre expressed in radians.

Rectangular plate. *A rectangular plate be sides b and h (Fig.9.) has a symmetry point (the intersection of the two symmetry axes) resulting that* ***the centroid of it is located in the middle of the distance between two parallel sides.***

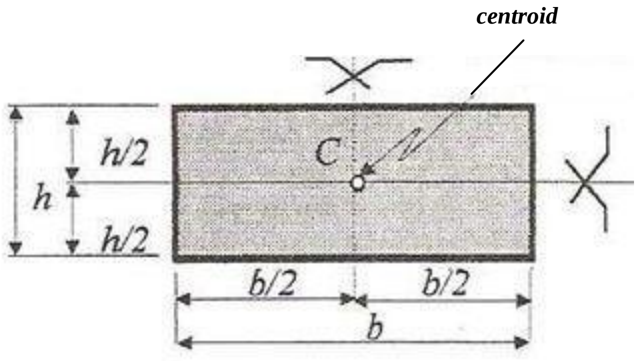


Fig.9.

Right angle triangular plate. *For to determine the position of the centroid we shall choose the reference system with the axes collinear with the two orthogonal sides. We remark that the triangle has the same position with respect the two axes, namely we shall calculate only a single coordinate (for example y_C) and changing x with y and b with h we have the other coordinate.*

$$y_C = \frac{\int_A y.dA}{\int_A dA}$$

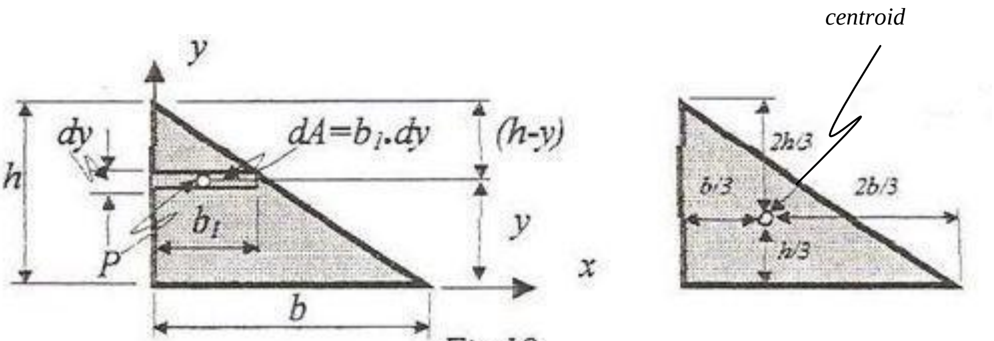


Fig.10.

The denominator is the area of the triangle and the numerator will be calculated choosing first an infinitesimal element of area dA. For to calculate easier this statically moment we shall transform the double integral in another simple integral. The elementary area must have the same proprieties as a particle as the area and the statically moment with respect the axis Ox. These conditions can be met if the elementary area has the shape of a bar parallel with Ox. Consequently we shall choose in point P by y coordinate an elementary area with the thickness dy that can assimilate with a rectangle by b_1 *base. The area of this element will be:*

$$dA = b_1 \cdot dy$$

where expressing b_1 *function of y (writing a likeness relationship between the two formed triangles: one with* b_1 *base and (h-y) height and the other the given triangle):*

$$\frac{b_1}{h-y} = \frac{b}{h} \longrightarrow b_1 = \frac{b}{h}(h-y)$$

Removing in the integral we find the coordinate of the centroid:

$$y_C = \frac{\frac{b}{h}\int_0^h y(h-y)dy}{\frac{bh}{2}} = \frac{h}{3}$$

The coordinate x_C *will be:*

$$x_C = b/3$$

Finally we can say: the centroid of a right angle triangle is located ***at a distance equal to a third from the length of the side measured from the other side.***

Circular sector plate. *We shall consider a sector of a circle by radius R and the angle in the centre* φ*. Like the circular arch, the circular sector is a symmetrical shape having as symmetry axis the bisecting line of the angle in the centre. Consequently the centroid of it will be located on this* *symmetry axis. we shall calculate the distance OC from the centre of the circle to the centroid of the sector. Choosing the Ox axis collinear with the symmetry axis we have:*

$$OC = x_C = \frac{\int_A x dA}{\int_A dA}$$

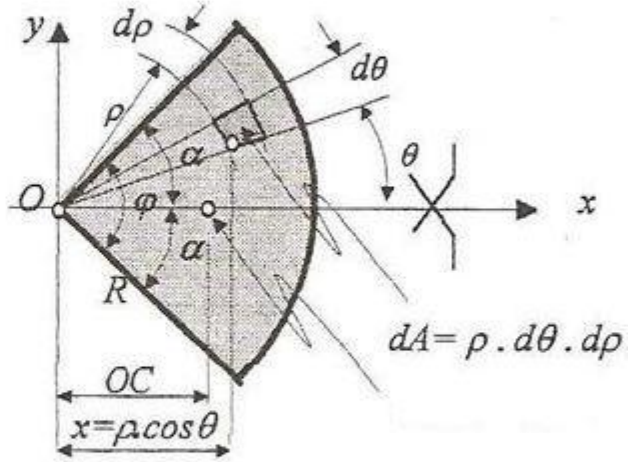


Fig.11.

where the denominator is the area of the sector:

$$A = \frac{\varphi(rad)}{2} R^2 = \alpha_{(rad)} \cdot R^2$$

and for to calculate the numerator we shall express x and dA in polar coordinates. Assimilating the elementary area with a small rectangle we obtain:

$$dA = \rho \cdot d\theta \cdot d\rho$$

Removing in the relationship of OC we find:

$$x_C = \frac{\int_A \rho.\cos\theta.\rho.d\theta.d\rho}{\alpha R^2} = \frac{\int_0^R \rho^2 d\rho \int_{-\alpha}^{\alpha} \cos\theta d\theta}{\alpha R^2} = \frac{2}{3} R \frac{\sin\alpha}{a}$$

namely the centroid of the circular sector ***is located on the bisecting line of the angle in the centre at a distance OC = 2. R. sinα / 3α from the centre of the circle.***

Three circular sectors are used more often in problems: the entire circle, the semicircle and the quarter of the circle. If for the entire circle the centroid is located in the centre of the circle, for the semicircle and for the quarter of the circle marking ***e*** *the eccentricity of the centroid with respect to the diameter of the semicircle or of the radius of the quarter of the circle will results:*

$$e = R\frac{4}{3\pi}$$

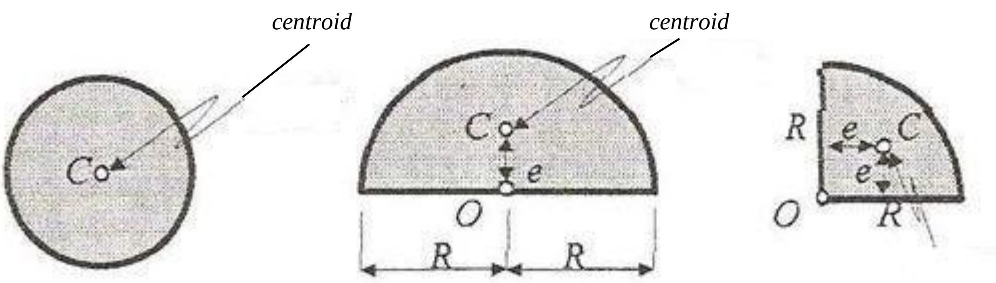


Fig.12.

3.7. Sample problems

Problem 1. *Determine the position of the centroid for the homogeneous bar from the figure 13.*

Solution. Step 1. *We may see that the bar can be considered as a sum of three simple bars: a quarter of circle, a rectilinear bar and a semicircle. For each of these bars we know the length and the position of the centroid. The lengths of the bars are:*

$$l_1 = l_{AB} = \frac{\pi R_1}{4} = 0.5\pi a \,;\; l_2 = l_{BC} = 4a \,;\; l_3 = l_{CD} = \frac{\pi R_2}{2} = 1.5\pi a$$

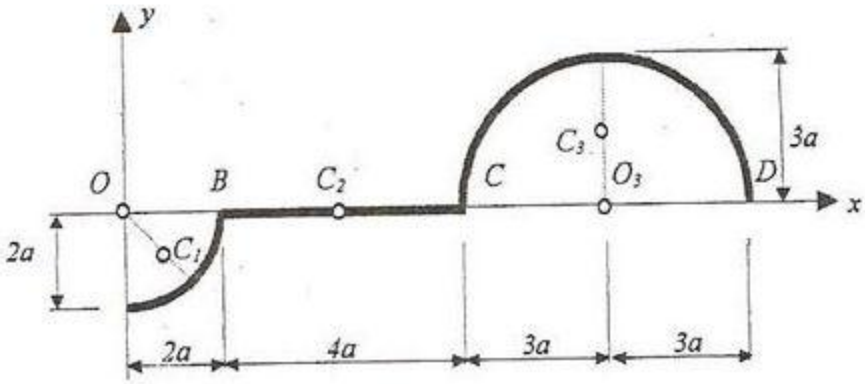


Fig.13.

Step 2. *We shall choose the reference system with the origin in the center of the circle corresponding to the quarter of the circle AB and horizontal and vertical axes.*

Step 3. *We shall determine the coordinates of the centroides of the simple bodies with respect to the chosen system of reference. Are obtained:*

$$OC_1 = R_1 \frac{\sin\alpha_1}{\alpha_1} = 2a\frac{\sin\frac{\pi}{4}}{\frac{\pi}{4}} = \frac{4\sqrt{2}}{\pi}a;$$

$$x_1 = OC_1 \cdot \cos\frac{\pi}{4} = \frac{4}{\pi}a; \; y_1 = -OC_1 \cdot \sin\frac{\pi}{4} = -\frac{4}{\pi}a;$$

$$x_2 = OB + \frac{BC}{2} = 4a \; ; \; y_2 = 0 \; ;$$

$$O_3C_3 = R_3 \frac{\sin\alpha_3}{\alpha_3} = 3a\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{6}{\pi}a;$$

$$x_3 = OB + BC + CO_3 = 9a \; ; \; O_3C_3 = \frac{6}{\pi}a;$$

Step 4. *The calculation of the total length and the statically moments is made in the following table:*

Bara	l_i	x_i	y_i	$l_i . x_i$	$l_i . y_i$
AB	$0.5\pi a$	$4a/\pi$	$-4a/\pi$	$2a^2$	$-2a^2$
BC	$4a$	$4a$	0	$16a^2$	0
CD	$1,5\pi a$	$9a$	$6a/\pi$	$13,5\pi a^2$	$9a^2$
Σ	$10,28a$			$60,39a^2$	$7a^2$

Step 5. *The coordinates of the centroid are:*

$$x_G = \frac{\Sigma l_i x_i}{\Sigma l_i} = \frac{60,39a^2}{10,28a} = 5,87a \; ;$$

$$y_G = \frac{\Sigma l_i y_i}{\Sigma l_i} = \frac{7a^2}{10,28a} = 0,68a.$$

Problem 2. *Determine the position of the centroid for the homogeneous plane plate from the figure 14.*

Solution. Step 1. *We shall divide the plate in simple plates for which we know the areas and the positions of the centroides. The division is not unique, but we shall choose that division in which the bodies are added as in the figure 14. b., ie two rectangles and a triangle.*

The areas of them are:

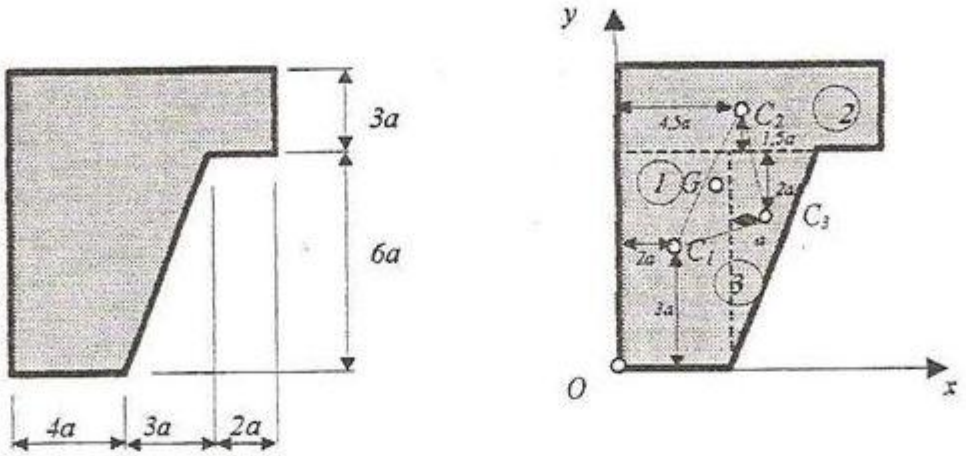


Fig.14.

$$A_1 = 4a\,.\,6a = 24a^2\ ;\ A_2 = 9a\,.\,3a = 27a^2\ ;\ A_3 = \frac{3a.6a}{2} = 9a^2$$

Step 2. *The reference system will be taken such as the entire plate to be located in the first frame. In this way all the coordinates will be positive.*

Step 3. *For to determine these coordinates, after which we mark the centroides of the simple plates we calculate the distances from these centers to the sides of the rectangles, of the triangle. Finally we determine the coordinates as sums (or differences) of distances from the centers to the axes, measuring parallel to the corresponding axes:*

$$x_1 = 2a\ ;\ y_1 = 3a\ ;\ x_2\ 4{,}5a\ ;\ y_2 = 6a + 1{,}5a = 7{,}5a\ ;\ x_3 = 4a + a = 5a\ ;$$
$$y_3 = 6a - 2a = 4a\,.$$

We may remark that for the triangle we should calculate the coordinate y_3 directly as 2/3 from the height of the triangle measured from the top of it.

Step 4. *We shall calculate the statically moments and the area of the plate in the following table:*

Fig	A_i	x_i	y_i	$A_i x_i$	$A_i y_i$
1	$24a^2$	$2a$	$3a$	$48a^3$	$72a^3$
2	$27a^2$	$4{,}5a$	$7{,}5a$	$121{,}5a^3$	$202{,}5a^3$
3	$9a^2$	$5a$	$4a$	$45a^3$	$36a^3$
Σ	$60a^2$			$214{,}5a^3$	$310{,}5a^3$

Step 5. *The coordinates of the centroid will be computed with the relations:*

$$x_G = \frac{\Sigma A_i x_i}{\Sigma A_i} = \frac{214{,}5a^3}{60a^2} = 3{,}57a\ ;$$

$$y_G = \frac{\Sigma A_i y_i}{\Sigma A_i} = \frac{310{,}5a^3}{60a^2} = 5{,}175a$$

Step 6. *If we represent the centroid of the plate , because it is made through the addition of the three simple plates, this centroid is located in the inside of the triangle obtained uniting the three centers of the simple plates.*

Problem 3. *Determine the centroid of the homogeneous plane plate from the figure 15.a.*

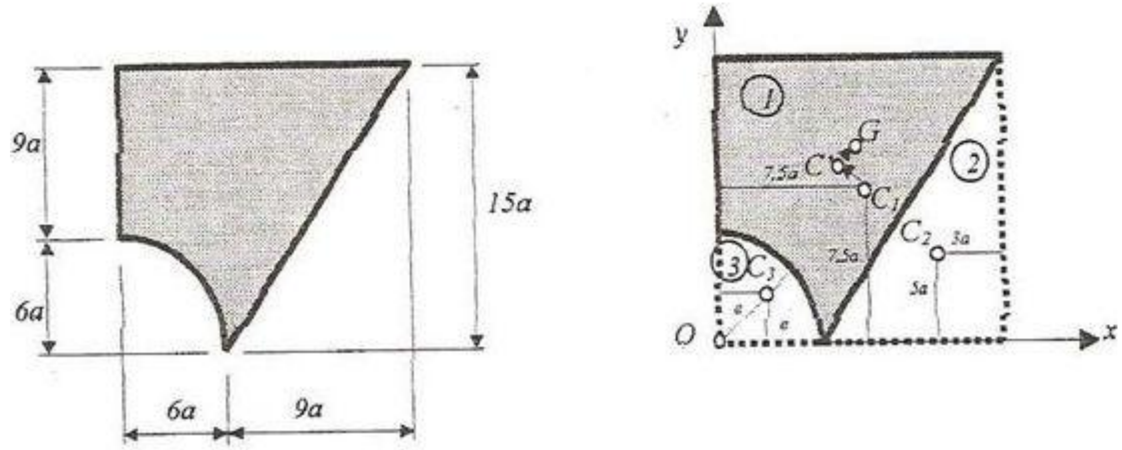


Fig.15.

Solution. *We shall choose as option of division of this plate to subtract a triangle and a quarter of circle from the square with 15a as side. As system of reference we shall choose the system having the axes tangent to the body, in this way the body will be located in the first frame.*

The areas of the three simple plates will be (the areas of the bodies 2 and 3 will be considered negative):

$$A_1 = 15a \, . \, 15a = 225a^2 \ ; A_2 = -\frac{1}{2} 9a \, . \, 15a = -67{,}5a^2 \ ;$$

$$A_3 = -\frac{\pi(6a)^2}{4} = -28{,}5a^2$$

The positions of the centroides of the simple plates will be determined with respect to the sides (for the rectangle and the triangle) and with respect to the radii (for the quarter of the circle):

$$e = \frac{4R_3}{3\pi} = \frac{4{,}6a}{3\pi} = 2{,}55a$$

With these distances the coordinates of the centroides are the next:

Fig.	A_i	X_i	Y_i	Ax_i	Ay_i
1 ◻	$225a^2$	7,5a	7,5a	$1687{,}5a^3$	$1687{,}5a^3$
2 △	$-67{,}5a^2$	12a	5a	$-810a^3$	$-337{,}5a^3$
3 ◿	$-28{,}5a^2$	2,55a	2,55a	$-72a^3$	$-72a^3$
Σ	$129a^2$			$805.5a^3$	$1278a^3$

The coordinates of the centroid of the entire body are obtained with the relations:

$$x_G = \frac{\Sigma A_i x_i}{\Sigma A_i} = \frac{805{,}5a^3}{129a^2} = 6{,}24a\,;$$

$$y_G = \frac{\Sigma A_i y_i}{\Sigma A_i} = \frac{1278a^3}{129a^2} = 9{,}9a$$

***Remark**: The centroid of the square is in C_1. Subtracting the triangle the center moves in C', and after subtracting the quarter of the circle the centroid passes in final position G.*

***Problem 4.** Determine the position of the centroid of the homogeneous plate from the figure 16a.*

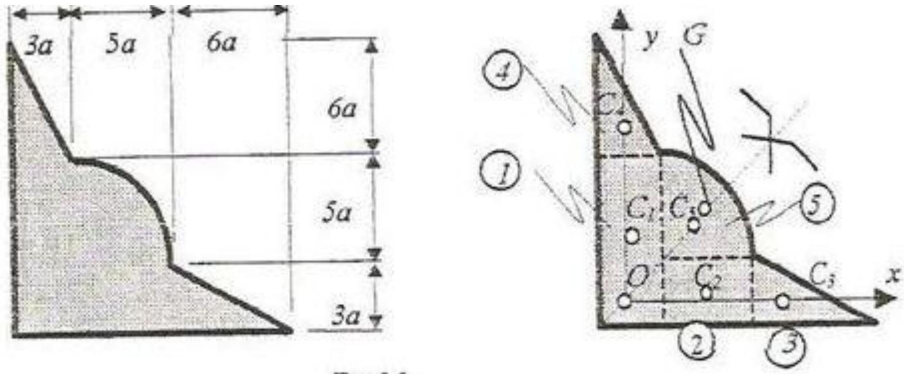


Fig.16.

***Solution.** We remark that the plate is symmetrical having as symmetry axis the bisecting line of the 90° angle. After which we divide the body in five simple plates(two rectangles, two right angle triangles and one quarter of circle) we shall choose as system of reference the system having the origin on the symmetry axis and passing through the centers of the two triangles. This system will make to have:*

$x_G = y_G$

namely we shall calculate only one single coordinate and two coordinates of the simple bodies will be zero. Consequently the calculation is simpler. At the other hand, because the centers of the simple bodies are located in the first frame all the other coordinates will be positives.

The table of computation is the next:

Fig	A_i	X_i	$A_i x_i$
1	$3a \cdot 8a = 24a^2$	$3a/2 - a = 0{,}5a$	$12a^3$
2	$5a \cdot 3a = 15a^2$	$3a + 5a/2 - a = 4{,}5a$	$67{,}5a^3$
3	$6a \cdot 3a/2 = 9a^2$	$8a + 6a/3 - a = 9a$	$81a^3$
4	$9a^2$	$3a/3 - a = 0$	0
5	$\pi(5a)^2/4 = 19{,}62a^2$	$3a + 4{,}5a/3\pi - a = 4{,}12a$	$80{,}89a^3$
Σ	$76{,}62a^2$		$241{,}39a^3$

Finally we have the coordinates of the centroid of the body:

$$x_G = \frac{\Sigma A_i x_i}{\Sigma A_i} = \frac{241{,}39a^3}{76{,}62a^2} = 3{,}15a.$$

Problem 5. *Determine the position of the centroid of the homogeneous bar from the figure 17.*

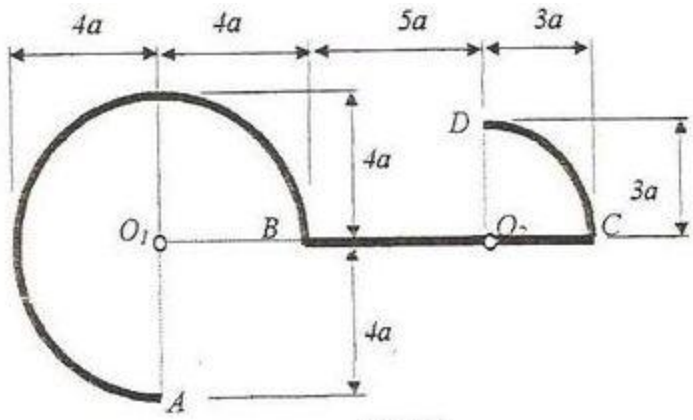


Fig.17.

Problem 6. *For the given homogeneous plate from the figure 18 determine the position of the centroid.*

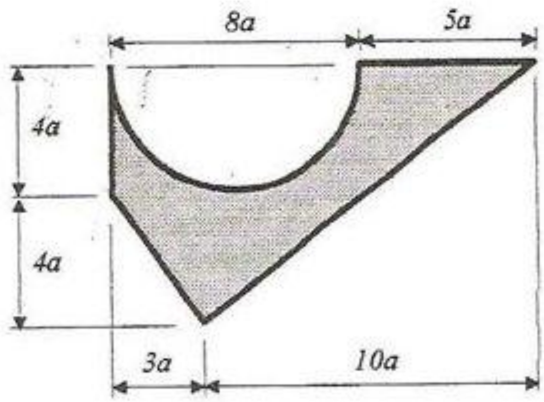


Fig.18.

Problem 7. *Determine the position of the centroid for the homogeneous plane plate from the figure 19.*

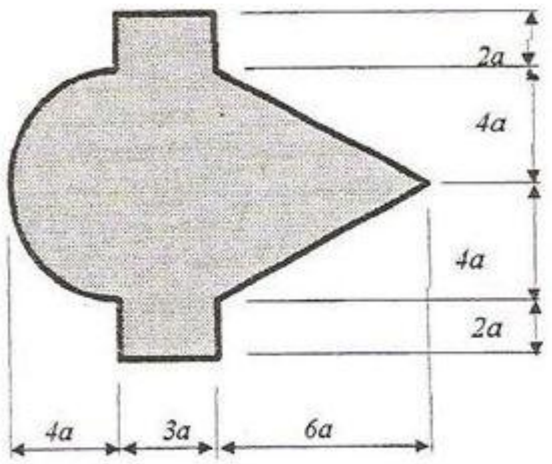


Fig.19.

3.8. Pappus – Guldin theorems.

In this section we shall state and demonstrate two theorems referring to the area and the volume of rotation bodies.

Theorem I. *The area of the surface obtained through rotation of a segment of a plane curved line around an axis located in the plane of the line, but without to intersect the axis, is equal to the product between the length of the segment of line and the length of the circle described by the centroid of the line around the axis:*

$$A = 2\pi \cdot d_C \cdot L$$

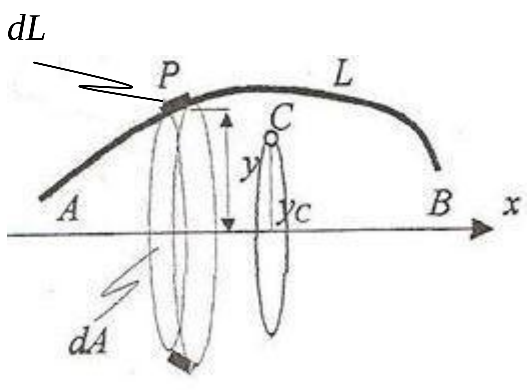


Fig.20.

To prove we shall consider the rotation axis as Ox axis and in the any point P from the segment of line an infinitesimal length **dl**. *If we rotate this small segment, that may be assimilated with a small straight line, will result a conical surface having the area:*

$$dA = 2\pi \cdot y \cdot dl$$

Summing (integrating) all these areas is obtained the area of the surface of rotation:

$$A = \int_A dA = 2\pi \int_L y dl = 2\pi \cdot y_C \cdot L$$

because the last integral is the statically moment of the segment of line with respect to the rotation axis.

Theorem II. *The volume of a body obtained through the rotation of a plane homogeneous surface around an axis located in the plane of the surface, but without to intersect the axis, is equal to the product between the area of the surface and the length of the circle described by the centroid of the surface around the axis:*

$$V = 2\pi \cdot d_C \cdot A$$

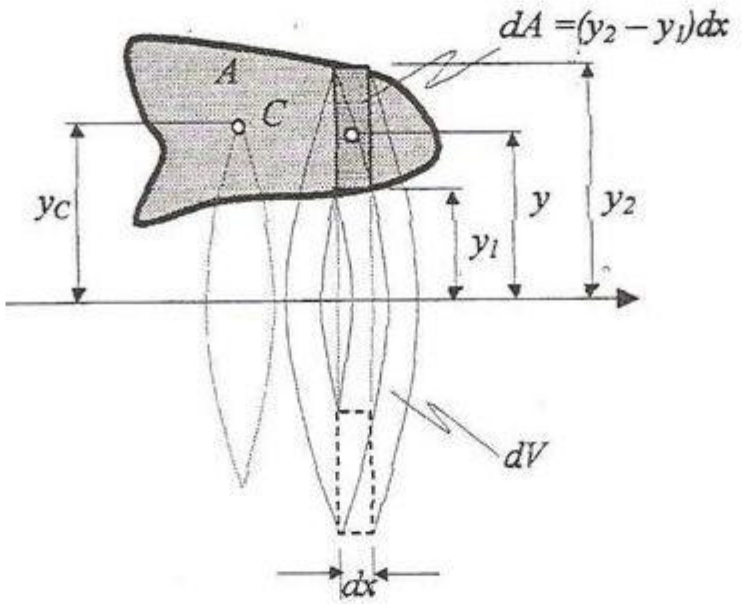


Fig.21.

We shall consider the rotation axis as the Ox axis. The elementary (infinitesimal) surface dA will be considered as a rectangle with the width dx and the height ($y_2 - y_1$). Rotating this elementary area around the axis is obtained a volume having the shape of a cylinder by the radius y_2 from which is subtracted another cylinder by the radius y_1. The volume of this elementary body will be:

$$dV = \pi \cdot y_2^2 \cdot dx - \pi \cdot y_1^2 \cdot dx = \pi (y_2^2 - y_1^2)\, dx$$

that can be expressed in the following way:

$$dV = 2\pi \frac{y_2 + y_1}{2} (y_2 - y_1)\, dx = 2\pi y\, dA$$

because the semi sum of the two coordinates y_1 and y_2 is the coordinate of the centroid of the elementary surface dA. Summing all this volumes results the volume of the rotation body:

$$V = \int_V dV = 2\pi \int_A y dA = 2\pi y_C\, dA$$

where the last integral is the statically moment of the surface with respect to the rotation axis.

Chapter 4. Statics of the particle.

4.1. Introduction.

By definition the particle is a body with very small dimensions (that can be neglected) or a body at which the dimensions are not interested in the studied problem. If the particle may have any position in space we say that it is a ***free particle.*** *But if the particle cannot have any position in space, due to restrictions of geometrical nature, we say that it is a* ***constrained particle*** *or a* ***restricted particle****.*

In this chapter we shall study the particle in the state of rest considering first the free particle, after the constrained particle.

4.2. Equilibrium of the free particle.

Consider a free particle. The position of it may be expressed with an any reference system using three scalar coordinates named generally ***position parameters****. If the particle is free it may have any position in space, so the position parameters can modify their values independently one with respect to the others and these parameters will be* ***independent position parameters****.*

At the other hand for to change the position of the particle from an any position to another any position (for example P_1 and P_2) we may perform three particular independent motions. ***Independent possibilities of motion*** *of the particle for to modify its position are called* ***degrees of freedom****. We remark that a free particle in space has three degrees of freedom and this number is equal to the number of the independent scalar position parameters:*

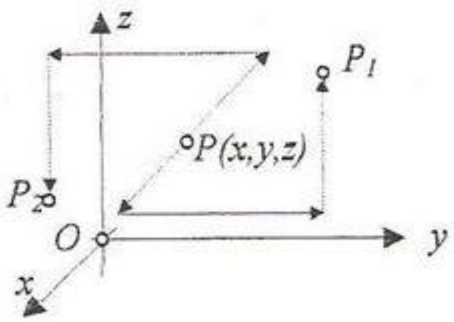


Fig.1

$$N_{DF} = N_{ISPP}$$

where N_{DF} is the number of the degrees of freedom and N_{ISPP} is the number of the independent scalar position parameters.

In plane problems (in two dimensions) the position of the particle is defined by two position parameters and for to change its position we have to make two independent motions, consequently the particle has two degrees of freedom in plane.

The previous relationship, if it remains true for the constrained particle, shows us that for to determine the number of the degrees of freedom we have two ways: or we determine the number of the independent scalar position parameters, or we determine the number of the independent possibilities of motion for to change its position.

Let be a free particle in rest. We shall consider that about the particle is acting a system of concurrent forces. The problem is the following: under what conditions the particle in rest will remain in rest under the action of the concurrent system of forces? Because the particle is in rest is obviously that the system of forces must have zero effect about the particle. But because the simplest equivalent system of the given system of forces is the resultant force we have the condition:

$$\overline{R} = 0$$

that is the vector condition for that the particle to remain in rest. In the same time, the state when the system of forces is equivalent to zero is called ***equilibrium.*** *So for a particle to remain in rest the system of forces must be in* *equilibrium. We shall say that **the particle is in equilibrium.** The scalar conditions of equilibrium of the particle will be:*

$$\Sigma X_i = 0 \; ; \; \Sigma Y_i = 0 \; ; \; \Sigma Z_i = 0 \; ;$$

Function of the nature of the unknowns of the problem we distinguish three kinds of problems:

- ***The direct problem** in which are known all forces that act about the particle and is asked the equilibrium position. Because the equilibrium position is defined by three scalar independent position parameters, and the number of the scalar equilibrium conditions is three also, the problem has unique solution.*
- ***The inverse problem** in which is known the equilibrium position of the particle and is asked the system of forces that give us that position. Generally this kind of problems has an infinity number of solutions.*
- ***The mixed problem** is the problem in which we know a part from the scalar position parameters and a part from the forces acting about the particle and is asked to determine the other part of the position parameters and forces for to have equilibrium of the particle.*

4.3. Sample problems.

***Problem 1.** A small ring P of negligible weight is connected with three ideal springs by three fixed points situated in the tops of a right angle triangle like in the figure 2. Knowing that the initial length of the springs are negligible and the elasticity coefficients of them are: $k_1 = k$, $k_2 = 2k$, $k_3 = 3k$, determine the equilibrium position of the particle in the plane of the triangle.*

***Solution.** The three springs in contact with the particle not prevent the possibility of changing the position of the particle. Consequently we have a free particle. At the other hand if we choose a system of reference xOy, the position of the particle will be defined by two coordinates that may have independent values one another. The three springs not impose restrictions in the possibility to modify the position of the particle. They are bodies that transmit forces to the particle. For to determine these forces we remark that these forces are tensions only (pull the particle) because the initial length of the springs are zero, so they are stretched. In conclusion the forces from the springs are directed on the direction from the particle P to the three points O, A and B . The magnitudes of these forces are proportional with the distances from P to the three points, consequently we have:*

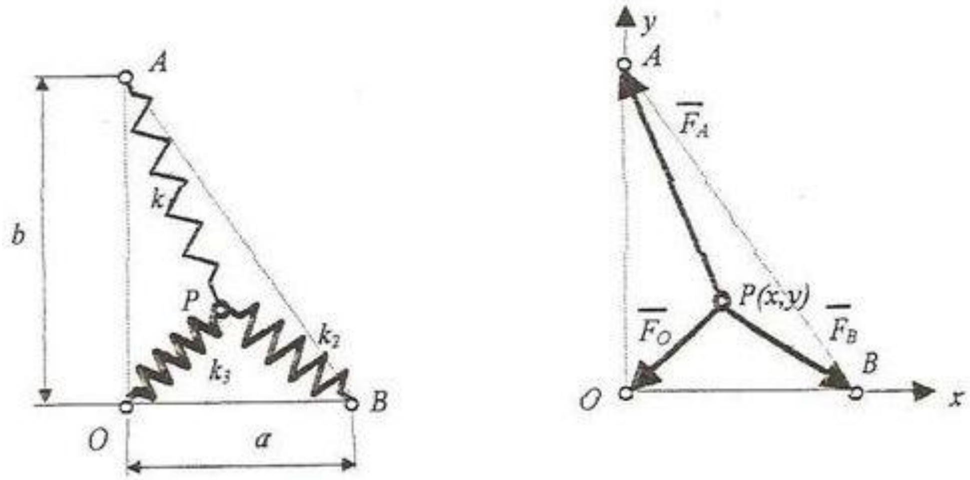


Fig.2.

$$\overline{F_A} = k_1 . \overline{PA} \; ; \; \overline{F_B} = k_2 . \overline{PB} \; ; \; \overline{F_O} = k_3 . \overline{PO}$$

The equilibrium equations of the particle P are obtained projecting the three forces on the two axes of the reference system. At calculation of these projections we will consider that in fact we project the position vectors of the three points O, A and B with respect to the point P. We have:

$$\Sigma X_i = 0 \; ; \; -F_{Ax} + F_{Bx} - F_{Ox} = 0 \; ;$$
$$\Sigma Y_i = 0 \; ; \; F_{Ay} - F_{By} - F_{Oy} = 0 \; ;$$

or:

$$\Sigma X_i = 0 \; ; \; -kx + 2k(a-x) - 2kx = 0;$$
$$\Sigma Y_i = 0 \; ; \; k(b-y) - 2ky - 3ky = 0.$$

Solving the two equations are obtained the coordinates:

$$x = \frac{a}{3} \; ; \; y = \frac{b}{6}$$

representing the equilibrium potion of the particle.

Problem 2. *A particle M by weight G is suspended with two ideal wires, which passing over two very small pulleys (with negligible radius) without friction, by two other weight P and Q as in figure 3. Knowing the positions of the two small pulleys and the fact that the equilibrium of the particle is made in the vertical plane determine the position of equilibrium of the particle M.*

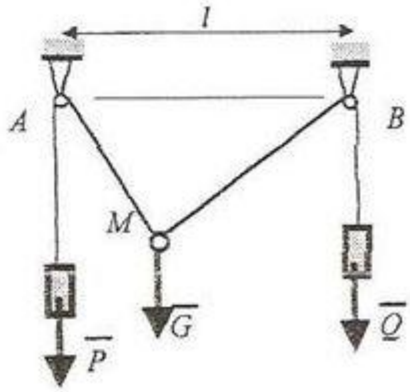


Fig.3.

4.4. Constraints. Axiom of the constraints.

By definition ***any geometrical restriction requires to the particle in the possibility of change its position*** *is called* **constraint**. *For example if a particle is joined by a fixed point with an ideal string (does not broken, not stretched, not resist to bending) by length* ***l****, then the particle does not live the sphere with the center in the fixed point and the radius* ***l****.*

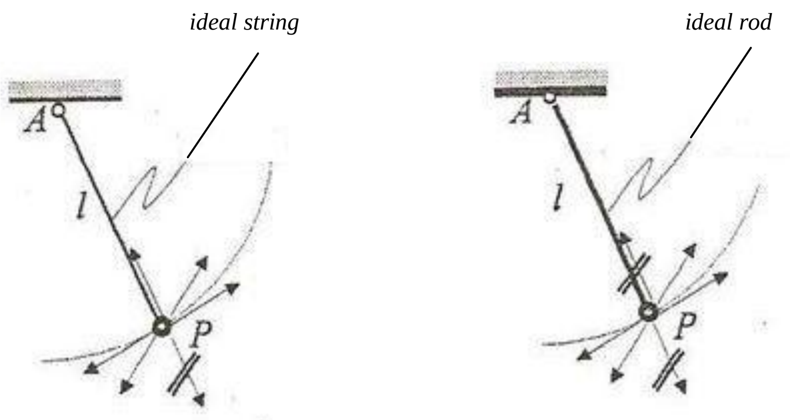


Fig.4.

If the string is replaced with a rigid rod that may perform rotations about the fixed point, then the particle is required to remain on the surface of the sphere with the center in the fixed point and the radius equal to the length of the rod. We remark that in the first example and in the second one the particle loses the possibilities of motion. In the first example the string *eliminates possibility of motion on the direction of the string only in one single sense, namely in the outside of the sphere, while in the second case the rod eliminates the possibilities of motion in the both senses on de direction of the rod. The particle will keep the possibilities of motion in the tangential plane to the sphere (perpendicular on the direction of the string or of the rod) on two directions on the both senses.*

We remark that ***the constraints remove degrees of freedom*** *of the particle. Also we remark that the constraints may remove possibilities of motion in a sense on a direction or in the both senses. If the constraint removes the possibility of motion in a single sense on a direction it is called* ***unilaterally constraint****, and if it removes the possibility of motion in the both senses on a direction then it is called* ***bilaterally constraint****.*

Now, if we consider a Cartesian system of reference with the origin in the fixed point, then the coordinates of the particle have to verify the relation:

$$x^2 + y^2 + z^2 - l^2 \leq 0$$

in the case of the string and:

$$x^2 + y^2 + z^2 - l^2 = 0$$

in the case of the rod. Consequently an unilaterally constraint will be expressed with an inequality and a bilaterally constraint with an equality:

$f(x,y,z) \leq 0$	*unilaterally constraint*
$f(x,y,z) = 0$	*bilaterally constraint*

The last relation is the equation of a surface. These being relations between the position parameters of the particle results that the constraint decrees the number of the independent , scalar position parameters, therefore the relation between the number of the degrees of freedom and the number of the independent scalar position parameters remains unchanged in the case of the constrained particle.

Let to give another example namely to suppose a small ring on a circular frame in vertical plane. It is easy to remark that the frame *eliminates the possibilities of motion on the direction of the radius of the circle and on normal direction on the plane of the circle remaining as single possibility of motion the slipping on the tangent direction to the frame.*

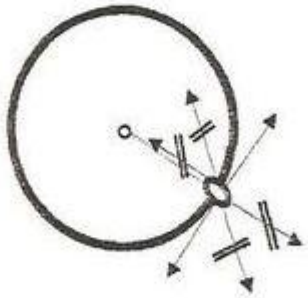


Fig.5.

Results from these examples that there are constraints that remove one degree of freedom or two or more degrees of freedom. The constraints which remove one degree of freedom are called ***simple constraints****, and those which remove more degrees of freedom are called* ***multiple constraints****.*

Remarking that the simple constraint is expressed using a single equation or un-equation, then the multiple constraint is expressed using more equations or un-equations. Therefore a multiple constraint may be considered as a combination of more simple constraints.

Let be consider once more the example made as a circular frame. To suppose also that the small ring has G as its weight.

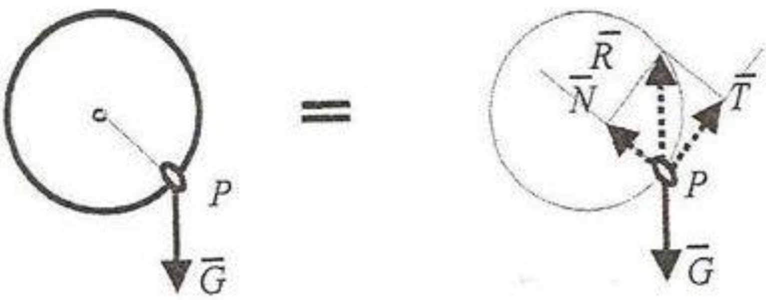


Fig.6.

If we locate the ring in a close position to the lover point of the circular frame we remark that we may find positions in which the particle is in rest. But the particle is acted by its weight G, so it should be, in accordance to *the principle of the action of the force, in motion on vertical direction with an acceleration proportional to the force G. Because the particle is in rest this mean that the action of the weight is canceled by another force that comes from the action of the frame about the particle. Consequently if we remove the frame the position of rest of the particle may be obtained under the action of G and another vertical force, marked R, having the same direction, magnitude and opposite sense as G. But removing the frame the particle may be considered as a free particle and we can state* ***the axiom of constraints****:* ***any constraint may be replaced with a force called reaction force without to change the state (of rest) of the particle****.*

If we decompose the reaction force R in two components: one on the normal direction to the tangential plane in the connection (on the direction of the removed displacement by the constraint) and one in tangential plane, then we may write:

$$\overline{R} = \overline{N} + \overline{T}$$

where N is called ***normal reaction force*** *and T is called* ***friction force****.*

Depending on the existence or not of the friction force we have two kinds of constraints: ***ideal constraint*** *(without friction force):*

$$\overline{T} = 0 \qquad\qquad \overline{R} = \overline{N}$$

namely the reaction force is the normal reaction force and ***real constraint*** *(with friction).*

4.5. Equilibrium of the particle with ideal constraints.

To suppose that a particle is acted by a system of concurrent forces and has some ideal constraints. We shall see to find the conditions in which the particle in rest remains in rest under the action of the forces and the constraints.

For to determine these conditions, first, using the axiom of the constraints we shall remove the ideal constraints with the corresponding *reaction forces. The reaction force of a simple ideal constraint will have the following characteristics:*

- ***The magnitude** is unknown. It will be determined from the conditions to keep the initial state of the particle. In this case the initial state is the rest, or for the forces the equilibrium.*
- ***The direction** of the reaction force is normal on the tangential plane to the surface representing the constraint. This direction is that of the removed displacement by the simple constraint, consequently the direction of the reaction force is known.*
- *For the unilateral constraints **the sense** of the reaction force is inversely as the sense of the removed displacement, and for the bilateral constraint the sense is unknown. But this is not an independent unknown because on a direction there are only two senses, and for to determine the true sense we shall choose a sense (arbitrary) and from computation will results the sense of the reaction force as sign of the magnitude (the sign plus meaning that we sense is the chosen sense and the sign minus meaning that the reaction force has opposite sense).*
- ***The point of application** is the particle.*

We may remark that a simple constraint removes one degree of freedom but introduces in calculation one scalar unknown, the magnitude of the reaction force. Because a simple constraint eliminates one independent, scalar position parameter, in fact a simple constraint does not change the number of the unknowns, it changes only the nature of the unknown, in place of a scalar position parameter we have as unknown a force (the reaction force). Using the axiom of constraints the particle becomes a free particle acted by two kinds of forces: one a system of given forces (active forces, loads) and the other a system of reaction forces (passive forces). The condition as this particle to be in rest, or in equilibrium, is that:

$$\overline{R_g} + \overline{R_r} = 0$$

where we have marked R_g *the resultant force of the given forces and* R_r *the resultant force of the reaction forces. The scalar conditions of equilibrium are:*

$$\Sigma X_{gi} + \Sigma X_{rj} = 0$$
$$\Sigma Y_{gi} + \Sigma Y_{rj} = 0$$
$$\Sigma Z_{gi} + \Sigma Z_{rj} = 0$$

We remark that if we don't make difference between the forces then the conditions are the same as for the free particle.

If we know the simple constraints as equations of the corresponding surfaces then the normal reaction forces can be expressed using the equation of the gradient:

$$\overline{N_j} = \lambda_j \cdot grad\, f_j = \lambda_j \cdot \left(\frac{\partial f_j}{\partial x}\overline{i} + \frac{\partial f_j}{\partial y}\overline{j} + \frac{\partial f_j}{\partial z}\overline{k}\right)$$

With this relation the scalar conditions of equilibrium of the particle with ideal constraints are:

$$\Sigma X_{di} + \Sigma \lambda_j \cdot \frac{\partial f_j}{\partial x} = 0;$$
$$\Sigma Y_{di} + \Sigma \lambda_j \cdot \frac{\partial f_j}{\partial y} = 0;$$
$$\Sigma Z_{di} + \Sigma \lambda_j \cdot \frac{\partial f_j}{\partial z} = 0;$$

As for the free particle we have three kinds of problems: direct problem, inverse problem and mixed problem.

4.6.Sample problems.

Problem 3. *A simple crane is represented as in the figure 7. Calculate the forces from the two rods MA and MB which connect the punctual pulley to the vertical wall. It is know that the pulley is without friction and the ideal string has at one end a weight G that have to lift and at the other end is wrapped on the drum of an engine.*

Solution. *The small pulley is considered a particle because all forces will be applied about it. Because the particle cannot occupy any position in space we have a constrained particle. Also because the particle is in connection with more bodies (two rods and one ideal string) the question is all the bodies are or not constraints. For to know which bodies are constraints we shall make in the following way: all the bodies are considered removed least one, and we check if it prevents the possibility of motion of the particle for to change its position. Then we repeat this with another body.*

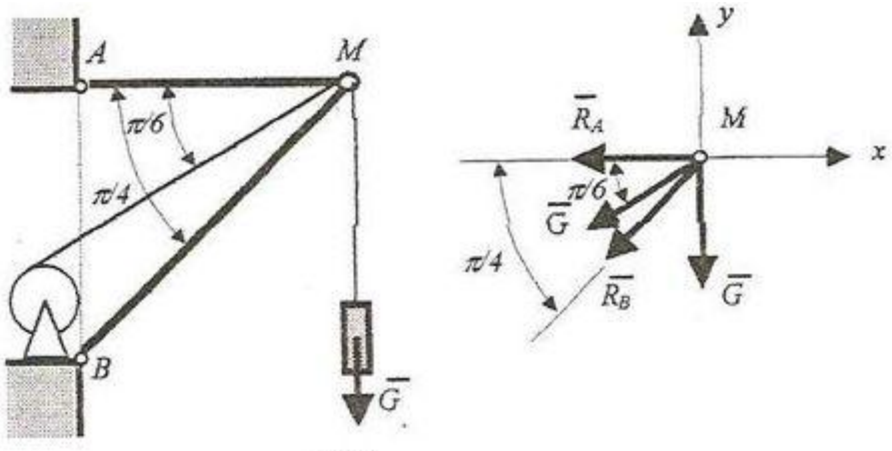


Fig.7.

If we remove all the bodies and we keep only the rod AM we see that it is a constraint because the particle cannot be moved on the direction of the rod. In the same way is for the rod BM. But if we remove the rods and remains only the ideal string we see that this does not oppose to the motion of the particle for to change its position, so the string is not a constraint it is only a body that exerts a force about the particle. These forces are only tensions and they are equals to the forces from the other ends of the strings.

Removing the constraints and replacing them with reaction forces (in this problem they have arbitrary senses because the two rods are bilateral constraints) on the directions of the two rods and replacing the string with two tensions on the directions of the string having the magnitudes equal to G is obtained a free particle acted by a system of concurrent forces from which a part (the given forces) are known and the other part (the reaction forces) are unknown. This scheme of forces is called the ***free particle diagram****.*

We shall express the equilibrium of this particle with two equilibrium equations:

$$\Sigma X_i = 0\ ;\ -R_A - G\frac{\sqrt{3}}{2} - R_B\frac{\sqrt{2}}{2} = 0;$$

$$\Sigma Y_i = 0\ ;\ -G - G\frac{1}{2} - R_B\frac{\sqrt{2}}{2} = 0;$$

Solving this system of two equations with two unknowns result the reaction forces:

$$R_A = 0{,}635\,G\ ;\ R_B = -2{,}12\,G.$$

The sign minus of the reaction force R_B shows that this reaction force has opposite sense as we have considered in the solution of the problem, the reaction force R_A has the true sense because it results with plus.

Problem 4. *A rod AM having the possibility of rotation about its end A is maintained in vertical position under the action of one horizontal force F = 100daN that acts in the point M and, due to a cable anchored in point B. Knowing the length of the rod and the position of the point B determine the the forces from the rod and the cable.*

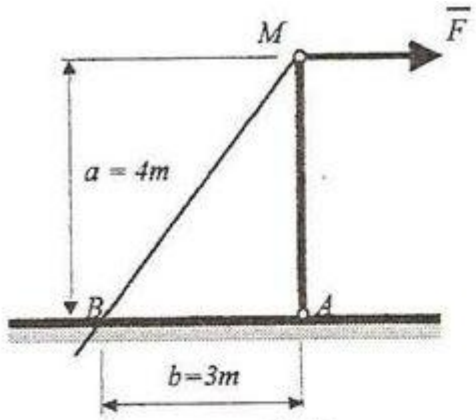


Fig.8.

4.7. Laws of friction. Equilibrium of the particle with constraints with friction.

In the previous section we have studied the equilibrium of the particle with ideal constraints. But as we know the constraints are real, with friction. We shall consider again the example with the small ring on the vertical circular frame. We have seen that in different positions of the ring, located in the neighborhood of the lower point of the frame, the ring acted only by its weight can have different positions of equilibrium. But if these positions are higher and higher on the frame, at an instant, the particle cannot be in rest on the frame and it will descend and it will find an equilibrium position in a lower position. At the other hand as we move up the particle on the circle so the component T of the reaction force will be bigger as magnitude. This thing means that in any conditions the component T (the friction force) cannot exceed a certain value as limit. We shall mark this value T_{max}. This value is determined experimentally and is subject to three laws which are known as the ***laws of friction*** *or as the* ***laws of Coulomb****. These laws are:*

- *1)* ***The magnitude of the maximum friction force does not depend by the magnitudes of the surfaces in contact****. This law makes that these laws to be true for bodies also at which the surface in contact may have a certain area.*

- *2)* ***The magnitude of the maximum friction force is proportional with the normal reaction force on the surface in contact.***
- *3)* ***The magnitude of the maximum friction force depends by the nature of surfaces in contact.***

These three laws are contained in the following formula:

$$T_{max} = \mu . N$$

where μ is called **friction coefficient**, *it is a non-dimensional value and it is determined experimentally.*

From all these results that the friction force has the following characteristics:

- *The magnitude is unknown but it is smaller than its maximum value* T_{max};
- *The direction is in the tangential plane to the support surface;*
- *the sense is opposite to the sense of motion or the tendency of motion;*
- *The point of application is the particle.*

To consider now a particle in rest acted by a system of forces and having constraints with friction. We want to determine the conditions in which the particle in rest remains in rest under the action of the forces and the constraints. Firs we shall replace the constraints with the corresponding reaction forces (using the axiom of constraints), in this way we obtain a free particle acted by two systems of forces: given forces and reaction forces. The scalar conditions of equilibrium will by the same as for the particle with ideal constraints but we add the conditions as the friction forces remain smaller than the maximum values of them:

$$\Sigma X_{di} + \Sigma X_{lj} = 0;$$
$$\Sigma Y_{di} + \Sigma Y_{lj} = 0;$$
$$\Sigma Z_{di} + \Sigma Z_{lj} = 0;$$
$$T_k \leq \mu_k . N_k \quad (k = 1,2)$$

The conditions k = 1,2 is necessary because if k = 0 we have not constraints with friction and if k Δ 3the particle is fixed by its constraints and the particle has not any possibilities of motion so we have not friction and the constraints have ideal behavior.

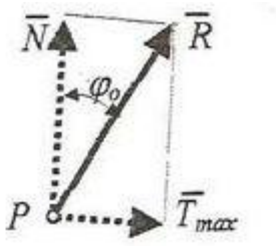


Fig.9.

*To consider now the limit state of the particle when the friction force has its maximum value. We mark φ_o the angle made among the ideal reaction force and the real reaction force. This angle is called **maximum friction angle**. The tangent of this angle may be expressed from one right angle triangle:*

$$tg\ \varphi_o = \frac{T_{max}}{N}$$

From the relationship that defines the value of the maximum friction force results the equality:

$$tg\ \varphi_o = \mu$$

This relationship will allows to determine experimentally the coefficient of the friction.

4.8. Sample problems.

***Problem 5**. Determine the equilibrium position of the particle P having the weight G on a horizontal rough surface (the friction coefficient is μ) and linked to the weight Q with an ideal string that passes without friction over the punctual pulley. Are known G = 100daN, Q = 20daN, μ = 0,1 and the position of the pulley.*

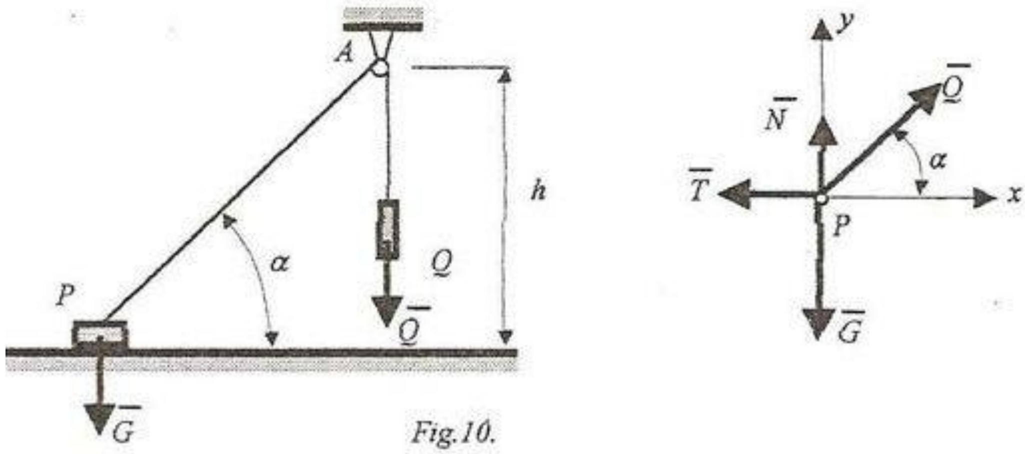


Fig.10.

Solution. *The particle has only one constraint with friction (the horizontal rough surface) because the string is not a constraint. Having one single constraint the particle has one degree of freedom, so it has one possibility of motion to slide on the horizontal plane. Because the force Q the particle has the tendency to move on the right sense.*

We make the free particle diagram replacing the rough surface with two components of reaction : one the normal reaction force directed up and o horizontal friction force (tangent to the rough surface) directed in left side (opposite to the tendency of motion under the action of the given forces). The string is replaced with a tension equal as magnitude of the Q force hanging by the string.

Being a particle with one degree of freedom, one single scalar position parameter will define completely its equilibrium position in plane. Knowing the position of the pulley we shall choose this parameter as the angle made by the string with the horizontal direction (angle α).

The scalar conditions of equilibrium will be:

$$\Sigma X_i = 0 \; ; \; -T + Q\cos\alpha = 0 \; ;$$
$$\Sigma Y_i = 0 \; ; \; N + Q\sin\alpha - G = 0 \; ;$$
$$T \leq \mu N$$

From the first two equations we have:

$$T = Q\cos\alpha \; ; \; N = G - Q\sin\alpha$$

which removed in the un-equality results:

$$Q\cos\alpha \leq \mu\,(G - Q\sin\alpha)$$

or:

$$Q\,(\cos\alpha + \mu\sin\alpha) \leq \mu G$$

We may write:

$$\mu = tg\,\varphi_o = 0{,}1 \longrightarrow \varphi_o = arctg\,\mu = 5^{\circ}\,50'$$

Replacing the friction coefficient with the tangent of the maximum friction angle the previously un-equality may be written in the following form:

$$\frac{Q}{\cos\varphi_o}(\cos\alpha . \cos\varphi_o + \sin\alpha . \sin\varphi_o) \leq tg\varphi_o . G$$

or:

$$\cos(\alpha - \varphi_o) \leq \frac{G}{Q}\sin\varphi_o$$

Solving the un-equality we find:

$$\alpha \geq \arccos(\frac{G}{Q}\sin\varphi_o) + \varphi_o = 59^o30' + 5^o50' = 65^o20'$$

namely the particle will have the equilibrium position if the angle made by the string PA with the horizontal direction is bigger than $65^o20'$, so if it is closer by the vertical line passing through the point A.

Problem 6. *Determine the equilibrium position of the particle P by weight G = 10daN that there is on a cylindrical rough surface by the radius R = 1,00m and the coefficient of friction μ = 0,05.*

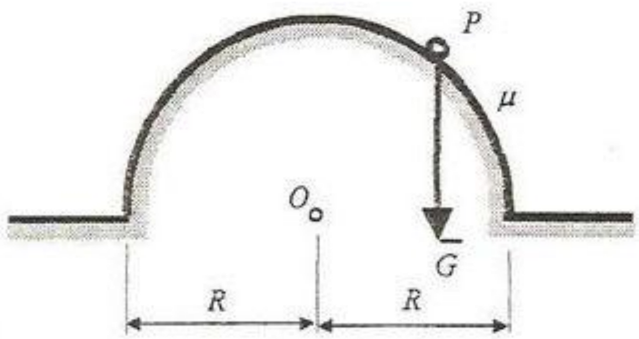


Fig.11.

Chapter 5. Statics of the rigid body.

5.1. Introduction.

The rigid body is the non-deformable body namely the body at which the distance among two any points remains unchanged indifferent to the actions about it.

*If the body may have any position in space we have a **free rigid body**. But if the body cannot have any position in space because restrictions imposed to its points then it is a **constrained body**.*

In this chapter we shall study the conditions in which a body in rest remains in rest under the action of the forces and the connections with other bodies.

5.2.Equilibrium of the free rigid body.

Suppose a free rigid body. For to define the position a free body with respect to a system of reference we shall make in the following way: first we define the position of one point from the body, for example the point A (in fact we fix this point). We see that if we fix a point from the body, the body is not fixed because it has possibilities to rotate about this point without to modify the coordinates of this point (the position parameters of the point). We shall define the position of another point from the body in fact we fix another point B at the distance $\boldsymbol{l}_1$ *from A. We see again that the body is not fixed because it may perform a rotation about the straight line passing through these two points. Consequently we shall consider defined (fixed) the third point using its coordinates and the two distances* $\boldsymbol{l}_2$ *and* $\boldsymbol{l}_3$ *with respect to the other two points* *and we see that in this case the body is completely defined in a unique position in space. Finally we may say that the position of a free body in space is defined through three points namely through nine scalar independent position parameters (the nine coordinates of the three points with respect to the chosen system of reference).*

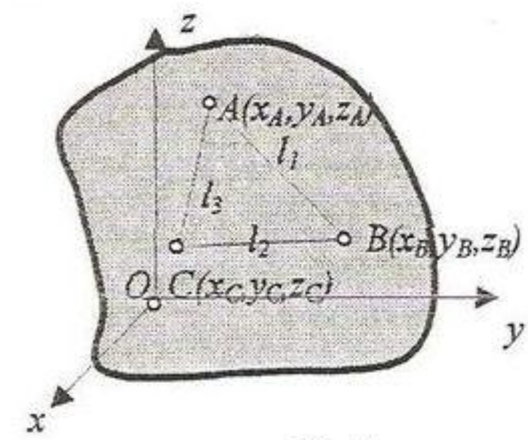


Fig.1.

But between these nine coordinates and the three distances between the three points we have the following three relationships:

$$l_1^2 = (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 ;$$
$$l_2^2 = (x_C - x_B)^2 + (y_C - y_B)^2 + (z_C - z_B)^2 ;$$
$$l_3^2 = (x_A - x_C)^2 + (y_A - y_C)^2 + (z_A - z_C)^2 .$$

From all these results that ***the position of a free body is defined by six scalar independent position parameters****.*

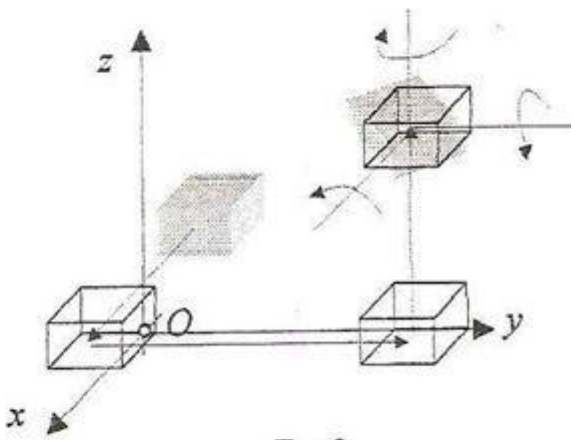


Fig.2.

*Suppose now a body (having parallelipipedium shape for to see easier the way in which it moves) in an any position (here with the sides parallel with the axes of the system). For to change its position (from an any position in another any position) there are six independent possibilities of motion namely three translations parallel to the three axes of the system and three rotations about three axes parallel to the axes of the system of reference. We have defined the independent possibilities of motion as degrees of freedom, so **the free rigid body has six degrees of freedom**.*

The relation between the number of degrees of freedom and the number of the independent scalar position parameters is also true in this case as for the particle:

$$N_{DF} = N_{SIPP}$$

*Let to consider now the plane problem of the rigid body (the body in two dimensions). We can see that the position of the body is defined completely defining the position of two points of the body. Knowing the mutual positions of the two points (the distance **l**) we may write:*

$$l^2 = (x_A - x_B)^2 + (y_A - y_B)^2$$

*resulting that **the position of a free rigid body in plane is defined by three scalar independent position parameters**.*

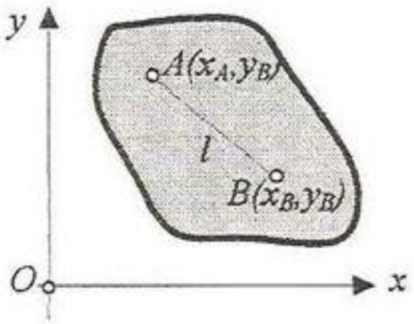


Fig.3.

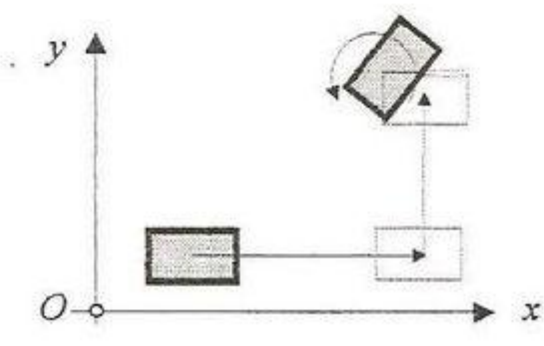


Fig.4.

If we want to change the position of the body we have three independent possibilities of motion (two translations parallel to the axes of a *system of reference and a rotation about a point). We say that* ***the rigid body in plane has three degrees of freedom****.*

Now, suppose a free rigid body in rest. We want to find the conditions in which the body remains in rest under the action of a system of forces. This may be obtained if the system of forces has zero effect about the body. We have seen that for a system of forces to have no effect about a body iy has to be in ***equilibrium****. We shall say in this case that* ***the rigid body is in equilibrium****. The conditions of equilibrium are:*

$$\bar{R} = 0\ ;\ \bar{M}_O = 0$$

or:

$$\Sigma \bar{F}_i = 0\ ;\ \Sigma \bar{M}_{Oi} = 0$$

The scalar conditions of equilibrium are obtained projecting the previous vector equations on the axes of a convenient system of reference:

$$\Sigma X_i = 0\ ;\ \Sigma Y_i = 0\ ;\ \Sigma Z_i = 0\ ;$$
$$\Sigma M_{xi} = 0\ ;\ \Sigma M_{yi} = 0\ ;\ \Sigma M_{zi} = 0\ .$$

In plane problem (in two dimensions) when the body is acted by a system of coplanar system of forces in the plane of the body these conditions are:

$$\Sigma X_i = 0\ ;\ \Sigma Y_i = 0\ ;\ \Sigma M_{Oi} = 0$$

namely we shall use two equations of forces and one moment equation. In fact these conditions express the condition as the forces acting about the body do not cause horizontal or vertical translations and rotations about any points from plane.

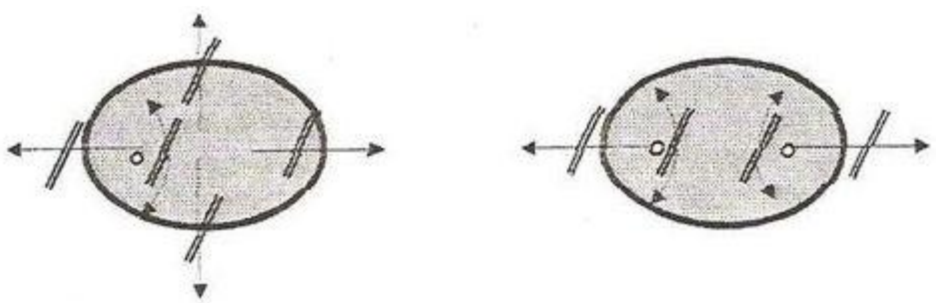


Fig.5.

But the same effect can be expressed if the body do not perform horizontal translation and two rotations about two points located on the same horizontal straight line. We see that if the body doesn't rotate about two points then it doesn't perform any vertical translation. From this results that the equilibrium of the body can be expressed with one force equation and two moment equations about two different points:

$$\Sigma X_i = 0 \; ; \; \Sigma M_{Ai} = 0 \; ; \; \Sigma M_{Oi} = 0$$

This propriety can be obtained in another way also. Suppose that the system of forces is made from vertical forces only (Y_i). It is obviously that one equilibrium condition is that the resultant moment about any point (for example point A) to be zero:

$$\Sigma M_{Ai} = \Sigma Y_i \, . \, d_i = 0 \; ;$$

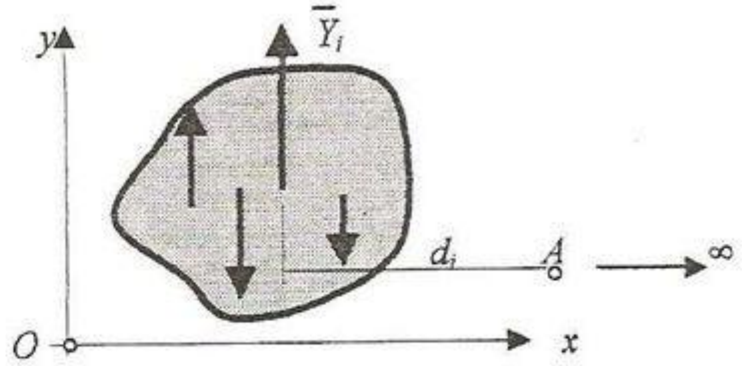


Fig.6.

If the point A is one located at large distance from the body, more bigger as the distances between the forces of the system then we may write the relation:

$$\Sigma M_{Ai} = (\Sigma Y_i) \, . \, d \approx 0$$

*because the distances may be considered equals (it is true if the distance **d** tends to infinity). At the limit we can say that the condition of equilibrium is true if we have:*

$$\Sigma Y_i = 0$$

so ***a moment equation is equivalent to a projection equation****.*

But we have an indeterminacy. Consequently we shall analyze this case.

As we know in the case of reduction of a system of forces in plane we have three cases of reduction (equilibrium, couple and unique resultant force). If we use moment equations it is obviously that one single moment equation eliminates the possibility to reduce the system to a couple. This mean that we have only two possibilities of reduction namely equilibrium or unique resultant force. Suppose that the system is reduced to a unique resultant force.

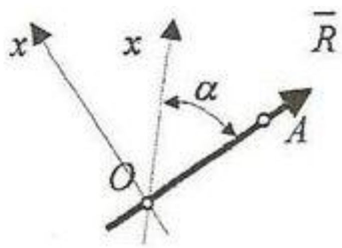


Fig.7.

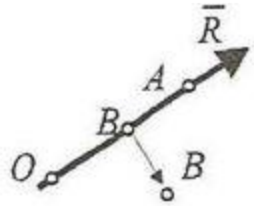


Fig.8.

We write the moment equations about the points O and A. These equations are checked in the case of unique resultant force if the two points are situated on the support line of the resultant force. The projection equation on the Ox axis can be checked if, and only if, this axis is perpendicular on the support line of the resultant force. For as these equations to be conditions of equilibrium, so the resultant force to be equal to zero, it is enough as the Ox axis to be different as perpendicular on the straight line passing through the points about which we considered the moment equations. We remark that in this case the projection equation is checked only if R = 0. Results that this way to express the equilibrium (with one projection equation and two moment equations) can be used with the condition that the projection direction to be different as the perpendicular on the straight line passing through the points of calculation the moments:

$$\Sigma X_i = 0\ ;\ \Sigma M_{Ai} = 0\ ;\ \Sigma M_{Oi} = 0\ ;\ Ox \measuredangle AO$$

Like one projection equation may be removed with one moment equation, in the same way we may remove the both projection equations with moment equations obtaining the scalar conditions of equilibrium with three moment equations about three different points. Also in this case we have to eliminate the case when these conditions do not express equilibrium. We remark that if the system of forces is reduced to a unique resultant force the three equations are checked if the three points are situated on the support line of the resultant force. It is enough one point from the three to be out of the support line and the equilibrium condition is checked only if R = 0. Results that the equilibrium may be expressed with three moment equations if the three points, about which are written the moment equations, are not collinear:

$$\Sigma M_{Ai} = 0 \ ; \ \Sigma M_{Bi} = 0 \ ; \ \Sigma M_{Oi} = 0 \ ;$$

O,A,B non-collinear

5.3. Ideal constraints of the rigid body in plane (in two dimensions).

In this section we shall show the ideal constraints of the rigid body in plane because we are interested only by the fixed rigid body in plane case in which all the constraints may be considered ideal. We shall study only the constraints for the plane problem, but in each case we shall make remarks about the constraints in space and their behavior.

The ideal constraints are classified function the number of the degrees of freedom removed by them. In this way we have three constraints: **simple support, hinged support (hinge) and fixed support**. *For each constraint we shall present how are made, which are the schemes and the mechanical equivalent of them.*

Simple support. *By definition the simple support is the ideal constraint (punctual and without friction) that remove one degree of freedom. This constraint was studied at the particle as the simple constraint. It can be made as an ideal string joined a point of the body to a fixed point, or as a rigid rod having possibilities of rotation about its ends (a pendulum, or resting in a point on a rigid and fixed surface (the surface of another body). In all these*

Fig.9.

cases is removed the possibility of motion on the direction of the string, or of the rod, or of the normal direction on the tangent in the point of support. In this way the simple support removes, always, ***one degree of freedom****. The other two possibilities of motion are free (here the translation on the normal direction and the rotation).*

The simple support has two corresponding schemes. The first, represented in the figure 10., is represented as a small triangle not fixed by the external support surface and where the normal direction on the two small parallel lines is the direction of the removed displacement and it is called ***direction of the simple support****. This scheme is considered bilaterally constraint and it will be used as scheme for the simple support in this part of the theoretical mechanics. The second scheme is represented as a small pendulum (generally it has infinitesimal length) as in the figure 11. This scheme will be used in space and it has the same properties as the previous scheme.*

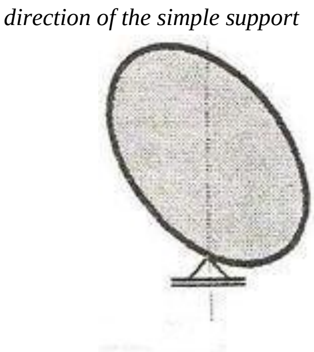


Fig.10.

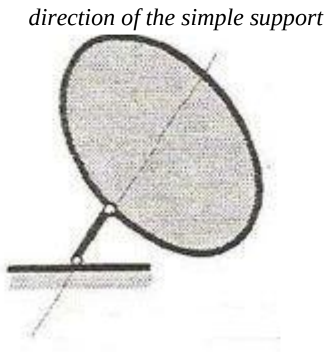


Fig.11.

Using the axiom of constraints the simple support can be replaced with a reaction force having the following characteristics:

- *The magnitude is unknown and it will be calculated from the condition of equilibrium of the rigid body;*

- *The direction is the direction of the simple support (the direction of the restricted displacement);*
- *The sense is unknown but it is not an independent unknown because we choose an any sense and the true sense results as sign from the computation of the magnitude.*

Fig.12.

We remark that a simple support removes one degree of freedom and ***introduces in calculation one scalar unknown as force****.*

Hinged support *or the* ***hinge.*** *By definition the hinged support is the constraint that fix a point of the body. This constraint is a double constraint in plane* ***removing two degrees of freedom****, the two translations on the two directions, the body keeps only the rotation about the fixed point. The hinged support can be obtained suspending a body in a point with two ideal wires, or replacing the two strings with two pendulums.*

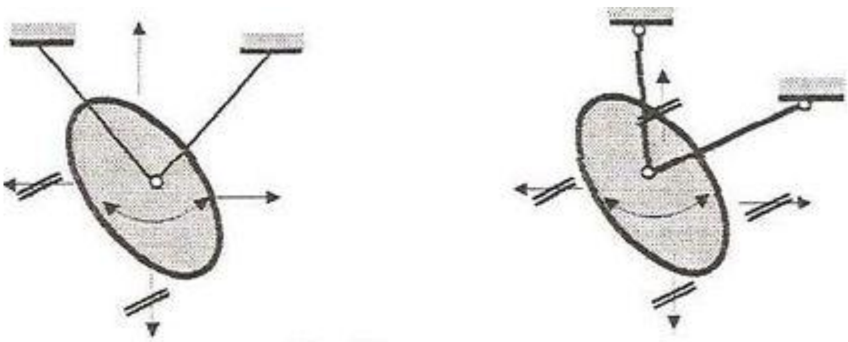


Fig.13.

The scheme used in theoretical mechanics, in plane, for this constraint is a small triangle joined by the supporting surface (Fig.14.), but may be used also a scheme from two small pendulums (Fig.15.).

Fig.14.

Fig.15.

Using the axiom of constraints the hinge may be removed with a reaction force with the following characteristics:

- *The magnitude is unknown and it will be calculated from the conditions of equilibrium of the body;*
- *The direction is unknown;*
- *The sense is unknown but is not an independent unknown, resulting as sign from the calculation of the magnitude of the reaction force.*

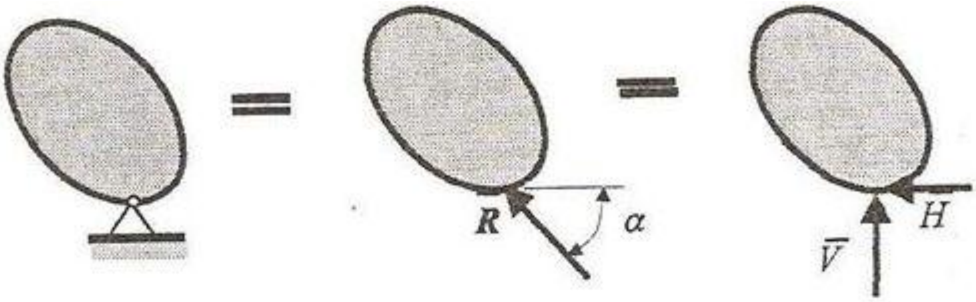


Fig.16.

In the problems we prefer to work, in place of a force having two unknown, two unknown reaction forces on known directions (generally on horizontal and vertical directions). Finally we see that a hinge removes two degrees of freedom and introduces in computation two unknown forces.

In space we have more kinds if hinged supports: spherical hinged support that fixes a point of the body and removes three degrees of freedom, the cylindrical hinged support with fixed direction (the bearing) that removes four degrees of freedom and the cylindrical hinged support with variable direction that removes two degrees of freedom.

***Fixed support** is the ideal constraint that removes all the degrees of freedom. This constraint is made introducing the body in other fixed* *body or welding the body to the external bodies. This constraint may be obtained also combining three pendulums so that they remove all degrees of freedom of the body.*

We remark that this constraint cannot be made in a single point. Consequently for to be an ideal constraint it is chosen a theoretical point of the fixation (that generally is the centroid of the connection surface).

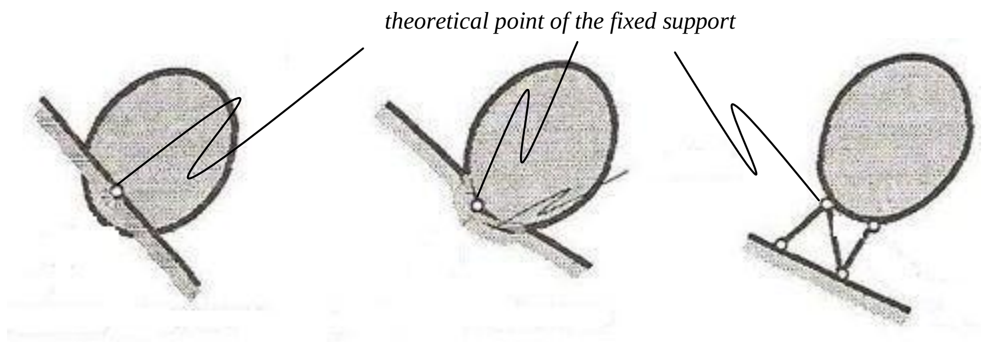


Fig.17.

Because the bodies used in this part of the mechanics will be represented as bars this theoretical point of fixation is the intersection point between the axis of the rod and the surface of fixation. The scheme of this constraint is represented in the figure 18.

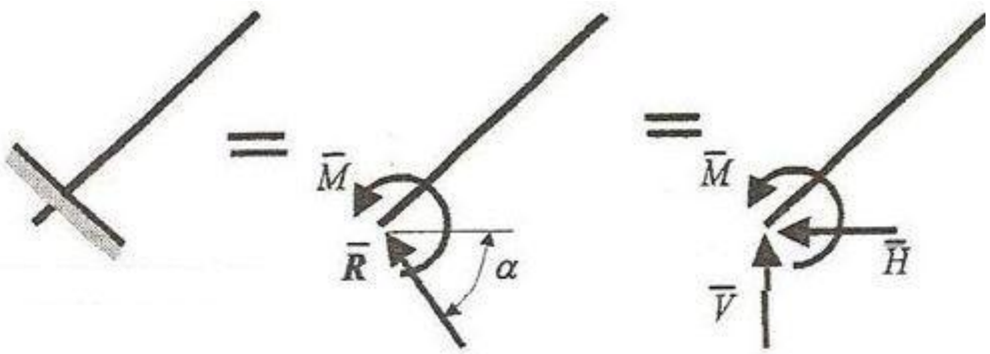


Fig.18.

Being a non punctual constraint it will be replaced with a system of reaction forces. Reducing these forces in the theoretic point of fixation we have the force-couple system of the reactions made from two components: a *reaction force having its magnitude and its direction unknowns and a moment reaction with unknown magnitude. For to simplify the using of this reactions we shall use two unknown reaction forces on two known directions and an unknown couple (moment).*

The fixed support removes three degrees of freedom and introduces in the computation three unknown reactions.
The fixed support in space removes six degrees of freedom.

5.4. Statically determined and stable rigid body

The intention to use the constraint is to fix the body. In this part of the theoretical mechanics we shall use the constraint for to fix the bodies but, this have to make so that using the equilibrium equations to can determine the reactions from the constraints. We shall say that if a body is fixed and has in the constraints the same number of scalar unknowns as the number of the scalar independent equilibrium equations the body is ***statically determined and stable****. Consequently the statically determined and stable body checks two conditions:* ***the first*** *is that:*

$$N_E = N_U$$

where N_E is the number of the scalar independent equilibrium equations and N_U is the number of the scalar unknowns in the constraints of the body, and ***the second*** *is: the rigid body to be fixed by its constraints. In fact the first condition is a quantitative condition namely the number of equations to be the same as the number of the unknowns, and the second is a qualitative condition as the system of equations to be compatible.*

Knowing that for a free rigid body in plane we can write only three scalar independent equilibrium equations we shall show the immobilization schemes of the rigid body in plane.

Using only simple supports and knowing that one simple support removes one degree of freedom and introduces in computation one scalar unknown, for to fix the body we need ***three simple supports****.*

If we want to fix a body using only hinged supports we see that one hinge doesn't fix the body and two hinges fixes the body but have four unknowns in them. In this way we shall combine ***one hinged support with a simple support*** *obtaining a fixed body with three unknowns in the constraints.*

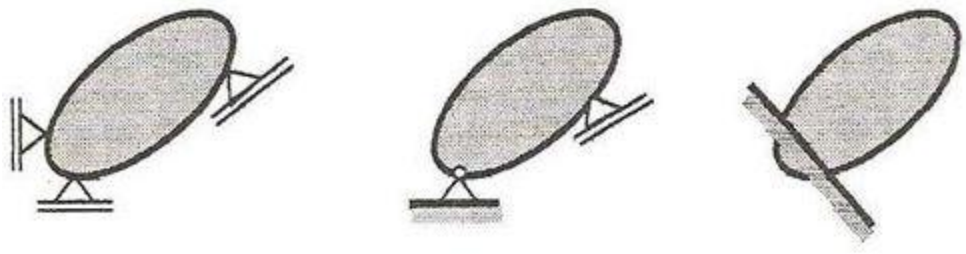


Fig.19.

Finally the body may be fixed with ***one fixed support*** *that introduces in equations three unknowns.*

If between the number of the equations and the number of the unknowns is the relation:

$$N_E < N_U$$

then the body is said to be a ***statically non-determined body****, and if there is the relation:*

$$N_E > N_U$$

then the body remains with degrees of freedom and we say it is a ***mechanism****.*

There are cases when the quantitative condition is checked but the qualitative condition doesn't, the body keeping degrees of freedom. These cases are called ***critical forms****. The first case of critical form is that when the* ***three simple supports have parallel directions****. We remark that the possibility of translation on the normal direction on the directions of the simple supports is not blocked by the constraints.*

Another critical form is obtained when ***the directions of the three simple supports are concurrent in the same point****. Knowing that the displacements of the points of a rigid body in a rotation about a point are perpendicular on the radii from the rotation center, the three simple supports*

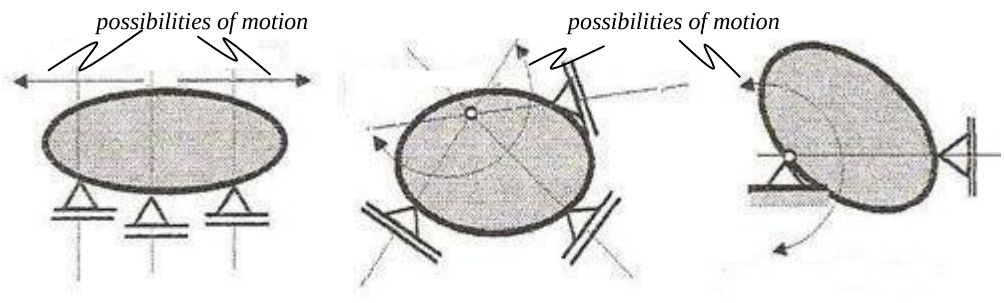


Fig.20.

don't prevent these displacements, so the body keeps the possibility of rotation about that point.

*Finally another critical form is obtained if **the direction of the simple support passes through the hinge** because in this case the body keeps the possibility of rotation about the hinge.*

It is obviously that the fixed support doesn't produces critical form.

We can see that in the three cases of critical forms the body checks condition of statically determination (the quantitative condition), the condition of mechanism (having possibilities of motion) and the condition of statically non-determination (because we have only two independent equations with three unknowns).

5.5.Loads.

*The loads are active systems of forces (generally given systems of forces) that act the bodies. Depending by the way of action about the bodies we have two kinds of loads: **concentrated loads** and **distributed loads**. Also the loads can be: **forces** or **moments** (couples).*

- ***Concentrated force**. Is a force that acts in a point on the rigid body. This force has force effect and moment effect also calculated as in the previously chapters.*
- ***Concentrated moment**. Is the load equivalent to a system of forces reduced to a couple. This load acts in a point about the rigid body and it is represented as in the figure 21. because for the rigid body the moment of a couple is a free vector, in the problems involving rigid bodies doesn't interested its point of application. The main characteristic of this load is*

Fig.21

that it has not force effect (it is not considered in the projection equations) and the moment effect is the same about all points in plane (so in each moment equation will have the same value and the same sign).

For the distributed loads we shall show only the distributed forces. These forces are classified function the law of distribution. We shall show only three kinds of distributed forces, the usual cases used in problems.

- ***Uniformly distributed force**. This load is a system of forces having continuous and uniformly distribution along a straight line. The schemes of these forces are represented in the figure 22. The scheme will present the following elements: **p** – the intensity of the load, **l** – the length and the direction of distribution, the direction and sense of action of the force (For example in the figure 22.a. is represented a uniformly distributed force, vertical action, having **p** as its intensity, distributed on the horizontal length **l**). As we can see in the figure 22 are represented three systems of parallel forces and the last is a system of concurrent forces.*

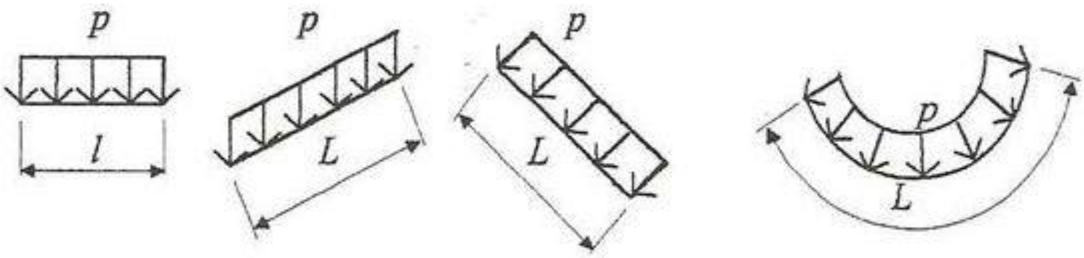


Fig.22.

Being a system of forces, the uniformly distributed force can be replaced with a resultant force having as magnitude the product between the intensity of it and the length of distribution:

$$R = p \cdot l \quad or \quad R = p \cdot L$$

The direction and the sense of the resultant is the same of the components of the system and the position is in the middle of the distribution length.

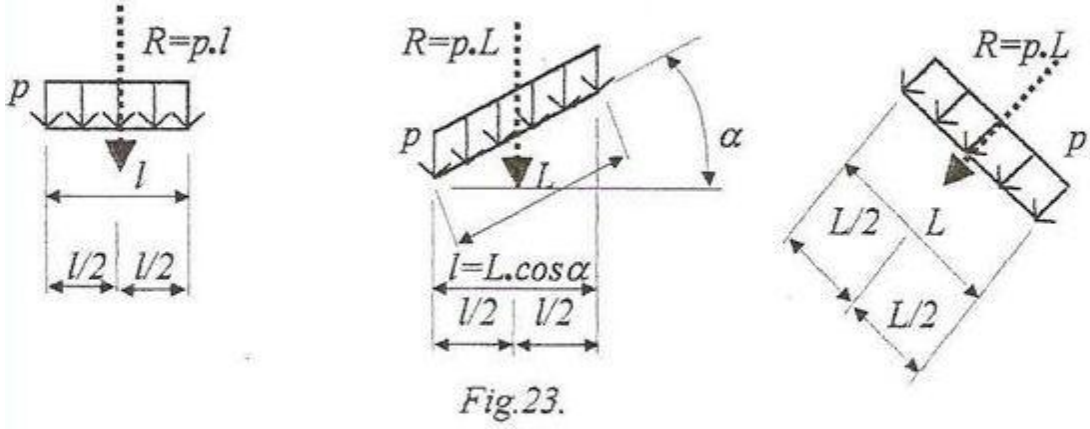


Fig.23.

- ***Triangular distributed force****. As in the case of the uniformly distributed force we have a system of continuous distribution. The law of variation of this distribution is triangular so a linear variation starting from zero. The schemes of these forces contain the following elements:* ***q*** *– the maximum intensity of the load,* ***l*** *– the length of distribution, and the direction and sense of action of the force.*

The system of forces may be replaced with its resultant force having the magnitude:

$$R = \frac{pl}{2} \quad or \quad R = \frac{pL}{2}$$

having the direction and sense of the components of the load. The position of this resultant is at a third from the length of distribution measured from the base of the triangle representing the scheme of this load, or at two thirds from the top of the triangle.

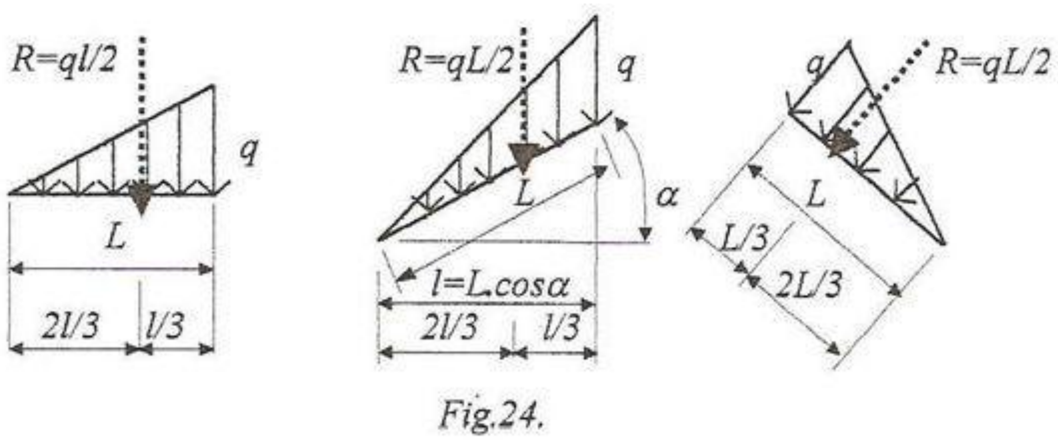


Fig.24.

- ***Trapezoidal distributed force***. *This load is a system of forces distributed linearly as the triangular distribution but the first value is not zero. The simplest way to work with this force is that to consider a superposition (an addition) of two triangular forces: one with the maximum value* $\boldsymbol{q_1}$ *and the second with the maximum value* $\mathbf{q_2}$ *and with the same length of distribution.*

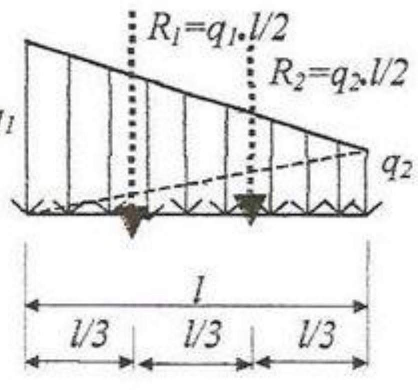


Fig.25.

The two resultant forces corresponding to the two triangular distributed forces will be located at a third from the length of distribution from the two ends of the load.

If in the scheme of the load is not represented the sense of action we shall consider the force acting toward the body.

5.6. *Steps to solve the reactions from the constraints of a statically and stable rigid body .*

In the solution of a problem in which we determine the reactions from the constraints of a statically and stable rigid body are passed the following steps:

- *1)We check if the body is or not statically determined and stable, namely it has or not three unknowns in the constraints and it is fixed or not;*

- 2)We remove the constraints with the corresponding reactions;
- 3)We arrange the loads, namely the forces are divided into two components on the convenient directions (generally on horizontal and vertical directions) and the distributed forces are replaced with the corresponding resultant forces. Finally we shall obtain a free rigid body acted by concentrated forces and couples (a part of them known forces and couples – the loads and the other part unknown forces and couples - the reactions) representing ***the free body diagram***;
- 4)We write three equilibrium equations. If the body has not fixed support then we prefer to write two moment equations about two constraints and one projection equation but if the body has fixed support we prefer to write two projection equations and a moment equation about the fixed support:
- 5)We solve the system of the three equations resulting the reactions from the constraints. These reactions are represented , with the real senses, on the scheme representing the free body diagram;
- 6)We check the solution writing one equation non-used for the solution.

5.7.Sample problems.

Problem 1. *Calculate the reactions from the constraints of the body represented in figure 26.*

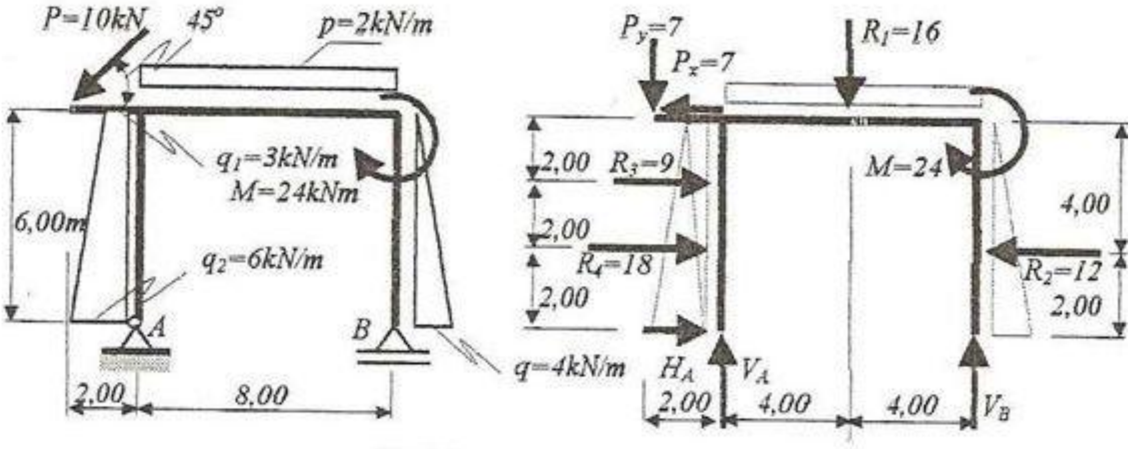


Fig.26.

Solution. Step 1. *The body is statically determined and stable because it has a hinge and a simple support, and the direction of the simple support doesn't pass through the hinge.*

Step 2. *We shall draw the body without to change its shape and the dimensions and without constraints, namely the free body. The constraints will be replaced with the corresponding unknown reactions: in A two forces, one horizontal H_A and the other vertical V_A and in B one vertical reaction force V_B.*

Step 3. *The loads will be arranged **.** In this way the concentrated force P will be decomposed in two components, one horizontal and the other vertical:*

$$P_x = P \,.\, cos\alpha = 10 \,.\, cos\, 45^o = 7\ kN;$$
$$P_y = P \,.\, sin\alpha = 10 \,.\, sin\, 45^o = 7\ kN,$$

The distributed force is replaced with its vertical resultant force R_1 acting in the middle of the distribution interval:

$$R_1 = 2 \,.\, 8 = 16\ kN,$$

The triangular distributed force is replaced with the horizontal resultant force R_2 acting at one third from the length of distribution measured from the point B (where is the base of the triangle representing the scheme of the load):

$$R_2 = \frac{4.6}{2} = 12\ kN,$$

The trapezoidal distributed force will be replaced by two horizontal resultant forces R_3 and R_4 acting each at a third from the distribution interval from the ends of the force:

$$R_3 = \frac{3.6}{2} = 9\ kN; \quad R_4 = \frac{6.6}{2} = 18kN.$$

*We remark that we have now a free body acted by concentrated forces and couples from which a part are known (the given forces – the loads) and a part unknown forces (the reaction forces). This is **the free body diagram**.*

Step 4. *We shall write three equilibrium equations. We prefer to wrote two moment equations about the two constraints (with respect to these points cancel some moments of the unknowns) and one projection equation on the horizontal direction. This direction is collinear with the straight line that passes through the two points about which we write the moment equations and consequently it is not perpendicular on that direction. The equations will be:*

$$\Sigma X_i\ ;\ H_A + 18 + 9 - 7 - 12 = 0;$$
$$\Sigma M_{Ai} = 0\ ;\ -18.2 - 9.4 + 7.6 + 7.\,2 - 16.4 - 24 + 12.2 + V_B.8 = 0\ ;$$
$$\Sigma M_{Bi} = 0\ ;\ -V_A.8 - 18.2 - 9.4 + 7.6 + 7.\,10 + 16.4 - 24 + 12.2 = 0\ ;$$

Step 5. *We solve the system of three equations with three unknowns (we remark that the three equations are independent equations each of them containing one unknown). We obtain the following values:*

$$H_A = -8\ kN\ ;\ V_A = 13\ kN\ ;\ V_B = 10\ kN.$$

The sign (-) for the reaction H_A means that the chosen sense of this reaction force is not the real sense, it is opposite as we have taken in the free body diagram.

Step 6. *We shall make the verification of the correct writing of the equations and their solutions writing another equation that was not used for to solve the problem. This equation will be a projection on the vertical direction:*

$$\Sigma Y_i = 0 \; ; \; V_A - 7 - 16 + V_B = 0$$

This equation confirms the values of the reactions.

Problem 2. *Calculate the reactions from the fixed support A for the body represented in the figure 27.*

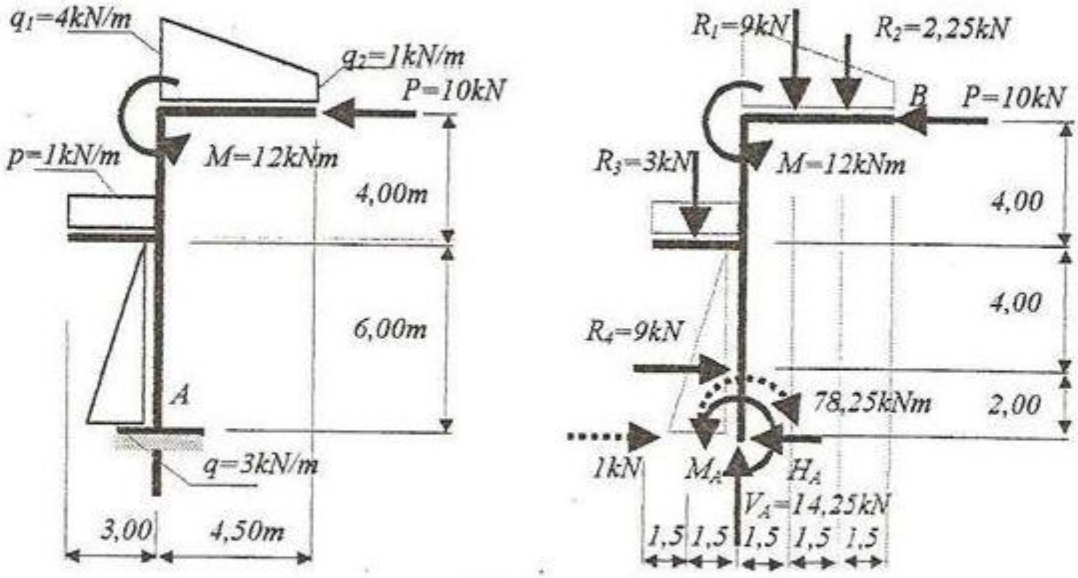


Fig.27.

Solution. *We shall make as in the previously problem. Firs we make the free body diagram replacing the fixed support with the three reactions (two forces and a concentrated moment) and the distributed force replaced with the corresponding resultant force.*

The three equilibrium equations will be two projections and one moment about the fixed support. In this way each equation will contain one single unknown. We have:

$$\Sigma X_i = 0 \; ; \; 9 - H_A - 10 = 0 \; ; \;\longrightarrow\; H_A = -1 \text{ kN}$$
$$\Sigma Y_i = 0 \; ; \; V_A - 3 - 9 - 2{,}25 = 0 \; ; \;\longrightarrow\; V_A = 14{,}25 \text{ kN}$$
$$\Sigma M_{Ai} = 0 \; ; \; M_A - 9 \,.\, 2 + 3 \,.\, 1{,}5 + 12 - 9 \,.\, 1{,}5 - 2{,}25 \,.\, 3 + 10 \,.\, 10 = 0 \; ;$$
$$\longrightarrow\; M_A = -78{,}25 \text{ kNm.}$$

Writing in this way the equations each equation will be solve independently. The negative signs of the results means that those reactions are in opposite senses, namely the horizontal reaction force will be directed toward the right and the concentrated moment will have clockwise sense. We shall represent these senses with dotted lines.

We make the verification using a moment equation about an any point, for example with respect to the point of application of the force P:

$$\Sigma M_{Bi} = 0 \; ; \; 2{,}25.1{,}5 + 9.3 + 12 + 3.6 + 9.8 + 1.10 - 78{,}25 - 14{,}25.4{,}5 = 0$$

The computation is correct.

Problem 3. *Calculate the reactions from the three simple supports for the body from the figure 28.*

Solution. *Making as in the previous two problems we shall obtain the free body diagram replacing the three simple supports with one reaction forces each on the directions of the simple supports and replacing the distributed forces with corresponding resultant forces.*

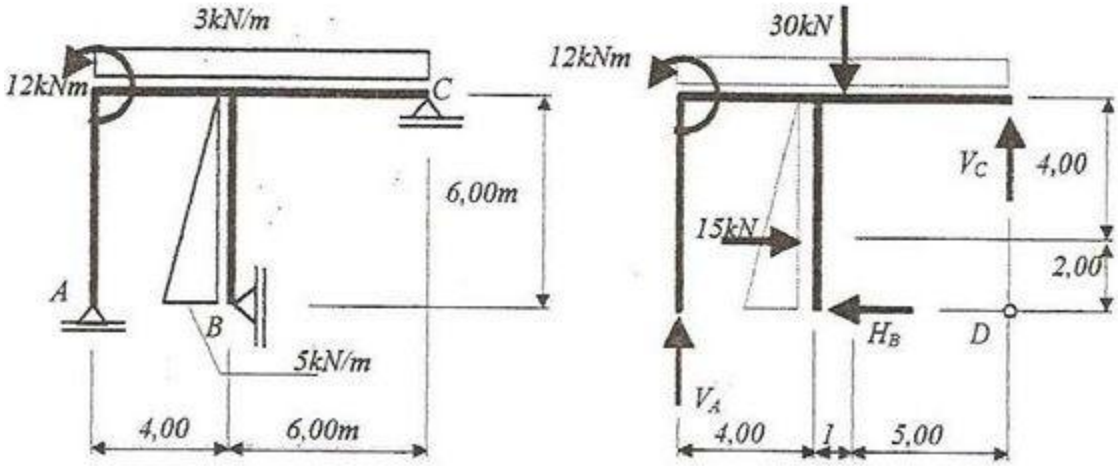


Fig.28.

In this case, as in the previous cases, for to express the equilibrium conditions we may use one of the previous way. For to obtain independent equations we prefer to write three moment equations about the three intersection points where the directions of the three simple supports intersect two by two. In this problem, because two simple supports have parallel directions we shall use two moment equations (about the points A and D) and one projection equation on the horizontal.

$$\Sigma X_i = 0\ ;\ 15 - H_B = 0\ ;\ \longrightarrow H_B = 15kN;$$
$$\Sigma M_{Ai} = 0\ ;\ 12 - 15\,.\,2 - 30\,.\,5 + V_C\,.\,10 = 0\ ;\ \longrightarrow V_C = 16{,}8\ kN;$$
$$\Sigma M_{Di} = 0\ ;\ -V_A\,.\,10 + 12 - 15.\,2 + 30\,.\,5 = 0\ ;\ \longrightarrow V_A = 13{,}2\ kN$$

Because we have independent equations the solution of the system is find very easy, the unknowns resulting one by one from each equation.

For to check the solution we shall use one projection equation on vertical direction:

$$\Sigma Y_i = 0\ ;\ 13{,}2 - 30 + 16{,}8 = 0.$$

Problems 4,5, 6. *Determine the reactions from the constraints of the bodies represented in the figures 29, 30 and 31.*

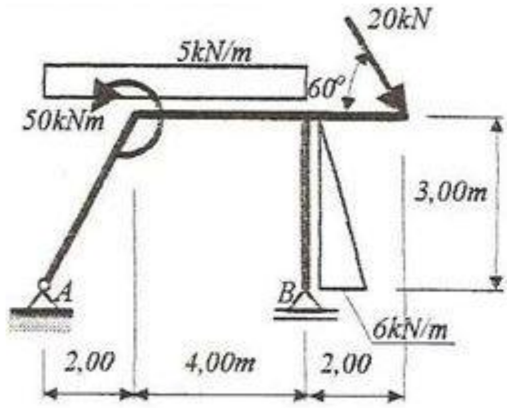


Fig.29 .

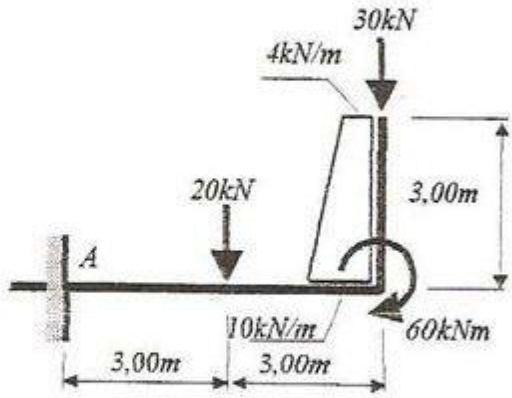


Fig.30.

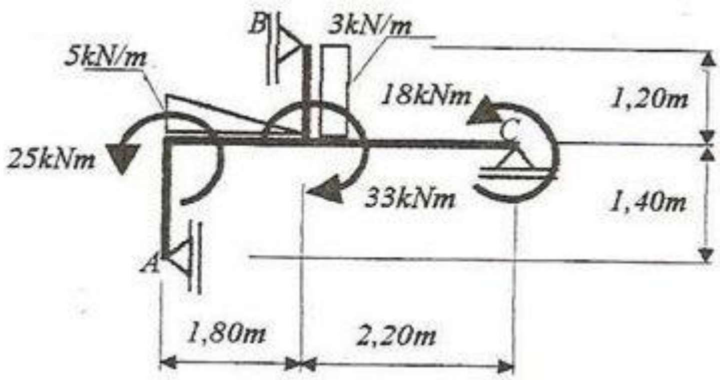


Fig.31

Chapter 6. Systems of rigid bodies.

6.1. Introduction.

In the previous chapter we have studied the equilibrium of one single rigid body. But in reality the bodies are in interaction between them and they form ***systems of rigid bodies*** *or* ***structures****. The definition of a system of bodies is the following:* ***an ensemble of rigid bodies in mechanical interaction****.*

Besides the constraints used to fix a body to the outdoors, namely the three supports, for the systems of bodies we shall have constraints, connections between the bodies of the system. These connections will be named ***internal constraints****.*

6.2.Internal connections.

We shall have two kinds of internal connections: ***simple internal connection*** *(as a simple support) and* ***internal hinges****. Between the bodies we shall not have fixed connections because if two bodies are fixed one by the other they form one single body.*

- ***The simple internal connection*** *is the constraint between two bodies that* ***removes one degree of freedom*** *for the ensemble of the two bodies. This connection is represented as a small pendulum between the two bodies.*

The removed displacement is the independent translation motion of each body on the direction of the pendulum, this allowing the translation together of the two bodies, the independent translation motion of the *bodies on the normal direction on the pendulum and the independent rotations of them.*

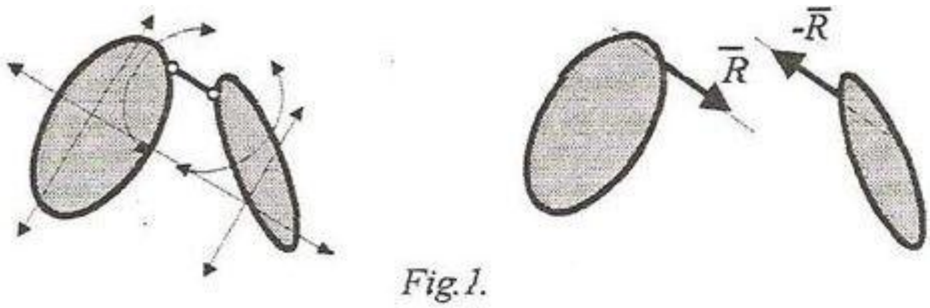


Fig.1.

Using the axiom of the constraints we shall replace the simple connection with a pair of two forces with the same magnitudes, opposite senses on the direction of the pendulum (principle of the action and the reaction). The magnitude of this force is unknown, the direction is known (the direction of the pendulum, or the direction of the removed displacement), the sense is unknown but not an independent unknown because it will results as sign after which we have taken an arbitrary sense for the reaction force. We remark that this connection ***removes one degree of freedom*** *and* ***introduces in calculation one scalar unknown*** *(the magnitude of the reaction force).*

• ***Internal hinge.*** *This connection can be made in two forms:* ***simple internal hinge*** *and* ***multiple internal hinge****. Because the multiple internal hinge can be expressed using a combination of simple internal hinges we shall analyze first the simple internal hinge and after we shall present the way to use a multiple internal hinge.*

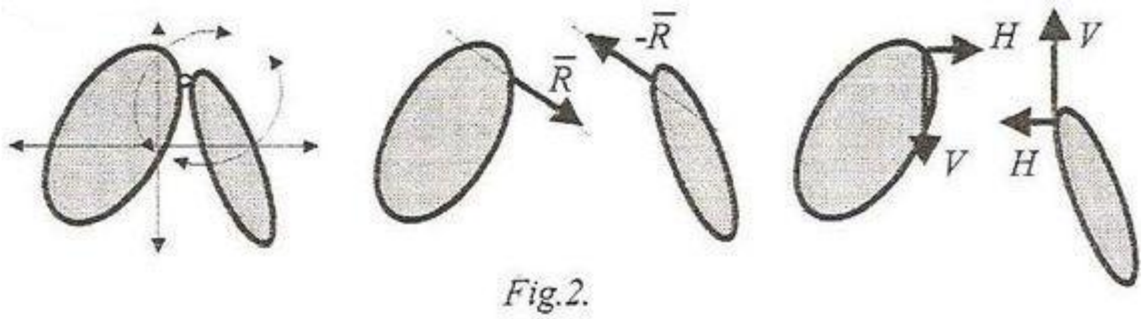


Fig.2.

By definition the simple internal hinge is the ideal connection between two bodies that ***fix a point of one body to a point of the other body****.*

The scheme used for to represent this connection is a small circle in the connection point.

This connection ***removes two degrees of freedom*** *for the ensemble of the two bodies allowing only the translation motion together on two perpendicular directions and the independent rotations of the bodies about the common point. We replace this connection with a pair of two reaction forces with the same magnitudes and directions but with opposite senses. The magnitude and the direction of the reaction force are unknowns and the sense is also unknown but not an independent unknown because it will result as sign. Results that an internal simple hinge* ***removes two degrees of freedom and introduces two scalar unknowns*** *(the magnitude and direction of the reaction force).*

Generally we prefer as the two unknowns to be two unknown forces on two known directions (horizontal and vertical).

As we may see these two connections have the same behavior as the external corresponding supports: the simple support and the hinged support.

The multiple internal hinge is equivalent with (N_c – 1) simple internal hinges, where N_c is the number of the bodies connected by that hinge. This may be seen if we consider a "principal body" to which are connected with simple internal hinges the other bodies coming in that multiple internal hinge.

Fig.3.

Finally we can say that at a system of bodies we use three kinds of forces: given forces or active forces (the loads), reaction forces in the external constraints and reaction forces in the internal connections.

6.3.Equilibrium theorems.

The primary objective of this chapter is that to express the equilibrium of the systems of bodies and to determine the reaction forces from the external and internal constraints and connections.

For to solve this problem we may use an important propriety of the systems that can be stated in the following way: ***if a system of rigid bodies is in equilibrium (or in rest) then all bodies from the system are ion equilibrium (in rest).*** *The reverse statement is also true, namely:* ***if all bodies of a system of rigid bodies are in equilibrium (in rest) then the entire system is in equilibrium (in rest).*** *Using this propriety in fact we have solved the problem of the determination of the reaction forces from the internal and external constraints. This is because the system is decomposed in the component bodies replacing the constraints with the corresponding reaction forces and it is enough to study the equilibrium of each body as separated bodies.*

This way to study the equilibrium of a system of rigid bodies has the deficiency that in all problems we have to determine all the reactions from the internal connections even we are not interested to determine them. There are problems in that we are interested by the external reaction forces only. For to eliminate from computation (if it is possible) of the reaction forces from the internal connections we shall use two theorems called ***equilibrium theorems*** *of the systems of rigid bodies.*

Before that we shall analyze the behavior of the internal reaction forces corresponding to two any bodies from a system. To consider two bodies (I and J) from a system having one any internal connection, for example one internal simple hinge in the point A.

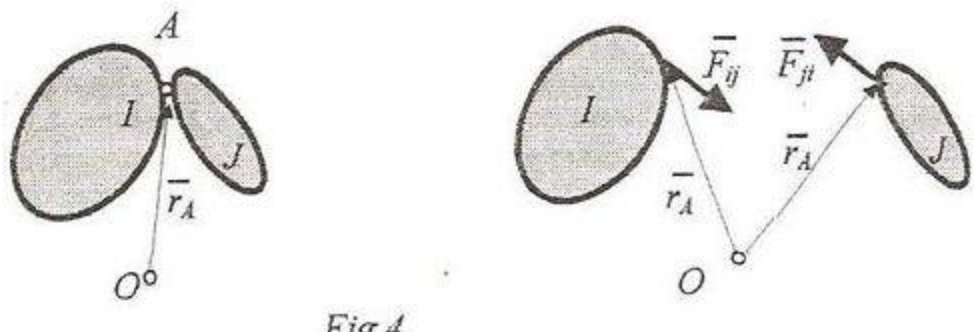


Fig.4.

After which the connection is removed we shall have a pair of two unknown reaction forces with opposite senses:

$$\overline{F}_{ij} = -\overline{F}_{ji}$$

or:

$$\overline{F}_{ij} + \overline{F}_{ji} = 0$$

If we calculate the moments of these two forces about an any point O we shall obtain:

$$\overline{r}_A \times \overline{F}_{ij} + \overline{r}_A \times \overline{F}_{ji} = 0$$

• *The first theorem is called* ***theorem of solidification*** *and has the following statement:* ***if a system of rigid bodies is in equilibrium then the system considered as a single rigid body have to be in equilibrium too****. This theorem may have the following statement also:* ***for a system of rigid bodies the force-couple system of all external forces (given and reactions) have to be equal to zero****.*

To demonstrate this theorem it is enough to express (using the previous propriety) the equilibrium of each body of the system (using the force-couple system of the corresponding forces):

$$\text{corpul 1.} \quad \begin{cases} \overline{F}_{d1} + \overline{R}_1 + \sum_{j=1}^{n} \overline{F}_{1j} = 0 \\ \overline{M}_O(\overline{F}_{d1}) + \overline{M}_O(\overline{R}_1) + \sum_{j=1}^{n} \overline{M}_O(\overline{F}_{1j}) = 0 \end{cases}$$

$$\vdots$$

$$\text{corpul I.} \quad \begin{cases} \overline{F}_{di} + \overline{R}_i + \sum_{j=1}^{n} \overline{F}_{ij} = 0 \\ \overline{M}_O(\overline{F}_{di}) + \overline{M}_O(\overline{R}_i) + \sum_{j=1}^{n} \overline{M}_O(\overline{F}_{ij}) = 0 \end{cases}$$

$$\vdots$$

where we have marked $\overline{F}_{gi}$ and $\overline{M}_O(\overline{F}_{gi})$ the resultant force and the resultant moment about the point O of the given forces acting about the body I, $\overline{R}_i$ and $\overline{M}_O(\overline{R}_i)$ the resultant force and the resultant moment about the point O of the reaction forces from the external constraints from the body I, and the resultant force and the resultant moment about the same point O of the internal reaction forces from the connections of the body I with all the other bodies from the system.

If we add all force equations and also all the moment equations we shall obtain the following two equations:

$$\begin{cases} \sum_{i=1}^{n}\overline{F}_{di} + \sum_{i=1}^{n}\overline{R}_i + \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{F}_{ij} = 0 \\ \sum_{i=1}^{n}\overline{M}_O(\overline{F}_{di}) + \sum_{i=1}^{n}\overline{M}_O(\overline{R}_i) + \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{M}_O(\overline{F}_{ij}) = 0 \end{cases}$$

Because the internal reaction forces are equals two by two and with opposite senses, the total sums (the double sums) of these forces and their moments about an any point are equal to zero:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{F}_{ij} = 0$$
$$\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{M}_O(\overline{F}_{ij}) = 0$$

resulting finally two vector equations:

$$\Sigma\overline{F}_{ext} = 0$$
$$\Sigma\overline{M}_{Oext} = 0$$

that represent the vector conditions of equilibrium of a rigid body having the loads and external constraints of the system of rigid bodies.

- *The second theorem is called* ***theorem of the equilibrium of component parts*** *and has the following statement:* ***if a system of rigid bodies is in equilibrium then any part of the system is in equilibrium under the action of the forces corresponding to that part.***

This theorem is proved as the first but the sums are partial sums and they are made for the corresponding part of the system. We see that in this case not all the internal forces disappear from the conditions of equilibrium. In these conditions remain the internal forces between the considered part and the other parts of the system.

Corresponding to these two theorems and the enounced propriety we shall develop four methods to solve the systems of rigid bodies. These methods are: method of the equilibrium of the component bodies, method of solidification, method of the equilibrium of component parts and the mixed method.

6.4.Statically determined and stable systems.

We shall say that a system of rigid bodies is ***statically determined and stable*** *if* ***it is fixed and the number of the scalar unknowns from the constraints is equal to the number of the scalar, independent equilibrium equations for to express the equilibrium of the system.***

We remark that for as a system of bodies to be statically determined and stable it must meet two conditions, one quantitative:

$$N_E = N_U$$

where N_E is the number of the scalar independent equilibrium equations, and N_U the number of the scalar unknowns from the constraints of the system, Knowing that for each body, in plane (in two dimensions) can be written three scalar independent equations, results for the total number of equations:

$$N_E = 3 \cdot N_b$$

where N_b is the number of bodies from the system. Also we know that one fixed support introduces three scalar unknowns, one hinge (external hinged support or internal simple hinge) introduces two scalar unknowns, and one simple support (and one internal simple connection) introduces one scalar unknown, results for the number of the unknowns:

$$N_U = 3 \cdot n_{fs} + 2 \cdot n_h + n_{ss}$$

where n_{fs} is the number of the fixed supports, n_h is the number of the hinges and n_{ss} is the number of the simple supports. With this the quantitative condition of statically determination and stability is:

$$3 \cdot N_b = 3 \cdot n_{fs} + 2 \cdot n_h + n_{ss}$$

The qualitative condition of statically determination and stability is that to place the constraints so that the system to be fixed, restrained. We can see that this condition is in fact the same as the quantitative condition because for to fix a system of bodies the number of the degrees of freedom must be equal to zero. But one free body has, in plane, three degrees of freedom, so if the system has not constraint it has $3 \cdot N_b$ degrees of freedom. But the constraints remove degrees of freedom , namely : one fixed support removes three degrees of freedom, one hinge two , and one simple support one degree of freedom, results that the total number of degrees of freedom for a constrained system of rigid bodies is:

$$N_{df} = 3 \cdot N_b - (3 \cdot n_{fs} + 2 \cdot n_h + n_{ss}) = 0$$

We remark that the two conditions are expressed using the same relation. So the second condition may be expressed in the following way: ***the constraints and connections of the system must be located so that the system to be a fixed one****. This condition is necessary because for the same constraints we may have more situations. For example, for the system of two rigid bodies from the figure 5., having one fixed support, one internal simple hinge and one simple support we may have three situations, namely: a) in figure 5.a. the system is statically determined and stable because the body I having one fixed support is fixed so the internal hinge becomes a fixed hinge (has the same behavior as a hinged support) and the body II with one fixed hinge and one simple support is also a fixed body;*

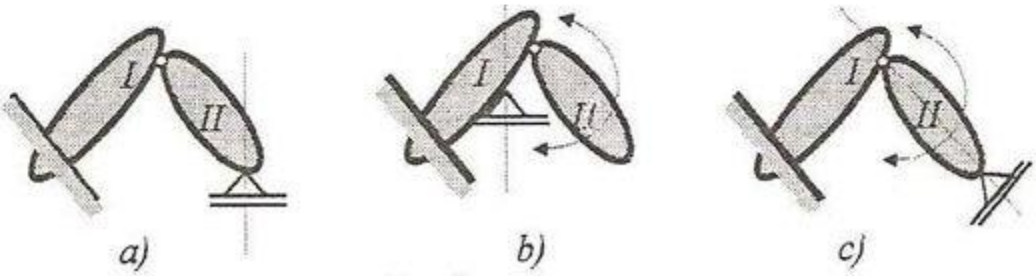


Fig.5.

b) in figure 5.b. the system is not statically determined because the body II has not enough constraints for to be fixed (it has only one fixed hinge that cannot fixes the body, it has possibility to rotates about the hinge), but the body I has too much constraints for to by statically determined;

c) in figure 5.c. although the system seems to be statically determined, it is a critical form because the direction of the simple support passes through the hinge allowing rotations about the hinge.

Immobility verification of the system is made checking each body separately if it is or not fixed (the three cases from the rigid bodies) knowing that one internal hinge becomes a fixed hinge if it is in connection with one fixed body. Besides the three cases may be also other situations but all of them can be reduced to the three cases presented at the rigid body. A common situation is that when we have a system made from two bodies each of them with one hinged support and an internal hinge between them. This system is called ***three hinged frame****. If the three hinges are not collinear then this system is statically determined and stable. The fact that this system is fixed may be checked very easy reducing to the second case of fixation of a rigid body by considering one body as a pendulum (rigid body with two hinges) that has the same mechanical effect as a simple support, so the second body is fixed. In the same way the first body is also fixed and therefore the three hinged frame is statically determined if the direction of the simple support does not pass through the hinge, namely the three hinges are not collinear.*

Fig.6.

6.5.The method of the equilibrium of the component bodies.

This method is based on the enounced propriety namely that if the system is in equilibrium then all bodies of the system are in equilibrium also. Consequently the system is divided in the component bodies that under the action of the corresponding loads, of the reactions from the external constraints and of the internal forces from the internal connections have to be in equilibrium.

The method is used in the following way:

- *1)First we check the statically determination and stability of the system, in fact if the system can be solved or not;*
- *2)We divide the system in the component bodies without to change their shape and dimensions. Each body will be loaded with the corresponding external loads (the given forces) conveniently arranged, with the reaction forces replacing the external constraints and with the internal forces (pairs and opposite as senses acting about the bodies connected by the internal connection) corresponding to the internal connections. If the internal connection is a multiple hinge we shall make in the following way: one body connected by that hinge is considered as "principal body" and all the other bodies will be considered connected to it. We shall obtain a scheme as in the figure 7.*

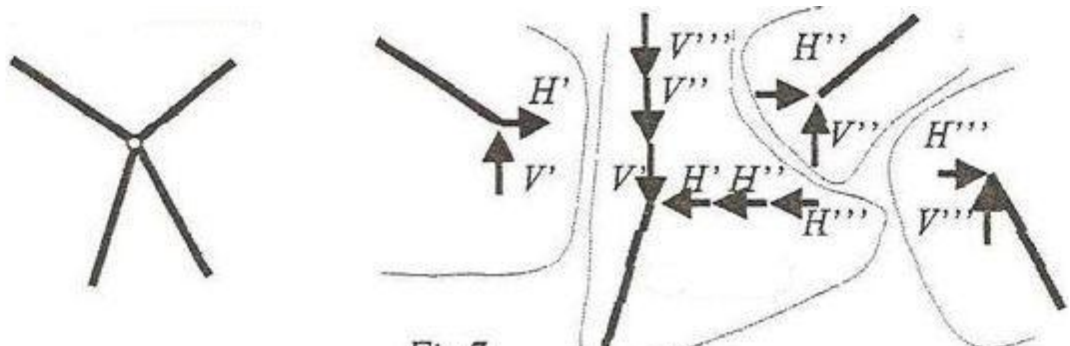


Fig.7.

The resulted scheme is the ***free body diagram*** *corresponding to this method.*

- *3)Be written three equilibrium equations for each body as in the case of the rigid body. Generally if we can the moment equations will be written with respect to the external constraints. In this way we shall obtain $3n_b$ equations with the same number of unknowns;*
- *4)Resolve the system of equations;*
- *5)We check the solution writing one equation (non used for to solve the system) for each body.*

6.6. Sample problems.

Problem 1. *Is given the system of bodies from the figure 8. Calculate the reactions from the external constraints and the internal forces from the internal connections.*

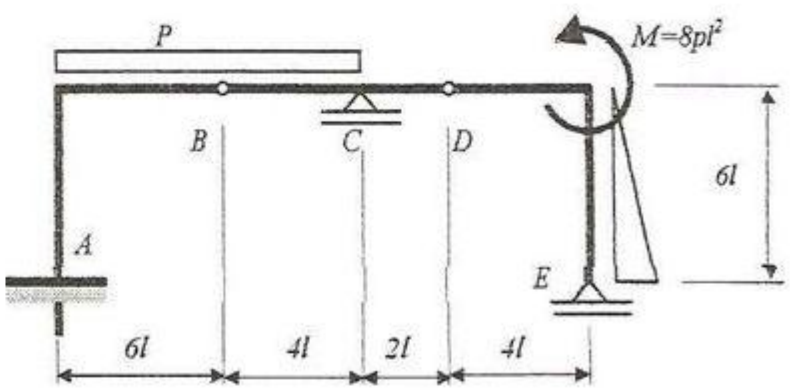


Fig.8.

Solution. Step 1. *The system of bodies is made from three bodies, has a fixed support in A, two simple supports in C and E and two internal simple hinges in B and D. The quantitative condition of statically determination is:*

$$3 \cdot 3 = 3 \cdot 1 + 2 \cdot 2 + 2$$

The qualitative condition of immobilization is checked in the following way: The body AB is fixed having a fixed support consequently the internal hinge from B becomes a fixed hinge, in this way the body BCD is also fixed having a fixed hinge (in B) and a simple support (the direction of this simple support does not pass through the fixed hinge).The internal hinge from D becomes a fixed hinge and finally the body DE having a fixed hinge and a simple support is also a fixed body.

Step 2. *We divide the system in the component bodies removing the two internal hinges from B and D.*

We load each body with the corresponding given loads: the force R_1 for the body I, the force R_2 for the body II and the concentrated moment and the force R_3 for the body III. The external constraints and the internal connections are removed with the corresponding reaction forces. Is obtained the free body diagrams from the figure 9. , representing the three free bodies loaded with concentrated forces and couples on *the convenient directions, from which a part are known (the active forces) and the other part unknown (the reactions).*

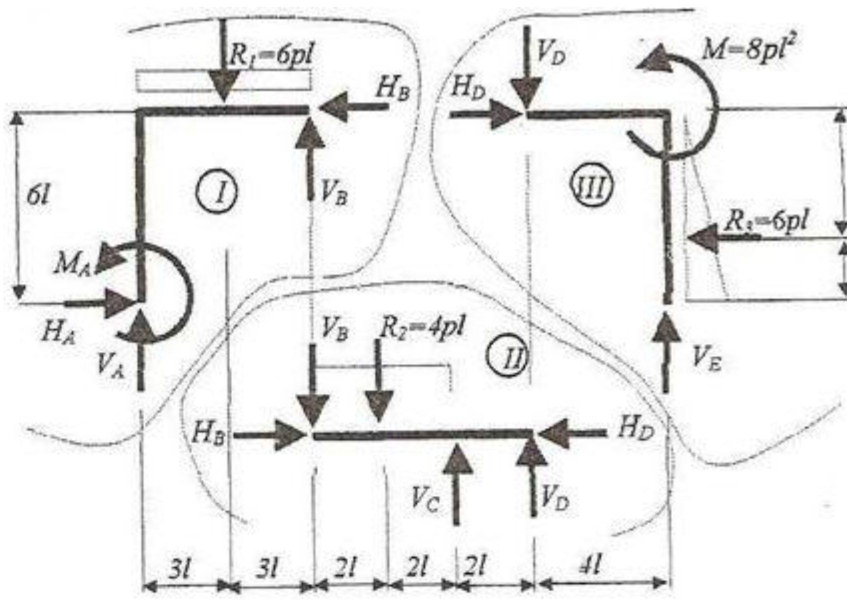


Fig.9.

Step 3. *For each body we shall write three equilibrium equations, namely: for the body I , that has a fixed support, we shall prefer two projections and a moment equation about the fixed support, and for the other two bodies we prefer two moment equations and one projection. The equations will be:*

-for the body I:

$$\Sigma X_i = 0 \; ; \; H_A - H_B = 0 \; ;$$
$$\Sigma Y_i = 0 \; ; \; V_A - 6pl + V_B = 0 \; ;$$
$$\Sigma M_{Ai} = 0 \; ; \; M_A - 6pl \, . \, 3l + V_B \, . \, 6l + H_B \, . \, 6l = 0 \; ;$$

-for the body II:

$$\Sigma X_i = 0 \; ; \; H_B - H_D = 0 \; ;$$
$$\Sigma M_{Bi} = 0 \; ; \; -4pl \, . \, 2l + V_C \, . \, 4l + V_D \, . \, 6l = 0 \; ;$$
$$\Sigma M_{Ci} = 0 \; ; \; V_B \, . \, 4l + 4pl \, . \, 2l + V_D \, . \, 2l = 0 \; ;$$

-for the body III:

$$\Sigma X_i = 0 \; ; \; H_D - 6pl = 0;$$
$$\Sigma M_{Di} = 0; \; 8pl^2 - 6pl \, . \, 4l + V_E \, . \, 4l = 0 \; ;$$
$$\Sigma M_{Ei} = 0; \; V_D \, . \, 4l - H_D \, . \, 6l + 8pl^2 + 6pl \, . \, 2l = 0$$

Step 4. *We have a system of nine equations with the same number of unknowns that will give us the following solutions:*

$$H_D = 6pl \; ; \; V_E = 4pl \; ; \; V_D = 4pl \; ; \; H_B = 6pl \; ; \; V_C = -4pl; \; V_B = -4pl \; ;$$
$$H_A = 6pl \; ; \; V_A = 10pl \; ; \; M_A = 6pl^2.$$

We remark that the solution of the system begins from the body III that has only three unknowns, so it can be solved independently by the other bodies of the system. Then fellows body II and the last body solved is the body I.

The found reactions will be represented on the scheme of the free bodies with the real senses, namely we shall change the senses of the two reactions with minus signs.

Step 5. *The verification of the results is made for each body with an equation non used for the solution:*

-body I:

$$\Sigma M_{Bi} = 0;\ M_A + H_A \cdot 6l - V_A \cdot 6l + 6pl \cdot 3l = 0;$$

-body II:

$$\Sigma Y_i = 0;\ -V_B - 4pl + V_C + V_D = 0;$$

-body III:

$$\Sigma Y_i = 0;\ -V_D + V_E = 0.$$

Problem 2. *Calculate the reactions and the internal forces from the internal connections for the system of bodies from the figure 10.*

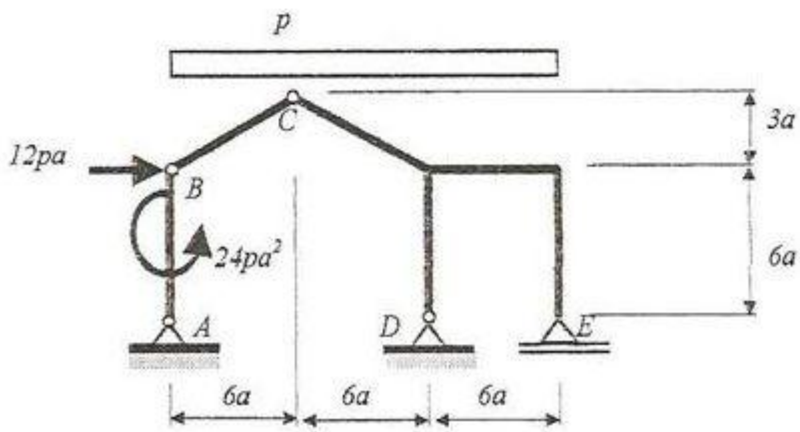


Fig.10.

Solution. *The system of bodies is made from three bodies (AB, BC, and CDE) and it is statically determined and stable:*

$$3 \cdot 3 = 3 \cdot 0 + 2 \cdot 4 + 1$$

because it has four hinges and one simple support. The system is fixed because the body CDE having one hinged support and one simple support is fixed, the internal hinge from C becomes a fixed hinge and the first two bodies with two fixed hinges and one internal between them (non collinear hinges) form a three hinged frame that is fixed also.

We divide the system in the three component bodies, load each of them with the corresponding given loads and with the reactions from the external and internal constraints and connections. The force 12pa acting in the internal hinge B may be placed on any of the two bosies caming in the internal hinge B.

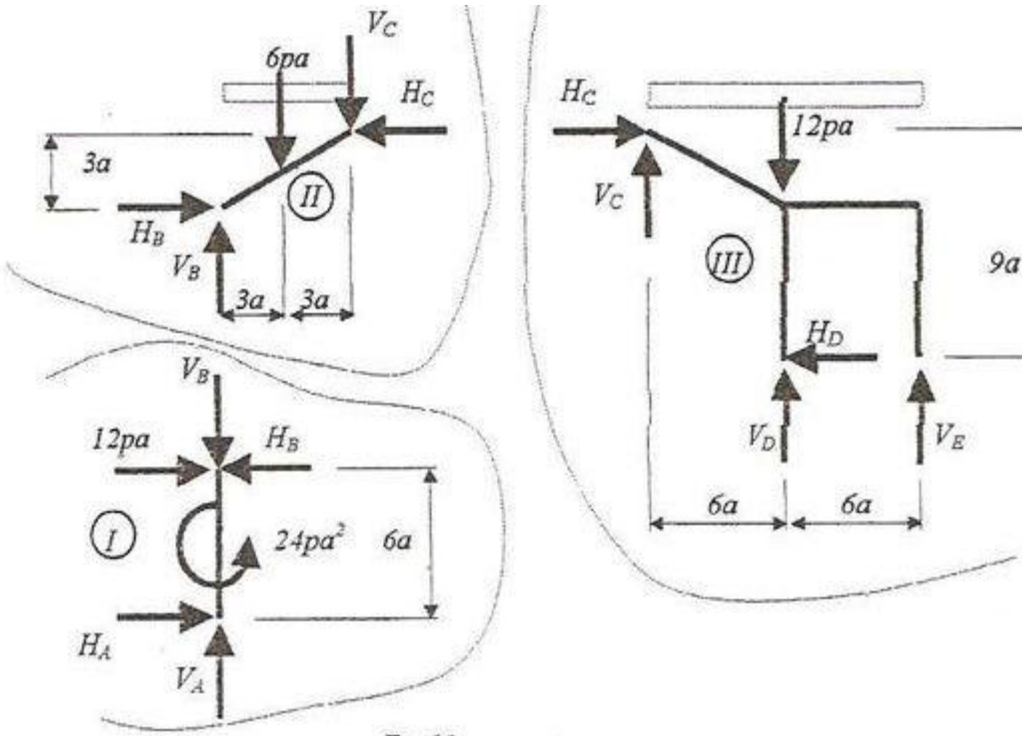


Fig.11.

The equilibrium equations for the three bodies are:

$$I \quad \begin{cases} \Sigma Y_i = 0\ ;\ V_A - V_B = 0\ ; \\ \Sigma M_{Ai} = 0\ ;\ 24pa^2 + H_B\,.\,6a - 12pa\,.\,6a = 0\ ; \\ \Sigma M_{Bi} = 0\ ;\ 24pa^2 + H_A\,.\,6a\ = 0\ ; \end{cases}$$

At this body the projection equation cannot be made on the horizontal direction because the straight line passing through the two points A and B (about which we write the moment equations) is vertically.

$$II \quad \begin{cases} \Sigma X_i = 0\ ;\ H_B - H_C = 0\ ; \\ \Sigma M_{Bi} = 0\ ;\ -6pa\,.\,3a - V_C\,.\,6a\ + H_C\,.\,3a = 0\ ; \\ \Sigma M_{Ci} = 0\ ;\ H_B\,.\,3a - V_B\,.\,6a\ + 6pa\,.\,3a\ = 0\ ; \end{cases}$$

$$III \quad \begin{cases} \Sigma X_i = 0\ ;\ H_C - H_D = 0\ ; \\ \Sigma M_{Di} = 0\ ;\ -H_C\,.\,9a - V_C\,.\,6a\ + V_E\,.\,6a = 0\ ; \\ \Sigma M_{Ei} = 0\ ;\ -H_C\,.\,9a - V_C\,.\,12a\ + 12pa\,.\,6a - V_D\,.\,6a\ = 0\ ; \end{cases}$$

For to solve the system of nine equations with the same number of unknowns we remark that the first two bodies have six unknowns, therefore we may solve these two bodies independently be the third. We have finally:

$$V_A = 7pa\ ;\ H_A = -4pa\ ;\ V_B\ = 7pa\ ;\ H_B\ = 8pa\ ;\ V_C = pa\ ;\ H_C = 8pa;$$
$$V_D = -2pa\ ;\ H_D = 8pa\ ;\ V_E = 13pa.$$

The verification is made with the three equations:

$$\sum_{I} X_i = 0 \; ; \; H_A - 12pa - H_B = 0 \; ;$$
$$\sum_{II} Y_i = 0 \; ; \; V_B - 6pa - V_C = 0 \; ;$$
$$\sum_{III} Y_i = 0 \; ; \; V_C - 12pa + V_D + V_E = 0 \; ;$$

Problem 3. *Calculate the reactions from the constraints and the internal forces from the connections for the system from the figure 12.*

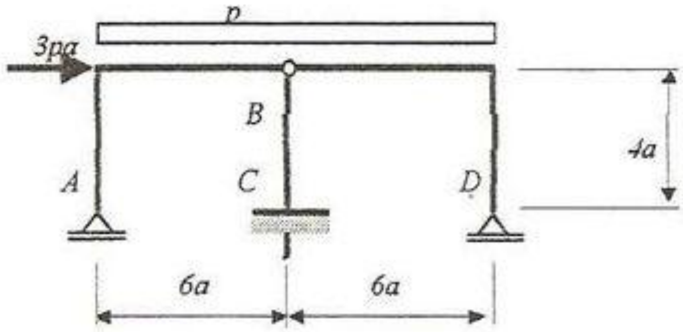


Fig.12.

Solution. *The system is made from three bodies and has a fixed support, two simple hinges and one double hinge (this hinge makes the connection between three bodies) in B. Therefore it is statically determined:*

$$3 \, . \, 3 = 3 \, . \, 1 + 2 \, . \, 2 + 2$$

The body BC having one fixed support is fixed so the internal hinge from B becomes a fixed hinge and the two bodies AB and BD having a fixed hinge and one simple support are fixed also.

We shall divide the system in the component bodies and for the hinge from B we shall consider the body BC as principal body, and the other two bodies are joined to this body. We have the scheme of the free bodies in the figure 13.

The equilibrium equations are:

$$I \quad \begin{cases} \Sigma X_i = 0 \; ; \; 3pa - H_{B1} = 0 \; ; \\ \Sigma M_{Ai} = 0 \; ; \; -3pa \, . \, 4a - 6pa \, . \, 3a + H_{B1} \, . \, 4a + V_{B1} \, . \, 6a = 0 \; ; \\ \Sigma M_{Bi} = 0 \; ; \; - V_A \, . \, 6a + 6pa \, . \, 3a = 0 \; ; \end{cases}$$

$$II \quad \begin{cases} \Sigma X_i = 0 \; ; \; H_{B1} - H_{B2} - H_C = 0 \; ; \\ \Sigma Y_i = 0 \; ; \; V_C - V_{B2} - V_{B1} = 0 \; ; \\ \Sigma M_{Ci} = 0 \; ; \; M_C + H_{B2} \, . \, 4a - H_{B1} \, . \, 4a = 0 \; ; \end{cases}$$

$$III \quad \begin{cases} \Sigma X_i = 0 \; ; \; H_{B2} = 0 \; ; \\ \Sigma M_{Bi} = 0 \; ; \; - 6pa \, . \, 3a + V_D \, . \, 6a = 0 \; ; \\ \Sigma M_{Di} = 0 \; ; \; - H_{B2} \, . \, 4a - V_{B2} \, . \, 6a + 6pa \, . \, 3a = 0 \end{cases}$$

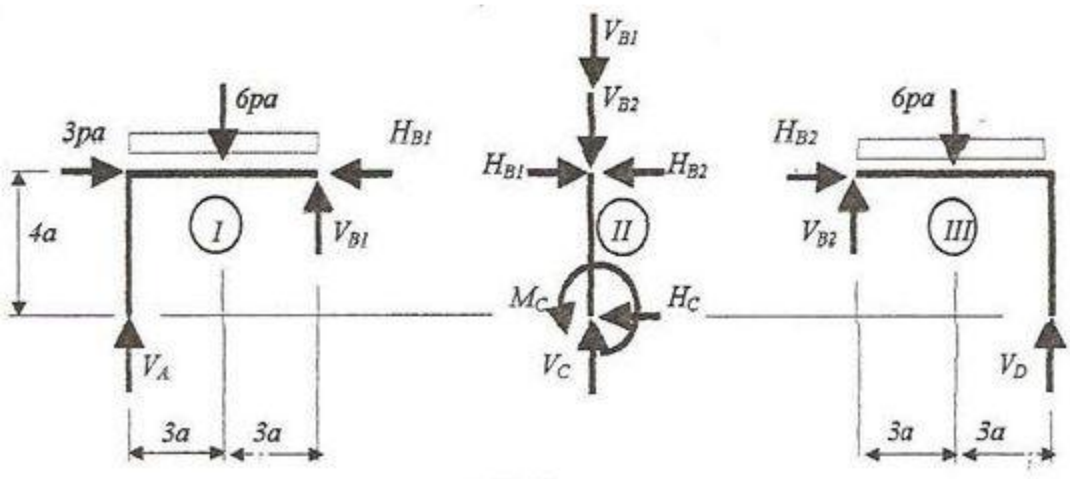


Fig.13.

Solving the system of equations we have:

$V_A = 3pa$; $H_{B1} = 3pa$; $V_{B1} = 3pa$; $H_{B2} = 0$; $V_{B2} = 3pa$; $H_C = 3pa$;
$V_C = 6pa$; $M_C = 12pa^2$; $V_D = 3pa$.

For the verification we will use the equations:

$\sum_{I} Y_i = 0$; $V_A - 6pa + V_{B1} = 0$; $\sum_{II} M_{Bi} = 0$; $M_C - H_C \cdot 4a = 0$;
$\sum_{III} Y_i = 0$; $V_{B2} - 6pa + V_D = 0$.

Problems 4, 5, 6. *Calculate the reactions from the external and internal constraints and connections for the systems of rigid bodies from the figures 14, 15, and 16.*

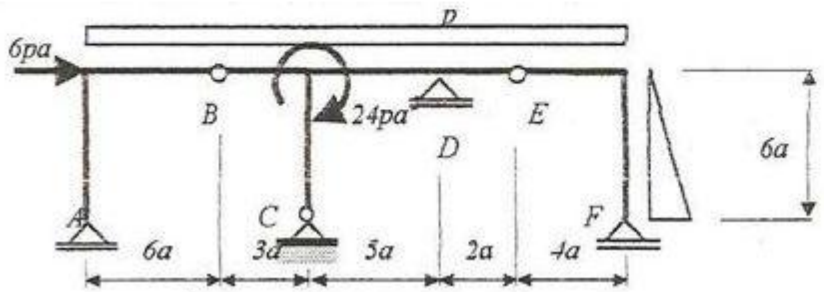


Fig.14.

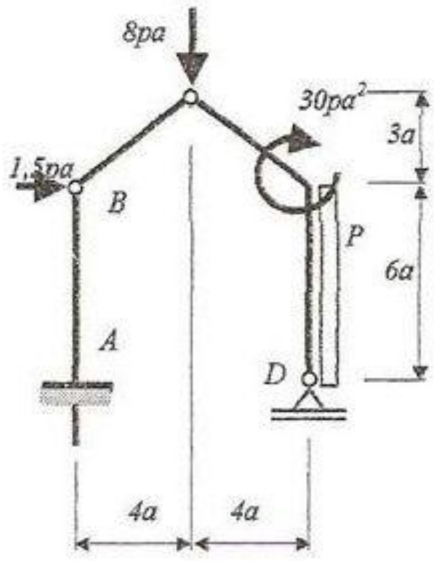


Fig.15.

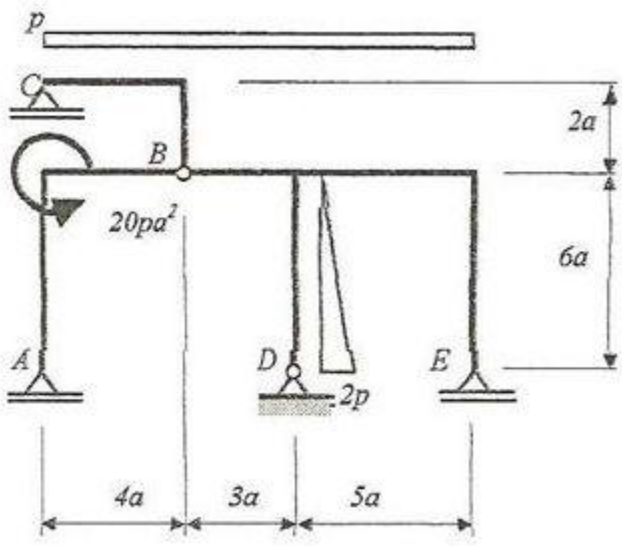


Fig.16.

6.7. Structural units.

Before to present the following three methods of solving the systems of rigid bodies we will define some elements regarding to the behavior and the role of the bodies and the parts of the system. We shall define as ***structural units, the bodies or parts of the system having the clear role in the structure.*** *In particular we are interested in two aspects of the behavior of the bodies and parts of the system: immobilization of the bodies and parts and the solution of the system of equations. We will have two kinds of bodies and parts: principal and secondary bodies and parts.*

The principal body *is that body of a system that has enough external constraints for to be fixed independently by the other bodies and parts of the system. Because of this fact the body has more than three unknowns in the constraints and connections and because of this, it cannot be solved independently by the other bodies and parts of the system. Consequently this body will be the first when we check the stability of the system and the last when we solve the system.*

The secondary body *is the body of the system that has not enough external constraints for to be a fixed body independently by the other bodies and parts of the system, but has only three scalar unknowns in its constraints and connections. Consequently this body may be solved* *independently by the other bodies or parts of the system and therefore it is the first in the solving the system and the last in the checking the stability of it.*

The principal part *is that part of a system of rigid bodies that has enough external constraints for to be fixed one independently by the other bodies or parts of the system. In this way this part has more than three times the number of bodies from that part and because of this it cannot be solved independently by the other parts or bodies from the system. Consequently this part will be the first in the checking the stability and the last in the solving the system.*

The secondary part *is the part of the system of bodies that has not enough external constraints for to be a fixed part independently by the other parts or bodies from the system but has only three times the number of bodies from the part scalar unknowns in its constraints and connections. In this way this part may be solved independently by the other parts or bodies from the system. Consequently this part will be the first in the solving the system and the last in the verification of the stability of it.*

Must be made the specification that the principal bodies and parts, because the external constraints (enough for to fix the part or the body) transfer the loads to the external bodies directly without to transfer to the other bodies or parts from the system, but the secondary parts and bodies transfer the loads to the other part and bodies also.

Some bodies may be parts simultaneously from the both kinds of parts.

In the figure 17 the body FGH is a principal body because has one hinged and one simple support. At the other hand this body has five scalar unknowns in his constraints and connections (two in the internal hinge F, two in the hinged support G and one in the simple support H), so it cannot be solved independently by the system.

The body AB is a secondary part having only one simple support as external constraint (without the internal hinge B it cannot be fixed), but has only three scalar unknowns in its constraint and connections, therefore it can be solved independently by the other parts or bodies of the system.

The part made from the bodies DEF and FGH is the principal part because has enough external constraints for to be fixed (the body FGH is the principal part so it is fixed, therefore the internal hinge from F becomes a fixed hinge and the body DEF having a fixed hinge and a simple

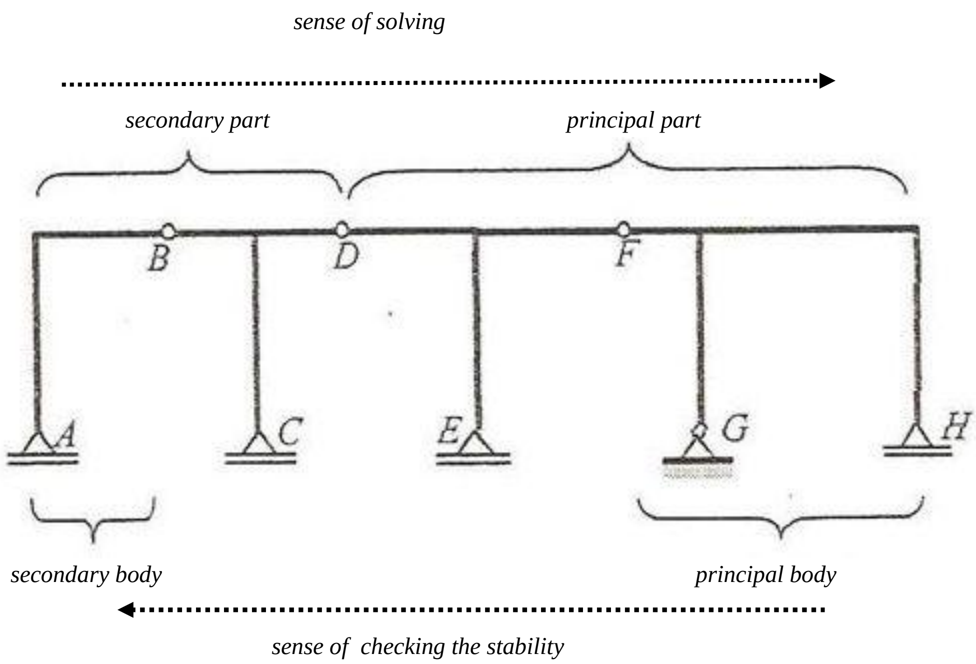


Fig.17.

support is also a fixed one). This part has in its constraints and connections eight scalar unknowns (two in D, two in F and two in G and one unknown in E and in H).

The part made from the first two bodies is a secondary part (has only two simple supports with the external bodies) and has only six scalar unknowns in the constraints and connections therefore it can be solved independently by the other parts or bodies from the system.

We may remark that the part made from the BCD, DEF, and FGH bodies is a principal part and the part made from the first three bodies (AB, BCD and DEF) is a secondary part.

The verification of the stability of the system is made starting from the principal body, or the principal part and is continued to the secondary part or body, and the solving of the system is started always from the secondary body or part and is finished with the principal body or part.

These knowledge will do that with the checking the statically determination and stability of the system to know the body or the part from that we shall start the solving of the system, in this way the equations may be written in order to solve them and consequently the results from the first equations may be used in the following equations the solution of the system making step by step.

6.8.Method of solidification.

This method is based on the theorem of solidification, meaning that it express the equilibrium of the system like the system is one single rigid body. The method is used when the number of the scalar unknowns from the external constraints is equal to three.

This method is used in the following way: 1) first we check the condition of statically determination and stability and if the number of the external unknowns is equal to three; 2) is made the free body diagram as the system is one body, namely the internal connections are blocked; 3) be written three equilibrium equations; 4) resolve the system of the three equations; 5) is made the verification writing one equation unused for to solve the system.

As we can see the method is used as for the rigid body.

6.9. Sample problems.

Problem 7. *Calculate the reactions from the external constraints for the system represented in the figure 18.*

hinges).

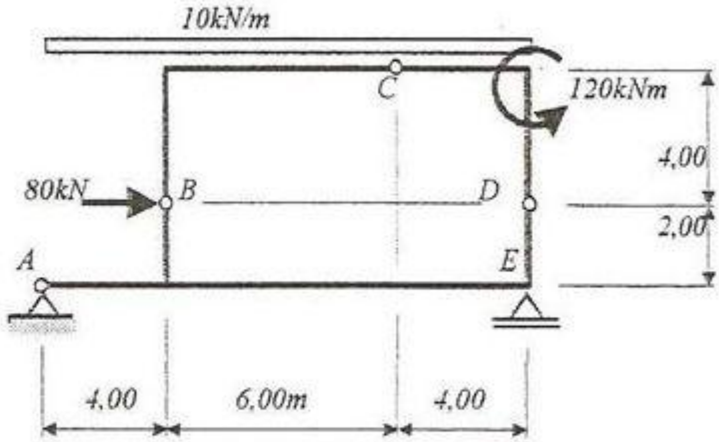


Fig.18.

Solution. *The system is statically determined and stable being made from three bodies having four hinges and one simple support and it is fixed because the body ABCD that is a principal body so it is fixed by its hinged support from A and simple support from E, and the part made from the bodies BC and CD (secondary part of the system) has two fixed hinges (in B and D) and an internal hinge in C (non collinear*

The free body diagram is obtained replacing only the external constraints, without to touch the internal connections that are considered blocked. The scheme is as for one single body.

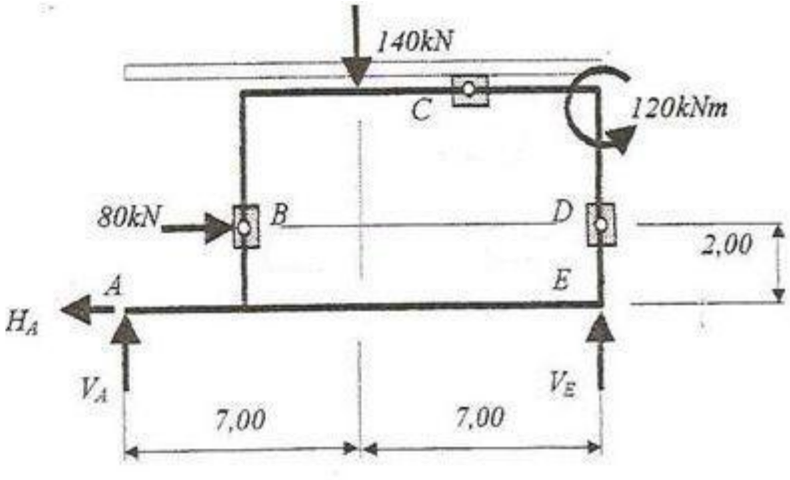


Fig.19.

The equilibrium equations and the resulted reactions are:

$$\Sigma X_i = 0\ ;\ 80 - H_A = 0\ ; \qquad H_A = 80\ kN;$$
$$\Sigma M_{Ai} = 0\ ;\ -80\ .\ 2 - 140\ .\ 7 + 120 + V_E\ .\ 14 = 0\ ; \qquad V_E = 72{,}85\ kN;$$
$$\Sigma M_{Ei} = 0\ ;\ -V_A\ .\ 14 - 80\ .\ 2 + 140\ .\ 7 + 120 = 0\ ; \qquad V_A = 67{,}15\ kN$$

The verification is made with the equation:

$$\Sigma Y_i = 0\ ;\ V_A - 140 + V_E = 0\ .$$

Problem 8. *Calculate the reactions from the external constraints for the system represented in the figure 20.*

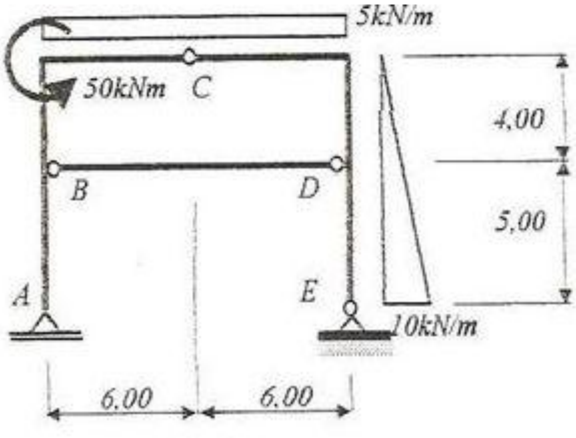


Fig.20.

6.10. Mixed method

One of the most efficient method in the solving of the systems of rigid bodies is the mixed method that uses simultaneously for to determine the reactions the both theorems of equilibrium. This method allows to solve the system of bodies entirely (to determine all the reactions from the external and internal constraints and connections).

For the systems of rigid bodies with open outlines this method can used for to determine the reactions from the external constraints without to divide the system in bodies or parts.

One ***system of bodies with open outlines*** *is that system of bodies in which for to arrive, from one point, in the same point, have to pass twice the same line. For a system with closed outlines we can arrive in the same point without to pass twice the same line.*

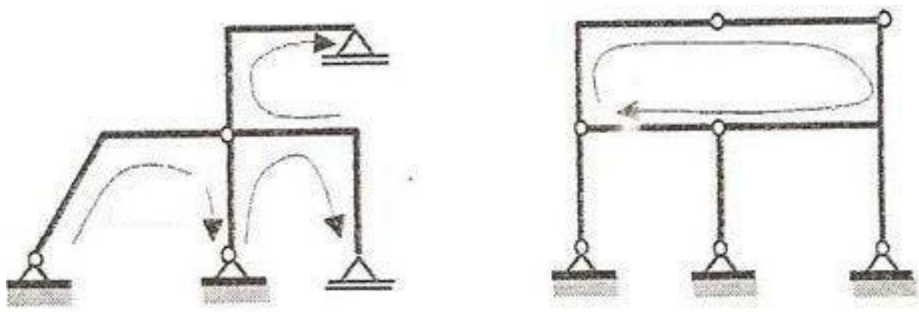


Fig.21.

Supposing that the internal connections are only internal hinges, the systems with open outlines have the following two proprieties:

1) the number of the external unknowns is equal always with:

$$N_{eu} = 3 + (N_b - 1)$$

where N_{eu} is the number of the external unknowns from the constraints of the system, and N_b is the number of the bodies from the system.

This propriety is easy remarked if we do a statically determined and stable system starting from one body. In this way for to have a fixed body we need or three simple supports, or one hinged support and one simple support or one fixed support. Therefore the first body will have always *three unknowns in the external constraints. The following body is join to the first with one internal hinge, and for to fix it is necessary to have a simple support also. The following bodies, for do not obtain a closed outline will be fixed as the second body, namely each of them introducing in the system one simple support so one external unknown. From these considerations results the previously relation.*

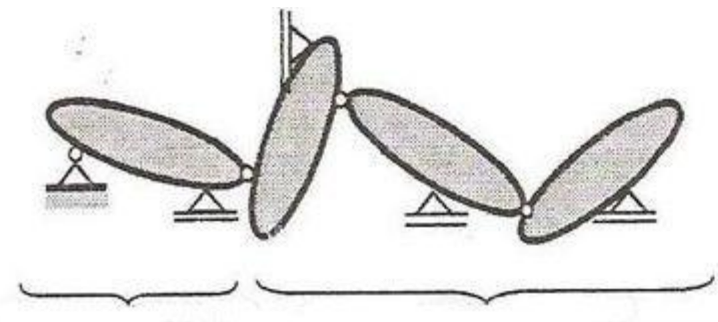


Fig.22.

It is obviously that the external constraints can be grouped in many other ways, but the equivalent of them remains unchanged.

2) the number of the internal simple hinges is always equal to:

$$N_{ish} = N_b - 1$$

where N_{ish} is the number of the internal simple hinges. Also this propriety can be easy remarked in the figure 22.

Now, suppose a system of rigid bodies with open outlines for which we want to determine the reactions from the external constraints. Using the theorem of solidification we can write three equilibrium equations for the entire system as one single body. As the number of the external unknowns is higher than three, we need to write other (N_b – 1) equations for to solve the problem. These equations will be obtained using the theorem of the equilibrium of component parts. Namely is taken one part (one subsystem) of the system and is expressed the equilibrium of it (considering this part as one single body). When one part is separated by the system the internal hinge that joins this part to the other part of the system is removed with two internal unknowns. It is obviously that in this way the number of the unknowns increases. But if we *express the equilibrium with one moment equation about this internal hinge then the two unknowns from this hinge do not appear in this equation and the number of the unknowns in the equilibrium equations remains the same.*

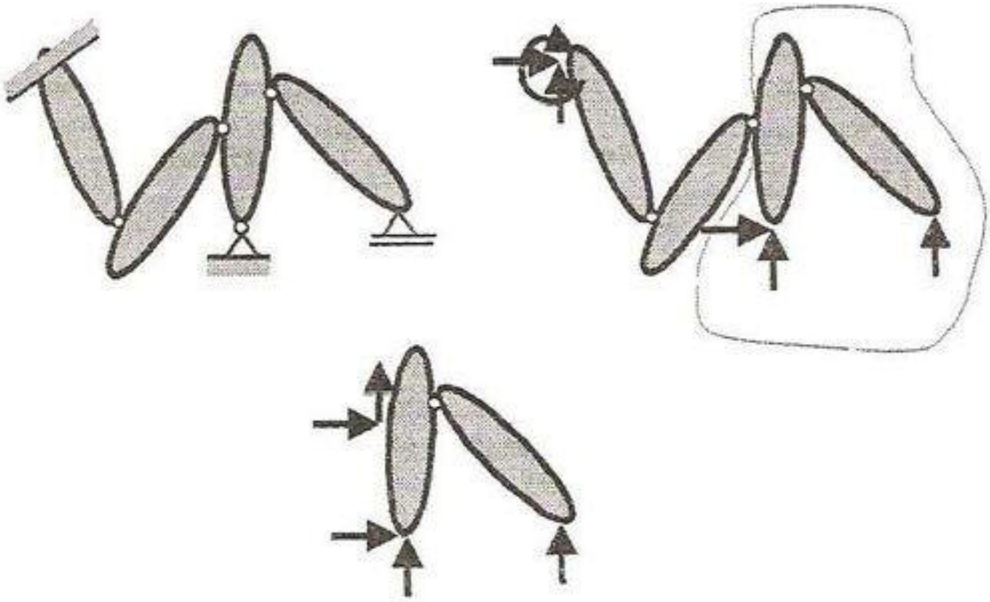


Fig.23.

But for this equation is not necessary to undo the system in parts.

We may remark that in one simple internal hinge we cannot write equilibrium equations for the both parts because the sum of the two equations of equilibrium for the two parts of the system represents one equation for the entire system that is not an independent equation to those three already written equations.

Further noting that we have the same number of supplementary unknowns, as three, than the number of the internal simple hinges, results that we shall write one single moment equation about each internal simple hinge.

At the other hand, if the three equations for the entire system represent the conditions as the system does not have translations or rotations as one body, the moment equations about the internal hinges are the conditions as the parts do not have rotations about the other parts of the system.

The method will be used in the following way:

- *1)is verified if the system of bodies is or not statically determined and stable, when we shall determine the principal and secondary parts of the system;*
- *2)is made the free body diagram for this method, containing the entire system loaded with the given forces arranged as the system is decomposed in parts, and the external constraints replaced with the corresponding reactions;*
- *3)we write three equilibrium equations for the entire system as it is one single rigid body;*
- *4)we write (N_b – 1) moment equations about each internal hinge for one part of the system. We shall choose that part which contains the secondary bodies or parts;*
- *5)is solved the system of equations;*
- *6)is made the verification using one equation for the entire system.*

6.11. Sample problems.

Problem 9. *Calculate the reactions from the external constraints for the system represented in the figure 24.*

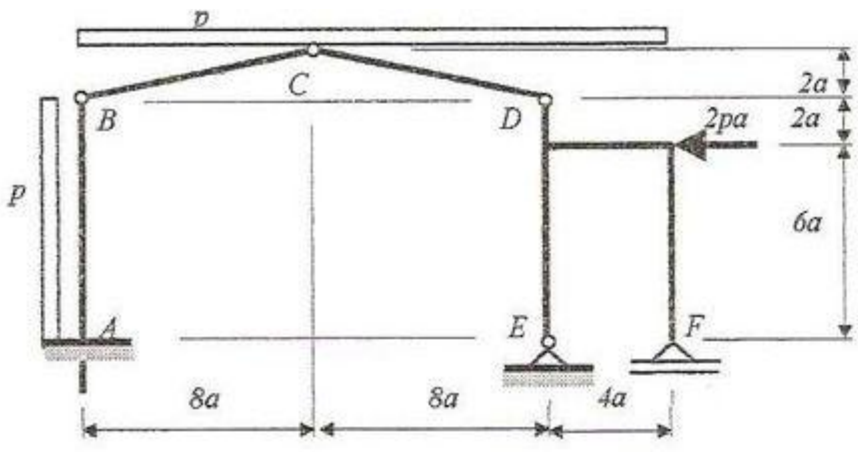


Fig.24.

Solution. *The system is made from four bodies and is statically determined and stable because it has one fixed support, four hinges and one simple support:*

$3 . 4 = 3 . 1 + 2 . 4 + 1$

The first body (AB) and the last (DEF) are fixed bodies. The internal hinges from B and D become fixed hinges and the system BCD is in this way a triple hinged frame that is also fixed. We have two principal bodies (AB and DEF) and a secondary part (BCD).

The free body diagram is represented in the figure 25.

The equilibrium equations for the entire system considered as one body are:

$$\Sigma X_i = 0\ ;\ H_A + 8pa - 2pa - H_E = 0;$$
$$\Sigma M_{Ei} = 0\ ;\ M_A - V_A \,.\, 16a - 8pa \,.\, 4a + 8pa \,.\, 12a + 8pa \,.\, 4a - 4pa \,.\, 2a + $$
$$+ 2pa \,.\, 6a + V_F \,.\, 4a = 0\ ;$$
$$\Sigma M_{Fi} = 0\ ;\ M_A - V_A \,.\, 20a - 8pa \,.\, 4a + 8pa \,.\, 16a + 8pa \,.\, 8a + 4pa \,.\, 2a + $$
$$+ 2pa \,.\, 6a - V_E \,.\, 4a = 0\ ;$$

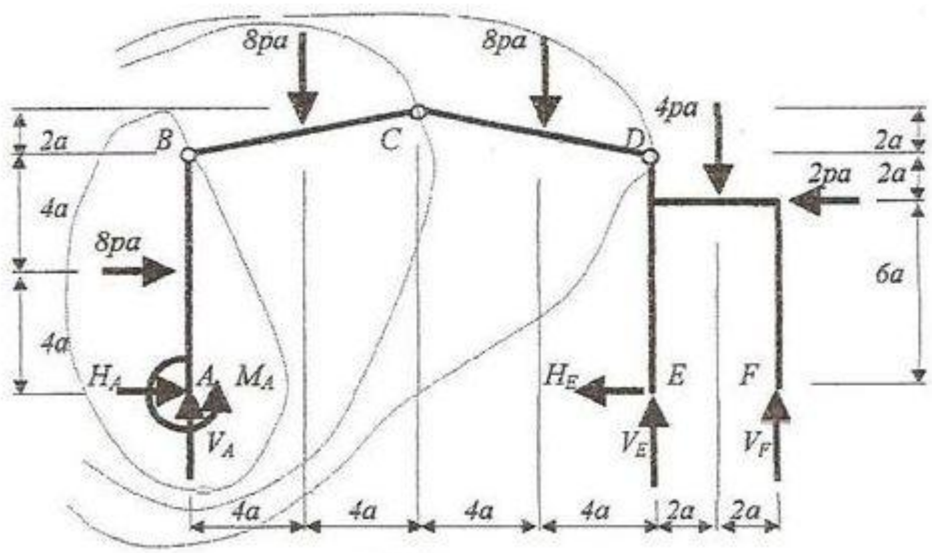


Fig.25.

We shall write other three equations for the parts of the system and we shall choose that all the three equations for the same part namely the part that does not contain the body DEF (this body is taken as principal body for the previously three equations). We shall choose the left part of the system for to write these equations:

$$\sum_{AB} M_{Bi} = \sum^{left} M_{Bi} = 0\ ;\ 8pa \,.\, 4a + M_A + H_A \,.\, 8a = 0\ ;$$

$$\sum_{ABC} M_{Ci} = \sum^{left} M_{Ci} = 0\ ;\ 8pa \,.\, 4a + 8pa \,.\, 6a + M_A + $$
$$+ H_A \,.\, 10a - V_A \,.\, 8a = 0\ ;$$

$$\sum_{ABCD} M_{Di} = \sum^{left} M_{Di} = 0\ ;\ 8pa \,.\, 4a + 8pa \,.\, 12a + 8pa \,.\, 4a + M_A + $$
$$+ H_A \,.\, 8a - V_A \,.\, 16a = 0\ ;$$

We remark that the last three equations contain the same three unknowns therefore we can solve this system of three equations. Result the values:

$$V_A = 8pa\ ;\ H_A = 8pa\ ;\ M_A = -96pa^2.$$

Removing these three values in the first three equations written for the entire system we shall obtain the following three unknowns:

$$H_E = 14pa\ ;\ V_E = -19pa\ ;\ V_F = 31pa.$$

The checking of these results is made using the projection on the vertical direction for the entire system:

$$\Sigma Y_i = 0\ ;\ V_A - 8pa - 8pa + V_E - 4pa + V_F = 0.$$

Problem 10. *Calculate the reactions in the external constraints of the system represented in the figure 26.*

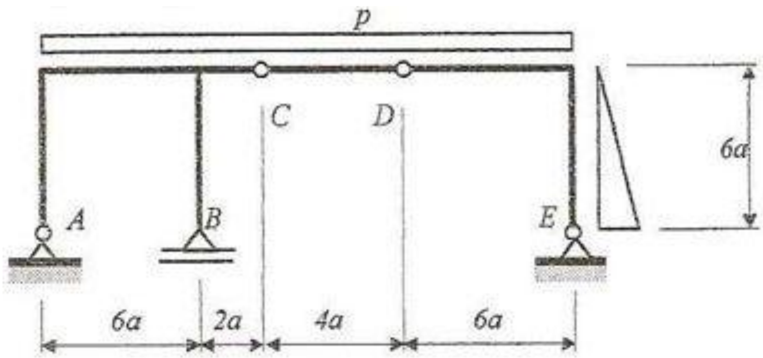


Fig.26.

6.12. Method of the equilibrium of the component parts.

This method uses, for to determine the reactions of a system of rigid bodies, the theorem of the equilibrium of the component parts. The method allows to obtain all the unknowns from the external constraints and the internal connections indifferent to the proprieties of the system.

Generally this method is the general method to solve a system of bodies, thus if through parts we understand bodies then this method is in fact the method of the equilibrium of the component bodies, but through parts we shall consider the entire system then the method is in fact the mixed method.

In the using of this method should consider that the equations to be independent equations and also the equations to contain all the unknowns of the system that remove the constraints and internal connections.

The method is used in the following way:

- *1)is checked the condition of statically determination and stability of the system;*
- *2)the system is divided in convenient parts and they are loaded with the reactions corresponding to the removed constraints and internal connections and with the given, active loads conveniently arranged for to express the conditions of equilibrium;*
- *3) are written the equilibrium equations corresponding to the equilibrium of each parts concerning as the number of equations to be equal to the number of the unknowns from the removed constraints and connections;*
- *4)we solve the obtained system of equations;*
- *5)is verified the solution using, for each part, an equation non used for the solution.*

6.13. Sample problems.

Problem 11. *Calculate the reactions from the external constraints for the frame from the figure 27.*

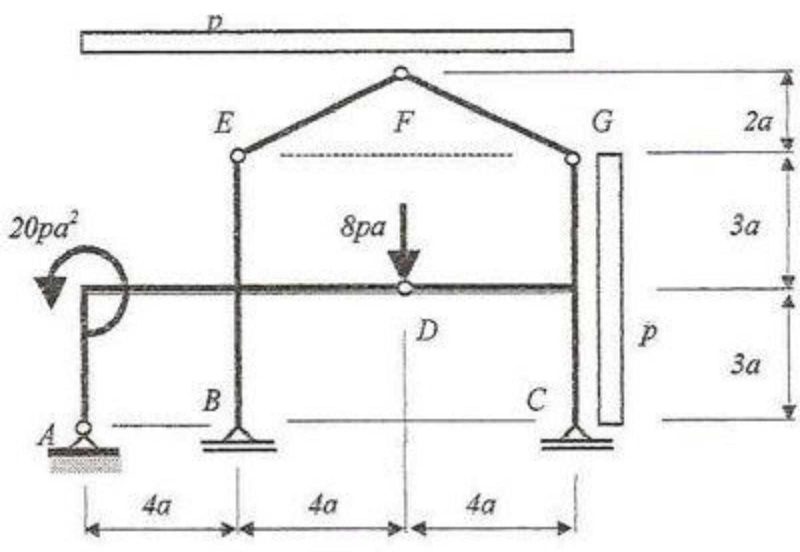


Fig.27.

Solution. *The system is statically determined and stable because is made from four bodies (ABDE, DCG, EF and FG) and has five simple hinges and two simple supports. At the other hand the body ABED is fixed because it has one hinged support and one simple support and the internal hinges from D and E become fixed hinges. The body DCG will be fixed having the fixed hinge in D and one simple support in C. This body together with the body ABDE forms a principal part of the system.*

The two bodies EF and FG together form a triple hinged frame, so they are fixed and represent the secondary part of the system.

Because it is asked to determine only the reactions from the external constraints should be ideal to use the mixed method, but because the system has one closed outline this method cannot be used in the shown way, so it is necessary to divide the system in parts. Consequently the method of the equilibrium of the component parts is recommended to use for to determine the reactions of this system. It is obviously this because we have four external unknowns and the number of total unknowns of the system is twelve. If we undo the system in two parts removing two internal hinges we will have another four unknowns besides the four external reactions. Results that using the method of equilibrium of component parts we shall determine only eight unknowns in place of twelve as in the method of the equilibrium of the component bodies.

Therefore the system is divided in that way to solve easier the system of equations, namely we shall divide in the secondary part that can be solved independently by the rest of the system and the principal part that will be solved after we shall determine the unknowns from the corresponding internal hinges in connections with the secondary part.

The free body diagrams of the secondary part is represented in the figure 28.

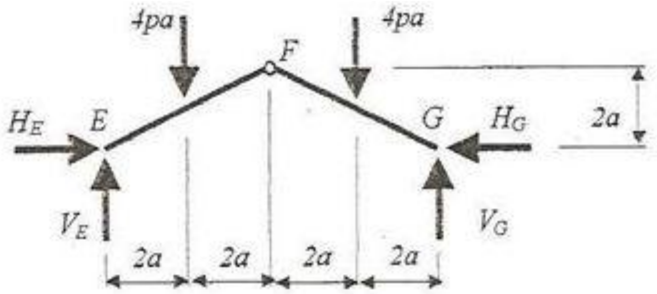


Fig.28.

This part is a system with open outlines and so we can solve using the mixed method for to determine the four unknowns. We shall make the remark that the equations for the entire subsystem are in fact equations for a part of the entire initial system. The equilibrium equations will be:

$$\Sigma X_i = 0\ ;\ H_E - H_G = 0\ ;$$
$$\Sigma M_{Ei} = 0\ ;\ -4pa \cdot 2a - 4pa \cdot 6a + V_G \cdot 8a = 0\ ;$$
$$\Sigma M_{Gi} = 0\ ;\ -V_E \cdot 8a + 4pa \cdot 6a + 4pa \cdot 2a = 0\ ;$$
$$\Sigma M_{Fi}^{left} = 0\ ;\ 4pa \cdot 2a + H_E \cdot 2a - V_E \cdot 4a = 0\ .$$

Solving this system of equations we have the values:

$$V_E = 4pa\ ;\ V_G = 4pa\ ;\ H_E = H_G = 4pa.$$

The verification of these results is made with one projection equation:

$$\Sigma Y_i = 0\ ;\ V_E - 4pa - 4pa + V_G = 0\ .$$

These reaction forces will be considered loads for the principal part represented in the figure 29 as the free body diagram.

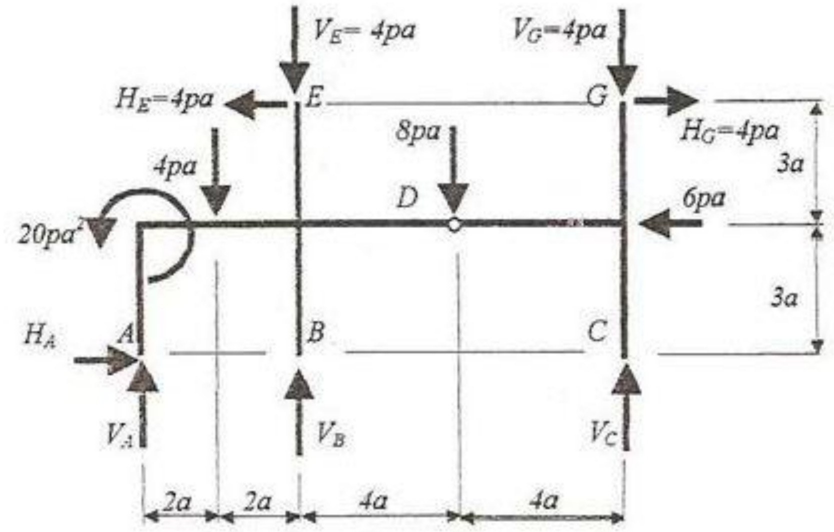


Fig.29.

This part is also a system with open outlines and consequently we shall solve in the same way as the secondary part using the mixed method. The equilibrium equations will be:

$$\Sigma X_i = 0 ; H_A - 4pa + 4pa - 6pa = 0 ;$$
$$\Sigma M_{Ai} = 0 ; 20pa^2 - 4pa \cdot 2a + V_B \cdot 4a - 4pa \cdot 4a + 4pa \cdot 6a - 4pa \cdot 6a -$$
$$- 8pa \cdot 8a - 4pa \cdot 12a + 6pa \cdot 3a + V_C \cdot 12a = 0 ;$$
$$\Sigma M_{Bi} = 0 ; -V_A \cdot 4a + 20pa^2 + 4pa \cdot 2a + 4pa \cdot 6a - 4pa \cdot 6a -$$
$$-8pa \cdot 4a - 4pa \cdot 8a + 6pa \cdot 3a + V_C \cdot 8a = 0 ;$$

$$\Sigma M_{Di}^{right} = 0 ; V_C \cdot 4a - 4pa \cdot 3a - 4pa \cdot 4a = 0 .$$

Solving the system of equations we have:

$$V_A = 7{,}5pa ; H_A = 6pa ; V_B = 7pa ; V_C = 5{,}5pa.$$

The verification will be made with the equation:

$$\Sigma Y_i = 0 ; V_A + V_B + V - 4pa - 4pa - 8pa - 4pa = 0 .$$

Problem 12. *Calculate the reactions from the external constraints for the system represented in the figure 30.*

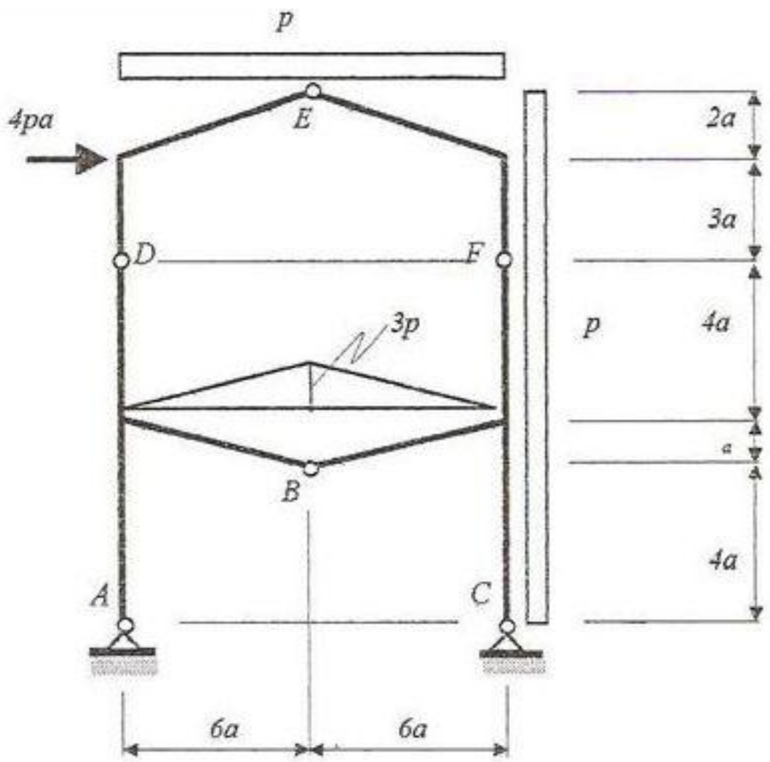


Fig.30.

6.14. Symmetrical systems of rigid bodies.

A very large part of the systems of rigid bodies are symmetrical systems and it is obviously that this propriety can be used in the calculation of the reactions.

We shall say that a system is symmetrical if its shape is symmetrical and ha s symmetrical external constraints and internal connections. In this chapter we study only the systems having internal hinges on the symmetry axis (but the proprieties of these kind of systems are true also for the other kind of symmetrical systems).

The loads will be classified in three, namely: ***symmetrical loads****,* ***antisymmetric loads*** *and* ***any loads****.*

The symmetrical loads are those loads that overlapping identical if we bend the system of bodies on the symmetry axis.

The antisymetric loads are those loads that cancel identical if we bend the system ob bodies on the symmetry axis.

symmetrical loads *antisymetric loads*

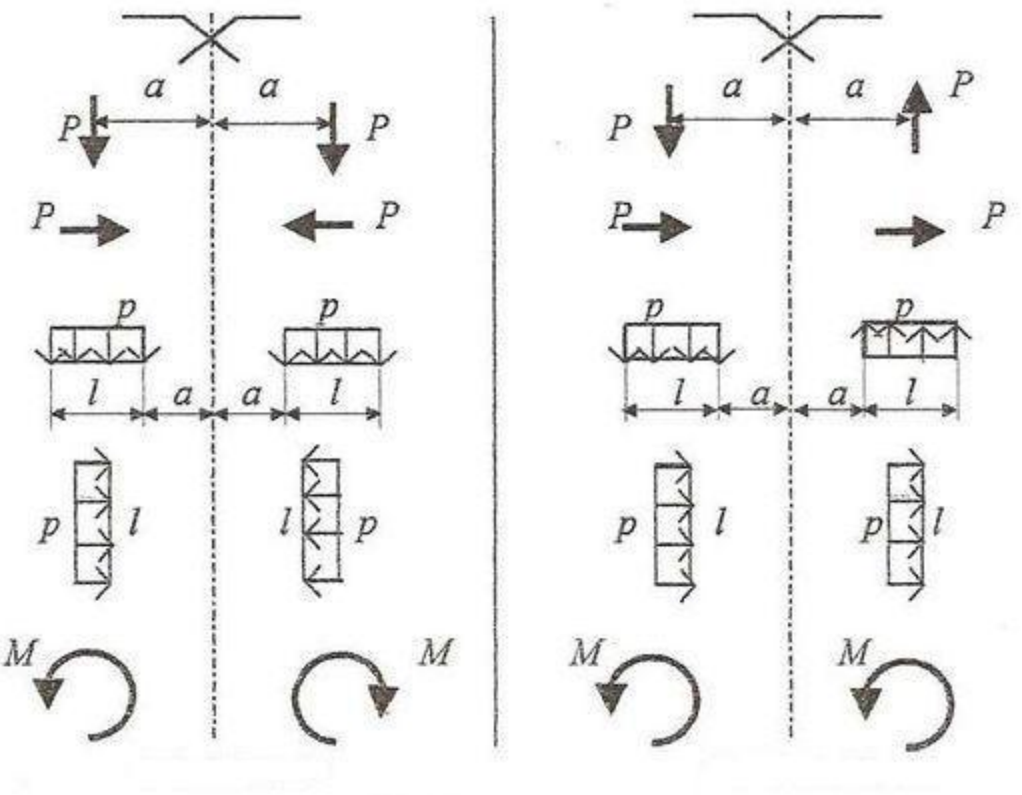


Fig.31.

From the elementary conditions of equilibrium results that ***if a symmetrical system of bodies is loaded with symmetrical loads then the reactions are also symmetrical.*** *This propriety makes as for this kind of systems we have need to use only a half number of the equilibrium equations.*

In the same way we can enounced that ***if the symmetrical system of bodies is loaded with antisymmetric loads then the reactions are also antisymetric.*** *also in this case we shall use one half of the number of equilibrium equations.*

If the loads are any, this can be divided in two components of loads: one symmetrical and one antisymmetric load. Then using the principle of superposition the reactions are obtained adding the reactions resulted from the two kinds of loads. The way to divide the loads in the two components is represented in the figure 32.

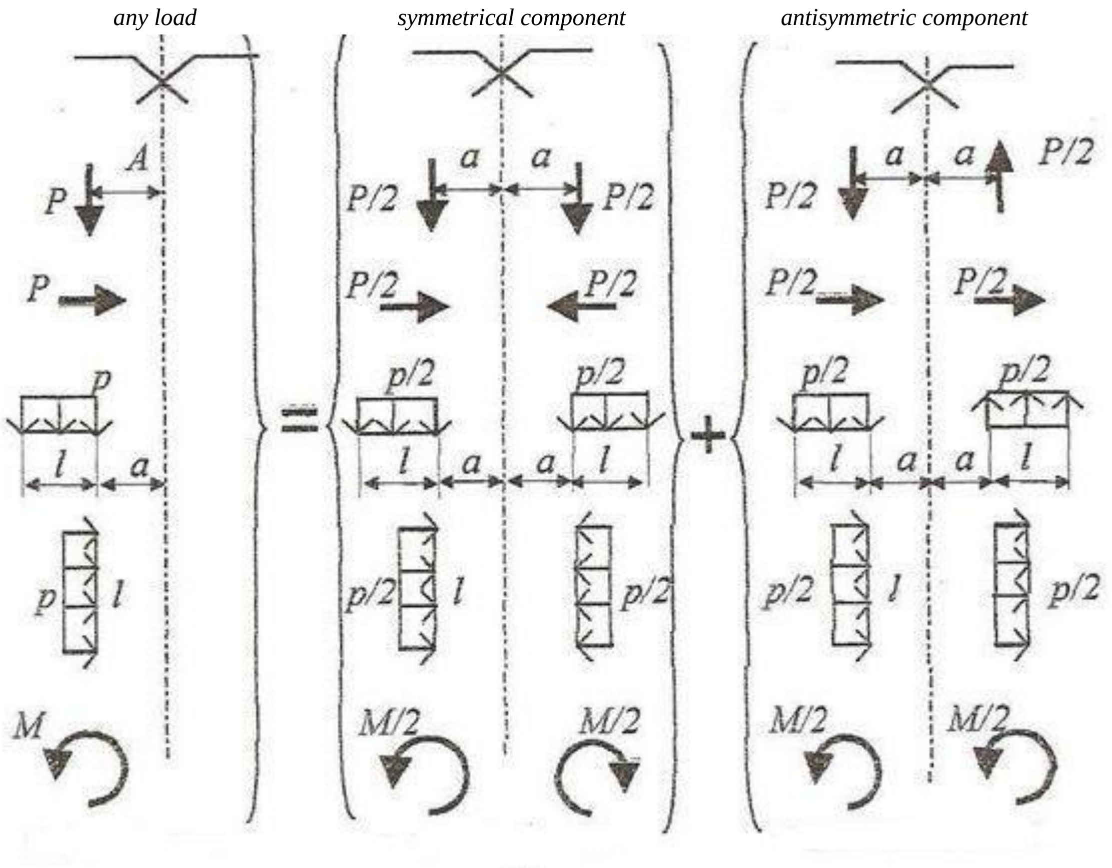


Fig.32.

Because the proprieties of symmetrical systems for the systems with internal hinges on the symmetry axis we can solve the system using one half of the system (semi-system).

For to use the semi-system we have to know that if the load is symmetrical than the internal hinge from the symmetry axis is equivalent to one simple internal connection perpendicular on the symmetry axis. This results from the fact that the other component of the internal reaction force cannot be symmetrical. Consequently in the semi-system corresponding to the symmetrical loads the internal hinge from the symmetry axis is replaced with a simple support perpendicular on the symmetry axis.

But if the load is antisymmetric then the internal hinge from the symmetry axis is equivalent to one simple internal connection collinear to the symmetry axis because the perpendicular component of the reaction in this kind of hinges cannot be antisymetric. Therefore in the semi-system *corresponding to the antisymetric loads the internal hinge from the symmetry axis will be removed with one simple support collinear to the symmetry axis.*

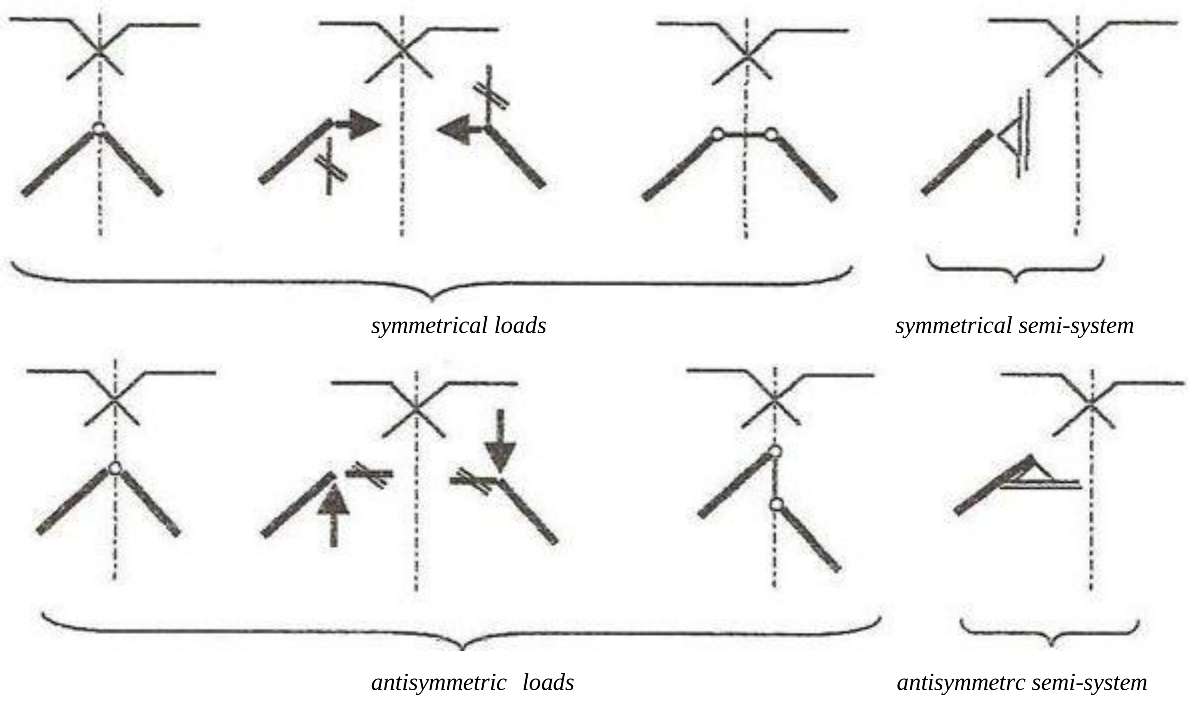


Fig.33.

6.15. Sample problems.

Problem 13. *Using the proprieties of the symmetrical systems of rigid bodies calculate the reactions from the external constraints of the system represented in the figure 34.*

Solution. *As we can see the system is symmetrical and it is loaded with symmetrical loads. This means that For to solve this system (to calculate the six external unknown) we can write only three equilibrium equations (resulting all the six unknowns):*

$$V_A = V_{A'}\,;\; V_B = V_{B'}\,;\; H_B = H_{B'}$$

But as we have seen we can solve the system using a symmetrical semi system (having one horizontal simple support in point D).

The free body diagram of this semi system is in fact a system with open outlines, so it can be solved using the mixed method computing only the external reactions (of the semi system). The body AC is secondary body therefore it can be solved independently, so we shall write one equation for this body:

$$\Sigma M_{C_1}^{left} = 0\,;\; 1{,}5pl\,.\,0{,}75l - V_A\,.\,1{,}5l = 0\,;\; \longrightarrow V_A = 0{,}75pl;$$

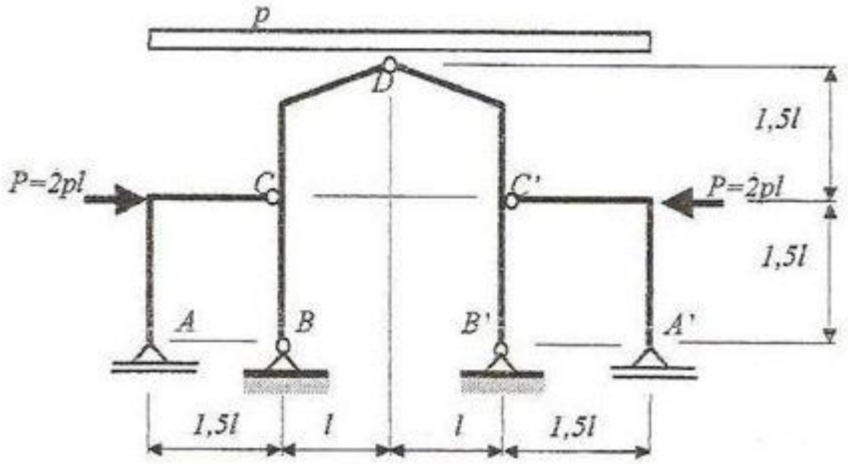


Fig.34.

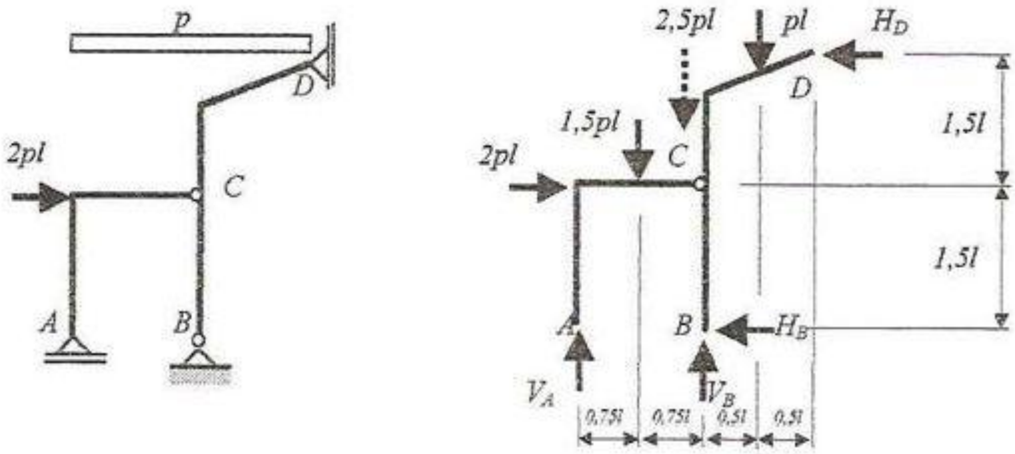


Fig.35.

For the entire system we can write:

$$\Sigma Y_i = 0\ ;\ V_A + V_B - 2{,}5pl = 0\ ; \longrightarrow V_B = 1{,}75pl;$$
$$\Sigma M_{Di} = 0\ ;\ -V_A\,.\,2{,}5l - V_B\,.\,l - H_B\,.\,3l + 2pl\,.\,1{,}5l + 2{,}5pl\,.\,1{,}25l = 0$$
$$\longrightarrow H_B = 0{,}833pl.$$

As we can see the external reactions of the system (the entire system) may be determined without to write all the equilibrium equations for the semi system, but the calculation of the reaction from D make possible to check the computation. In this way writing the third equation for the entire semi system we shall find:

$$\Sigma M_{Bi} = 0\ ;\ -V_A\,.\,1{,}5l + H_D\,.\,3l - 2pl\,.\,1{,}5l + 2{,}5pl\,.\,0{,}25l = 0$$
$$\longrightarrow H_D = 1{,}16pl.$$

In these equations (for the entire semi system) we have used the resultant force of the given distributed load and not the two resultants corresponding to the two bodies.

The verification is made with the following equation:

$$\Sigma X_i = 0\ ;\ 2pl - H_B - H_D = 0\ .$$

Problem 14. *For the system from the figure 36 calculate the reactions from the external constraints.*

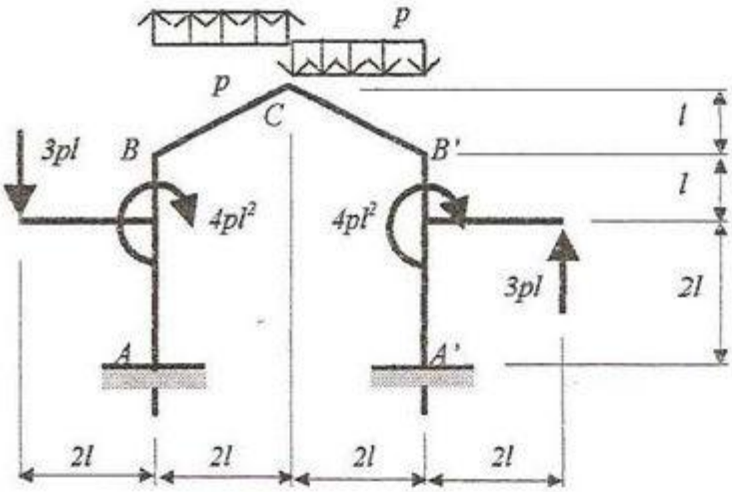


Fig.36.

Solution. *The system is statically determined and stable and also it is symmetrical with an antisymmetric load. Will result that the reactions will be also antisymmetric. The conclusion is that for to determine the six external reactions it is enough to write only three equilibrium equations. These equations, if they are written for the entire system have to chosen to contain all the unknowns. For example the projection on the vertical direction is not independently equation and it does not contain unknowns if we accept the equality:* $V_A = -V_{A'}$ *(the condition of the antisymmetry of the loads).*

But we can work using the antisymmetric semi system having a vertical simple support in the point C. The free body diagram of this semisystem is represented in the figure 37.

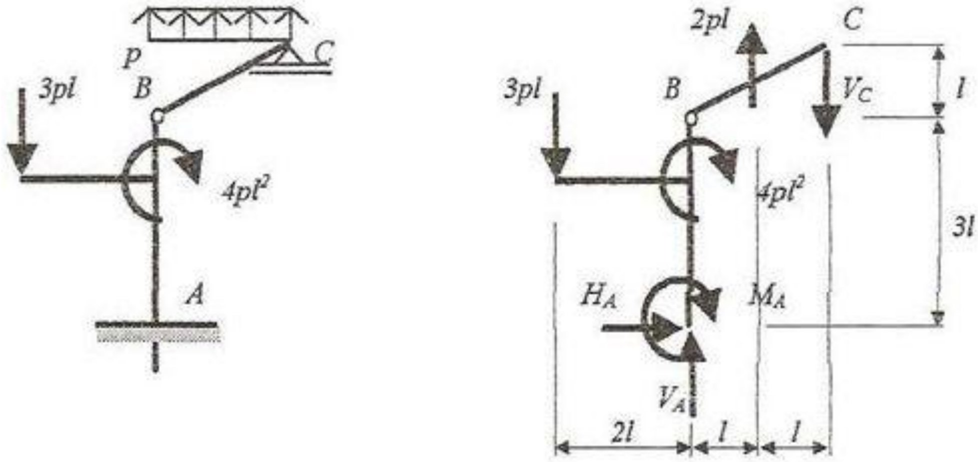


Fig.37.

Because this system is with open outlines we can use the mixed method and if we want to determine only the reactions from the fixed support A we can choose the equilibrium equations so that to result only these unknowns. But for to check the solution of the computation we shall solve entirely the semi system that has the body BC as secondary body. The equilibrium equations will be:

$$\Sigma M_{Bi}^{right} = 0 \;; 2pl \cdot l - V_C \cdot 2l = 0; \longrightarrow V_C = pl;$$
$$\Sigma X_i = 0 \;; H_A = 0 \;; \longrightarrow H_A = 0 \;;$$
$$\Sigma Y_i = 0 \;; -3pl + V_A + 2pl - V_C = 0 \;; \longrightarrow V_A = 2pl;$$
$$\Sigma M_{Ai} = 0 \;; 3pl \cdot 2l - 4pl^2 - M_A + 2pl \cdot l - V_C \cdot 2l = 0 \;; \longrightarrow M_A = 2pl^2.$$

The verification is made with the equation:

$$\Sigma M_{Ci} = 0 \;; 3pl \cdot 4l - 4pl^2 - M_A + 2pl \cdot l - V_C \cdot 2l + H_A \cdot 4l = 0$$

Problem 15. *Calculate the reactions from the external constraints using the proprieties of the symmetrical system represented in the figure 38.*

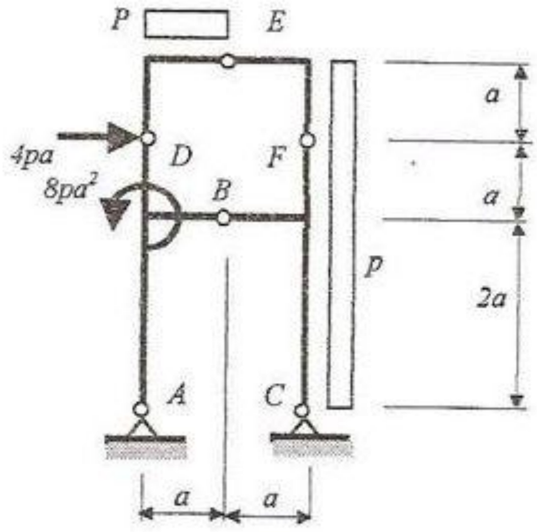


Fig.38.

Solution. *We remark that the system is symmetrical but the load is an any load, consequently if we want to use the proprieties of the symmetrical systems we need to divide the load in the two components: one symmetrical and one antisymmetrical.*

The scheme of the decomposition is represented in the figure 39. Because the system is with closed outlines for to solve easier we need to decompose the system in two parts with open outlines. In place to divide the system in principal and secondary parts we shall divide in semi systems. The two semi systems corresponding to the two components of load (symmetrical and antisymmetrical) are represented in the figure 40.

The solving of the symmetrical semi system is made as for a system with open outlines:

$$\Sigma M_{Di}^{DE} = 0\ ;\ H_E^s \,.\, a - 0{,}5pa \,.\, 0{,}5a - 0{,}5pa \,.\, 0{,}5a = 0\ ; \rightarrow H_E^s = 0{,}5pa;$$
$$\Sigma Y_i = 0\ ;\ V_A^s - 0{,}5pa = 0\ ;\ \rightarrow V_A^s = 0{,}5pa;$$
$$\Sigma M_{Ai} = 0\ ;\ -1{,}5pa \,.\, 1{,}5a + 4pa^2 + H_B^s \,.\, 2a - 2pa \,.\, 3a - 0{,}5pa \,.\, 3{,}5a -$$
$$- 0{,}5pa \,.\, 0{,}5a + H_E^s \,.\, 4a = 0\ ;\ \rightarrow H_B^s = 2{,}125pa;$$
$$\Sigma M_{Bi} = 0\ ;\ -V_A^s \,.\, a - H_A^s \,.\, 2a + 1{,}5pa \,.\, 0{,}5a + 4pa^2 - 2pa \,.\, a -$$
$$- 0{,}5pa \,.\, 1{,}5a + 0{,}5pa \,.\, 0{,}5a + H_E^s \,.\, 2a = 0\ \rightarrow H_A^s = 1{,}375pa$$

The verification is made with:

$$\Sigma X_i = 0\ ;\ - H_E^s + 0{,}5pa + 2pa - H_B^s + 1{,}5pa - H_A^s = 0\ .$$

For the antiymmetrical semi system we shall write the following equilibrium equations:

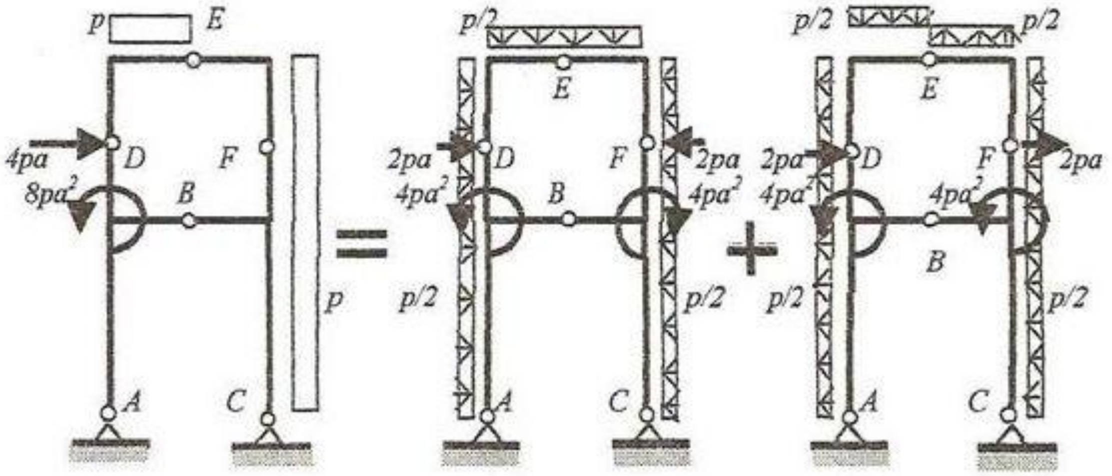
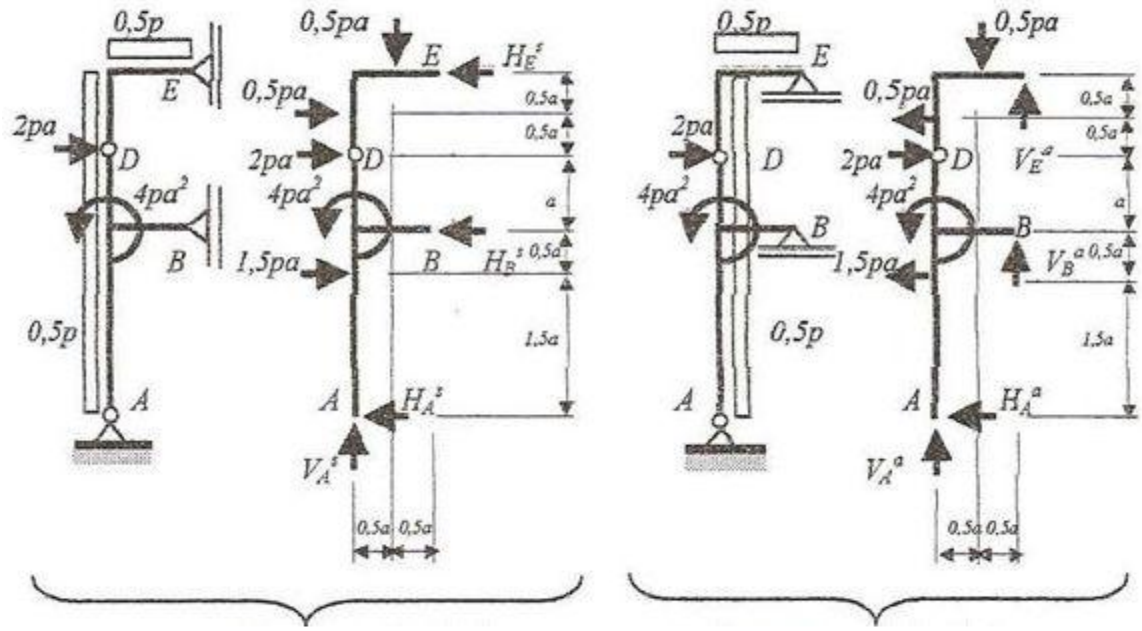


Fig.39.



symmetrical semi system *antisymmetrical semi system*

Fig.40.

$$\Sigma M_{Di}^{DE} = 0\ ;\ V_E^a \cdot a - 0{,}5pa \cdot 0{,}5a + 0{,}5pa \cdot 0{,}5a = 0\ ; \qquad V_E^a = 0;$$
$$\Sigma X_i = 0\ ;\ -0{,}5pa + 2pa - 1{,}5pa - H_A^a = 0\ ; \rightarrow H_A^a = 0;$$
$$\Sigma M_{Ai} = 0\ ;\ V_E^a \cdot a + V_B^a \cdot a - 0{,}5pa \cdot 0{,}5a + 0{,}5pa \cdot 3{,}5a - 2pa \cdot 3a + 4pa^2 + 1{,}5pa \cdot 1{,}5a = 0\ ; \rightarrow V_B^a = -1{,}75pa;$$
$$\Sigma M_{Bi} = 0\ ;\ -V_A^a \cdot a - H_A^a \cdot 2a - 1{,}5pa \cdot 0{,}5a + 4pa^2 - 2pa \cdot a + 0{,}5pa \cdot 1{,}5a + 0{,}5pa \cdot 0{,}5a = 0 \rightarrow V_A^a = 2{,}25pa$$

The checking is made with the equation:

$$\Sigma Y_i = 0\ ;\ V_A^a + V_B^a + V_E^a - 0{,}5pa = 0.$$

The final reactions of the system are obtained summing the reactions corresponding for the two components of loads:

$$V_A = V_A^s + V_A^a = 2{,}75pa\ ;\ H_A = H_A^s + H_A^a = 1{,}375pa;$$
$$V_C = V_C^s - V_C^a = -1{,}75pa\ ;\ H_C = H_C^s - H_C^a = 1{,}375pa.$$

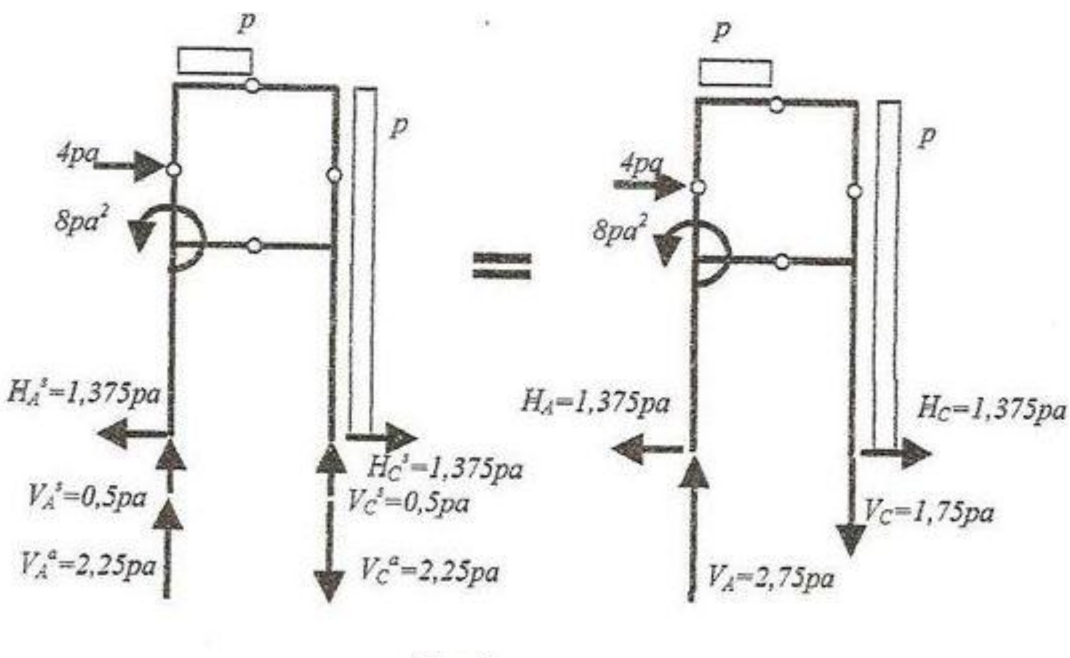


Fig.41.

Problems 16, 17, 18. *Calculate, using the proprieties of the symmetrical systems, the reactions from the external constraints for the systems from the figures 42, 43 and 44.*

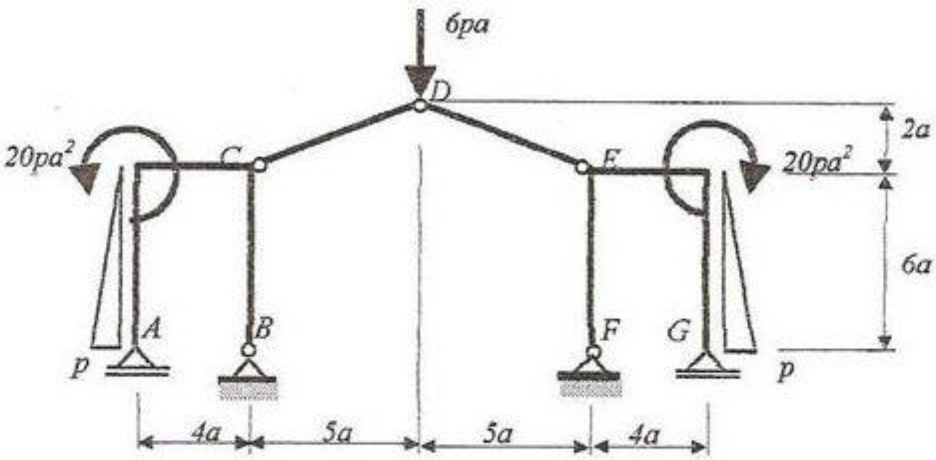


Fig.42.

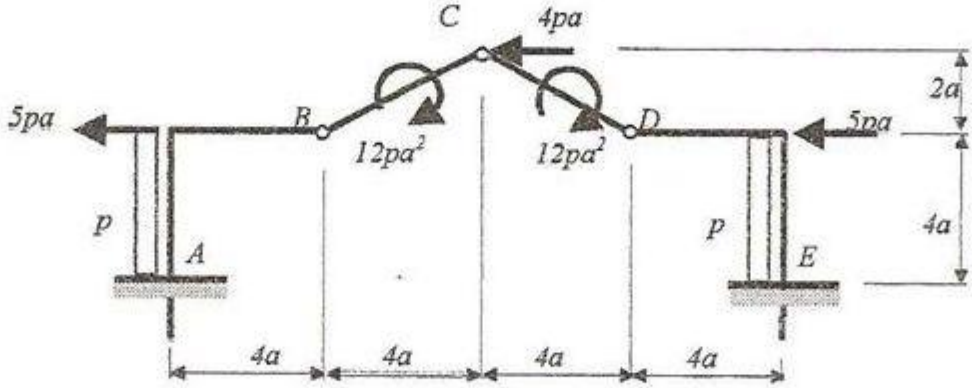


Fig.43.

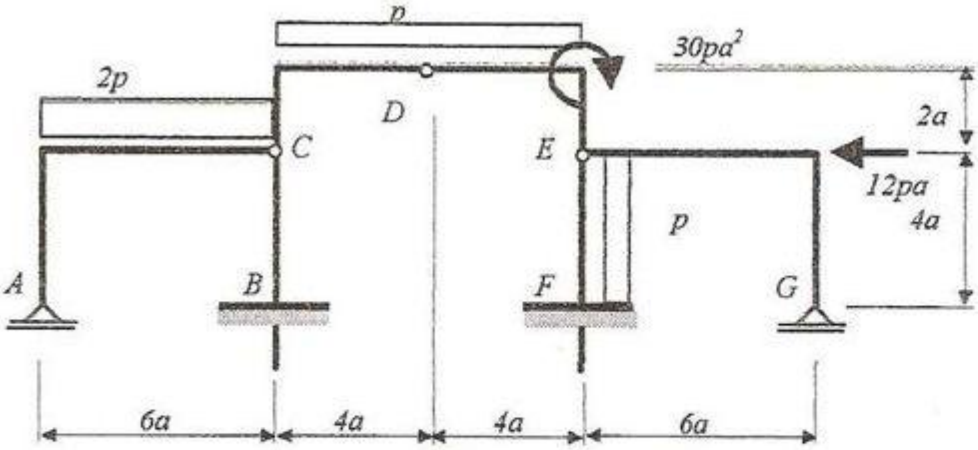


Fig.44.

Chapter7. Trusses

7.1. Introduction.

The trusses represent one of the most efficient engineering structures answering to two requests: one practical, engineering referring to the strength of the structure and the second of the economical solution to use the material in the structure.

The truss is a particular system of rigid bodies in which the bodies are straight rods, hinged at their ends and having as external constraints only simple supports and hinged supports. Generally the bars that form the truss are called ***members****, and the internal hinges are called* ***joints****.*

The trusses may be classified function different criteria and they may have different shapes.

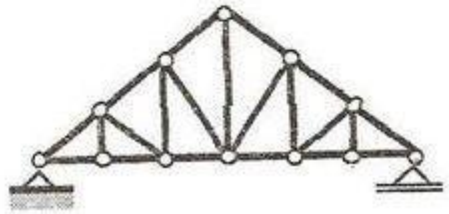
1) After the number of the scalar unknowns from the constraints and connections the trusses are divided in: ***statically determinate and stable trusses*** *and* ***statically non determinate trusses.*** *The statically determinate trusses are the systems at which the number of the scalar unknowns from the constraints and connections is equal to the number of the independent scalar equilibrium equations that we can write for to express the equilibrtium of the system.*

2) After the configuration of the truss we have: ***trusses in space*** *and* ***trusses in plane****. The plane trusses are those trusses at which the configuration is plane and the loads are acting also in the same plane.*

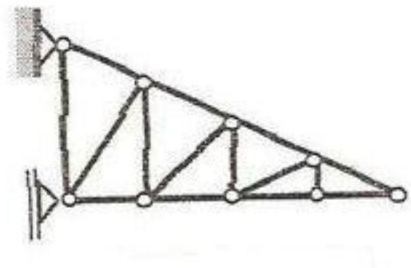
3)After how they are made we have: ***simple trusses****,* ***compound trusses*** *and* ***complex trusses****. The simple trusses, using a simple definition are those trusses that are made joining triangles side by side. The compound trusses are obtained overlapping the simple trusses and the complex trusses are obtained overlapping triangles.*

From these kimnd of trusses in this chapter we shhall study only the simple, plane and statically determined trusses.

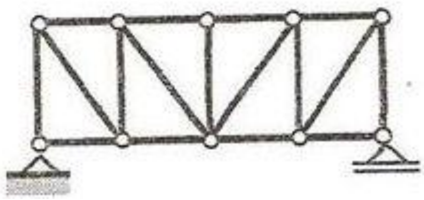
The shapes of the trusses are very different, but the must usual are the triangular, rectangular or with parallel sides, poligonal and lenticular.



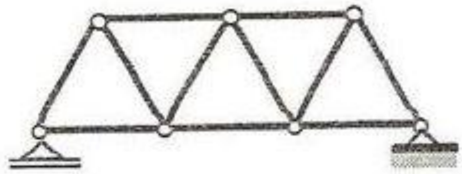
triangular truss



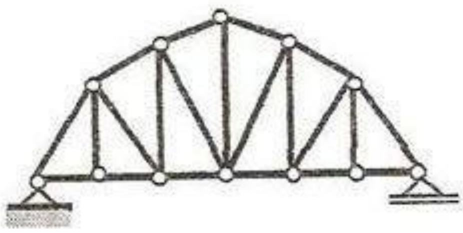
triangular truss



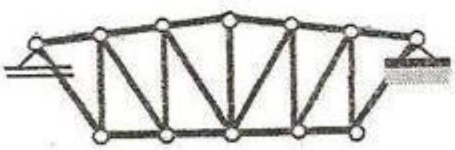
rectangular truss



truss with parallel sides



poligonal truss



lenticular truss

Fig.1.

As we can see all the trusses from the figure 1 are simple trusses.

Being a system of bodies the expression of their equilibrium can be made as for an any system of rigid bodies, namely using the methods from the previously chapter. But as we can remark on the exemples from the figure 1. the expression of the equilibrium of a truss considered as a system of rigid bodies goes to a system of equations with a very large number of equations. Even if these equations are very simple however the volum of the computation necessary for to solve the system is very large. This makes that or we use a program of the computer or we simplify the calculation so that the system of equations to be not very large. For this reason are considered some simplifications that although the results are approximate with respect to the real structure, the calculation is made easier yet.

7.2.Simplifying assumptions.

The simplifying assumptions considered generally for the trusses are the following:

- *1)The bars (the members) of the truss are straight. This hypothesis is generally respected.*

- *2)The mambers have the cross section are negligeable with respect to the length of them. They are centered in the joints that are considered internal hinges. In reality this hypothesis is not fully met because although the members have small cross sections with respect to their lenghts and are centered in the joints (the axes of the members are intersected in the same point in a joint), the joint is not a hinge is a restraint connection between the members. In reality the contact in the joint is made or as a welding connection or a riveted one or the joint is a continous connection (as for the concrete trusses).*

- *3)The external loads are only concentrated forces and they are applied only in the joints. This hypothesis can be made for the usefully loads (those for the trusses are made) using the indirect transmission of the loads but not for the weight of the members that acts distributed on the length of them. If the weights of the members cannot be neglected then they will be concentrated in the adjoining joints (in each joint will be considered half of the weight of the member).*

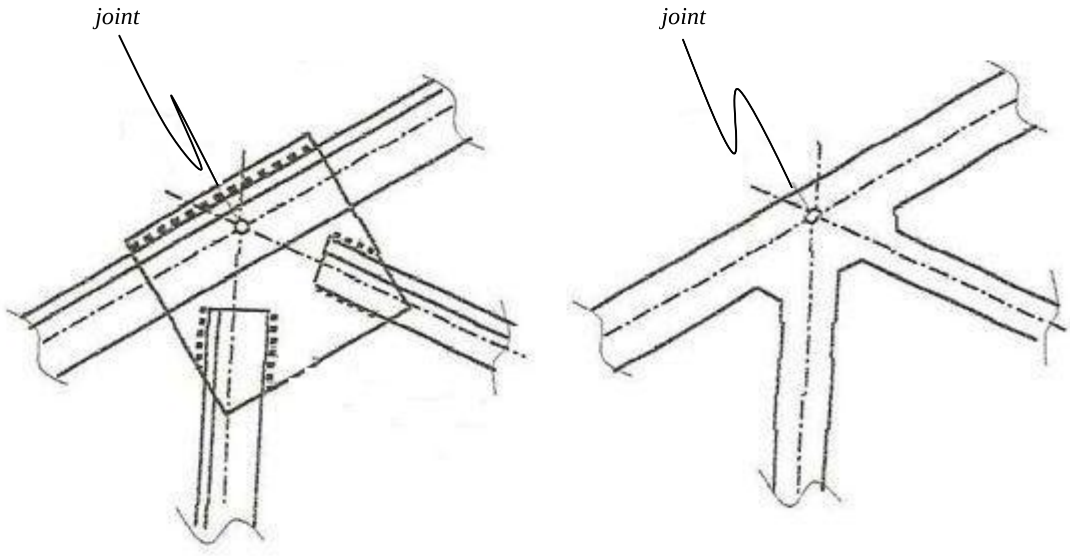


Fig.2.

We make the remark that the external constraints are considered located in the joints also.

Considering the three hypothesis let be one any member from a truss situated between the joints ***i*** *and* ***j****.*

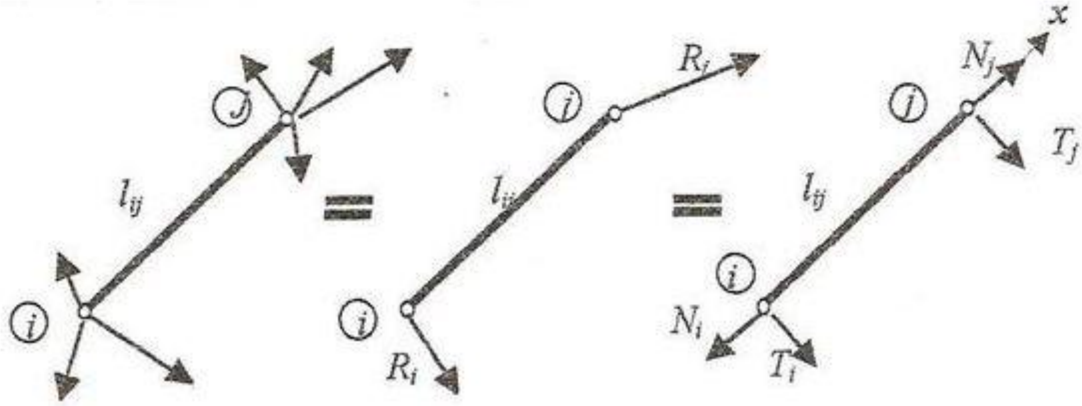


Fig.3.

All the forces, given and reaction, will have the points of application in the two joints. Being concurrent forces they can be removed with their resultant forces acting in the two points. We decompose the two resultants in two components: one on the direction of the axis of the member ant the second on the normal direction on the member, marked respectively N_i *and* N_j *and also* T_i *and* T_j*.*

If we study the equilibrium of the system, using the propriety of the systems (for as system to be in equilibrium each body from the system have to be in equilibrium), the member ***ij*** *have to be in equilibrium. The scalar conditions of equilibrium are:*

$$\Sigma X_i = 0 \; ; \; N_i = N_j \; ;$$
$$\Sigma M_i = 0 \; ; \; T_j \cdot l_{ij} = 0 \; ; \; \rightarrow T_j = 0 \; ;$$
$$\Sigma M_j = 0 \; ; \; T_i \cdot l_{ij} = 0 \; ; \; \rightarrow T_i = 0 \; ;$$

From the three equations results that ***each member of the truss are acted by axial forces*** *called* ***internal forces****. We shall note these forces* ***:***

$$N_i = N_j = N_{ij}$$

namely each member is acted by one single axial force (internal force), unknown as magnitude and sense.

This makes that the system of bodies, the truss, in which the double hinged bars are the bodies and the joints are the internal connections, to be viewed from another perspective. We remark that all the forces, given and reactions, act about the joints, so, we can consider that in fact the joints are the material support of the system, they can be considered ***particles****. At the other hand, the members, which are in fact pendulums (non loaded double hinged rods) are acted by one unknown force (axial force) therefore they can be considered* ***simple internal connections****. Consequently the truss can be considered , using the previously hypothesis, a system of particles joined among them with simple internal connections.*

This way to see a truss allows as to develop specific methods for these kind of systems, methods that allows to determine the reactions from the external constraints and the internal axial forces from the members.

7.3. Notations, names, conventions of signs

In strength of materials we agreed that if the bar is tensioned, stretched, then the internal axial force to be considered positive, and if the bar is compressed then it will be considered negative.

Supposing a member between two joints, separated by them, we remark that if the member is tensioned then the internal force goes out from the joint (pull the joint), and if the member is compressed then the internal force goes in the joint (push the joint).

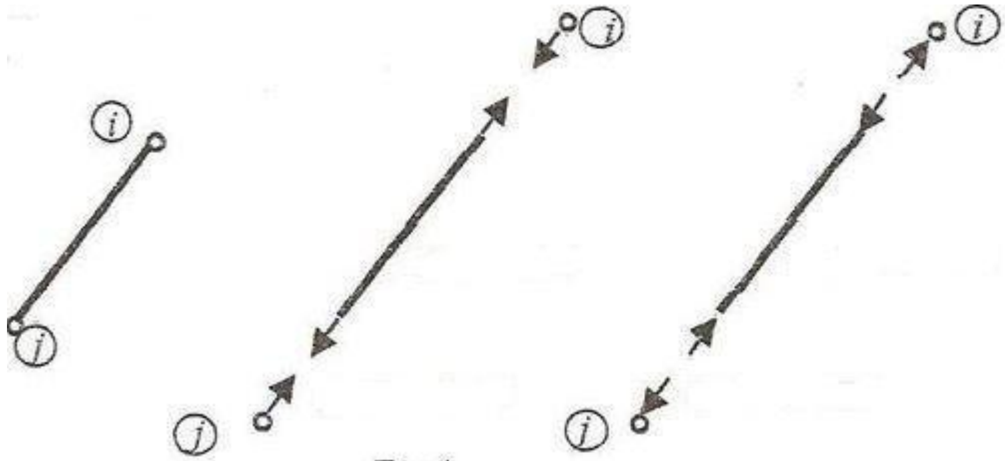


Fig.4.

A simple truss (developed on horizontal direction) has the following components with the following names:

-the members that bordering in upper part of the truss form the ***upper chord****;*

-the members that bordering the lower part of the truss form the ***lower chord****;*

-the inclined members are named ***diagonals****;*

-the space from two rows of joints is called ***panel****.*

7.4. Statically determination of the truss

The checking of the possibility to solve of a system is the verification of the condition of statically determination and stability. This condition in the case of an any system has two aspects: the quantitative aspect, namely the verification of the condition as the number of the scalar independent equilibrium equations (N_e) to be equal to the number of the scalar unknowns from the constraints and connections (N_u):

$$N_e = N_u$$

and the qualitative aspect as the constraints and connections to fix the system.

Because the truss is considered to be a system of particles the number of equilibrium equations, in plane problem is:

$$N_e = 2n$$

where ***n*** *is the number of joints.*

The number of the scalar unknowns is equal to:

$$N_u = r + i$$

where ***r*** *is the number of the reactions from the external constraints and* ***i*** *is the number of the unknown internal forces. If we have a simple truss then the number of the reactions from the external constraints is equal to three, and the number of the internal forces is equal to the number of the members (m) :*

$$N_u = 3 + m$$

Results the condition of the statically determination of a simple plane truss:

$$2n = 3 + m$$

For the simple plane trusses the existence of the three external reactions makes, from this point of view, that the truss to have the *behavior of a rigid body. This means that the truss have to be geometric non-deformable. Let to analyze how we can obtain a geometric non-deformable structure using joints and members. First we shall consider a member with two joints at its ends. The following joint can be joined to the two joints with two members forming, in this way, a triangle that is a non-deformable shape. Still, we add each joint using two members and in this way each time we have a non-deformable structure. In conclusion, if the truss is made from triangles side by side, then the truss will result geometric non-deformable.*

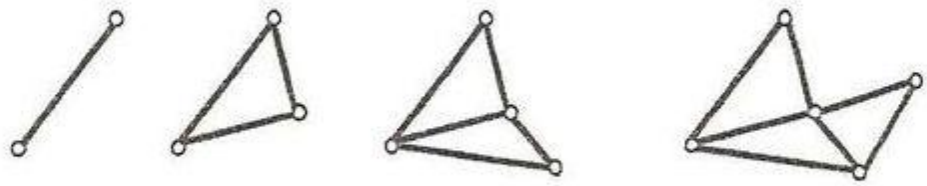


Fig.5.

7.5.Method of joints.

The methods of solving the trusses are in essence the same as the methods used for to solve the systems of bodies, but adapted for the systems of particles.

From the large number of the methods we shall study in this chapter only two methods, from which the must known is the ***method of joints****.*

This method has two variants:

1) ***The general version of the method of joints****. The method is based on the propriety of the systems of bodies: if a system is in equilibrium then all bodies (here all particles, namely the joints) are in equilibrium. Consequently the method corresponds to the method of the equilibrium of the component bodies. In this version isolate all joints of the truss loading each of them with the given forces, the external reaction forces and the internal forces which replace the members (these are pairs, equals and with opposite senses having the directions of the members). Expressing the equilibrium of each joint using two equilibrium equations is obtained one system of* ***2n*** *equations with the same number of unknowns. Finally solving the system of equations we find the external reaction forces and all internal forces corresponding to the members of the truss.*

Generally, because the large number of the joints, the solution of the system of equations can be obtained using a computer. Consequently this version is suitable for computer programming. As we can see is not any restriction in the use of this method at any kind of trusses. (simple, compound or complex)

*2) **Simplified version of the method of joints**. This version is used for to solve the simple trusses. It is in fact the mixed method used for the systems of bodies adapted to the systems of particles. In the first stage calculate the external reactions considering the truss as a simple rigid body (theorem of solidification). Next we express the equilibrium of each joint, the joints being chosen so that each joint to have only two unknowns which can be solved using the two corresponding equilibrium equations.*

For to solve complete a simple truss we shall pass the following steps:

- *1) Is checked the condition of statically determination and stability;*
- *2) Is calculated the cosines of the directions of the inclined members and is numerated the joints of the truss in an any order;*
- *3) Is calculated the reactions from the external constraints considering the truss one single body, namely writing three equilibrium equations. Is made the verification of the obtained results. The results are represented on **the scheme of the results**;*
- *4) Is chosen one joint with two members. Always at the simple trusses there is one (or more) joint with two members because the last joint is connected to the structure in the process to obtain the geometric non-deformable system with only two members. This joint is isolated and is loaded with the external forces (now all know forces) and the two members are removed with unknown internal forces. For the signs of the results to be the same as those considered in the strength of materials, the unknown internal forces will be considered **always** tensions (pull the joint, go out the joint);*
- *5) Is writing two equilibrium equations namely:*

$$\Sigma X_i = 0 ; \quad \Sigma Y_i = 0 .$$

We remind that if the angles of the inclined members are measured with the horizontal direction then the projection equations are:

$\Sigma X_i = \Sigma\,horizontal\ forces + \Sigma\,inclined\ forces$ x $\cos\alpha_i$

$\Sigma Y_i = \Sigma\,vertical\ forces + \Sigma\,inclined\ forces$ x $\sin\alpha_I$

For the signs of the projections on the horizontal direction we agree as all forces with the sense directed to right will be considered with (+) sign and all forces directed to left will be considered with (-) sign.

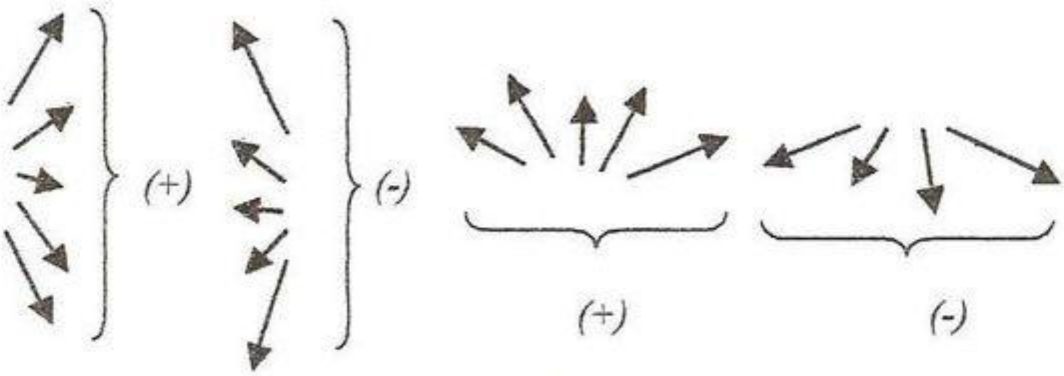


Fig. 6.

For the vertical forces we agree that all the forces directed with the sense up will be considered with the (+) sign and all forces directed down with the sign (-).

The system of two equations with two unknowns is solved are obtained the two internal forces and they are represented on the scheme of the results. Thus if the sign of the result is plus the internal force is tension namely it is represented pulling the joints (go out from the joints), and if the result is minus then the internal force is compression namely it push the joint (go in the joints);

- *6) Is searching one joint with two members with unknown internal forces. The joint may have more members but only two with unknown internal forces. The joint is isolated and is loaded with the external known forces , and the members are replaced with known and unknown internal forces (the unknown forces are tensions).*

After that is made as in the step 5) and is continued with the step 6) and the cycle is resumed until we finish all the joints.

• *7) The last two joints contain three equations of checking. These equations are obtained because we have introduced three equilibrium equations for the entire system (equations from another method).*

We make the remark that on the scheme of the results we may work using the signs of the internal forces.

7.6. Sample problems.

***Problem 1**. Calculate the internal forces from the members of the truss represented in the figure 7.*

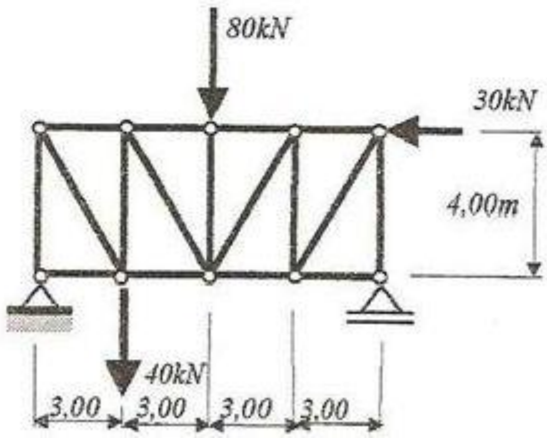


Fig.7.

***Solution.** First we shall check the condition of statically determination and stability of the truss. For this, we remark that the truss has 10 joints and 17 members (4 in the upper side, 4 in the lower side, 4 diagonals and 5 vertical members). Results:*

$$2 \cdot 10 = 3 + 17$$

Also it is a truss made from triangles side by side, so it is geometric non-deformable and having one hinged support and one simple support it is fixed.

All the inclined members make the same angle α with the horizontal direction and we have:

$$cos\alpha = \frac{3}{5} = 0,6 \; ; \; sin\alpha = \frac{4}{5} = 0,8$$

We number the joints of the truss and we calculate the reactions from the external constraints considering the entire truss one rigid body (Fig. 8). The equilibrium equations are:

$\Sigma X_i = 0 \, ; \, H_1 - 30 = 0 \, ; \longrightarrow H_1 = 30 \; kN;$
$\Sigma M_{1i} = 0 \, ; \, -40 \, . \, 3 - 80 \, . \, 6 + 30 \, . \, 4 + V_9 \, . \, 12 = 0 \, ; \longrightarrow V_9 = 40 \; kN;$
$\Sigma M_{9i} = 0 \, ; \, - V_1 \, . \, 12 + 40 \, . \, 9 + 80 \, . \, 6 + 30 \, . \, 4 = 0 \, ; \longrightarrow V_1 = 80 \; kN.$

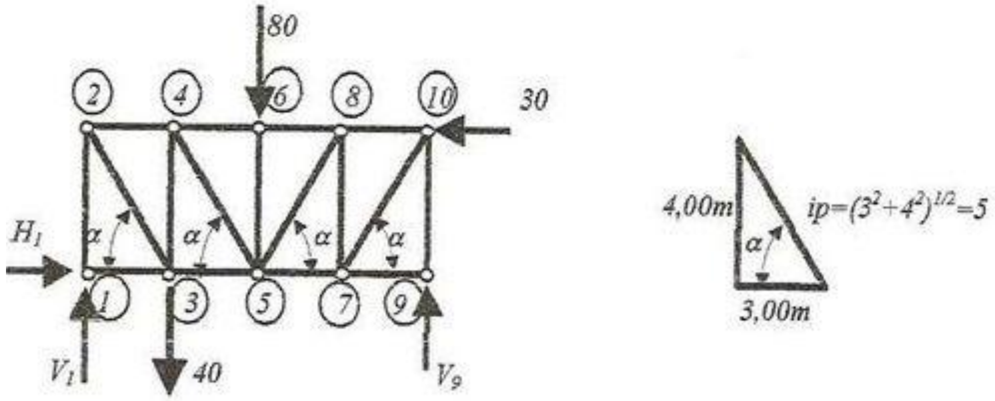


Fig.8.

We shall check these reactions using the equation:

$\Sigma Y_i = 0 \, ; \, V_1 - 40 - 80 + V_9 = 0 \, .$

Having all the external forces known we pass to the isolation of the joints. We choose one joint with two members. We remark that we have two joints with two members namely the joints 1 and 9. We shall take the joint 1 and it will isolate. We load the joint with the two known reaction forces from the hinged support and the two members are replaced with unknown internal forces. As we agreed these forces will be tensions (go out from the joint, pull the joint).

JOINT 1

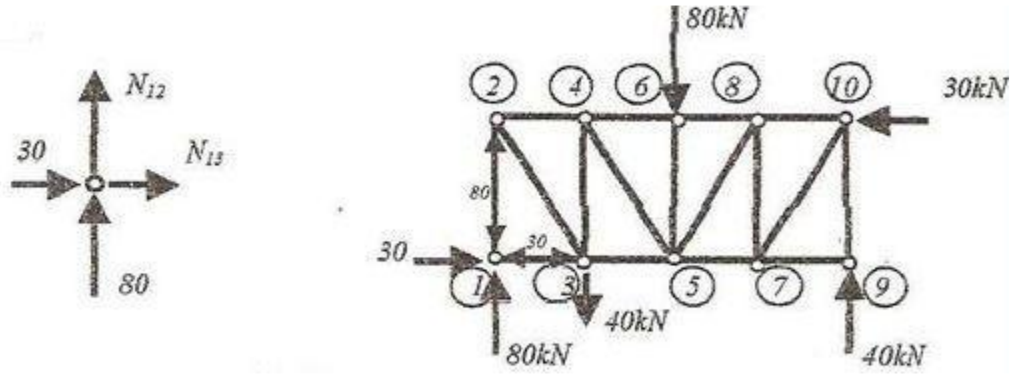


Fig.9.

We write two equilibrium equations for each joint namely:

$\Sigma X_i = 0 \, ; \, \Sigma Y_i = 0 \, .$

For the joint 1 we have:

$$\begin{cases} 30 + N_{13} = 0 \\ 80 + N_{12} = 0 \end{cases}$$

Solving the equations we shall obtain the values of the two internal forces:

$N_{13} = -30$; $N_{12} = -80.$

The both internal forces have resulted with minus senses so they are compressions (go in the joints) and they will be represented on the scheme of the results as in the figure 9. We remark that in this way the scheme does not contain signs, the signs have been converted in senses.

Still we shall choose another joint, but this time it will be with two members with unknown internal forces. This joint will be chosen in the neighborhood of the solved joints and in this case it will be the joint 2. Isolate this joint and we shall make as for the previously joint, namely we load the joint with the external forces (here we have not these kind of forces), and the members are removed with internal forces. The known internal forces are represented with their senses and the unknown forces as tensions.

JOINT 2.

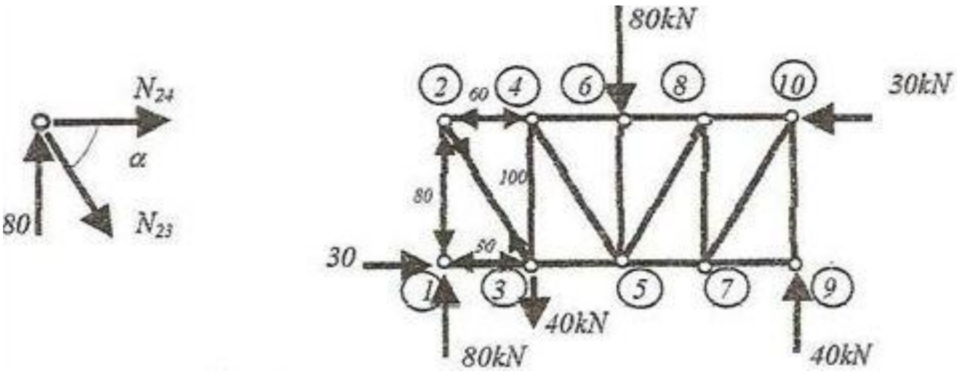


Fig.10.

The equilibrium equations for this joint are:

$$\begin{cases} N_{24} + N_{23} \cdot \cos\alpha = 0 ; \\ 80 - N_{23} \cdot \sin\alpha = 0 \end{cases}$$

Solving we have:

$N_{23} = 100; N_{24} = 60.$

The both internal forces are positive so they are tensions namely they are represented pulling the joints on the scheme of the results (as in the figure 10).

The next joint with two unknown internal forces is the joint 3.

The equilibrium equations are:

$$\begin{cases} 30 - 100 \cdot \cos\alpha + N_{35} = 0 ; \\ N_{34} - 40 + 100 \cdot \sin\alpha = 0 \end{cases}$$

with the solution:

$N_{35} = 30$; $N_{34} = -40$

JOINT 3

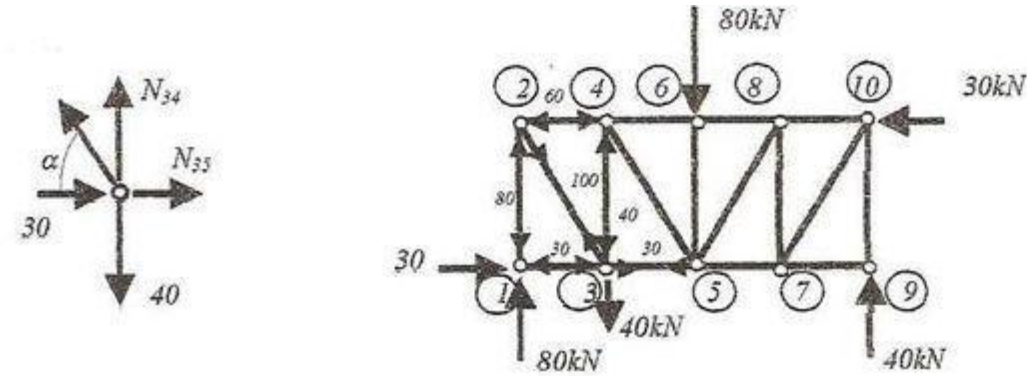


Fig.11.

the internal forces which are represented on the scheme of the results as in the figure 11.

The following joint is the joint 4.

JOINT 4

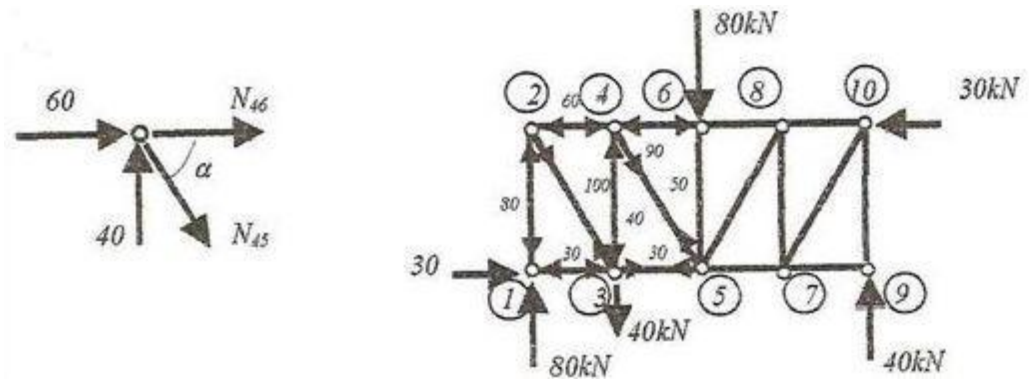


Fig.12.

We have the equations:

$$\begin{cases} 60 + N_{46} + N_{45} \cdot \cos\alpha = 0\,; \\ 40 - N_{45} \cdot \sin\alpha = 0 \end{cases}$$

from which result the internal forces.

$$N_{45} = 50\,;\ N_{46} = -90.$$

and which are represented on the scheme of the results in the figure 12.

The following joint is the joint 6 represented in the figure 13. The equilibrium equations are:

$$\begin{cases} 90 + N_{68} = 0\,; \\ -80 - N_{56} = 0 \end{cases}$$

Solving this system of equations we have:

JOINT 6

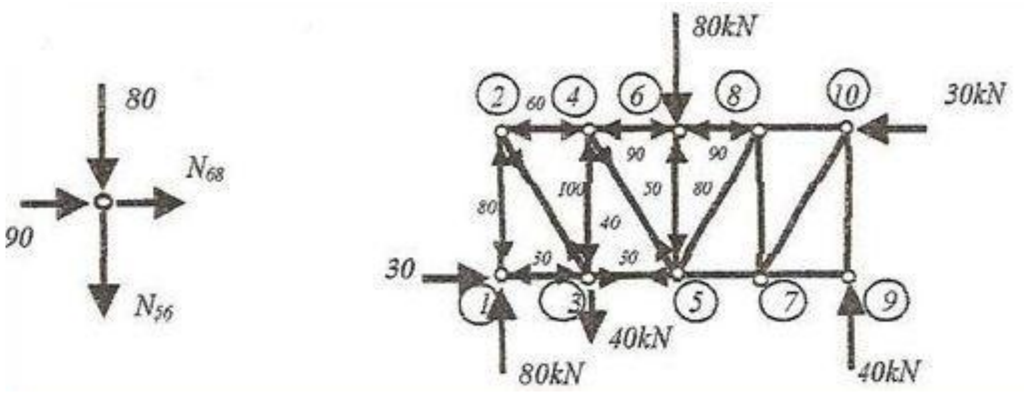


Fig.13.

$$N_{68} = -90\ ;\ N_{56} = -80.$$

those are represented on the scheme of the results in the figure 13.

The following joint with two unknown internal forces is the joint 5, represented in the figure 14. We shall write the equilibrium equations for this joint and results the system:

JOINT 5

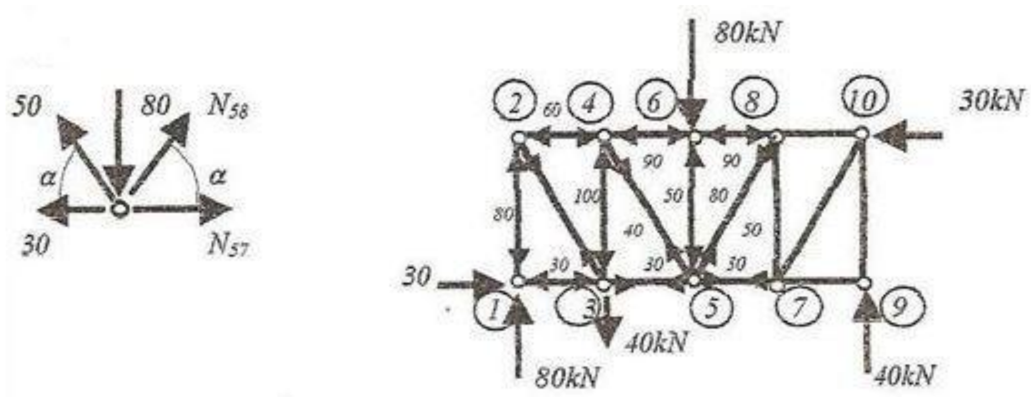


Fig.14.

$$\begin{cases} -30 - 50\cdot\cos\alpha + N_{57} + N_{58}\cdot\cos\alpha = 0; \\ 50\cdot\sin\alpha - 80 + N_{58}\cdot\sin\alpha = 0 \end{cases}$$

with the solutions:

$$N_{58} = 50\ ;\ N_{57} = 30$$

represented in the figure 14 on the scheme of the results.

The following joint isolated is the joint 8 that is represented in the figure 15. and for that we have the equations:

JOINT 8

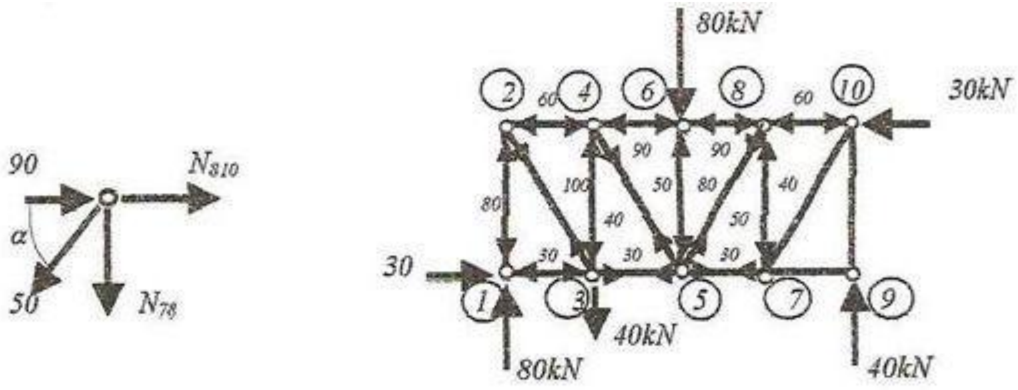


Fig.15.

$$\begin{cases} 90 - 50 \cdot \cos\alpha + N_{810} = 0; \\ -50 \cdot \sin\alpha - N_{78} = 0 \end{cases}$$

with the solutions:

$$N_{78} = -40\ ;\ N_{810} = -60.$$

which are represented in the figure 15 on the scheme of the results. We can remark that in this stage we have arrived to the last three joints, consequently we may choose any joint from them because each of them has only two unknown internal forces.

We choose the joint 7.

JOINT 7

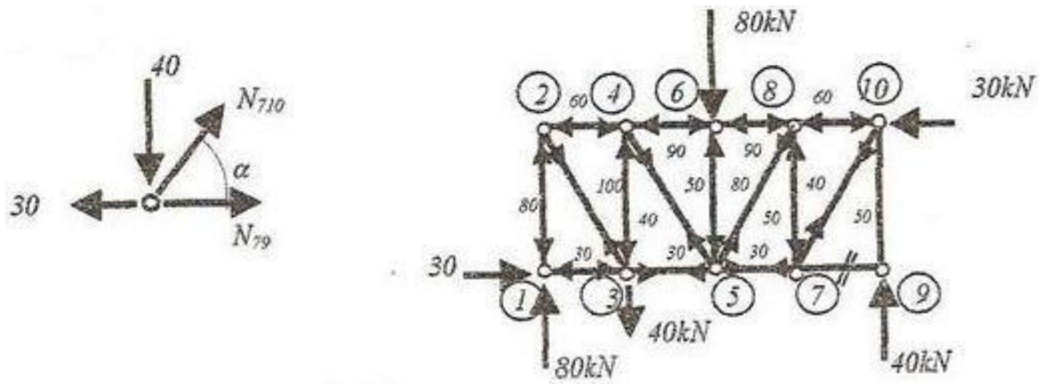


Fig.16.

We write the equations:

$$\begin{cases} -30 + N_{710} \cdot \cos\alpha + N_{79} = 0; \\ -40 + N_{710} \cdot \sin\alpha = 0 \end{cases}$$

Solving the system results the internal forces:

$$N_{710} = 50\ ;\ N_{79} = 0$$

which are represented on the scheme of the results in the figure 16. Now on the last two joints we have only one single unknown internal force, namely from the four equations of equilibrium that we can write only one equation is used for solve. The other three equations will be used for verification.

We shall take the joint 9 (represented in the figure 17), for which we have the equations:

$$\begin{cases} 0 = 0 \\ 40 + N_{910} = 0 \end{cases}$$

where the first equation is one of verification and the second will give the value:

$$N_{910} = -40.$$

JOINT 9

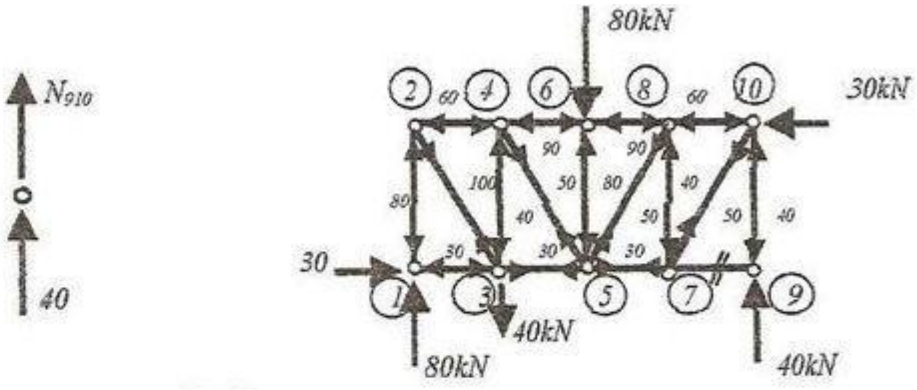


Fig.17.

that is represented on the scheme of the results and the entire truss is solved. The last joint is entirely for verification. We write the equations:

$$\begin{cases} 60 - 50 \cdot \cos\alpha - 30 = 0; \\ -50 \cdot \sin\alpha + 40 = 0 \end{cases}$$

JOINT 10

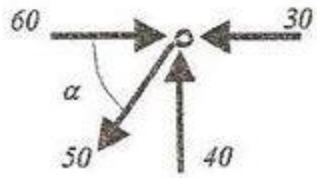


Fig. 18.

The last two equations shows as that the solution of the truss is correct.

Problem 2. *Using the method of joints calculate the internal forces in the members of the truss represented in the figure 19.*

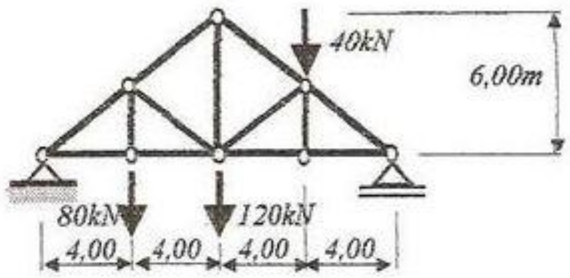


Fig.19.

7.7. Joints with particular loads.

In certain situations of loads of the joints we can determine some of the internal forces without to isolate the joint and without to express the equilibrium of it. Generally we meet six cases of particular loads of the joints, from which three are with zero internal forces. These three cases with zero internal forces are:

1) Unloaded joint with two members. The both members will have zero internal forces;

2) Joint with two members loaded with a force collinear with one member (from the two). The second member has zero internal force and the first (collinear with the given force) has the internal force equal to the given force and with the same effect about the joint;

3) Unloaded joint with three members from which two are collinear. The third member has zero internal force and the two collinear members have the same internal forces and with the same effect about the joint;

The following three cases of particular loaded joints are:

4) Joint with two members loaded with two forces collinear each with one member (from the two). The internal forces from the two members are equal with the two given forces and with the same effect about the joint;

5) Joint with three members, two collinear members, loaded with a force on the direction of the third member. The third member has the internal force equal to the given force and with the same effect about the joint and the two collinear members will have the internal forces equal and with the same effect about the joint;

6) Unloaded joint with four members, two by two collinear. The collinear members will have the same internal forces with the same effect about the joint.

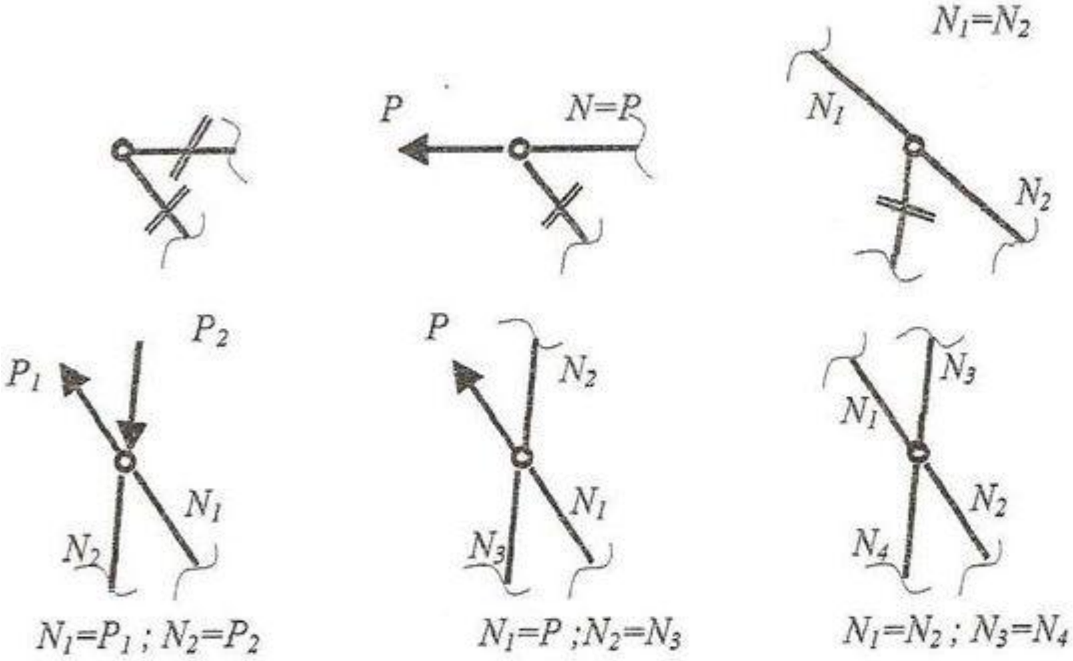


Fig.20.

We make the remark that the members with zero internal forces can be considered eliminated from the truss.

7.8. Method of sections.

The second method for to solve a truss studied in this chapter is the method of sections. As the method of joints this method has also two variants.

*1) **The generalized version of the method of sections (Szolga's method – developed by the author of this work)**. This version corresponds to the method of the equilibrium of the component parts for the systems of bodies with open outlines. In this method is made one complete section of the truss considering the truss as a system of bodies (the members) and as simple internal connections the unloaded independent members. Is expressed the equilibrium of each part as for a system of bodies with open outlines (the mixed method).*

2) ***The simplified version of the method of sections (Ritter's method)****. This version of the method is used at the simple trusses and it corresponds to the mixed method from the systems of bodies. In the first stage are calculated the external reactions considering the truss as one rigid body. In the second stage the truss is sectioned complete so that in the section to be no more than three members with unknown internal forces. If in the section we have three members with unknown internal forces then we have to check as they do not be all the three parallel or concurrent in the same point. We choose one part of the truss and is expressed the equilibrium of it with three equations considering the part as one single body. From these equations results the unknown internal forces. Because this method is used generally for to check the internal forces resulted from other methods and because it is a base for another method, is necessary as the three equations to be independent. From this reason the equilibrium equations are moment equations about the points in which two by two are intersected the directions of the unknown internal forces.*

For to use this method we shall pass the following steps:

- *1)Is verified if the truss is statically determined and stable;*
- *2)Are numbered the joints and are calculated the cosinuses of the inclined members in which we want to calculate the internal forces;*
- *3)Are calculated the reaction forces from the external constraints considering the truss as one single rigid body;*
- *4)Is made one complete section (so that to obtain two independent parts) through no more than three members with unknown internal forces (the section can be made through more than three members but the difference from three have to be members with known internal forces). If the section is made through three members with unknown internal forces then the three members have not be all the three parallel or concurrent in the same point;*
- *5)Is chosen one part resulted after the section, it is loaded with external forces (all known) and the sectioned members are replaced with internal forces;*
- *6)Are written three equilibrium equations for the chosen part as this part is a body. For to have independent equations we prefer to write three moment equations about the point in which two by two are*

- intersected the directions of the members with unknown internal forces. If two members with unknown internal forces are parallel one moment equation is removed with a projection on the perpendicular direction of the two parallel internal forces;
- 7)Are solved the equations;
- 8) the verification can be made or with one equation non used for to solve or with one equation for the other part of the truss.

7.9. Sample problems

Problem 3. *Calculate the internal forces from the marked members in the truss represented in the figure 21.*

Solution. *The truss is statically determined having 12 joints and 21 members:*

$2 \cdot 12 = 3 + 21$

and is made from triangles side by side.

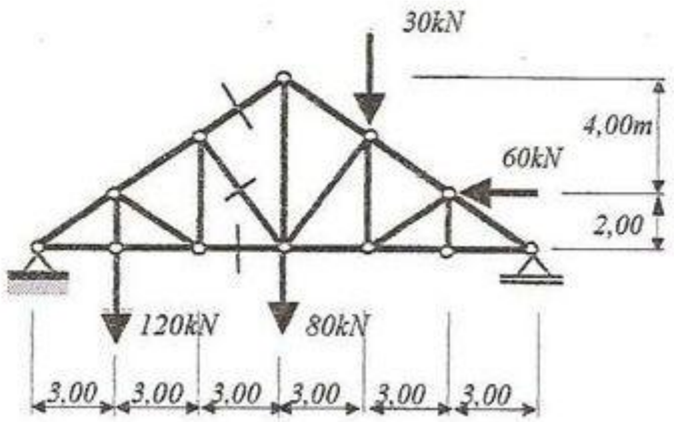


Fig.21.

We shall number the joints in an any order and we shall calculate the sinuses and cosinuses of the angles α and β that define the directions of the two members in which we shall calculate the internal forces.

$$\cos\alpha = \frac{3}{\sqrt{13}} = 0{,}832\ ;\ \sin\alpha = \frac{2}{\sqrt{13}} = 0{,}554;$$
$$\cos\beta = \frac{3}{5} = 0{,}6\ ;\ \sin\beta = \frac{4}{5} = 0{,}8$$

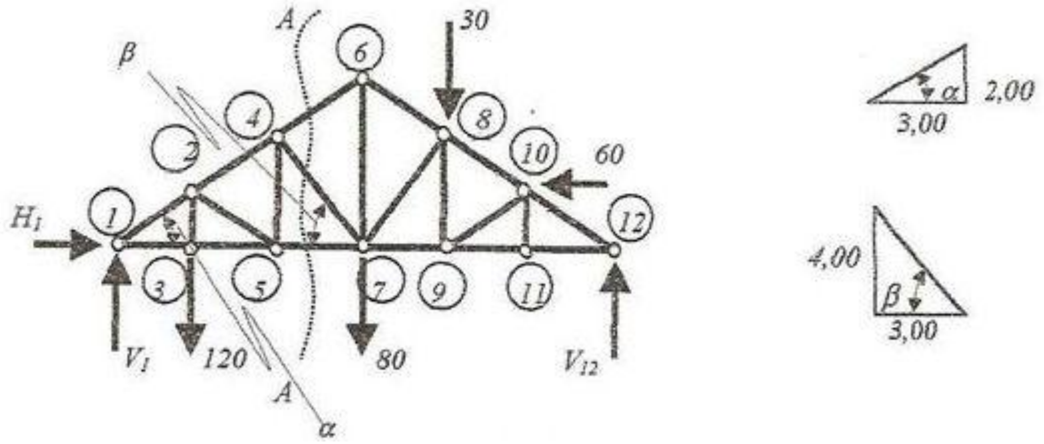


Fig.22.

We shall calculate the reactions from the external constraints using the equations:

$$\Sigma X_i = 0\ ;\ H_1 - 60 = 0; \longrightarrow H_1 = 60\ ;$$
$$\Sigma M_{1i} = 0; 120.3 + 80.9 + 30.12 - 60.2 - V_{12}.18 = 0 \longrightarrow V_{12} = 73{,}33$$
$$\Sigma M_{12i} = 0;\ V_1.18 - 120.15 - 80.9 - 30.6 - 60.2 = 0 \longrightarrow V_1 = 156{,}66$$

The checking is made with the equation:

$$\Sigma Y_i = 0;\ V_1 - 120 - 80 - 30 + V_{12} = 0\ .$$

We shall perform a section A-A that cuts the three marked members in which we want to determine the internal forces, section that meets the conditions namely it is a complete section cutting only three members with unknown internal forces which are not all the three parallel or concurrent in the same point.

Because the left side of the section has less number of forces we shall choose this part for to express the equilibrium. The three sectioned members are removed with unknown internal forces (tensions - go out of the joints) resulting the scheme of forces from the figure 23.

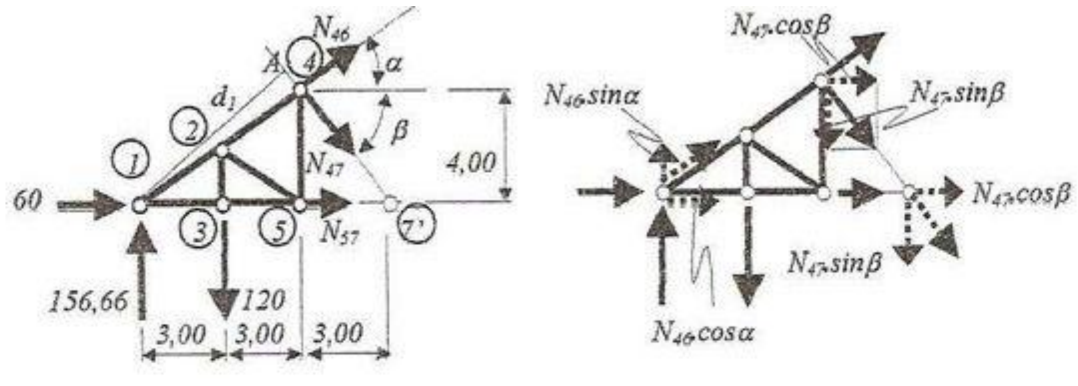


Fig.23.

We shall write three equilibrium equations as this part is one single body. These equations may be any equations but we shall choose them so that to be independent equations, namely we shall write three moment equations about the three point in which two by two the directions of the unknown forces are intersected namely with respect to the points 4, 1 and 7'. Mentioned that the point 7' has the same position, with respect to the joints of this part as the joint 7, joint that it is located on the right part. The equilibrium equations are:

$$\Sigma M_{4t} = 0 ;\ 156{,}66 \cdot 6 - 60 \cdot 4 - 120 \cdot 3 - N_{57} \cdot 4 = 0 \longrightarrow N_{57} = 85;$$
$$\Sigma M_{1t} = 0 ;\ 120 \cdot 3 + N_{47} \cdot d_1 = 0 ;$$

In this equation d_1 is the distance from the joint 1 (with respect to which we write the moment equation) to the direction of the force N_{47}. This distance is determined from the right angle triangle 1A7' being opposite to the angle β. We have:

$d_1 = 9 \cdot \sin\beta = 7{,}2$

Removing in the equation we have:

$N_{47} = -50$

We mention that we can calculate the moment of the internal force N_{47} making its resolution in the joint 4. With this the equation is :

$$\Sigma M_{1t} = 0 ;\ 120 \cdot 3 + N_{47} \cdot \cos\beta \cdot 4 + N_{47} \cdot \sin\beta \cdot 6 = 0 ;$$

The same propriety (theorem of Varignon) may be used more efficient if the unknown internal force slides until it arrives with its point of application on the horizontal line (or vertical) passing through the point about we calculate the moment, here the point 7'. If we decompose the force in two components then the equation becomes:

$$\Sigma M_{1t} = 0 ;\ 120 \cdot 3 + N_{47} \cdot \sin\beta \cdot 9 = 0 ;$$

that is the same equation as those obtained in the previously ways.

The third independent equilibrium equation, decomposing the force N_{46} in the joint 1, is:

$$\Sigma M_{7't} = 0 ;\ 156{,}66 \cdot 9 + N_{46} \cdot \sin\alpha \cdot 9 - 120 \cdot 6 = 0 \longrightarrow N_{46} = -138{,}37.$$

Problem 4. *Using the method of sections calculate the internal forces from the marked members for the truss represented in the figure 24.*

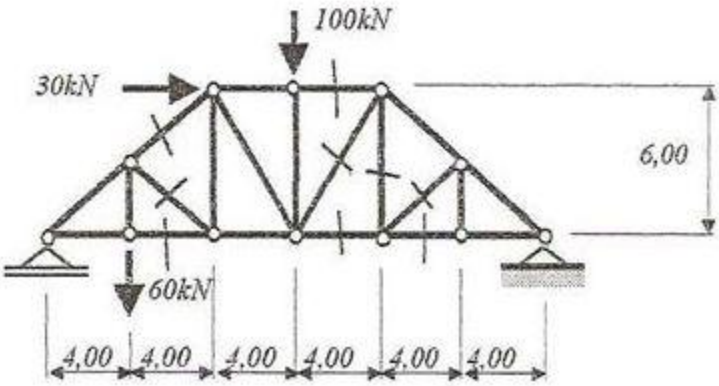


Fig.24.

KINEMATICS

Chapter 8. Kinematics of the particle

8.1.Introduction

As we have presented in the first chapter of this course, ***Kinematics*** *is that part of the Theoretical Mechanics that deals with the study of the* ***mechanical motion*** *without to consider the forces and the masses of the bodies in motion, namely studies the geometry of the motion. We remind that through mechanical motion we understand the changing the position of bodies (or parts from bodies) with respect to other bodies considered as reference systems.*

The reference system may be fixed or in motion. If the motion of the bodies is performed with respect to a fixed reference system (or that can be considered fixed system) we shall say that the motion is ***absolute motion****, but if the motion of the bodies is performed with respect to o moving reference system then the motion is called* ***relative motion****. All the elements of the absolute motion will be marked with the index* ***a****, and of the relative motion with the index* ***r****. But if in a problem we shall study the absolute motion only then we quit the index corresponding to the absolute motion.*

We should also noted that the elements of the absolute motion may be expressed with respect to a moving reference system and the elements of the relative motion with respect to a fixed reference system.

The reference systems used in the theoretical mechanics will be: ***Cartesian reference system*** *that will be generally considered fixed one (with three fixed points),* ***cylindrical reference system*** *having one fixed axis (with two fixed points),* ***spherical reference system*** *having one fixed point (the origin of the system) and* ***the Frenet's reference system*** *that is in motion together with the body (entirely in motion).*

Without to have one chapter dealing with the principal notions in kinematics we shall study this part of the theoretical mechanics in the next chapters: ***kinematics of the absolute motion of the particle, kinematics of***

the rigid body (in absolute motion), kinematics of the relative motion of the particle *and* ***kinematics of the systems (plane mechanisms).***

In this chapter we shall study the absolute motion of the particle (without to consider the forces and the mass of the particle) with respect to different reference systems.

In the kinematics we shall have to solve generally two problems: to determine the position of the particle (or of the body) in each instant of the motion, and to know how moves the particle (or the body).

For to define the position of the particle we can use the vector way (used in theoretical demonstrations generally) and the scalar way used in problems.

For to define how the motion is made we shall introduced two vector notions: ***velocity*** *and* ***acceleration****.*

8.2. Position of the particle. Trajectory

As we have seen the position of the particle can be expressed in vector way or in scalar way.

In the first case is used the **position vector***, that in absolute motion is represented with respect to a fixed point.*

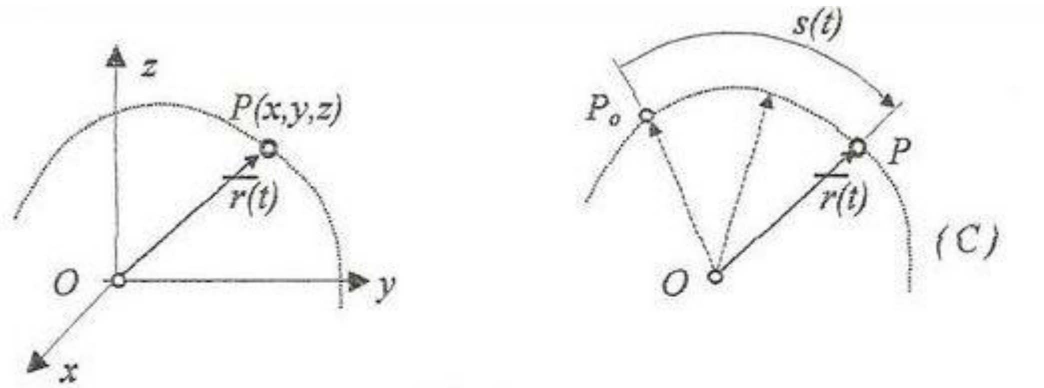


Fig.1.

Because the particle is in motion (changes its potion in time) the position vector is a function of time:

$$\overline{r} = \overline{r}(t)$$

This function of time, for represents a real motion will meet the following conditions: it is continuous (the particle cannot make instantaneous jumps), it is uniformly (the particle cannot have more positions simultaneously) and it is derivable.

If we want to express the position of the particle in scalar way we know that, with respect to a reference system, for example the Cartesian reference system, the position of the particle may be expressed using three coordinates (three scalar position parameters). These coordinates are functions of time also having the same conditions as the position vector:

$$x = x(t);\ y = y(t);\ z = z(t)$$

It is obviously that between the vector and the scalar expression of the potion we have the relation:

$$\overline{r(t)} = x(t)\mathbf{.}\,\overline{i} + y(t)\mathbf{.}\,\overline{j} + z(t)\mathbf{.}\,\overline{k}$$

*The position of the particle can be expressed in another way also: we define the curved line (C) on which moves the particle and defines the position of the particle using the distance on this line with respect to a given position from the line. The curved line on which the particle moves is called **trajectory** or **path** and by definition it is **the locus of the successively occupied positions of the particle in motion**. Noting that all positions from the trajectory can be defined using the position vector the trajectory may be defined also as **the locus of the position vector's peaks**.*

*If the parameter time has a given value, the position vector or the coordinates of the particle will be defined an **instantaneous position of the particle** (at a given instant). One of the important instantaneous position of the particle in the study of the motion is the **initial position**.*

8.3. Velocity and acceleration

*Let be a particle P in motion on an any trajectory. At the instant **t** of the motion the position of the particle will be defined by the position vector **r(t)**. At another instant t_1:*

$$t_1 = t + \Delta t$$

the position of the particle will be defined by the position vector:

$$\overline{r_1} = \overline{r(t_1)} = \overline{r(t} + \Delta t) = \overline{r} + \overline{\Delta r}$$

where Δr is the variation of the position vector in the Δt interval of time (Fig.2.).

We shall consider the following vector quantity defined by the relation:

$$\overline{v}_m = \frac{\Delta r(t)}{\Delta t}$$

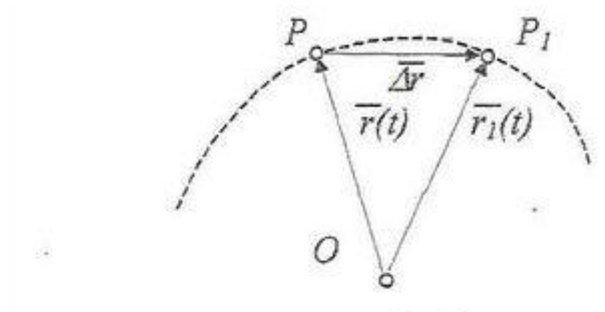


Fig.2.

*This vector is called **average velocity**. But we see that this vector does not correctly describe (than in particular cases) the kinds of motion. This rate, for example, if we consider a circular motion and the interval of time is equal to the time necessary to perform an entire circumference then the average velocity results equal to zero that is not true. Consequently this rate between the variation of the position vector and the corresponding interval of time is a feature of the motion only if the interval of time is very small (tends to zero). In this case we shall obtain the next vector:*

$$\overline{v} = \lim_{\Delta t \to 0} \overline{v}_m = \lim_{\Delta t \to 0} \frac{\Delta \overline{r}}{\Delta t} = \frac{d\overline{r}}{dt}$$

*This vector is called **instantaneous velocity** (at a given instant) and by definition is :**the first derivative with respect to time of the position vector**.*

For to simplify we shall mark the first derivative with respect to time with a point above the derivate vector:

$$\overline{v} = \dot{\overline{r}}$$

*For to simplify the names in the problems we shall call the instantaneous velocity simply **velocity**. We shall use also the name **instantaneous velocity** but for the velocity at a given instant of the motion.*

*Consider now the particle in the two positions corresponding to the two instants: **t** and $\boldsymbol{t_1}$. Because the velocities in these two positions are different, it is necessary, for to know the kind of motion of the particle to introduce a new notion that defines the variation of the velocity. We shall bring* *the two velocities from the two instants in a convenient point. The variation of the velocity (as vector) in the interval of time is marked:*

$$\overline{a}_m = \frac{\Delta \overline{v}}{\Delta t}$$

that is called ***average acceleration****. Because this vector does not describe well enough the kind of motion we shall define unother notion decreasing the interval of time, finaly obtaining the* ***instantaneous acceleration****:*

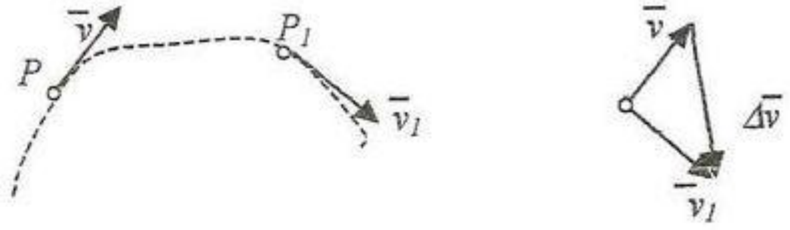


Fig.3.

$$\overline{a} = \lim_{\Delta t \to 0} \overline{a}_m = \lim_{\Delta t \to 0} \frac{\Delta \overline{v}}{\Delta t} = \frac{d\overline{v}}{dt} = \dot{\overline{v}} = \ddot{\overline{r}}$$

Consequently the ***instantaneous acceleration is the first derivative , with respect to time, of the velocity of the particle*** *or* ***the second derivative, with respect to time, of the position vector of the particle****.*

As we can see the second derivative with respect to time is marked with two points above the corresponding vector.

8.4. Kinematics of the particle in Cartesian coordinates

As we have seen in the previous sections the absolute motion of a particle can be studied using different reference systems. The simplest reference system is the ***Cartesian system of reference*** *considered as a* ***fixed system****.*

Consider a particle in motion (absolute motion) and a fixed Cartesian system of reference Oxyz.

The main property of this system can be expressed in the following way:

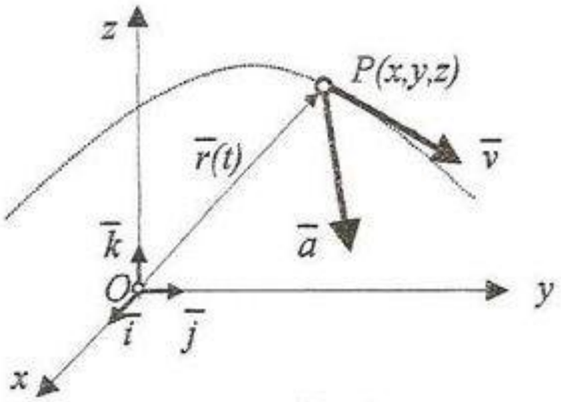


Fig.4.

$$\dot{\overline{i}} = \dot{\overline{j}} = \dot{\overline{k}} = 0.$$

The position of the particle may be defined in scalar way using the three coordinates:

$$x = x(t)\ ;\ y = y(t);\ z = z(t)$$

that are functions of time because the particle is in motion (change its position) with respect to the fixed reference system.

These coordinates are called ***the laws of motion in Cartesian coordinates*** *or* ***parametric equations of the motion in Cartesian coordinates****.*

The position of the particle can be expressed also using the position vector with respect to the origin of the reference system:

$$\overline{r} = \overline{r}(t)$$

Between this vector and the Cartesian coordinates we may write the well-known relation:

$$\overline{r}(t) = x(t).\overline{i} + y(t).\overline{j} + z(t).\overline{k}$$

If we eliminate the time parameter (t) from the three coordinates are obtained two equations:

$$\begin{cases} f(x,y,z) = 0\ ; \\ g(x,y,z) = 0 \end{cases}$$

*representing the equation of the **trajectory** (or the **path**) of the particle in Cartesian coordinates. We see that the trajectory is defined as the intersection of two fixed surfaces.*

For to know the kind of motion we shall express the velocity of the particle. Using the definition of the instantaneous velocity we find:

$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{dx}{dt}.\bar{i} + \frac{dy}{dt}.\bar{j} + \frac{dz}{dt}.\bar{k}$$

or:

$$\bar{v} = \dot{\bar{r}}(t) = \dot{x}(t)\,.\,\bar{i} + \dot{y}(t)\,.\,\bar{j} + \dot{z}(t)\,.\,\bar{k}$$

This means that the projections of the velocity on the axes of the reference system are:

$$v_x = \dot{x}\ ;\ v_y = \dot{y}\ ;\ v_z = \dot{z}$$

from which we obtain, using the well-known relations, the magnitude and the direction of the velocity in Cartesian coordinates:

$$|\bar{v}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\ ;\ \cos\alpha_v = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}};$$

$$\cos\beta_v = \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\ ;\ \cos\gamma_v = \frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}$$

We remark that the projections of the velocity on the fixed axes are equal to the first derivatives, with respect to time, of the corresponding coordinates.

Also we remark that in this reference system we have not any properties of the velocity resulted from the relations.

The second vector defining the kind of motion is the acceleration. From definition we have:

$$\bar{a} = \frac{d\bar{v}}{dt} = \dot{\bar{v}}$$

or removing function the Cartesian coordinates we obtain finally:

$$\bar{a} = \ddot{x}(t) \cdot \bar{i} + \ddot{y}(t) \cdot \bar{j} + \ddot{z}(t) \cdot \bar{k}$$

namely we have the following projections on the axes, magnitude and direction in Cartesian coordinates:

$$a_x = \ddot{x} = \dot{v}_x \,;\; a_y = \ddot{y} = \dot{v}_y \,;\; a_z = \ddot{z} = \dot{v}_z \,;$$

$$|\bar{a}| = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2} = \sqrt{\dot{v}_x^2 + \dot{v}_y^2 + \dot{v}_z^2} \,;$$

$$\cos\alpha_a = \frac{x}{|\bar{a}|} \,;\; \cos\beta_a = \frac{y}{|\bar{a}|} \,;\; \cos\gamma_a = \frac{z}{|\bar{a}|}$$

Results that: the projections of the acceleration on the axes of the Cartesian reference system are equal to the second derivatives, with respect to time, of the corresponding coordinates, or the first derivatives, with respect to time, of the corresponding projections of the velocity.

Also we remark the same thing: the projections and the characteristics of the acceleration may be calculated very easy but does not result any important properties of it.

8.5. Kinematics of the particle in cylindrical coordinates. Polar coordinates.

Let be a particle in absolute motion. For to express the motion of the particle we shall use a reference system that is chosen in the following way:

-the Oz axis is fixed, and for simplification we shall consider it vertical;

*-the Oρ axis, called **radial axis**, is taken so that the particle to be located, in any time of the motion in the Oρz reference plane. This axis, and obviously the plane Oρz, are in motion, namely in rotation motion about the fixed axis Oz;*

*-the On axis, called **normal axis**, is perpendicular on the Oρz plane and has the positive sense so that the reference system to be a right-hand system.*

The main characteristic of this system may be expressed in the following way:

$$\dot{\bar{k}} = 0.$$

namely the Oz axis is fixed, but the unit vectors of the other two axes, marked $\bar{i}_\rho$ *and* $\bar{i}_n$ *are functions of time:*

$$\bar{i}_\rho = \bar{i}_\rho(t);\ \bar{i}_n = \bar{i}_n(t)$$

The position of the particle with respect to this reference system will be defined with two coordinates: ρ *and z (the particle is located in* $O\rho z$ *plane).*

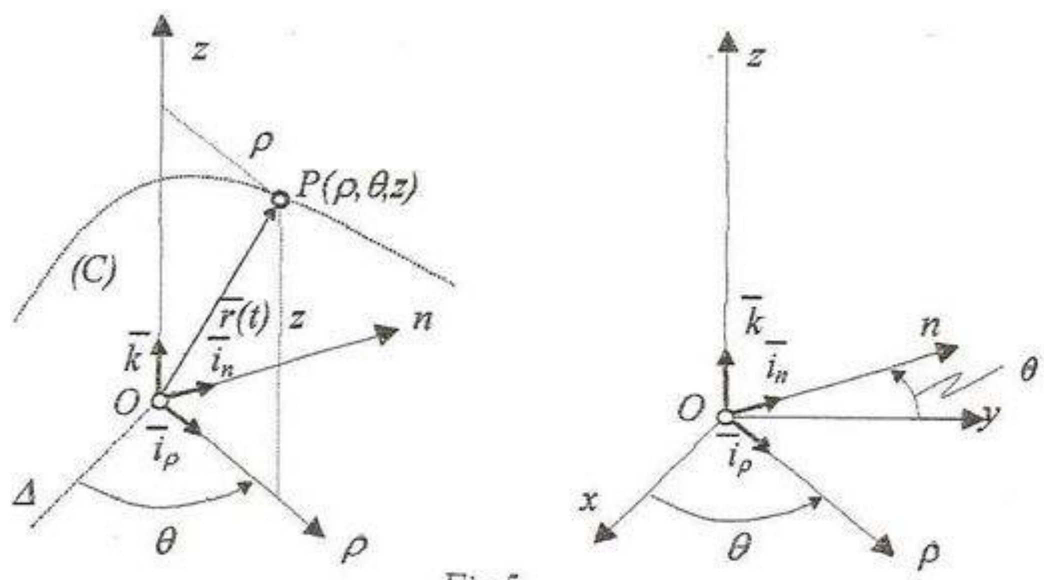


Fig.5.

But to define the position of the particle in space is necessary to define the position of the reference system with respect to another fixed system. This fact is made if it is known the angle θ *measured between a fixed plane from space (for example the* ΔOz *plane) and the moving plane* ρOz*. Results that the position of the particle in scalar way is given by three coordinates:*

$$\rho = \rho(t);\ \theta = \theta(t);\ z = z(t)$$

called ***cylindrical coordinates*** *or* ***equations of motion in cylindrical coordinates*** *and the* $O\rho nz$ *reference system is called* ***cylindrical reference system****.*

In vector way the position of the particle is defined using the position vector with respect to the fixed point O:

$$\overline{r} = \overline{r}(t)$$

Between the cylindrical coordinates and the position vector we have the relation:

$$\overline{r}(t) = \overline{\rho}(t) \cdot \overline{i_\rho}(t) + z(t) \cdot \overline{k}$$

Eliminating the parameter time from the parametric equations of the motion we shall obtain two equations:

$$f(\rho,\theta,z) = 0 \text{ and } g(\rho,\theta,z) = 0$$

representing the equations of the trajectory in cylindrical coordinates.

For to define the kind of motion we have need to use the derivatives of the unit vectors of the moving axes. For to calculate the derivatives we shall express these unit vectors with respect to a fixed reference system (for example the Oxyz Cartesian system having the Ox axis collinear with the straight line Δ). We have:

$$\overline{i_\rho}(t) = \cos\theta(t) \cdot \overline{i} + \sin\theta(t) \cdot \overline{j};$$
$$\overline{i_n}(t) = -\sin\theta(t) \cdot \overline{i} + \cos\theta(t) \cdot \overline{j}$$

Deriving with respect to time these two relations we have:

$$\overset{\circ}{\overline{i_\rho}} = -\overset{\circ}{\theta}.\sin\theta.\overline{i} + \overset{\circ}{\theta}.\cos\theta.\overline{j} = \overset{\circ}{\theta}\overline{i_n}$$
$$\overset{\circ}{\overline{i_n}} = -\overset{\circ}{\theta}.\cos\theta.\overline{i} - \overset{\circ}{\theta}.\sin\theta.\overline{j} = -\overset{\circ}{\theta}\overline{i_\rho}$$

Now we shall calculate the velocity using the definition of it:

$$\overline{v} = \overset{\circ}{\overline{r}} = \overset{\circ}{\rho} \cdot \overline{i_\rho} + \rho \cdot \overset{\circ}{\overline{i_\rho}} + \overset{\circ}{z} \cdot \overline{k}$$

that replacing the derivative of the unit vector becomes:

$$\overline{v} = \overset{\circ}{\rho} \cdot \overline{i_\rho} + \rho \cdot \overset{\circ}{\theta} \cdot \overline{i_n} + \overset{\circ}{z} \cdot \overline{k}$$

The projections, magnitude and the direction of the velocity in cylindrical coordinates will be:

$$v_\rho = \dot{\rho}\,;\; v_n = \rho\dot{\theta}\,;\; v_z = \dot{z}\,;\;\; v = \sqrt{\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2}\,;$$
$$\cos\alpha_v = \frac{\dot{\rho}}{|\bar{v}|}\,;\; \cos\beta_v = \frac{\rho\dot{\theta}}{|\bar{v}|}\,;\; \cos\gamma_v = \frac{\dot{z}}{|\bar{v}|}$$

The acceleration of the particle will be obtained deriving again, with respect to time, the velocity:

$$\bar{a} = \dot{\bar{v}} = \ddot{\rho}.\bar{i}_\rho + \dot{\rho}.\dot{\bar{i}}_\rho + \dot{\rho}.\dot{\theta}.\bar{i}_n + \rho.\ddot{\theta}.\bar{i}_n + \rho.\dot{\theta}.\dot{\bar{i}}_n + \ddot{z}.\bar{k}$$

Removing the derivatives of the unit vectors we have finally:

$$\bar{a} = (\ddot{\rho} - \rho\dot{\theta}^2)\,\bar{i}_\rho + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\,\bar{i}_\theta + \ddot{z}\,\bar{k}$$

namely the projections on the three axes, the magnitude and the direction of the acceleration in cylindrical coordinates will be:

$$a_\rho = \ddot{\rho} - \rho\dot{\theta}^2\,;\; a_n = 2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\,;\; a_z = \ddot{z}\,;$$
$$a = \sqrt{(\ddot{\rho} - \rho\dot{\theta}^2)^2 + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})^2 + \ddot{z}^2}\,;$$
$$\cos\alpha_a = \frac{\ddot{\rho} - \rho\dot{\theta}^2}{|a|}\,;\; \cos\beta_a = \frac{2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}}{|a|}\,;\; \cos\gamma_a = \frac{\ddot{z}}{|a|}$$

*If the particle performs a plane motion (the trajectory of the particle is located in a fixed plane) we may consider z = 0 and the cylindrical system of reference becomes a **polar reference system**. This system has the origin in the fixed point O and the axis Oρ is taken so that the particle to be located on this axis all the time of motion. Results:*

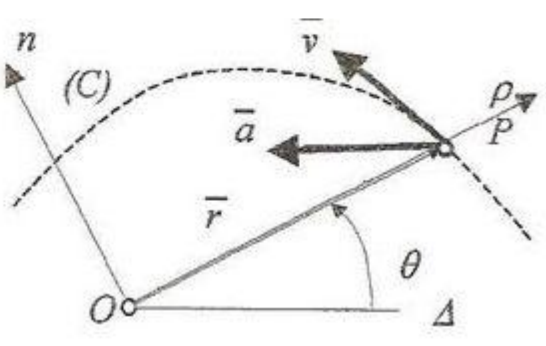


Fig.6.

$$\bar{r}(t) = r\,.\,\bar{i}_\rho$$

namely the motion of the particle in polar coordinates is obtained changing in the previous relations:

$\rho = r ; z = 0$

In this way the study of the motion in polar coordinates will be made with the following relations:

-polar coordinates:

$r = r(t); \theta = \theta(t)$

-the equation of the trajectory:

$f(r,\theta) = 0$

-the velocity:

$$\overline{v} = \dot{r} . \overline{i_\rho} + r . \dot{\theta} . \overline{i_n} ;$$

-the acceleration:

$$\overline{a} = (\ddot{r} - r . \dot{\theta}^2)\overline{i_\rho} + (2\dot{r} . \dot{\theta} + r . \ddot{\theta})\overline{i_n}$$

8.6. Kinematics of the particle in Frenet's system

*This reference system, called **natural system** also, is used only the cases when is known the trajectory of the particle.*

Consider a particle P in motion on a known trajectory (C).

We shall consider the following reference system:

-The origin of the system is taken in the point representing the particle;

*-The axis **P**$\boldsymbol{\tau}$, called **tangent axis**, will be tangent to the trajectory in point P and with the positive sense in the sense of motion;*

*-The axis **P**$\boldsymbol{\nu}$, called **normal axis**, will have the direction of the principal normal to the trajectory in point P. The positive sense of this axis will be directed toward the center of curvature of the trajectory;*

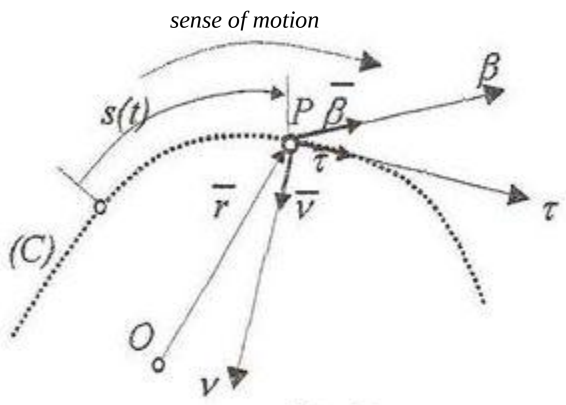


Fig.7.

*-The axis **Pβ**, called **binormal axis**, is perpendicular on the previous two axes and the positive sense is considered so that the three axes to make a right hand system.*

Because the particle is located in the origin of this system and the names of the axes are not used to define coordinates, we shall use the names of these axes for the names of the corresponding unit vectors. In this way we shall have the unit vectors of the three axes:

$$\bar{\tau} = \bar{\tau}(t)\ ;\ \bar{\nu} = \bar{\nu}(t)\ ;\ \bar{\beta} = \bar{\beta}(t).$$

The position of the particle (because we know the trajectory of it) may be defined using one scalar quantity:

$$s = s(t)$$

called curvilinear coordinate or natural coordinate and representing the space performed on the trajectory measured from a convenient position (generally the initial position) to the current position.

Because we study the absolute motion of the particle, for to define the velocity and acceleration we need to use the position vector with respect to a fixed point O. This vector, as the unit vectors, will be supposed to be functions of time through the natural coordinate:

$$\bar{r} = \bar{r}(t) = \bar{r}(s(t));$$
$$\bar{\tau} = \bar{\tau}(s(t));\ \bar{\nu} = \bar{\nu}(s(t));\ \bar{\beta} = \bar{\beta}(s(t))$$

For the velocity of the particle we have:

$$\bar{v} = \dot{\bar{r}}(s(t)) = \frac{d\bar{r}}{ds} \cdot \frac{ds}{dt} = \frac{d\bar{r}}{ds} \cdot \dot{s}$$

To see what is the derivative, with respect to the natural coordinate, of the position vector we shall consider two close positions of the particle P and P'.

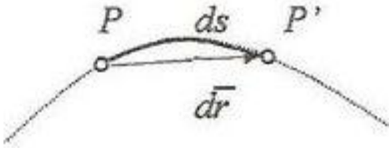


Fig.8.

Is noted that $d\bar{r}$ is the chord and ds is the arch between the two positions. At limits these tends to be equals in magnitude (so this derivative is a unit vector) and $d\bar{r}$ tends as direction to the direction of the tangent in point P. Results:

$$\frac{d\bar{r}}{ds} = \bar{\tau}$$

*This relation is called **the first Frenet's relation**.*
In this way the velocity of the particle will be:

$$\bar{v} = \dot{s} \cdot \bar{\tau}$$

namely the projections of the velocity are:

$$v_\tau = \dot{s} = v\ ;\ v_\nu = 0\ ;\ v_\beta = 0$$

From all these result the following proprieties of the velocity:

- *The magnitude of the velocity always is equal to the first derivative with respect to time of the law of motion on the trajectory;*
- *The velocity always is tangent to the trajectory;*
- *The velocity always is directed in the sense of motion.*

For the acceleration we shall use de definition of it:

$$\bar{a} = \dot{\bar{v}} = \ddot{s} \cdot \bar{\tau} + \dot{s} \cdot \dot{\bar{\tau}}$$

The derivative with respect to time of the tangent unit vector will be:

$$\dot{\bar{\tau}} = \frac{d\bar{\tau}}{ds} \cdot \frac{ds}{dt} = \frac{d\bar{\tau}}{ds} \cdot \dot{s}$$

For to calculate the derivative with respect to the natural coordinate of the unit vector we shall consider again the two close positions of the particle.

The unit vector in the two positions will be marked τ and τ'. The perpendicular directions on these two unit vectors are intersected in the center of curvature and the distances are equal with the radius of curvature. If ds tends to zero we can consider that the $PP'\Gamma$ is an isosceles triangle. Bringing one tangent unit vector in the point of application of the other (for example τ in point P') results the variation $d\tau$. It is formed in this point an other triangle like the $PP'\Gamma$ triangle, so we may write the relation:

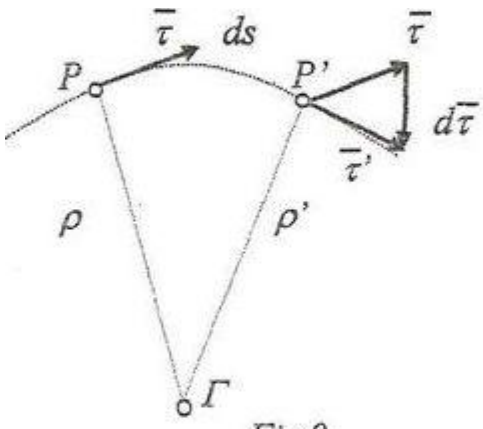


Fig.9.

$$\frac{|d\bar{\tau}|}{ds} = \frac{1}{\rho}$$

We remark that at limit $d\tau$ is perpendicular on the direction of the tangent, so results finally:

$$\frac{d\bar{\tau}}{ds} = \frac{1}{\rho} \cdot \bar{\nu}$$

that is the ***second Frenet's relation****.*

Now we replace in the expression of the acceleration and we find:

$$\bar{a} = \ddot{s}.\bar{\tau} + \frac{\dot{s}^2}{\rho}.\bar{\nu}$$

with the following projections on the three axes:

$$a_\tau = \ddot{s} = \dot{v}\ ;\ a_\nu = \frac{\dot{s}^2}{\rho} = \frac{v^2}{\rho}\ ;\ a_\beta = 0.$$

Results the next proprieties of the acceleration of a particle:

- *The acceleration has always two components: one tangent component and the other the normal component;*
- *The tangent component is due of variation of the magnitude of the velocity;*
- *The normal component is due of variation of the direction of the velocity;*
- *The single motion that can be performed with zero acceleration is the rectilinear uniformly motion.*

The magnitude of the acceleration is:

$$|\bar{a}| = \sqrt{\ddot{s}^2 + \frac{\dot{s}^4}{\rho^2}} = \sqrt{\dot{v}^2 + \frac{v^4}{\rho^2}}\ ;$$

and the direction with respect to the principal normal is given by:

$$tg\varphi = \frac{a_\tau}{a_\nu} = \frac{\ddot{s}}{\frac{\dot{s}^2}{\rho}} = \frac{\dot{v}\rho}{v^2}$$

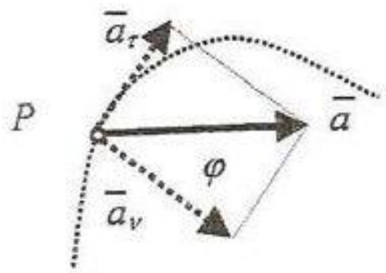


Fig.10.