A Class of Linear Boundary Value Problem for *k*-regular Functions in Clifford Analysis

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Abstract

In this paper, we introduce the linear boundary value problem for *k*-regular function, and give an unique solution for this problem by integral equation method and fixed-point theorem.

Keywords: *k*-regular function, Clifford analysis

1. Introduction

The boundary value problem is one of the important aspects in Clifford analysis. This problem on bounded domains has seen great achievements. [Wen, 1991; Huang, 1996; Zhang et al., 2001] have discussed Riemann-Hilbert boundary value problems of regular function on bounded domains. [Li, 2007] characterized boundary value problems of *k*- regular functions. In this paper, we introduce the *linear* boundary value problem of *k*-regular function, and give an unique solution to this problem by integral equation method and fixed-point theorem.

Let n be a positive integer, and $\{e_0, e_1, \cdots, e_n\}$ be basis for the Euclidean space $\mathbb{R}^{n+1}$. We denote by $\mathcal{A}$ the 2^n dimensional real Clifford algebra, which is generated by $\mathbb{R}^{n+1}$; denote the basis of $\mathcal{A}$ by $e_A = e_{\alpha_1,\alpha_2,\cdots,\alpha_h}$, $A = \{\alpha_1, \alpha_2, \cdots, \alpha_h\} \subseteq \{1, 2, \cdots, n\}$, $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_h \le n$. In particular, if $A = \emptyset$, $e_\phi = e_0$. So, for an arbitrary $u \in \mathcal{A}$, we have $u = \sum_A u_A e_A$ with $u_A \in \mathbb{R}$. In $\mathcal{A}$, we have

$$e_i^2 = -1, e_i e_j = -e_j e_i \text{ for } i \neq j, i, j = 1, 2, \cdots, n,$$

that is so-called combinative and incommutable multiplication rule of Clifford algebra. For $u \in \mathcal{A}$, we write $u^* = \sum_A (-1)^{\frac{|A|(|A|-1)}{2}} u_A e_A$, $u' = \sum_A (-1)^{\frac{|A|}{2}} u_A e_A$ and $|u|$ for its module, where $|A|$ is the cardinality of the index set A. Define $|u|^2 = \sum_A |u_A|^2$; $\bar{u}$ its conjugate with $\bar{u} = (u^*)'$, where $u^* = \sum_A (-1)^{\frac{|A|(|A|-1)}{2}} u_A e_A$, and $u' = \sum_A (-1)^{|A|} u_A e_A$. For u, $v \in \mathcal{A}$, we have

$$|u + v| \le |u| + |v|, |uv| \le 2^n |u||v|.$$

Let D be a region in $\mathbb{R}^{n+1}$. For a differentiable function $f : D \to \mathcal{A}$ with $f(x) = \sum_A f_A(x) e_A$, we say f is a *regular function* if

$$\bar{\partial} f = \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i} = \sum_{i=0}^{n} \sum_{A} e_i e_A \frac{\partial f_A}{\partial x_i} = 0,$$

and a *k-regular function* if $\bar{\partial}^k f = 0$, where the operator $\bar{\partial} = \sum_{j=0}^{n} \frac{\partial}{\partial_{x_j}} e_j$. Let $\Omega \subset \mathbb{R}^{n+1}$ be an unbounded domain with smooth oriented Liapunove boundary $\partial\Omega$, and Ω^c, the complementary set of Ω containing a non-empty open set. We denote the bounded Hölder continuous function on $\partial\Omega$ in order of $\beta (0 < \beta < 1)$ by $H(\partial\Omega, \beta)$. For $f \in H(\partial\Omega, \beta)$, we define its norm by

$$\|f\|_\beta = \sup_{t \in \partial\Omega} |f(t)| + \sup_{t_1 \neq t_2} \frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|}.$$

Then $H(\partial\Omega, \|\cdot\|_\beta)$ is a Banach space. And for f, $g \in H(\partial\Omega, \|\cdot\|_\beta)$, we have

$$\|f + g\|_\beta \le \|f\|_\beta + \|g\|_\beta, \ \|fg\|_\beta \le 2^n \|f\|_\beta \|g\|_\beta.$$

2. Main Result

In what follows, we denote by Ω a non-empty connected open set in $\mathbb{R}^{n+1}$ with smooth oriented Liapunove boundary $\partial\Omega$, and by w_n the area of unit ball in $\mathbb{R}^{n+1}$. We first give the linear boundary value problem for *k*-regular function.

Definition 2.1. Let $A(t)$, $B(t)$, $g_l(t) \in H(\partial\Omega, \beta)$, $1 \le l \le k$. Write $\Omega^+ = \Omega$, $\Omega^- = \mathbb{R}^{n+1} \backslash \bar{\Omega}$ with $\bar{\Omega} = \Omega \cup \partial\Omega$. If there exists some function ϕ such that

1) ϕ is a k-regular function on $\Omega^{\pm}$;

2)

$$\begin{cases} \phi^+(t)A(t) + \phi^-(t)B(t) = g_1(t) \\ \bar{\partial}\phi^+(t)A(t) + \bar{\partial}\phi^-(t)B(t) = g_2(t) \\ \qquad \vdots \\ \bar{\partial}^{k-1}\phi^+(t)A(t) + \bar{\partial}^{k-1}\phi^-(t)B(t) = g_k(t) \end{cases} \tag{1}$$

Then we say ϕ is a solution to the linear boundary problem. And this problem is also called linear boundary problem for k-regular function.

The following lemmas are borrowed from [Li, 2007]:

Lemma 2.1. Let $f(x)$ be a k-regular function on Ω. Then we have

$$f(x) = \sum_{m=0}^{k-1} \frac{1}{m!} x_0^m f_m(x), \tag{2}$$

where $f_m, m = 0, 1, \cdots, k-1$ are regular functions defined on Ω.

Lemma 2.2 Here we give Plemelj equation for regular function:

$$\phi_m^{\pm} = \pm\frac{1}{2}\varphi_m + \frac{1}{w_n}\int_{\partial\Omega} \frac{\overline{\tau - x}}{|\tau - x|^{n+1}} m(\tau)\varphi_m(\tau) d_{s_\tau}. \tag{3}$$

where $m(u)$ is the unit vector in $\partial\Omega$'s normal direction, and $\varphi_j \in H(\partial\Omega, \beta)$, $j = 0, 1, \cdots, k-1$. Then ϕ is a regular function on $\mathbb{R}^{n+1}\backslash\partial\Omega$.

The following lemma is borrowed from [Xu et al., 2008]

Lemma 2.3. Let $\phi \in H(\partial\Omega, \beta)$. Define a operator K on $H(\partial\Omega, \beta)$ by

$$(K\phi)(x) = \frac{1}{w_n}\int_{\partial\Omega} \frac{\overline{\tau - x}}{|\tau - x|^n} m(\tau)\phi(\tau) d_{s_\tau} \tag{4}$$

for $x \in \partial\Omega$. Then there exists some $C > 0$ such that $\|K \cdot\| \le C\| \cdot \|$ on $H(\partial\Omega, \beta)$.

Theorem 2.1. Let $A(t)$, $B(t)$, $g_l(t)$, $(1 \le l \le k) \in H(\partial\Omega, \beta)$. If

$$\begin{aligned} &\zeta = 2^n[(\frac{1}{2} + C)(\|A + B\|) + \|1 - B\|] \in (0, 1), \\ &\|g'_{m+1}\|_\beta \le M(1 - \zeta), \end{aligned} \tag{5}$$

where C is in Lemma 2.3, then the solution of the m-th equation in (1) is given by

$$\phi(x) = \sum_{m=0}^{k-1} \frac{1}{m!} x_1^m \phi_m(x)$$

with

$$\phi_m = \frac{1}{w_n}\int_{\partial\Omega} \frac{\overline{\tau - x}}{|\tau - x|^n} m(\tau)\varphi_m(\tau) d_{s_\tau}$$

for $m = 0,\ 2,\ \cdots,\ k-1$.

Proof. Substituting (2) into (1), we have

$$T\begin{pmatrix} \phi_0^+ \\ \vdots \\ \phi_{k-2}^+ \\ \phi_{k-1}^+ \end{pmatrix} A + T\begin{pmatrix} \phi_0^- \\ \vdots \\ \phi_{k-2}^- \\ \phi_{k-1}^- \end{pmatrix} B = \begin{pmatrix} g_1 \\ \vdots \\ g_{k-1} \\ g_k, \end{pmatrix} \tag{6}$$

where

$$T=\begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 1 & x_1 & \cdots & \frac{1}{(k-2)!}x_1^{k-2} \\ \vdots & \vdots & \vdots & & \\ 1 & x_1 & \frac{1}{2!}x_1^2 & \cdots & \frac{1}{(k-1)!}x_1^{k-1} \end{pmatrix}.$$

We rewrite (6) as

$$\begin{pmatrix} \phi_0^+ & A \\ \vdots & \vdots \\ \phi_{k-2}^+ & A \\ \phi_{k-1}^+ & A \end{pmatrix}+\begin{pmatrix} \phi_0^+ & B \\ \vdots & \vdots \\ \phi_{k-2}^+ & B \\ \phi_{k-1}^+ & B \end{pmatrix}=\begin{pmatrix} g_1' \\ \vdots \\ g_{k-1}' \\ g_k' \end{pmatrix} \text{ with } \begin{pmatrix} g_1' \\ \vdots \\ g_{k-1}' \\ g_k' \end{pmatrix}=T^{-1}\begin{pmatrix} g_1 \\ \vdots \\ g_{k-1} \\ g_k \end{pmatrix},$$

which is equivalent to

$$\begin{cases} \phi_0^+A+\phi_0^-B=g_1' \\ \phi_1^+A+\phi_1^-B=g_2' \\ \qquad\vdots \\ \phi_{k-1}^+A+\phi_{k-1}^-B=g_k,' \end{cases} \tag{7}$$

herein ϕ_m is a regular function given by

$$\phi_m=\frac{1}{w_n}\int_{\partial\Omega}\frac{\overline{\tau-x}}{|\tau-x|^n}m(\tau)\varphi_m(\tau)d_{s_\tau} \tag{8}$$

for $m=0,\cdots,k-1$. Next, to finish the proof, we only need to prove that ϕ_m $(0\le m\le k-1)$ given by (8) are solutions to (7). By substituting (3) into (7), we have

$$(\frac{1}{2}\varphi_m+K\varphi_m)A+(-\frac{1}{2})\varphi_m+K\varphi_mB=g_{m+1}' \quad m=0,\cdots,k-1. \tag{9}$$

Write

$$L\phi_m=(\frac{1}{2}\varphi_m+K\varphi_m)(A+B)+\varphi_m(1-B)-g_{m+1}',$$

then (9) can be rewritten as $L\phi_m=\phi_m$. Let $T=\{\varphi|\varphi\in H(\partial\Omega,\beta),||\varphi||_\beta\le M\}$. Then T is a closed subspace of $H(\partial\Omega,\beta)$. Since

$$\begin{aligned} ||L\varphi_m||_\beta &= ||(\frac{1}{2}\phi_m+K\phi_m)(A+B)+(1-B)\phi_m-g_{m+1}'|| \\ &\le 2^n[(\frac{1}{2}+C)(||A+B||)+||1-B||]||\phi_m||+||g_{m+1}'|| \\ &\le \zeta||\phi_m||+||g_{m+1}'|| \\ &\le M, \end{aligned} \tag{10}$$

F is a map on T. For $\phi_m',\phi_m''\in T$, we have

$$||L\phi_m'-L\phi_m''||\le\zeta||\phi_m'-\phi_m''||_\beta,$$

with $0<\zeta<1$, and thus L is a compression map on T. So, there is an unique fixed φ_m such that $L\varphi_m=\varphi_m$ by fixed point theorem, which implies that

$$\phi_m=\frac{1}{w_n}\int_{\partial\Omega}\frac{\overline{\tau-x}}{|\tau-x|^n}m(\tau)\varphi_m(\tau)d_{s_\tau}$$

is unique solution for the m-th equation in (7). This gives the proof.

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