INTEREST RATE RISK

– USING BENCHMARK SHIFTS IN A MULTI HIERARCHY PARADIGM

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ABSTRACT

This master thesis investigates the generic benchmark approach to measuring interest rate risk. First the background and market situation is described followed by an outline of the concept and meaning of measuring interest rate risk with generic benchmarks. Finally a single yield curve in an arbitrary currency is analyzed in the cases where linear interpolation and cubic interpolation technique is utilized. It is shown that in the single yield curve setting with linear interpolation or cubic interpolation the problem of finding interest rate scenarios can be formulated as convex optimization problems implying properties such as convexity and monotonicity. The analysis also shed light on the difference between linear interpolation and cubic interpolation technique for which scenario is generated and means to go about solving for the scenarios generated by the views imposed on the generic benchmark instruments. Further research on the topic of the generic benchmark approach that would advance the understanding of the model is suggested at the end of the paper. However at this stage it seems like using generic benchmark instruments for measuring interest rate risk is a consistent and computational viable option which not only measures the interest rate risk exposure but also provide a guidance in how to act in order to manage interest rate risk in a multi hierarchy paradigm.

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1. INTRODUCTION

In the financial crises of 2008 due to the collapse of the U.S subprime mortgage loan, traditional modeling of interest rate markets broke down. Common assumptions underpinning the prevalent models suffered from two crucial facts, the first one being that counter credit risk was not taken into account. Furthermore, the market unquestionably used the London inter-bank offer rate(LIBOR), as a proxy for the risk free interest rate. In the aftermath of the 2008 financial crises it became clear that both of these modeling assumptions had been far remote from the reality of the market during the 2008 financial crises. Today, nearly five years after the financial meltdown, the environment of the financial markets of interest rate products, resembles that preceding the crises of 2008. The conditions have recessed back to what has always been considered normal conditions, for example like the condition of the financial market of 2006. This means that the liquidity in the market is back to healthy levels where credit and default risks are considered small. Furthermore, the huge basis spreads between different interest rate have recessed to lower levels than the 2008 levels. Contrasting to the 2008 financial crises where the spread between the three month U.S treasury rate and the three month U.S dollar Libor rate peaked at a level of 450 basis points in October 2008 (White, 2012). This can be compared to the levels observed during normal market conditions where the spread fluctuates below 50 basis points, implicating a ten folded increase in the spread in October 2008.

Despite the fact that market conditions have recessed to what can be considered normal, market participant realize that going back to modeling interest rate products in the same manner as before the 2008 financial crises would only work as long as the market does not take a severe down turn. If the financial market were to take a severe down turn, the interest rate models in use before the 2008 financial crises would again be unable to handle such a situation. Therefore the market trend has been to develop new market models which are not only robust between financial crises but also during such crises. This has led to more complicated models were assumptions earlier considered reasonable now are deemed inadequate and replaced by extended models.

During this development many new problems and new situations arise which has to be addressed and solved in a novel way since there is no common market practice or research on the subject at hand. This master thesis is done in collaboration with Handelsbanken AB, a prominent Swedish bank where I will investigate a new way of looking at and modeling of interest rate risk in their new interest rate modeling framework. The purpose of this master thesis is thus to find a suitable interest rate risk modeling tool for their new interest rate modeling framework, which consists of multi-hierarchy term structures. More specifically the question is how one can measure and limit interest rate risk in the new multi hierarchy framework.

There are many properties an interest rate risk model must comply with. First of all, every major interest rate risk of importance should be captured by the model. Secondly, if some risk measure has been defined and the risk limit is met then it must be clear how to act in order to reduce the unauthorized interest rate risk exposure. Simply reporting a too high value at risk measure leaving the traders or bank clueless of how to reduce it is nonsensical. Thirdly one should be able to partition interest rate risks in the model into independent risks that can be used for interest rate risk limiting.

Traders in interest rate contracts are exposed to not only interest rate risk but also to credit risk, i.e., the risk that the counterparty will not fulfill his end of the contract. Furthermore, there can be an associated liquidity risk, i.e., the risk that a given interest rate derivative cannot be liquidated to cash or bought in the market. In such a case the theoretical value of the contract is of little use. This master thesis deals exclusively with interest rate risk in the multi hierarchy yield curves and not with either credit risk or liquidity risk. It should be pointed out that the multi hierarchy framework was amounted as a response to counter party credit risk but there is much more to say about credit and liquidity risk. Therefore, the focal point of this master thesis is on interest rate risk, but that is not to say that the three types of risks, credit risk, liquidity risk and interest rate risk are independent of each other or additive.

2. THEORETICAL FRAMEWORK

Interest rate product

An interest rate product or interest rate contract is a financial contract where two parties exchange cash flows at different points in time. The cash flows are determined at the entry of the contract but the amount of the cash flow may be unknown at the entry of the contract and commonly stipulated to be some function of market interest rates. The circumstances for the trades take various forms such as over-the-counter (OTC) or exchange-traded-derivatives (ETD) and may or may not involve a credit support annex (CSA). These conditions affect the liquidity of the contract being traded as well as the credit risk. Again this master thesis deals exclusively with interest rate risk and not with liquidity risk or credit risk.

Below follows a specification of some concepts and the interest rate contracts that are material for this master thesis, since they are used to construct the yield curves.

Let $s_0 = t_0 = 0$ be today and let T be the set of time points including today, i.e., $s_0, t_0 \in T \subseteq \mathbb{R}$, where $\mathbb{R}$ denotes all real numbers. Furthermore, let $\mathbb{R}_+$ denote all non-negative real numbers. Analogously, $\mathbb{R}_+^n$ denotes the n dimensional Euclidian space where all vector elements are non-negative, i.e., $(x_1, \dots, x_n) \in \mathbb{R}_+^n \iff (x_1, \dots, x_n) \in \mathbb{R}^n \; and \; x_k \geq 0 \; \forall k \in \{1, \dots, n\}$, where $\mathbb{R}^n$ denotes the n dimensional Euclidian space, $n \in \mathbb{N}_+$ where $\mathbb{N}_+$ denotes the set of non-negative integers. The currency of the following cash flows is immaterial for the discussion but may be thought of as Swedish krona unless otherwise stated.

Coupon paying bond & Zero coupon bond

A coupon paying bond is a contract that pays the fixed amount $c_1, \dots, c_n$ of money at time $t_1, \dots, t_n \in T: t_k \geq 0 \; \forall k \in \{1, \dots, n\}, n \in \mathbb{N}_+$, where the amount c_k is paid at time t_k, and the coupon paying bond costs $p(t, c)$ today, $t = (t_1, \dots, t_n)$, $c = (c_1, \dots, c_n)$. The final t_n is called the maturity of the bond.

A zero coupon bond is a coupon bond with only one fixed payment that is $c = (c_1)$ and the corresponding price is $p(t_1, c_1)$.

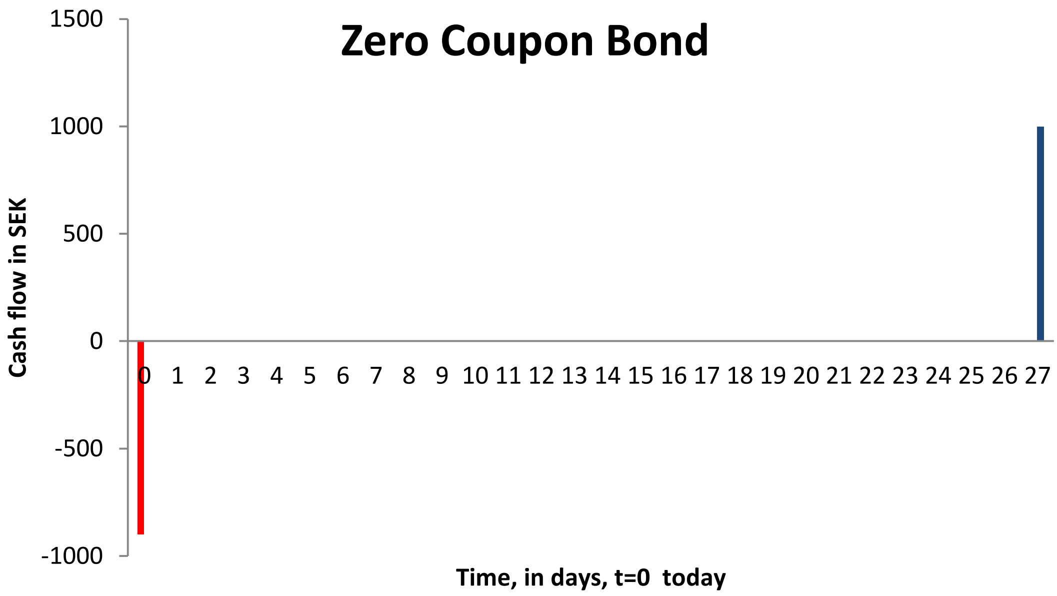

Figure 1. An example of a zero coupon bond with maturity 27 days from today with face value 1000 SEK and price 900 SEK today. The cash flows are on the y-axis and the time measured in days is on the x-axis.

Deposits

Deposits are money that one has deposited at another party. The amount K is deposited at time t_0 and the amount $K + k\colon k \geq 0$, is received at time t_1. The time between t_0 and t_1 is usually very short a week or less. For interest rate modeling purposes deposits can be viewed as zero coupon bonds with short maturity.

Interest rates

There is an equivalent way of quoting fixed cash flow payment. Define the effective interest rate of the zero coupon bond between time t_0 and t_1 as $r_{t_1} = \frac{c_1}{p(t_1,c_1)} - 1$. Then quoting the values $\left(r_{t_1},\, c_1\right)$ is equivalent to quoting the values $(p(t_1, c_1),\, c_1)$ which specifies the complete terms of the zero coupon bond. Furthermore, the law of one price dictates that $r_{t_1} = \frac{c_1}{p(t_1,c_1)} - 1$ is equal to $r_{t_1} = \frac{\lambda c_1}{p(t_1,\lambda c_1)} - 1 \;\forall\, \lambda, c_1 > 0$. Therefore the market practice is to talk about the interest rate r_{t_1}over the period t_0 and t_1.

The interest rate is not usually quoted as an effective rate, rather it is quoted as a rate per unit of time. In the financial literature it is often described in terms of continuous compounding which in this case would be the interest rate r solving $e^{rt_1} = 1 + r_{t_1}$, where t_1 is measured in years. The banking industry practice including Handelsbanken is to use daily compounding which in this case would be the interest rate r solving $\left(1+\frac{r}{n}\right)^n = r_{t_1} + 1$, where n is the number of days from t_0 to t_1. So far the cash flows have been fixed and known at the time of emission of the contract. However there are agreements where the payments, or equivalently, the interest rate of some or all of the cash flows are not known at the day when the parties enter the contract. These payments are called floating payments or floating legs, the corresponding interest rate of these floating payments are called floating rate. The floating rate is typical to be determined by the market at some future time point and is before that point in time considered as a random variable whose outcome of course will be known at the day the floating cash flow is due and most often some time before that.

If one can calculate the interest rate $r_t \; \forall t \in [0, T']: T' \in \mathbb{R}_+$, then the plot of the function r_t as a function of t on the interval $[0, T'] \ni t$, is called the yield curve or term structure.

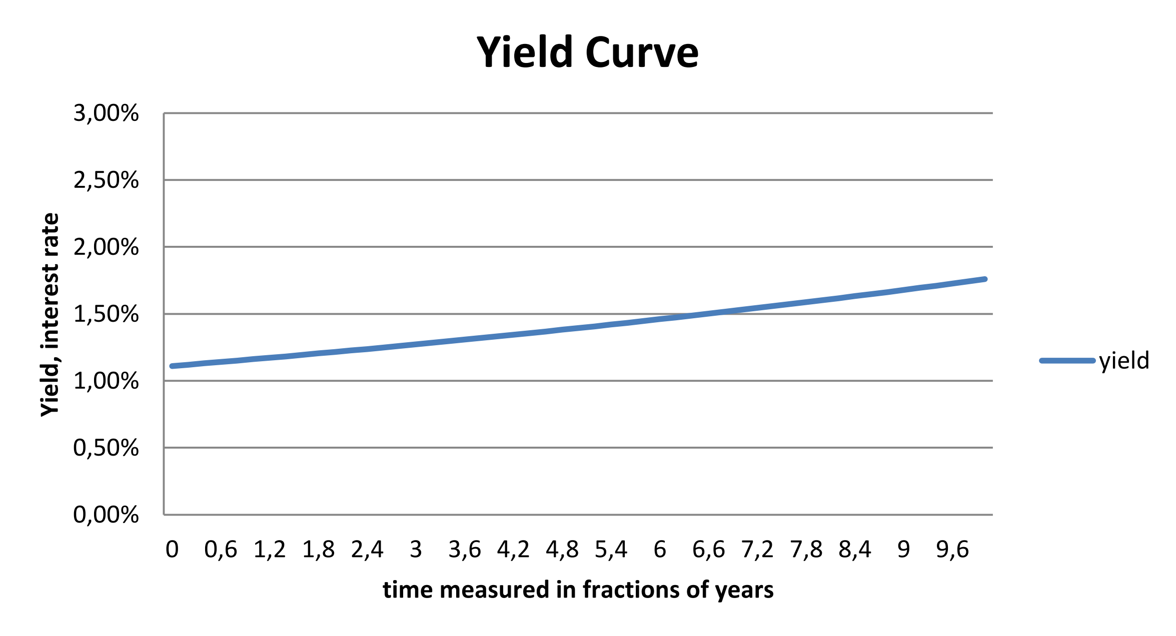

Figure 2. An example of a yield curve. The yield curve is upward sloping and is somewhat representative for the current situation with low interest rates. The interest rate is on the y-axis and the time measured in years on the x-axis.

Interest rate swap, IRS

An interest rate swap is a contract where two parties agree two exchange cash flow streams. The contract specifies a set of time points $t = (t_1, \dots, t_n): t_k \in T \; \forall k \in \{1, \dots, n\}, n \in \mathbb{N}_+$, a notional N, a fixed rate R, and a reference rate $L(t_{k-1}, t_k)$ at the emission of the contract. Time t_0, is today and at each of the time points in t, party A receives a floating payment. For each time $t_k: k \in \{1, \dots, n\}$, party A receives the floating payment $N \cdot (t_k - t_{k-1}) \cdot L(t_{k-1}, t_k)$, where $L(t_{k-1}, t_k)$ is the floating rate prevailing between the time points t_{k-1} and t_k. $L(t_{k-1}, t_k)$, could for example be the LIBOR-rate or the EURIBOR-rate. Party B, who pays the floating leg to party A, receive in return a fixed coupon payment $N \cdot R \cdot \delta_t$ at some time points in t but usually not at every time point in $t = (t_1, \dots, t_n)$, δ_t is the day count factor $t_k - t_i: k > i, and\ i, k \in \{1, \dots, n\}$. $N \cdot R \cdot \delta_t$ is usually paid annually while the floating coupon may be paid every 3 months or every 6 months typically.

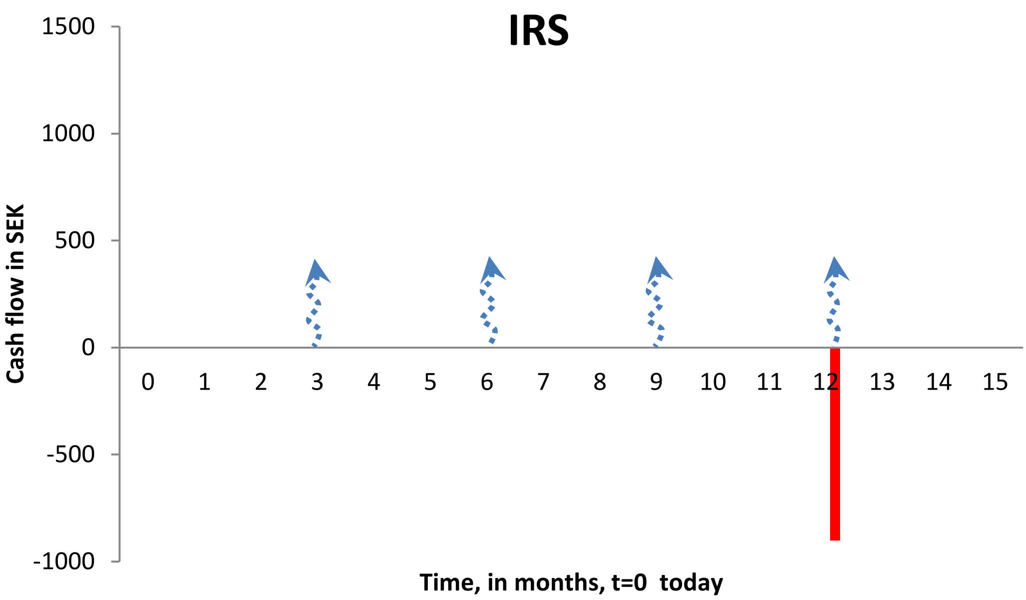

Figure 3. An example of an IRS. The return in SEK is on the y-axis and the time measured in months on the x-axis. The curly dotted blue arrow indicate that the cash flows are determined by some reference rate and thus unknown at the beginning of the contract while the solid red bar is stipulated at the emission of the contract. The reference rate could in this case be the 3 month STIBOR-rate.

Forward rate agreement, FRA

A forward rate agreement is a contract that lock the interest rate prevailing between the time points t_1 and t_2, today at time $t_0 = 0$. The notional N, the fixed rate R, and the floating reference rate $L(t_1, t_2)$ is specified at the emission of the contract at time t_0. The only cash flow is $N \cdot (L(t_1, t_2) - R) \cdot (t_2 - t_1)$ at time t_2, to the buyer of the contract, which may be positive or negative.

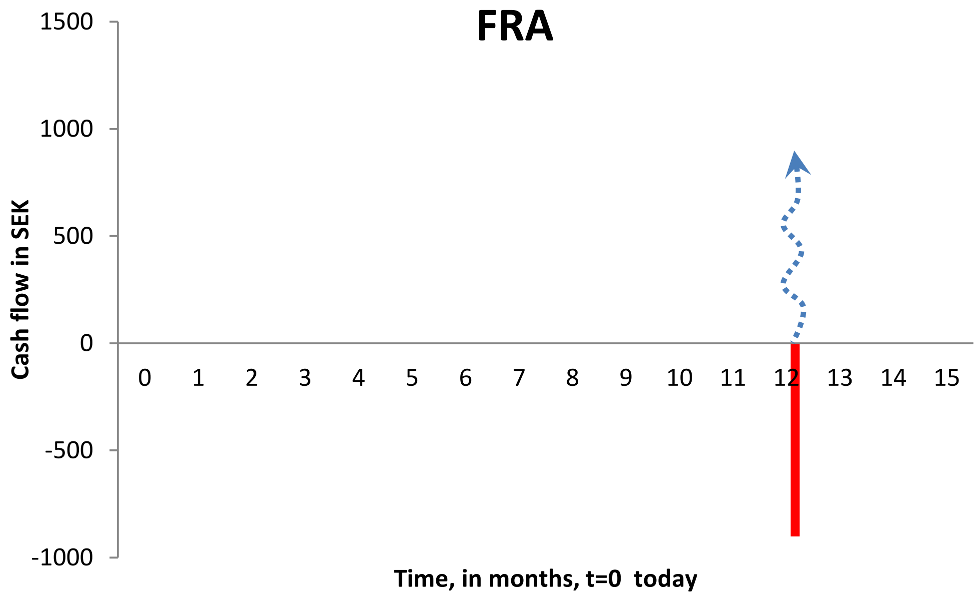

Figure 4. An example of an FRA. The return in SEK is on the y-axis and the time measured in months on the x-axis. The reference rate in this case could be the 6 month STIBOR-rate and will be unknown at inception of the contract and observed at time $t = 6$, the sixth month.

Interest rate basis swap, IRBS

An interest rate basis swap is a contract where two parties exchange floating for floating cash flows, i.e party A pays the reference rate $L_1(t_{k-1}, t_k)$ at time $t_k \in T$, $\forall\ k \in \{1, \dots, n-1\}, n \in \mathbb{N}_+$ plus a spread S on the notional, N. Hence the resulting cash flow that party A pays is $N \cdot (L_1(t_{k-1}, t_k) + S) \cdot (t_k - t_{k-1})$ for each $t_k \colon k \in \{1, \dots, n\}$. Party B receives the mentioned cash flow from party A and pays in return to party A the cash flow $N \cdot L_2(z_{k-1}, z_k) \cdot (z_k - z_{k-1})$ for each $z_k \in T \colon k \in \{1, \dots, n'\}, n' \in \mathbb{N}_+$ and $z_0 = t_0 = 0$. The spread S is set at inception of the contract on the market such that the contract has net present value zero for both parties at time $t_0 = 0$. $L_1(s, t), L_2(s, t)$ are two given reference rates. For example the 3 month LIBOR-rate against the 6 month LIBOR-rate.

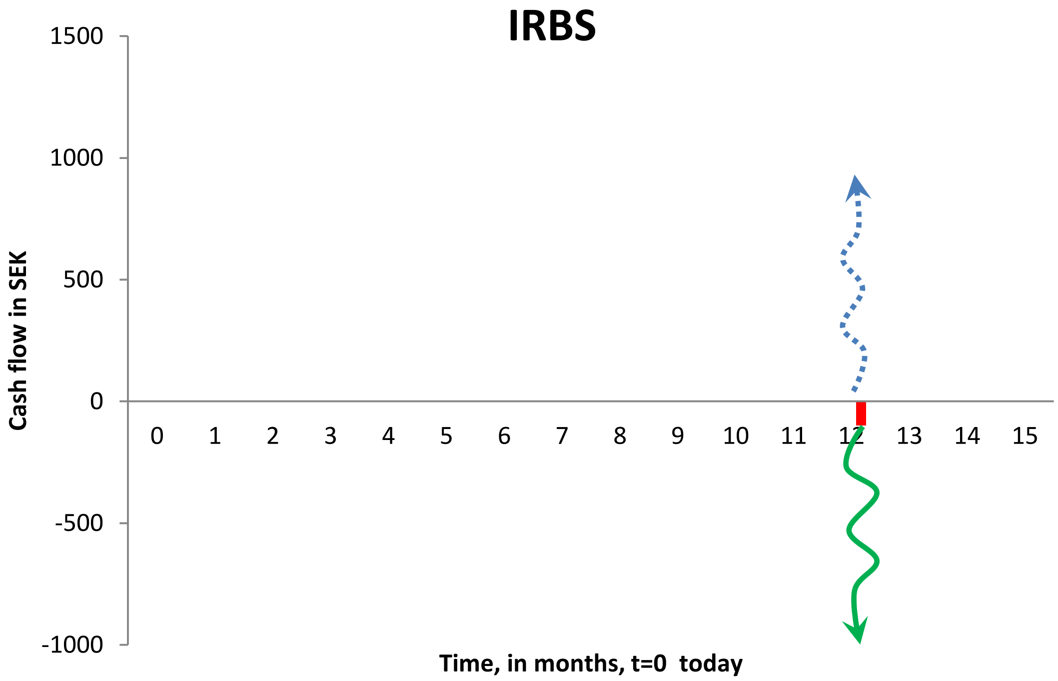

Figure 5. An example of an IRBS. The return in SEK is on the y-axis and the time measured in months on the x-axis. The reference rates in this case could be the 6 month STIBOR-rate and the 3 month STIBOR-rate. The green and dotted blue colored arrows are the floating legs to be exchanged, which are unknown at the inception of the contract. The red solid bar is the spread agreed upon at the inception of the contract and thus completely known throughout the term of the contract.

Cross currency swap, CCS

A cross currency swap involves two currencies, for simplicity we consider the currencies Euro and SEK. Party A pays an amount K_A at time t_0 in SEK and receives the amount K_B in Euro. For the time point $t_k \in T\colon k \in \{1, \dots, n\}, n \in \mathbb{N}_+$ party A pays $K_B \cdot L_B(t_{k-1}, t_k) \cdot (t_k - t_{k-1})$ in Euro and receives at the same time points the cash flow $K_A \cdot L_A(t_{k-1}, t_k) \cdot (t_k - t_{k-1})$ in SEK. Additionally, at time point t_n party A pays to party B the amount K_B in Euro and receives the amount K_A plus a spread $R \cdot K_A$ in SEK. Here $L_B(t_{k-1}, t_k)$ denotes the reference rate that is working upon K_B that is to be paid to party B. This could for example be the EURIBOR-rate between time points t_{k-1} and t_k. The notation is analogously for $L_A(t_{k-1}, t_k)$, this could for example be the STIBOR-rate between time points t_{k-1} and t_k.

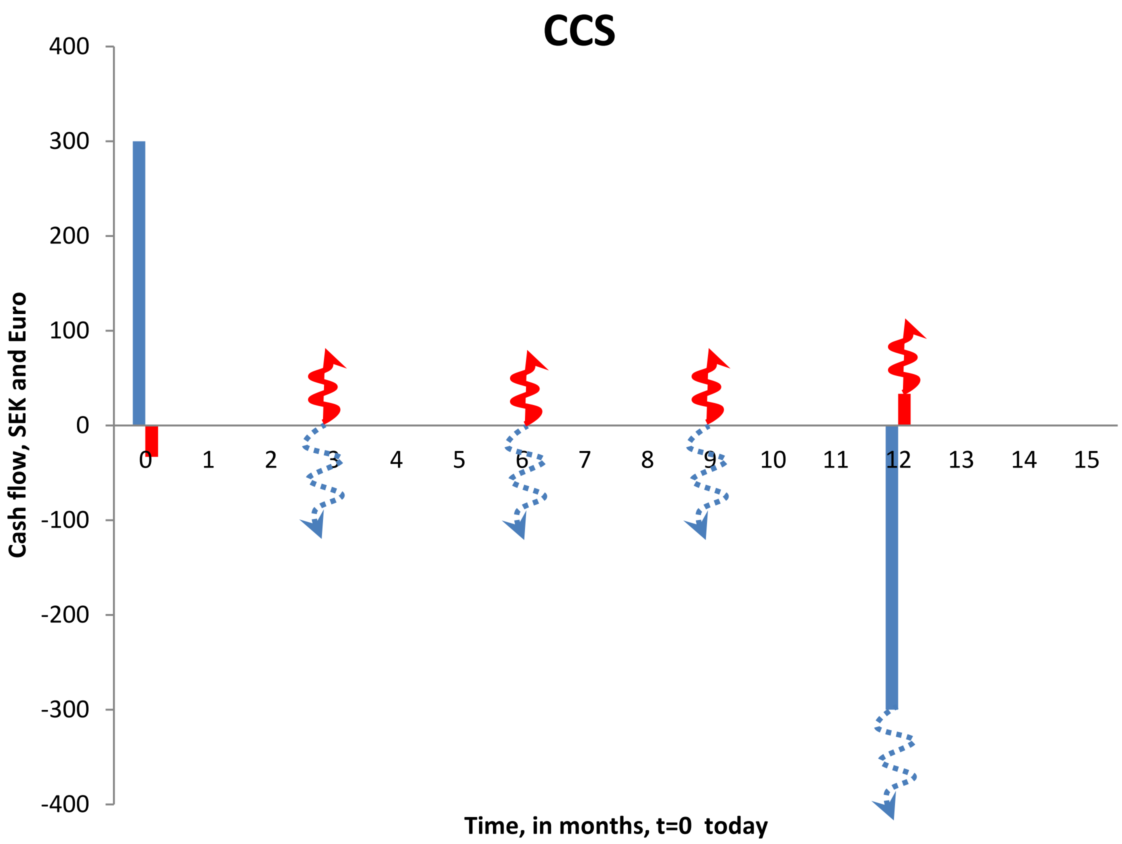

Figure 6. An example of a CCS. The return in SEK and Euro is on the y-axis and the time measured in months on the x-axis. The red cash flows are paid in Euro and the blue cash flows are paid in SEK. The solid bars are fixed cash flows known at the inception of the contract and the curly arrows are floating payments given by the notional, day count convention and the reference rates. The reference rates could in this case for example be the 3 months STIBOR –rate for the SEK currency and the EURIBOR-rate for the Euro currency. In this example the FX-rate between the SEK and Euro is about 9.

FX & FX-Forwards

The FX-rate between two currencies is the spot exchange rate between the two currencies. A FX-Forward is a contract that specifies today, time t_0, the foreign exchange rate, FXF, for some future date, t_1, and a Notional, N. At time t_0, the amount N measured in one of the currencies will then be exchanged at the forward rate between the two parties in the contract.

EONIA & EONIA-Swap

Euro overnight Index average (EONIA), is an effective overnight interest rate, computed as a weighted average of the unsecured lending transactions in the European interbank market and is computed by the European central bank.

An EONIA swap is an IRS where the reference rate, $L(t_{k-1}, t_k): t_k \in T \; \forall k \in \{1, \dots, n\}, n \in \mathbb{N}_+$, of the floating payment is the EONIA rate. The buyer of the contract receives the floating amount $\sum_{k=1}^{n} N \cdot L(t_{k-1}, t_k) \cdot (t_k - t_{k-1})$ at time t_n, where N is the notional, and the buyer pays the fixed amount $N \cdot R \cdot (t_n - t_0)$. Hence R is set at the emission of the contract and therefore fixed. R is set so that the present value of the swap is zero, i.e., no money changes hands at the inception of the contract. This contract then naturally constitutes several yield curve points for any given time point, t, see below.

3. THE MULTI HIERARCHY FRAMEWORK

Interest rate points

The above contracts and pertaining market prices are used together with a bootstrapping procedure to derive series

$$(r)_\alpha = (r_\alpha^1, \dots, r_\alpha^n) : \alpha \in \{E_t, S_t, 1f_t, 3f_t, 6f_t\}$$

of interest rates.

Euro currency spot rates

$$(r)_{E_t} = (r_{E_t}^1, \dots, r_{E_t}^{n_1})$$

SEK currency spot rates

$$(r)_{S_t} = (r_{S_t}^1, \dots, r_{S_t}^{n_2})$$

1 months forward SEK rates

$$(r)_{1f_t} = (r_{1f_t}^1, \dots, r_{1f_t}^{n_3})$$

3 months forward SEK rates

$$(r)_{3f_t} = (r_{3f_t}^1, \dots, r_{3f_t}^{n_4})$$

and 6 months forward SEK rates

$$(r)_{6f_t} = (r_{6f_t}^1, \dots, r_{6f_t}^{n_5})$$

of course $n, n_1, \dots, n_5 \in \mathbb{N}_+$. The tuple of all interest rate points are denoted by:

$$R(I_t) = ((r)_{E_t}, (r)_{S_t}, (r)_{1f_t}, (r)_{3f_t}, (r)_{6f_t})$$

cubic splines are used to derive the entire yield curves corresponding to the series of interest rates that is the set of yield curves $y_\alpha(s) : \alpha \in \{E_t, S_t, 1f_t, 3f_t, 6f_t\}, s \in \mathbb{R}_+$. Where $y_\alpha(s)$ corresponds to the points of $(r)_\alpha$, and $s \in [0, T]$, denotes the maturity.

Interest rate risk

An interest rate instrument, denoted I_t^i, or a portfolio, denoted P_t, at time t, of interest rate instruments with market value $P(I_t^i)$ and $NPV(P_t)$ respectively carries interest rate risk in the sense that the market value of $P(I_t^i)$ and $NPV(P_t)$ may change adversely. Possible fluctuations in the prices $P(I_t^i)$ or $NPV(P_t)$ are equivalent to fluctuations in the curves, $y_\alpha(s)$, which are used to value fixed cash flows by discounting them to present values and also determine the value of floating legs before discounting.

The multi-hierarchy yield curves

The yield curves used for valuing interest rate products by discounting fixed and floating legs are derived for a number of different currencies. The yield curve used for discounting a specific leg depends on the characteristics of the leg. For example if the leg is a floating leg the forward curve with corresponding forward time would be used to value the floating leg and then the estimated value would be discounted to present value. If the leg is fixed then a spot curve with the appropriate credit risk incorporated would directly be used to discount the cash flow. This line of thought, that is, the increase in granulation with respect to valuation method is what preceded and is the aim of the new multi hierarchy yield curves.

The base currency from which all yield curves are derived from is the Euro, using EONIA swaps. In this master thesis we will treat and discuss the dependency between the yield curves denominated in SEK and their derivation from the EONIA base curve. The generalization of results found can easily be implemented in any other yield curve tree with respect to another currency e.g. GBP, DKK or JPY.

Let I_t^M be the set of interest rate contract whose prices are quoted on the market at time t. By prices we mean either their nominal value in the associated currency or the interest rate, whichever is the standard market practice for quoting a given interest rate product.

The spot curve for the Euro currency, called the EONIA curve, is derived using 34 EONIA swaps. Denote this set at time t by

$$ES_t = \{ES_t^1, \dots, ES_t^{34}\}$$

Since we cannot know the future prices with certainty we naturally have that, $t \leq 0$, where $t_0 = 0$ is today. Usually $|ES_t| = 34$ but this may vary with $t \in \mathbb{R}$. By bootstrapping these quotes and the deposit terms at time t denoted

$$D_t = \{D_t^1, \dots, D_t^{n_7}\},$$

the yield curve

$$y_{E_t}(s) \colon s \in \mathbb{R}_+$$

is obtained for a given time t. The SEK spot curve, $y_{S_t}(s) \colon s \in \mathbb{R}_+$, is then derived using $y_{E_t}(s)$, the FX spot rates at time t denoted FXS_t, FX-forwards between SEK and Euro at time

t denoted $FXF_t = \{FXF_t^1, \dots, FXF_t^{n_8}\}$, and IRS denominated in SEK at time t denoted $IRSS_t = \{IRSS_t^1, \dots, IRSS_t^{n_9}\}$ and IRS in Euro at time t denoted $IRSE_t = \{IRSE_t^1, \dots, IRSE_t^{n_{10}}\}$, and CCS between SEK and Euro at time, t, denoted $CCS_t = \{CCS_t^1, \dots, CCS_t^{n_{11}}\}$. Where $n_7, n_8, n_9, n_{10}, n_{11} \in \mathbb{N}$, and simply denote the number of instruments available and utilized from the market. The relative importance of the instruments varies, with the cross currency swaps and forward rate agreements on the exchange rate playing a dominant part. This dependency can be illustrated as below:

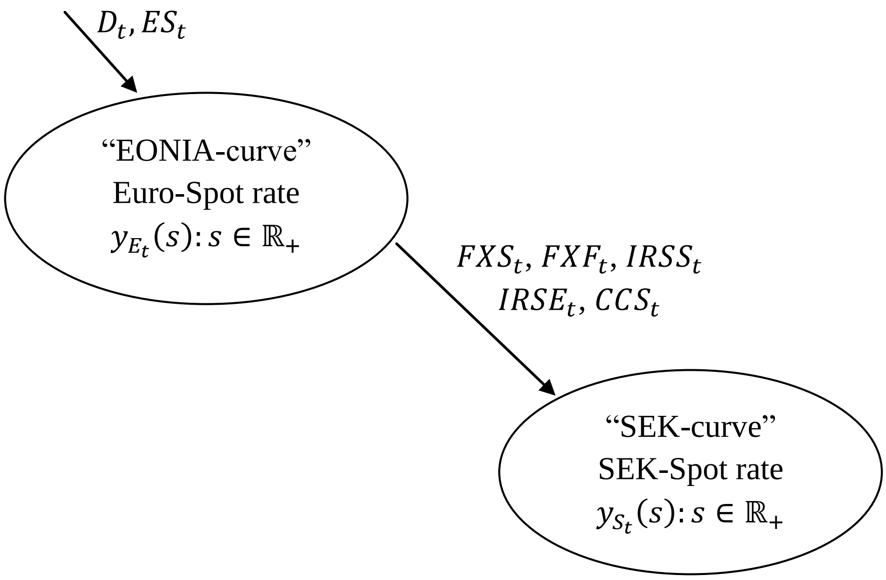

Figure 7. Illustration of how to derive the EONIA-curve and the SEK-curve.

Furthermore from the SEK-curve together with a set of FRA, IRBS and IRS a set of forward rate curves are generated in order to be able to value floating legs in interest rate instruments. The set of forward rate agreements used at time t is denoted by, $FRA_t = \{FRA_t^1, \dots, FRA_t^{n_{12}}\}$, the set of interest rate basis swaps denoted by $IRBS_t = \{IRBS_t^1, \dots, IRBS_t^{n_{13}}\}$, the set of interest rate swaps are $IRSS_t$ as above. This dependency can be illustrated as below

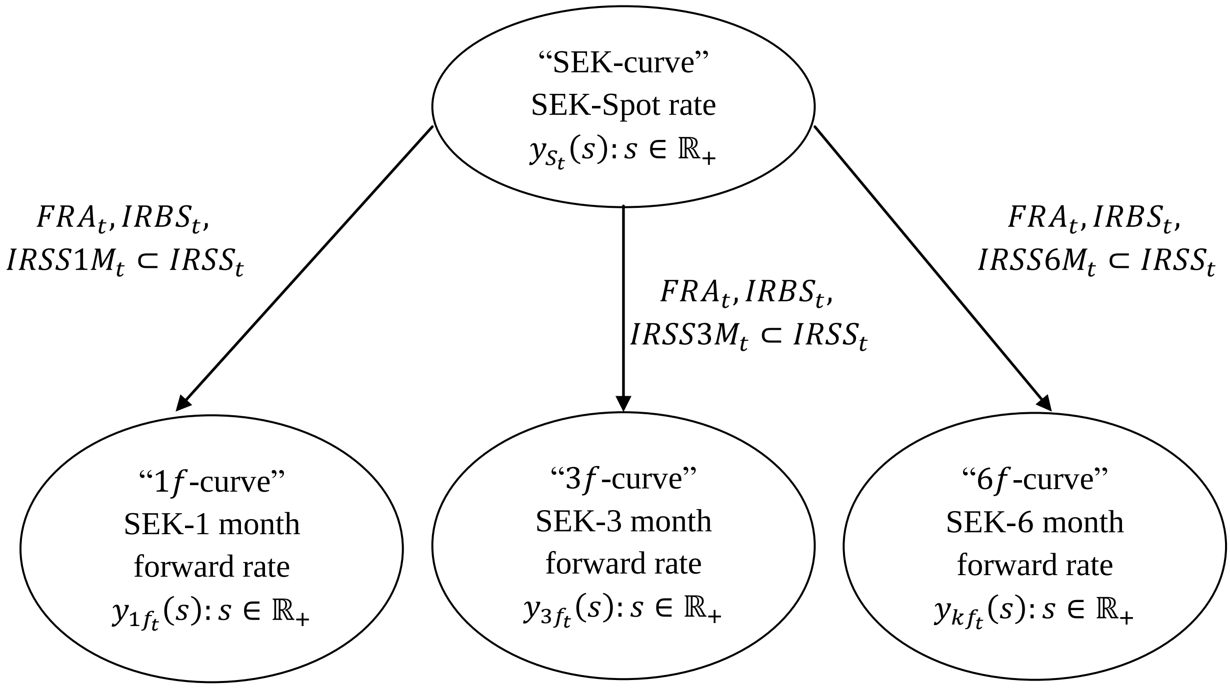

Figure 8. Illustration of how to derive the $1f, 3f, 6f$ -curves.

There are 3 forward curves, namely $1f, 3f, 6f$. $IRSSkM_t$ denotes the IRS denominated in SEK with k months interval of floating leg payments.

Denote by I_t all the instruments used to generate all yield curves at time t, $I_t = \{I_t^1, \dots, I_t^n\}$ where $n \in \mathbb{N}_+$, denote by Y_{I_t} the set of corresponding yield curves derived from I_t, $Y_{I_t} = \{y_\alpha(s) : \alpha \in \{E_t, S_t, 1f_t, 3f_t, 6f_t\}, s \in \mathbb{R}_+\}$.

Furthermore let AI_t denote all conceivable interest rate instruments at time, t, traded as well as non-traded ones, obviously:

$I_t = \{D_t, ES_t, FXS_t, FXF_t, IRSS_t, IRSE_t, CCS_t, IRBS_t, FRA_t\} \subset I_t^M \subset AI_t$.

The entire yield curve hierarchy pertaining to the SEK currency and dependencies upon the instruments generating these yield curves can be illustrated as below:

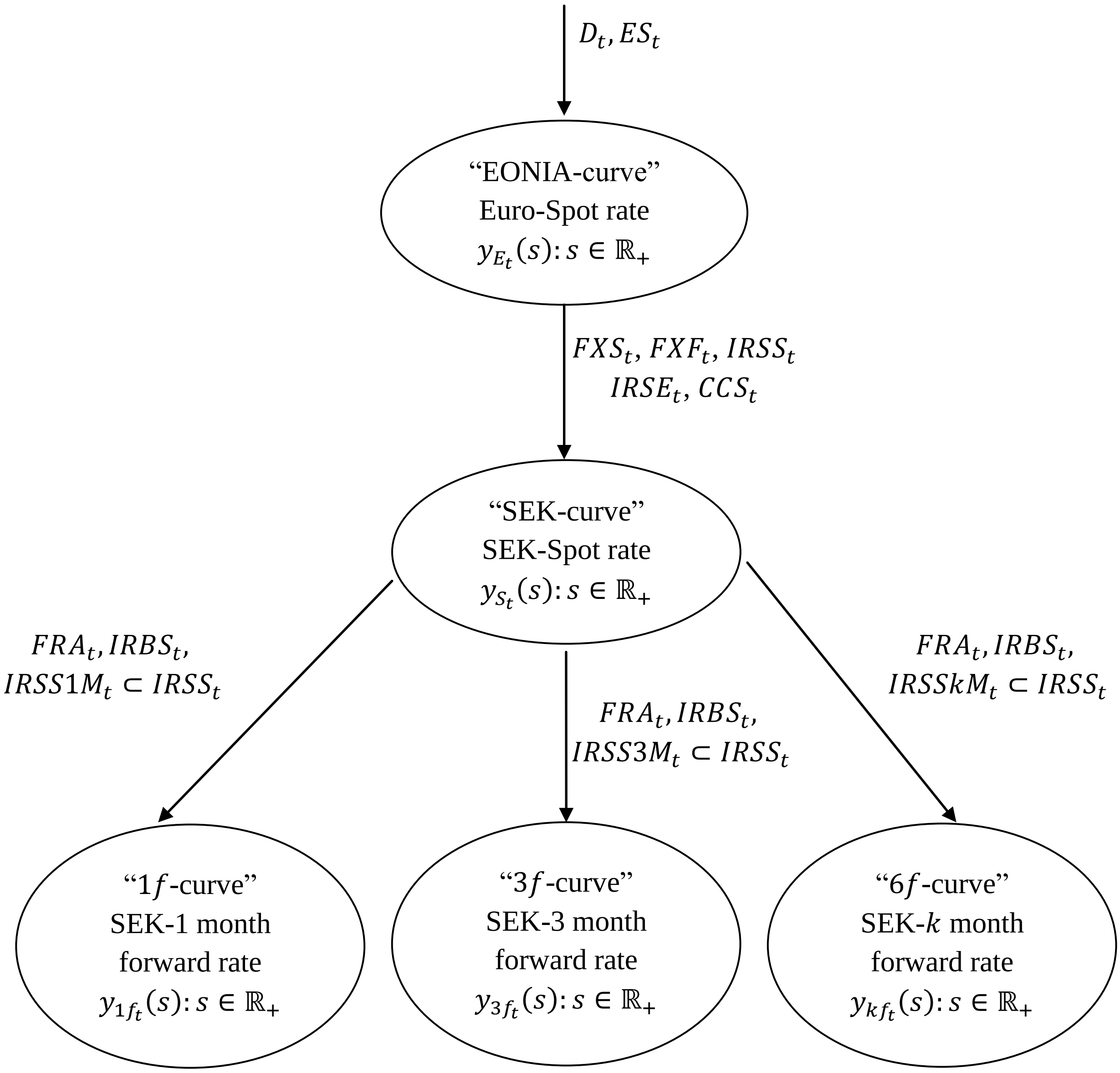

Figure 9. Illustration of the entire yield curve structure and constituting instruments.

4. THE MODEL

Risk modeling Approach

A benchmark approach is used for measuring and limiting risk exposure in the multi hierarchy yield curves framework. There are multiple possible models for the problem at hand. The benchmark approach means that the dynamics of the prices, $p(G_t) = \left(p(G_t^1), \dots, p\left(G_t^k\right)\right) = (p_1, \dots, p_k)$, where $k \in \mathbb{N}_+$, of a set of interest rate products, $G_t = \{G_t^1, \dots, G_t^k\} \subset AI_t$, are modeled to understand risks and movements in the price of a portfolio and individual contracts of interest rates. $p\left(G_t^i\right)$ is the price of the benchmark instruments G_t^i. The prices $p(G_t)$ may be scenarios or derived in some matter. $P(I_t) = (P(I_t^1), \dots, P(I_t^n)) = (P_1, \dots, P_n)$, will denotes the prices of market instruments, $I_t = (I_t^1, \dots, I_t^n) \subset I_t^M$. Another common approach to understand risks and portfolio dynamics is to set up models for the prices $P(I_t)$, or for the yield curves, Y_{I_t} or the interest rate points, $R(I_t) = (R_{I_t}^1, \dots, R_{I_t}^n)$. Schematically the modeling decision could be illustrated as follows:

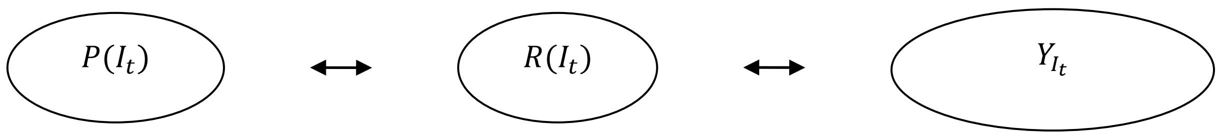

Figure 10. Illustration of different modeling approaches and how they relate.

Here $I_t = (I_t^1, \dots, I_t^n)$, again, is the tuple of interest rate contract constituting the yield curves and $P(I_t) = (P(I_t^1), \dots, P(I_t^n))$, where $P(I_t^k)$ is the price of contract, k, at time, t. The illustration conveys that given $P(I_t)$, one can construct $R(I_t)$ and given $R(I_t)$ one can get the prices $P(I_t)$. Similarly given $R(I_t)$ one can construct Y_{I_t} and given Y_{I_t} one can get the interest rate points, $R(I_t)$. Assumed is that the specification of the contracts I_t is known and an interpolation technique has been decided upon and is also known. Viewed in this way there are three natural main categories of models, with multiple sub-models, of interest rate products.

The first category, first bubble in Figure 10, corresponding to modeling the price dynamics, $P(I_t)$, of the contracts directly. The second category, second bubble, corresponding to model the discrete interest rate points, $R(I_t)$, and the last category, last bubble, corresponding to modeling the entire yield curve/curves, Y_{I_t} .

Complications of these three approaches are that I_t changes over times making it hard to get an understanding of the dynamics of the prices $P(I_t)$. Since not only $P(I_t)$ changes but also the instruments contained in I_t. The same fundamental change of the number of the interest rate points $R(I_t)$ holds as well as changes in the interest rates of $R(I_t)$.

The idea of the generic benchmarks approach is to achieve three different things at the same time. The first one being to reduce the dimension of the risk space to make it more tractable as well as managing concrete modeling situation with computational constraints. The second aim is that the model should convey an intuition and understanding of the interest rate risk components. Thirdly, the model should facilitate a consistent way of modeling interest rate risk in the extended multi hierarchy yield curve framework.

The generic benchmark idea is to define a set of generic benchmark instruments $g_t \subset G_t$, which is of lower dimension than I_t. The generic part means roughly that the definition of g_t be the "same" no matter were in time one examines g_t. This actually means that the very definition of g_t will change over time but one should essentially be unable to determine the time point t by examining, g_t. This property makes g_t generic and will be precisely defined later. These two aspects will achieve the first two requirements above. Making the risk space lower dimensional which makes the concrete modeling easier to handle and less computationally demanding; depending of course on how much the dimension is reduced. Secondly, by keeping g_t "constant" the impact of solely the interest rate risk should be made clear and contribute to a better understanding of such interest rate risks. The belief is that solid understanding of the lower dimensional risk space g_t will make it possible to form qualified views over risks and allow those views to propagate consistently into the risk space of $P(I_t)$, at any point in time.

Schematically, the benchmark approach developed in this master thesis can be illustrated by the following:

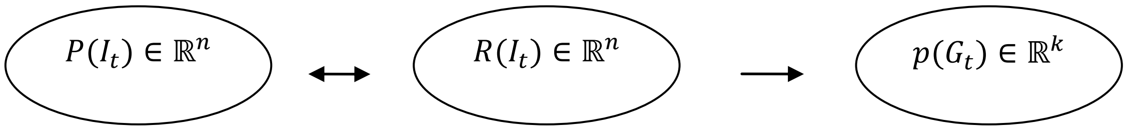

Figure 11. Illustration of the benchmark approach.

In Figure 11 the market instruments I_t and benchmark instruments G_t are given at any time t. $P(I_t)$ and $p(G_t)$ denote prices of the market instruments I_t and the prices of the benchmark instruments G_t respectively. The double arrow indicate as noted above that given the market instruments and there prices one can derive $R(I_t)$, and that given the interest points $R(I_t)$ one can derive the prices $P(I_t)$. The single arrow indicates that the relation only goes in one direction. In other words in the top row one see that given $R(I_t)$ one can compute the prices of any set of benchmark instruments, $p(G_t)$. However in general since $\dim\big(p(G_t)\big) = k < \mathrm{n} = \dim\big(P(I_t)\big)$ one cannot from $p(G_t)$ derive either $P(I_t)$ or $R(I_t)$. The reason for $\dim\big(P(I_t)\big) = \dim(R(I_t)) = n$ is that one doesn't add an instrument to I_t if it doesn't generate some information, that is can be used to derive some interest rate point. Also when one adds an instrument to I_t it can never be used to generate more than one interest rate points. So $\dim\big(P(I_t)\big) \geq \dim(R(I_t))$ but the assumption $\dim\big(P(I_t)\big) = \dim(R(I_t))$ fulfilled in practice.

Viewed mathematically the compression of $P(I_t)$ to $p(G_t)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^k$ by

$$\phi: \mathbb{R}^n \to \mathbb{R}^k,$$

where

$$\phi(\cdot) = \beta(\alpha(\cdot)).$$

Such that $\alpha(\cdot)$ is the function

$$\alpha: \mathbb{R}^n \to \mathbb{R}^n,$$

taking $P(I_t)$ to $R(I_t)$ and β is the function

$$\beta: \mathbb{R}^n \to \mathbb{R}^k,$$

mapping $R(I_t)$ to $p(G_t)$. α is one to one but β is not.

Given $P(I_t)$ the calculated prices $p(G_t)$ can be changed to the prices $p'(G_t) = S(G_t)$ and will be called a scenario for $p(G_t)$. The scenario prices $p'(G_t)$ can be any vector in $\mathbb{R}^k$. These new prices $S(G_t)$ of G_t correspond to some $P'(I_t)$, where $P'(I_t)$ is some hypothetical tuple of prices of the market interest rate instruments. Such that if $P'(I_t)$ was observed on the market they would generate the prices $p'(G_t)$ equal to those given by $S(G_t)$, i.e., $\phi(P'(I_t)) = p'(G_t)$. $P'(I_t)$ can be any vector in $\mathbb{R}^n$. It should be noted that there may be several distinct tuples $P'(I_t)$ that corresponds to a given $p'(G_t)$ in other words ϕ is not injective. Denote by $R'(I_t) = \alpha(P'(I_t))$. Then finding some $P'(I_t)$ for a given scenario $S(G_t)$ is equivalent to finding $R'(I_t)$. Since α is one to one by no arbitrage.

The idea is now to given $P(I_t)$ calculate the prices $p(G_t)$ and to shift those prices to get the interest rate scenario $S(G_t)$. Finally from $S(G_t)$ generate some $P'(I_t)$. In this way price scenarios are generated for I_t^M.

The interest rate points pertaining to $P'(I_t)$ is denoted by $R(I_t')$ and the corresponding yield curves are denoted by $y_{I_t'}(s)$.

The functions, α, β and consequently ϕ are only defined given some I_t, G_t and some interpolation technique $\mathbb{c}$, which are chosen in some manner. I_t, G_t and $\mathbb{c}$ should therefore be considered as parameters for the functions α, β and ϕ. This could be made explicit with the notation $\alpha(\cdot\,; I_t), \beta(\cdot\,; G_t, \mathbb{c})$ and $\phi(\cdot\,; I_t,\ G_t,\ \mathbb{c})$ but is refrained from, for the ease of reading.

Generic Benchmarks

The prices of I_t^M are by definition quoted on the market at time, t, and $I_t \subset I_t^M$ generates the interest rates points $R(I_t)$, and the interest rate points $R(I_t)$ generate the yield curves Y_{I_t}. Let $G_t = (G_t^1, \dots, G_t^n)$ where $n \in \mathbb{N}_+$ and let $p(G_t) = (p(G_t^1), \dots, p(G_t^n))$ be the prices of the instruments G_t at time, t, with respect to the yield curves generated from the prices $P(I_t)$, i.e. the prices $P(I_t^1), \dots, P(I_t^n)$ are used to calculate interest rate curves Y_{I_t}, which are used to value the instruments G_t. The tuple $G_t = (G_t^1, \dots, G_t^n)$ are as stated before called benchmark instruments at time, t, and G_t^k are called benchmark instrument k at time, t. Obviously $G_t^k \in AI_t\ \forall\ k$ but G_t^k may or may not be in I_t or I_t^M.

Denote by $\mathrm{c}(G_t^k) = (c_1, \dots, c_n)$, $n \in \mathbb{N}_+$, the tuple of floating or fixed, cash flows of the contract to be paid between time, t, to the end of the contract. Denote by $\mathrm{c}(G_t^k(s)) = (c_1(s), \dots, c_n(s))$ the same contract $\mathrm{c}(G_t^k)$ with the only difference that all floating reference rates are observed and referenced to at time point, s. Denote by $\Delta\mathrm{t}(G_t^k) = (\Delta t_1, \dots, \Delta t_n)$ the tuple of time points, where Δt_a is the time left before cash flow c_a is paid/received. Write, $G_{t_1}^k = G_{t_2}^k$ iff $\Delta\mathrm{t}(G_{t_1}^k) = \Delta\mathrm{t}(G_{t_2}^k)$ and $\mathrm{c}(G_{t_1}^k(t_1 + t_2)) = \mathrm{c}(G_{t_2}^k(t_1 + t_2))$. If $|G_t| = n \; \forall t$ and $G_{t_1}^k = G_{t_2}^k \; \forall \, k \in (1, \dots, n), t_1, t_2 \in \mathbb{R}$ then the set, G_t, is called generic benchmark instruments and to highlight this property, G_t is denoted by g_t, and analogously $g_t = (g_t^1, \dots, g_t^n)$. Essentially what it means is that g_t is a set of generic benchmark instruments if, for any $t \in \mathbb{R}$, g_t contains interest rate contracts stipulated in the same way with reference to the same floating rates and the same time remaining for every cash flow. The only difference being of course that the reference rate is referenced at different points in time for different g_t in the absolute sense but not in the relative sense.

There are several advantages of defining benchmark instruments and require them to be generic. The primary advantage of having generic benchmark instrument is that they always contain the “same” instruments while I_t changes over time as new contracts with different standards gain in popularity on the market or when existing contracts wane in popularity and stop being traded on the market. Such aspects make the price history of contract scarce and implies that it is impossible to calculate risks based on long enough empirical data. Secondly, other aspect, than interest rate risk will contaminate the data. Take for example the overnight rate, effectively the short rate, this interest rate may measure interest rate fluctuation in the short rate but it is not generic and will therefore be distorted by holidays. Normally the interest rate is really over one day and when that is the case comparing the price evolution would give a sense of the volatility of the short rate. The problem is that weekend and holidays make the short rate effectively run over several days and the increase in interest rate is not due to volatility of the short rate but instead to the fact that the maturity has been prolonged. Of course banks know this so it does not affect the pricing of interest rate products. However, it does affect risk management both in the situation of limits using benchmark instrument since in this case the risk measured would suddenly change while the limit would not, implying inconsistency. Furthermore, as explained above the measurement of interest rate risk in the short rate would be contaminated with noise so that other factors than interest rate risk comes in to effect which of course is detrimental when one wants to manage interest rate risk solely. Using generic benchmark instruments would alleviate these problems enabling to solely measure interest rate risk, measure the same type of interest rate risk over time which should facilitate communication and understanding of risk exposure. In the multi hierarchy yield curve structure the task of guarantee that a given risk model is consistent becomes increasingly difficult. Using generic benchmark shift does not introduce any immediate inconsistencies.

Restrictions on Benchmarks

Natural restrictions arise in the process of choosing generic benchmark instruments and are contingent upon the specific purpose of the interest rate risk assessment. The two situations at hand are:

1. risk assessment, meaning that a realistic assessment of the interest rate risks in the interest rate portfolio. This risk assessment is typically expressed as an empirical value at risk amount and will affect the capital requirement of the bank and give a realistic overview of the interest rate risk in the portfolio.
2. The second aim is to define limits on trades in interest rate product in order to guide trading activities and prevent excessive risk taking.

The former could be said to be reactive and the later proactive. In the case of risk assessment the requirements are that the generic benchmark instrument should be chosen so that they capture all conceivable interest rate risk, heuristically speaking the generic benchmark instruments should span the risk space. In the second case of defining limits there are more severe restrictions. Not only is it desirable that the interest rate risk space is spanned but in addition that the procedures are not to computationally demanding. This is due to the fact that limits need to be monitored intra-day, i.e., several times each hour preferably continuously. This means that the generic benchmarks should be relatively few, since the computational time grows with the number of generic instruments. In this vein it is advantageous not to add a generic benchmark instrument g_t^i to g_t if there already is a generic benchmark instrument g_t^j measuring the almost the same risk. Hence in some heuristic sense it is desirable that the instruments in g_t are unrelated. Furthermore the limits should measure independent risk such that traders who adjust one limit by changing his position should not inevitable change his risk exposure in another limit.

In conclusion there are two situations to handle in the benchmark model. Both should rely only on generic benchmark instruments. The requirements are summarized below.

Requirements for risk assessment, Value at Risk

1. Generic Instruments
2. Span most of the Risk Space

Requirements for Limits

1. Generic instruments
2. Span most of the Risk Space
3. Computational speed ⇔ Few benchmark instruments
4. Independent limits, Computational speed ⇔ Unrelated instruments

Defining the Risk space, $\Re(I_t)$, and Unrelatedness, $\langle \cdot \mid \cdot \rangle$

This section will make precise the above requirement put on the benchmarks including what is meant but Risk space and unrelated benchmark instruments.

Consider a portfolio, P_t, of interest rate products at time t consisting of ω_t^i units of contract $I_t^i \in I_t^M$. The value of the portfolio P_t at time t is denoted by $NPV(P_t) = \sum_{i=1}^{n} \omega_t^i P(I_t^i)$.
The difference in value, $\Delta NPV(P_t) = NPV(P_{t+\Delta t}) - NPV(P_t)$, of this portfolio between time t and $t + \Delta t$ such that $\Delta t > 0$, given that no trade has occurred in the portfolio between time t and time $t + \Delta t$, i.e., $\omega_t^i = \omega_s^i \; \forall \; s \in [t + \Delta t], i \in \{1, \dots, n\}$, is in part due to interest rate risk. The other factors influencing the price process is credit risk and the time value. The interest rate portfolio must consist of n contracts $I_{P_t} = \{I_t^1, \dots, I_t^n\}$, which are all elements of I_t^M. Hence any fluctuation in price of I_{P_t} must correspond to some price fluctuation in I_t^M. It may of course happen that there is some price fluctuation in the prices of the instruments in I_t^M to the latter time point $t + \Delta t$ but that $\Delta NPV(P_t) = 0$. This could happen if some interest rate products in P_t increase in value while other interest rate products decrease in value. Furthermore approximately every instruments in I_t^M is also in I_t so I_t^M and I_t are quite similar which is motivated by the fact that as much market information as possible is utilized when constructing the interest rate yield curves. This motivates the definition of the interest rate risk space. The interest rate risk space at time t is defined as the instantaneous price changes of the

instruments in I_t. These alternative prices, or the set of elements in the interest rate risk space at time t is denoted $\Re(I_t)$. In other words $\Re(I_t) = \mathbb{R}^n \ni P'(I_t)$.

It should be noted that the set of $\Re(I_t)$ at time t, need not be a plausible scenario for any short time interval or even possible in some qualitative sense. As an example suppose that I_t contains a zero coupon bond A, with face value 1000 Swedish krona and maturity one year from now. Furthermore, suppose that I_t also contains a bond B identical to A in every way except that the face value is 2000 Swedish krona. Then by to avoid no arbitrage the price of bond B must be roughly twice that of bond A. However, there are uncountable many scenarios in $\Re(I_t)$ that would allow arbitrage by trading the bonds A and B, assuming that the positions in A and B are real numbers.

If g_t is such that α is a bijective function then g_t is said to span the risk space, $\Re(I_t)$. In practice g_t will not span the risk space, $\Re(I_t)$, but the aim is that the loss of information is not large.

If ϕ is a smooth function such that ϕ is differentiable where the derivative is denoted by ϕ'. Denoting $\phi(x) = \phi(P_1, \dots, P_n) = \left(\phi_1(P_1, \dots, P_n), \dots, \phi_k(P_1, \dots, P_n)\right)^T$ it follows that:

$$\phi'(x) = \begin{bmatrix} \nabla\phi_1^T(x) \\ \vdots \\ \nabla\phi_k^T(x) \end{bmatrix},$$

Where the gradient $\nabla\phi_i(x)$ at point $x \in \mathbb{R}^n$ is:

$$\nabla\phi_i(x) = \left(\frac{\partial\phi_i}{\partial P_1}(x), \dots, \frac{\partial\phi_i}{\partial P_n}(x)\right)^T,$$

for $i = 1, \dots, k$.

Let e_j denote the unit vector in $\mathbb{R}^n$ with 1 in the $j{:}th$ component and zeros elsewhere and denote the scalar product by $\langle\cdot,\cdot\rangle$. If $\langle\nabla\phi_i(x), e_j\rangle = 0 \ \forall x \in \mathbb{R}^n$, then instrument I_t^j is said to be unrelated to the benchmark instrument g_t^i written $\langle g_t^i \middle| I_t^j \rangle = 0$. This means that the price $p(g_t^i)$ of g_t^i does not change as a response to small changes of the price $P(I_t^j)$ of I_t^j.

Denote by $U(g_t^i)$ the set of instruments in I_t which are unrelated to g_t^i. If $U\left(g_t^i\right)^c \cap U\left(g_t^h\right)^c = \emptyset$ then g_t^i and g_t^h are said to be unrelated.

To avoid redundancy in g_t is would be desirable that the benchmarks in g_t are pairwise unrelated. Furthermore for g_t to capture as much as possible of the interest rate risks there should not exists any instruments I_t^j which are unrelated to every benchmark in g_t.

Correspondence Scenario

Given a scenario $S(g_t)$ one wishes to find "the" corresponding future scenario for the real market instruments $P'(I_t)$. It is not a priori certain that there always exist a $P'(I_t)$, consistent with $S(g_t)$, which may be due to a poor design of g_t. Furthermore, if g_t have a proper design it will usually be the case that there exist several $P'(I_t)$ consistent with $S(g_t)$, since the dimension of g_t is usually smaller than I_t, i.e, $\dim(g_t) = k < n = \dim(I_t)$. Therefore, when talking about the scenario $P'(I_t)$ generated by $S(g_t)$ when such exists the scenario in mind is $P'(I_t)$ consistent with $S(g_t)$ and such that $|\alpha(P'(I_t)) - \alpha(P(I_t))|^2 = |R(I_t') - R(I_t)|^2$ is minimal. Here the norm is taken between the interest rate points of $R(I_t')$ and $R(I_t)$. The $P'(I_t)$ satisfying these conditions are called the correspondence scenario. The correspondence scenario has the important property that the correspondence scenario generated by $p(g_t)$ is indeed $P(I_t)$.

Benchmarks for limiting

The generic benchmarks $g_t = (g_t^1, \ldots, g_t^n)$ will be used for risk limiting and the aim is to chose them in a way so that they are pairwise unrelated. A subset of the unrelated generic benchmark instruments will pertain to a relevant trader or portfolio of interest rate instruments. Say that a trader manage a portfolio of interest rate instruments in the SEK currency, furthermore let the instruments be of a specified class e.g. fixed interest rates only. This means that the portfolio is sensitive to interest rate changes in the SEK spot rate curve. Then the corresponding generic benchmarks $g_t^{SEK} = (g_t^{SEK1}, \ldots, g_t^{SEKn})$ will be used to limit the risk exposure of the specific trader. The set g_t^{SEK} will be instruments that the trader would hypothetically be allowed to trade. The risk limit of the portfolio is with respect to the delta of the generic benchmark instruments in g_t^{SEK}, i.e., if P_t^{SEK} is the traders portfolio at time t then the risk limits is the vector $(l_1, \ldots, l_n)$ such that the delta of the portfolio, P_t^{SEK}, with respect to instrument g_t^{SEKk} may not exceed the l_k, which means that, $\frac{\Delta NPV(P_t^{SEK})}{\Delta p(g_t^{SEKk})} \leq l_k \; \forall k \in \{1, \dots, n\}$. Where $\Delta NPV(P_t^{SEK})$ is calculated as a function of the price changes in $p(g_t^{SEKk})$, as a result of the correspondence scenario.

Note that in any specific case there will most likely exist generic benchmark instruments in g_t say $g_t^i : g_t^i \notin g_t^{SEK}$ and $\frac{\Delta NPV(P_t^{SEK})}{\Delta p(g_t^i)} \neq 0$. This imply that the portfolio will change in value even if all the instruments in g_t^{SEK} does not change in value but other generic benchmark instruments do. In reality and for any common portfolio one would suspect that $\frac{\Delta NPV(P_t^{SEK})}{\Delta p(g_t^i)} \neq 0$ for some $i : g_t^i \notin g_t^{SEK}$. This has the consequence that one cannot talk about "SEK risk" which was the case before the multi hierarchy framework. The question about the "SEK risk" is simply not well defined anymore because of the fact that the interest rates share a close interdependency which is a natural consequence of the required consistency of today´s valuation framework.

It is clear that the bank as a whole manages all the necessary risks since each $g_t^k \in g_t$ is used as a limit for some trader and only for the trader who can trade in the hypothetical generic benchmark instrument. It is also clear that in this way there will be no double accounting for one specific interest rate risk.

A suggestion for what could be meant by a "SEK risk" is the largest decrease in value that would hit the trader with the P_t^{SEK} portfolio if the prices of the set g_t^{SEK} moved in some one of several predefined scenarios. The scenarios could for example be that all of the prices of g_t^{SEK} either increase by 10% or decrease by 10%. This last scenario is a reminiscent of the parallel shift scenarios that was common before the 2008 financial crises.

Benchmarks for Risk assessment

Given the benchmark instruments g_t, the prices of g_t, $p(g_t)$, can be calculated for any given $t : t \leq 0$. This means that the historical value of the generic benchmark instruments can be obtained. Given these historical prices an empirical value at risk can be calculated by generating the scenarios for the benchmark shift by setting scenario number k, denote it by S_k, equal to $S_k = p(g_t) + \Delta p_k$ where $\Delta p_k = p(g_{t-k}) - p(g_{t-k-1})$ hence $S_k = p(g_t) + p(g_{t-k}) - p(g_{t-k-1})$. The choices of k will in practice be $k \in \{0, -1, -2, \dots, -n\}$ where n

would typically be 365 representing the number of days of a year or 250 representing the number of trading days per year. The scenarios would then be used to find equally many correspondence scenarios, from which equally many empirical scenarios for the future value of the entire portfolio would be obtained.

Furthermore, it could be argued that weekends should be excluded from the historical prices since the price of $p(g_{t-k})$ is usually defined to be $p(g_{t-k}) \coloneqq p(g_{t-k-1})$ if the time point k is a holiday. This means that the historical price changes would be zero, $\Delta p_k = 0$ if k is on a holiday, that is, a none trading day. By the same pattern Δp_k would not measure a one day price shift if k is the first day following a holiday. The remedy would be to exclude any Δp_k if it measured on a holiday or is measured over some holiday then every Δp_k would measure interest rate changes over the period of one day which adds consistency.

Lastly the generic benchmark approach will allow the risk assessment to take into account dependencies in the interest rate process for example by basing the calculation of the Δp_k only on k´s such that $p(g_t)$ is similar to $p(g_{t-k})$, a natural way of comparing the similarities between $p(g_t)$ and $p(g_{t-k})$ could for example be to calculate $|p(g_t) - p(g_{t-k})|$ or $\max_i |p(g_t^i) - p(g_{t-k}^i)|$.

5. A MINIATURE ENVIRONMENT EXAMPLE WITH LINEAR INTERPOLATION

Example Setup

In this section a miniature interest yield curve example consisting of only one spot rate curve in one arbitrary currency is examined. Continuous compounding of interest rates is used together with linear interpolation technique in the yield curve. Consider the following set $I_0 = \{I_0^1, I_0^2, I_0^3, I_0^4\}$ of market contract at time $t = 0$, consisting of four zero coupon bonds in the SEK currency with maturity t_k of bond I_0^k where $k \in \{1,2,3,4\}$. Without loss of generality assume that the face value of the bonds is 1 krona. These four market contracts generate a set of four interest rate points $R(I_0) = (r_{I_0}^1, r_{I_0}^2, r_{I_0}^3, r_{I_0}^4)$, by $r_{I_0}^k = -\frac{\ln\left(P(I_0^k)\right)}{t_k}$, with $t_k = \frac{3k}{12}$, that is, bond I_0^1 having maturity 3 months from now and I_0^2 having maturity 6 months from now etcetera. In total the maturities are 3,6,9 and 12 months and

$$\left(P(I_0^1), P(I_0^2), P(I_0^3), P(I_0^4)\right) = (0.9975, 0.9945, 0.9903, 0.9841).$$

After rounded gives the interest rate points

$$\left(r_{I_0}^1, r_{I_0}^2, r_{I_0}^3, r_{I_0}^4\right) = (0.01, 0.011, 0.013, 0.016) = (1.0\%, 1.1\%, 1.3\%, 1.6\%)$$

Using linear interpolation and constant extrapolation gives the yield curve function:

$$y_{I_0}(s) = \begin{cases} r_{I_0}^1 \quad if \;\; 0 \le s < t_1, \\ r_{I_0}^k + (s - t_k)\left(\dfrac{r_{I_0}^{k+1} - r_{I_0}^k}{t_{k+1} - t_k}\right) \quad if \; t_k \le s < t_{k+1} \qquad : k \in \{1,2,3\}, \\ r_{I_0}^4 \quad if \;\; t_4 \le s. \end{cases}$$

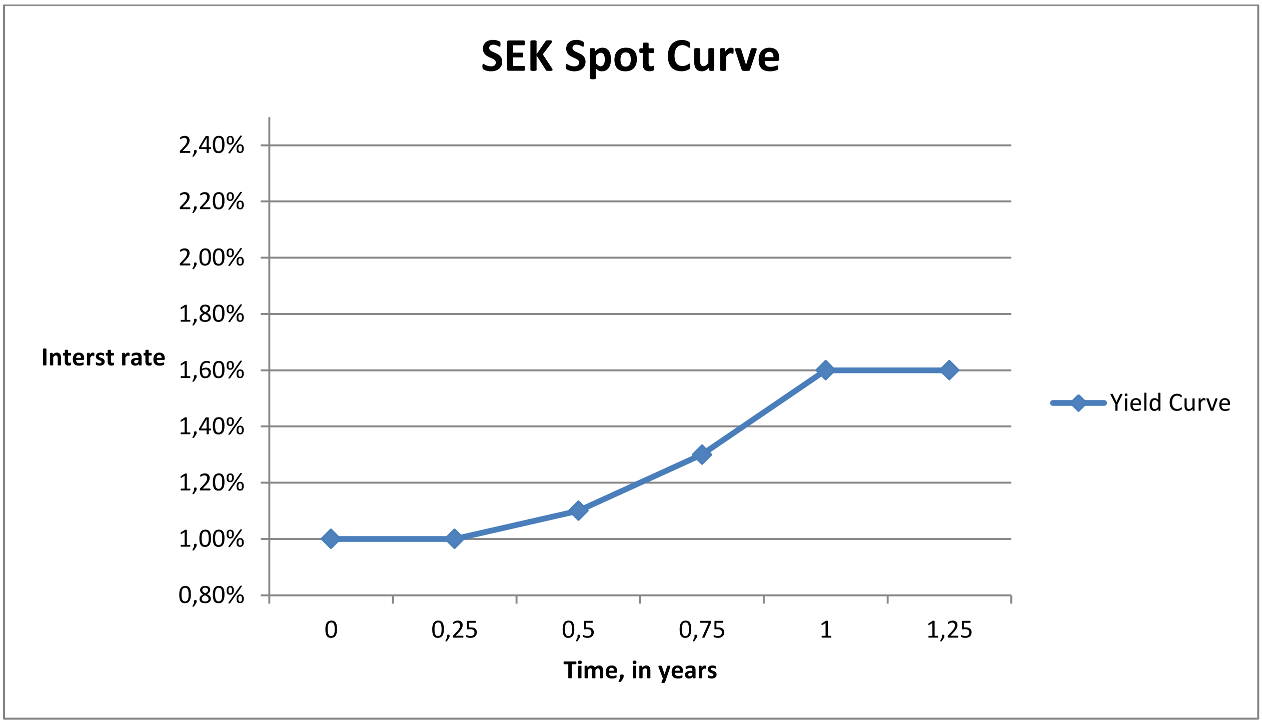

Figure 12. SEK spot curve in the miniature example from hypothetical market data. Time is measured in years on the x-axis an the interest rate is given on the y-axis as a function of time, i.e. the maturity.

Since $|I_0| = 4$ it is reasonable that $|g_0| \leq 4$. In general computational reasons would require that $|g_0| \ll |I_0|$ but it can be seen that not only the size of $|g_0|$ in relation to the size of $|I_0|$ mater but also the design of g_0.

Example 1. Poor Choice of Generic Benchmarks

Let $g_0 = \{g_0^1, g_0^2, g_0^3\}$ consist of three zero coupon bonds with face value 1 SEK and time left to maturity s_k for generic benchmark instrument g_0^k. With $(s_1, s_2, s_3) = (\frac{7}{12}, \frac{7.5}{12}, \frac{8}{12})$ corresponding to 7 months, 7.5 months and 8 months left to maturity. The prices of g_0 are $p(g_0) = \{p(g_0^1), p(g_0^2), p(g_0^3)\}$ where $p\left(g_0^k\right) = e^{-y_{I_0}(s_k)s_k}$. The prices for (g_0^1, g_0^2, g_0^3) is calculated to be $(0.9932,\ 0.9925,\ 0.9918)$ and the corresponding interest rates are $\left(y_{I_0}(s_1), y_{I_0}(s_2), y_{I_0}(s_3)\right) = (0.01167,\ 0.0120,\ 0.0123) = (1.167\%,\ 1.20\%,\ 1.23\%)$

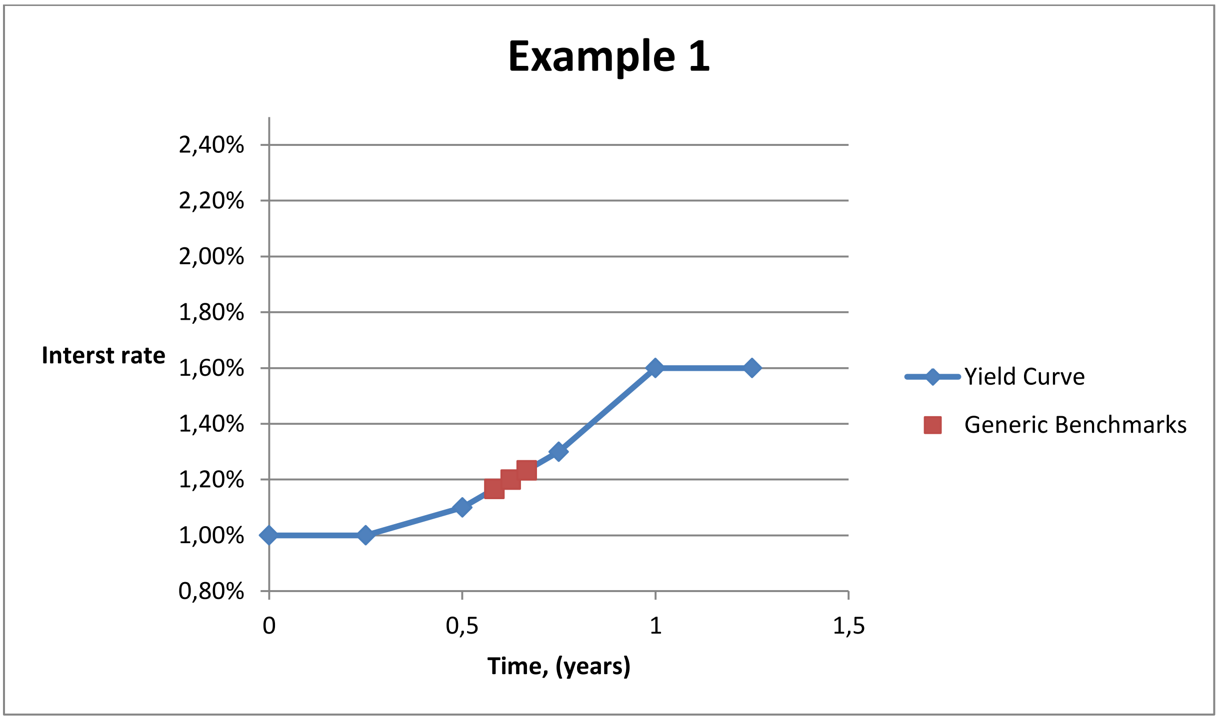

Figure 13. SEK spot curve in the miniature example from hypothetical market data and interest rate points of three generic benchmark instruments in example 1. Time is measured in years on the x-axis an the interest rate is given on the y-axis as a function of time, i.e. the maturity.

Let $S(g_0)$ denote a scenario for g_0 which could be $(p(g_0^a) = price_a, \ldots, p(g_0^n) = price_n, y_{I_0'}(s_a) = rate_a, \ldots, y_{I_0'}(s_n) = rate_n)$, that is, the scenario corresponds to $P'(I_0)$ such that the prices of g_0 under $P'(I_0)$ are those specified in the scenario and the interest rates of g_0 under $P'(I_0)$are those in the scenario. It is not necessary to form a view regarding all the generic benchmark instruments a scenario $S(g_0)$ may only specify beliefs regarding some of the generic benchmark instruments of g_0. Furthermore when it comes to zero coupon bond specifying a price for g_0^k or an interest rate for g_0^k is equivalent. That is why one can have views of the kind $y_{I_0'}(s_a) = rate_a$ when applicable to some g_0^i. Regardless of the formulation of the views, in the end $\phi(P'(I_0)) = p'(g_0)$ must hold.

Given a new scenario $S(g_0)$ where the new prices are forecasted or estimated for all $k = 1,2,3$ in Example 1 it is easy to see that there will exist infinitely many scenarios for the prices of I_0 such that $P'(I_0)$ complies with a given scenario, for every $k = 1,2,3,$ or none at all. This is because all of the interest rate points $y_{I_0'}(s)$ between the time points $\frac{6}{12}$ and $\frac{9}{12}$ are determined by $r_{I_0'}^2$ and $r_{I_0'}^3$that is two variables. Any scenario for g_0 give rise to three interest rate points in the interval corresponding to the time points of $r_{I_0}^2$ and $r_{I_0}^3$ which is t_2 and t_3.

The first two prices of the scenario for g_0 completely determines $r^2_{I'_0}$ and $r^3_{I'_0}$. The third price of g_0, $p'(g_0^3)$, generate an interest rate point which either does or does not lie on the interpolation line between $r^2_{I'_0}$ and $r^3_{I'_0}$ implying that there is no scenario $P'(I_0)$ consistent with the one imposed on g_0 or that the generated interest rate point corresponding to the third price of $S(g_0)$ does lie on the interpolation line between $r^2_{I'_0}$ and $r^3_{I'_0}$. Since $p'(g_0^3)$ only depends on $r^2_{I'_0}$ and $r^3_{I'_0}$ given that there exist a consistent scenario one realize if $R(I'_0) = (r^1_{I'_0}, r^2_{I'_0}, r^3_{I'_0}, r^4_{I'_0})$ is consistent with $S(g_0)$ then so is $R(I^*_0) = \{r^1_{I^*_0}, r^2_{I^*_0}, r^3_{I^*_0}, r^4_{I^*_0}\}$ where $r^3_{I^*_0} = r^3_{I'_0}, r^2_{I^*_0} = r^2_{I'_0}$ and $r^1_{I^*_0}, r^4_{I^*_0}$ is arbitrary. This situation illustrates that the set g_0 has poor design and that at least one generic benchmark should be removed or modified.

Example 2. Unrelated Generic Benchmarks & Pairwise Unrelatedness

Let $g_0 = \{g_0^1, g_0^2\}$ instead consist of two zero coupon bonds with face value 1 SEK and time left to maturity s_k for generic benchmark instrument g_0^k with $(s_1, s_2) = (\frac{4}{12}, \frac{10}{12})$ corresponding to 4 months and 10 months left to maturity. The prices of g_0 are $p(g_0) = (p(g_0^1), p(g_0^2))$ where $p(g_0^1) = e^{-y_{I_0}(s_1)s_1}$, and $p(g_0^2) = e^{-y_{I_0}(s_2)s_2}$. The prices for (g_0^1, g_0^2) is $(0.9967,\ 0.9884)$ and the corresponding interest rates are $\left(y_{I_0}(s_1), y_{I_0}(s_2)\right) = (0.0103,\ 0.0140) = (1.03\%,\ 1.40\%)$. The situation is illustrated below:

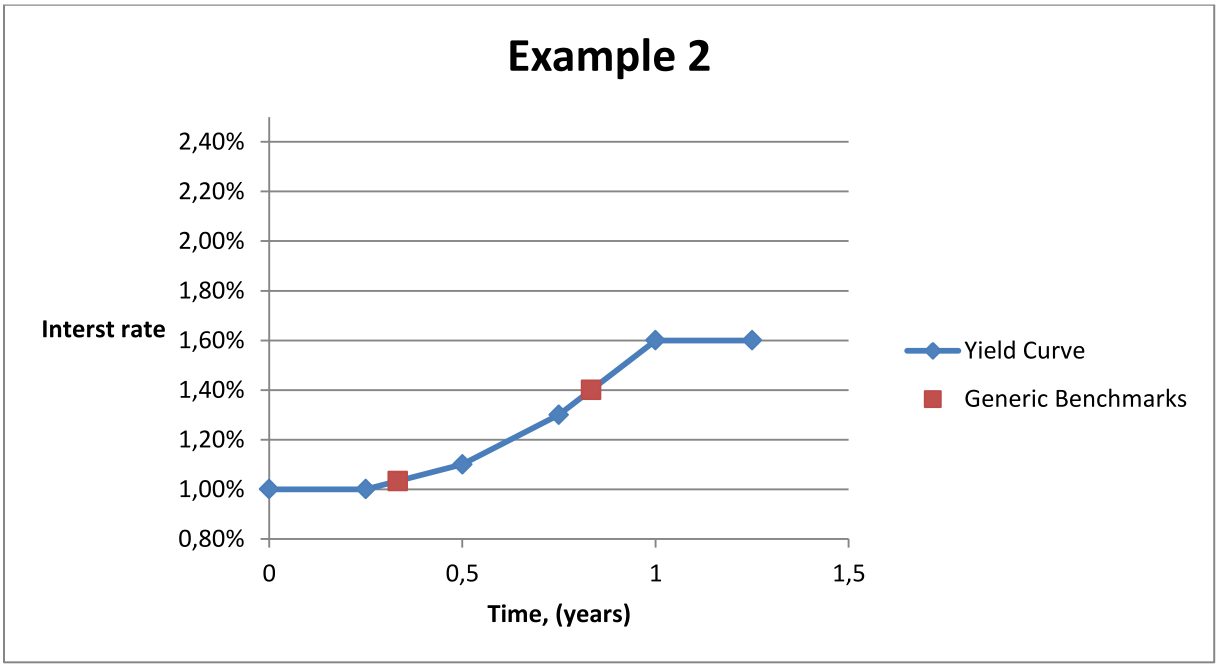

Figure 14. SEK spot curve in the miniature example from hypothetical market data and interest rate points of the two generic benchmark instruments in example 2.

Again, in the same vein, as in Example 1, there will exist infinitely many scenarios $P'(I_0)$ for the prices of I_0 such that $\phi_k(P'(I_0)) = p'\left(g_0^k\right) = price_k$ for any new price scenario, $S(g_0)$ of g_0. Given the prices of $S(g_0)$ the scenarios $P'(I_0)$, $R(I_0')$ of $P(I_0)$ are the interest rate points satisfying:

$$\begin{cases} price_1 = e^{-\left(r_{I_0'}^1 + (s_1 - t_1)\left(\frac{r_{I_0'}^2 - r_{I_0'}^1}{t_2 - t_1}\right)\right)s_1} \\ price_2 = e^{-\left(r_{I_0'}^3 + (s_2 - t_3)\left(\frac{r_{I_0'}^4 - r_{I_0'}^3}{t_4 - t_3}\right)\right)s_2} \end{cases}$$

This shows that $\langle g_0^1 | I_0^3 \rangle = \langle g_0^1 | I_0^4 \rangle = 0$ and $\langle g^1 | I_0^1 \rangle \neq 0, \langle g^1 | I_0^2 \rangle \neq 0$ hence $U(g^1) = \{I_0^3, I_0^4\}$ and $U(g^1)^c = \{I_0^1, I_0^2\}$.

Similarly, this also shows that $\langle g^2 | I_0^1 \rangle = \langle g^2 | I_0^2 \rangle = 0$ and $\langle g^2 | I_0^3 \rangle \neq 0, \langle g^2 | I_0^4 \rangle \neq 0$ hence $U(g^2) = \{I_0^1, I_0^2\}$ and $U(g^2)^c = \{I_0^3, I_0^4\}$.

Since $U(g^1)^c \cap U(g^2)^c = \emptyset$, g^1 and g^2 are unrelated. Since g^1 and g^2 are the only generic benchmark instruments g_0 is pairwise unrelated.

Monotonicity

The price function $\phi_k(P'(I_0)) = p'(g_k)$ is given by $p'(g_k) = e^{-y_{I_0'}(s_i)s_i}$ where $y_{I_0'}(s_i) = r_{I_0'}^k + (s_i - t_k)\left(\frac{r_{I_0'}^{k+1} - r_{I_0'}^k}{t_{k+1} - t_k}\right) \ : \ t_k \leq s_i < t_{k+1}$. Rearranging $y_{I_0'}(s_i)$ gives:

$$y_{I_0'}(s_i) = r_{I_0'}^k + (s_i - t_k) * \left(\frac{r_{I_0'}^{k+1} - r_{I_0'}^k}{t_{k+1} - t_k}\right) = \left(\frac{t_{k+1} - s_i}{t_{k+1} - t_k}\right) r_{I_0'}^k + \left(\frac{s_i - t_k}{t_{k+1} - t_k}\right) r_{I_0'}^{k+1}$$

Since $t_k \leq s_i < t_{k+1} \Rightarrow t_k < t_{k+1} \Rightarrow 0 < t_{k+1} - t_k$ and $0 < t_{k+1} - s_i$ and $0 \leq s_i - t_k$ it is true that $\left(\frac{t_{k+1} - s_i}{t_{k+1} - t_k}\right) > 0$ and $\left(\frac{s_i - t_k}{t_{k+1} - t_k}\right) \geq 0$. This means that $y_{I_0'}(s_i)$ as a function of $r_{I_0'}^{k+1}$ is monotone increasing and likewise $y_{I_0'}(s_i)$ as a function of $r_{I_0'}^k$ is monotone increasing. This implies that $\phi_k(P'(I_0)) = \alpha(R(I_0'))$ is monotone decreasing as function of any of the $r_{I_0'}^k \in I_0'$ since the exponential function is a strictly increasing function.

If a generic benchmark instrument in this miniature environment is coupon paying bond the monotonicity property still holds due to the linearity property of the pricing operator. Let $g_0^1, \dots, g_0^n$ be n distinct zero coupon bond and define g_0^{n+1} to be cash flow equivalent to the sum of $g_0^1, \dots, g_0^n$. Then $\phi_{n+1}(P'(I_0)) = \sum_{a=1}^{n} \phi_a(P'(I_0))$ hence $\phi_{n+1}(P'(I_0))$ is monotone increasing since each $\phi_a(P'(I_0))$ is monotone increasing for $a = 1, \dots, n$.

Example 3. Single Generic Benchmark Scenario

Suppose now that the I_0 remains the same as in Example 1 and 2, but $g_0 = \{g_0^1, g_0^2, g_0^3\}$ consist of three zero coupon bond with face value 1 SEK and maturities $s_1 = \frac{4}{12}, s_2 = \frac{7.5}{12}, s_3 = \frac{11}{12}$ corresponding to 4 months, 7,5 months and 11 months. The prices $p(g_k) = \phi(P(I_0))$ are given by

$\big(\phi_1(P(I_0)), \phi_2(P(I_0)), \phi_3(P(I_0))\big) = (e^{-y_{I_0}(s_i)s_i}, e^{-y_{I_0}(s_i)s_i}, e^{-y_{I_0}(s_i)s_i}) = (0.9932,$ $0.9925,\ 0.9884)$ and the corresponding interest rates are $\big(y_{I_0}(s_1), y_{I_0}(s_2), y_{I_0}(s_3)\big) = (0.0103,\ 0.0120,\ 0.0150) = (1.03\%,\ 1.20\%,\ 1.50\%)$. The situation is illustrated below:

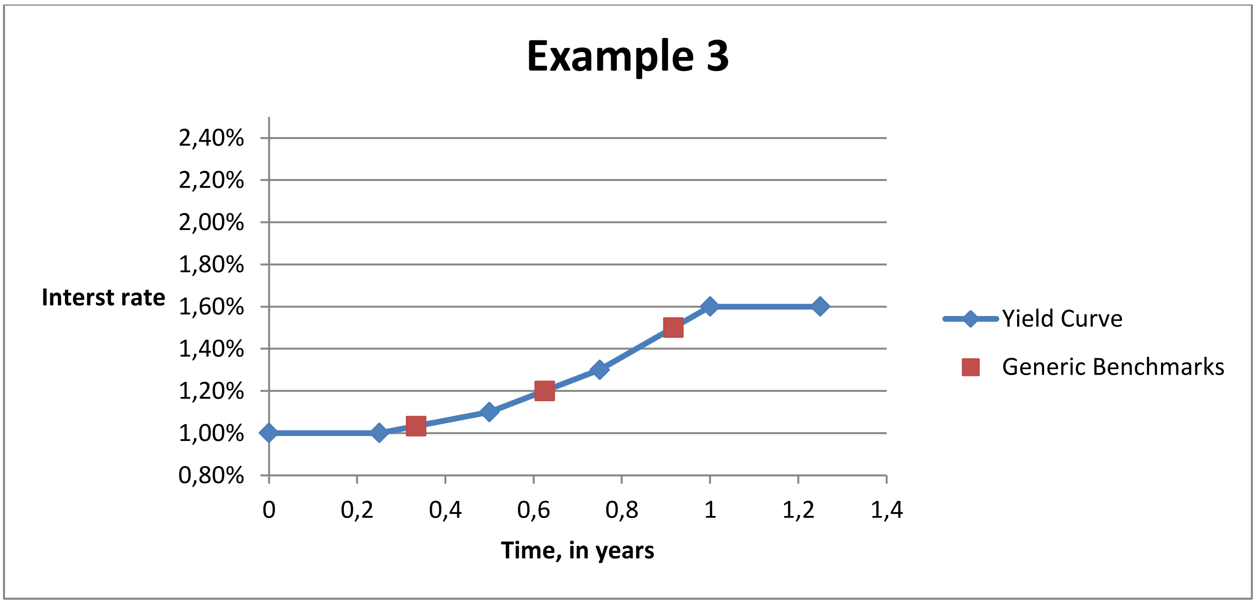

Figure 15. SEK spot curve for example 3, with generic benchmarks marked on the yield curve.

The Scenario is now that interest rate of g_0^2 will increase by 5 basis points from 1.20% to $rate_2 = y_{I_0'}(s_2) = 1.25\%$ which is equivalent to a price drop of g_0^2 to $price_2 =$

$\phi_2\big(P'(I_0)\big) = 0.9922$. No view is held regarding g_0^1 or g_0^3. The correspondence scenario of I_0 can then be expressed as:

$$\min_{R(I_0')} |R(I_0') - R(I_0)|^2$$
$$s.t.\ \ \beta_2(R(I_0'))\ = price_2$$

Since $\alpha(R(I_0'))\ = e^{-y_{I_0'}(s_k)s_k}$, one may solve for $y_{I_0'}(s_2)$ in the equation $e^{-y_{I_0}(s_2)s_2} = p'(g_0^2)$ with $R(I_0') = \left(r_{I_0'}^1, r_{I_0'}^2, r_{I_0'}^3, r_{I_0'}^4\right)^T$ as the unknown and $R(I_0) = \left(r_{I_0}^1, r_{I_0}^2, r_{I_0}^3, r_{I_0}^4\right)^T$. In this example it is obvious that $rate_2 = y_{I_0'}(s_2) = 1.25\%$ since it is the scenario. Hence the optimization problem can be written as

$$\min_{R(I_0')} |R(I_0') - R(I_0)|^2 \tag{1}$$
$$s.t.\ \ y_{I_0'}(s_2) = rate_2$$

More explicitly, using the expression for $y_{I_0'}(s_2)$ from the monotonicity section, this example takes the form:

$$\min_{R(I_0')} |R(I_0') - R(I_0)|^2 \tag{2}$$
$$s.t.\ \ \left(\frac{t_3 - s_2}{t_3 - t_2}\right) r_{I_0'}^2 + \left(\frac{s_2 - t_2}{t_3 - t_2}\right) r_{I_0'}^{3\prime} = rate_2$$

Thus, this optimization problem is nothing more than a quadratic optimization problem in $(r_{I_0'}^1, r_{I_0'}^2, r_{I_0'}^3, r_{I_0'}^4)$ with equality constraints and can be solved easily. Recall that $R(I_0') = \left(r_{I_0'}^1, r_{I_0'}^2, r_{I_0'}^3, r_{I_0'}^4\right)^T$ and $R(I_0) = (r_{I_0}^1, r_{I_0}^2, r_{I_0}^3, r_{I_0}^4)^T$.

Putting $\Delta R = (\Delta R_1, \Delta R_2, \Delta R_3, \Delta R_4)^T = R(I_0') - R(I_0) = \left(r_{I_0'}^1, r_{I_0'}^2, r_{I_0'}^3, r_{I_0'}^4\right)^T - \left(r_{I_0}^1, r_{I_0}^2, r_{I_0}^3, r_{I_0}^4\right)^T$ and rewriting the constraint as

$$\left(\frac{t_3 - s_2}{t_3 - t_2}\right) r_{I_0'}^2 + \left(\frac{s_2 - t_2}{t_3 - t_2}\right) r_{I_0'}^3 = rate_2$$
$$\Leftrightarrow \left(\frac{t_3 - s_2}{t_3 - t_2}\right)\left(r_{I_0'}^2 - r_{I_0}^2\right) + \left(\frac{s_2 - t_2}{t_3 - t_2}\right)(r_{I_0'}^3 - r_{I_0}^3) + \left(\frac{t_3 - s_2}{t_3 - t_2}\right) r_{I_0}^2 + \left(\frac{s_2 - t_2}{t_3 - t_2}\right) r_{I_0}^3 = rate_2$$
$$\Leftrightarrow \left(\frac{t_3 - s_2}{t_3 - t_2}\right)\left(r_{I_0'}^2 - r_{I_0}^2\right) + \left(\frac{s_2 - t_2}{t_3 - t_2}\right)(r_{I_0'}^3 - r_{I_0}^3) + \frac{(t_3 - s_2) r_{I_0}^2 + (s_2 - t_2) r_{I_0}^3}{t_3 - t_2} = rate_2$$

$$\Leftrightarrow \left(\frac{t_3 - s_2}{t_3 - t_2}\right)\left(r_{I_0'}^2 - r_{I_0}^2\right) + \left(\frac{s_2 - t_2}{t_3 - t_2}\right)(r_{I_0'}^3 - r_{I_0}^3) = rate_2 - \frac{(t_3 - s_2)r_{I_0}^2 + (s_2 - t_2)r_{I_0}^3}{t_3 - t_2}$$

$$\Leftrightarrow \left(\frac{t_3 - s_2}{t_3 - t_2}\right)\Delta R_2 + \left(\frac{s_2 - t_2}{t_3 - t_2}\right)\Delta R_3 = rate_2 - \frac{(t_3 - s_2)r_{I_0}^2 + (s_2 - t_2)r_{I_0}^3}{t_3 - t_2}$$

$$\Leftrightarrow \left(\frac{t_3 - s_2}{t_3 - t_2}\right)\Delta R_2 + \left(\frac{s_2 - t_2}{t_3 - t_2}\right)\Delta R_3 = \Delta rate_2,$$

Since $\Delta rate_2 = rate_2 - \frac{(t_3 - s_2)r_{I_0}^2 + (s_2 - t_2)r_{I_0}^3}{t_3 - t_2}$. With $\Delta R = R(I_0') - R(I_0)$ one can instead solve:

$$\min_{\Delta R} |\Delta R|^2 \tag{3}$$

$$s.t \quad \left(\frac{t_3 - s_2}{t_3 - t_2}\right)\Delta R_2 + \left(\frac{s_2 - t_2}{t_3 - t_2}\right)\Delta R_3 = \Delta rate_2$$

It is easy to see that $\Delta R_1 = \Delta R_4 = 0$. Heuristically one would expect that the best way to change ΔR so that $\left(\frac{t_3 - s_2}{t_3 - t_2}\right)\Delta R_2 + \left(\frac{s_2 - t_2}{t_3 - t_2}\right)\Delta R_3 = \Delta rate_2$ but at the same time keeping $|\Delta R|^2$ as small as possible would be to view $\Delta rate_2$ as a function of ΔR and move in the direction of the gradient of this function. The gradient of $\Delta rate_2(\Delta R_1, \Delta R_2, \Delta R_3, \Delta R_4)$ is $(0, \left(\frac{t_3 - s_2}{t_3 - t_2}\right), \left(\frac{s_2 - t_2}{t_3 - t_2}\right), 0)$ the only remaining uncertainty is how far in the direction of the gradient one should move. But that is simply the length α such that:

$$\left(\frac{t_3 - s_2}{t_3 - t_2}\right)\left(\frac{t_3 - s_2}{t_3 - t_2}\right)\alpha + \left(\frac{s_2 - t_2}{t_3 - t_2}\right)\left(\frac{s_2 - t_2}{t_3 - t_2}\right)\alpha = \Delta rate_2 \Rightarrow$$

$$\Rightarrow \alpha = \frac{\Delta rate_2}{\left(\frac{t_3 - s_2}{t_3 - t_2}\right)\left(\frac{t_3 - s_2}{t_3 - t_2}\right) + \left(\frac{s_2 - t_2}{t_3 - t_2}\right)\left(\frac{s_2 - t_2}{t_3 - t_2}\right)}$$

The solution, ΔR_s, would then be:

$$\Delta R_s = \alpha\left(0, \left(\frac{t_3 - s_2}{t_3 - t_2}\right), \left(\frac{s_2 - t_2}{t_3 - t_2}\right), 0\right)^T = \frac{\Delta rate_2}{\left(\frac{t_3 - s_2}{t_3 - t_2}\right)\left(\frac{t_3 - s_2}{t_3 - t_2}\right) + \left(\frac{s_2 - t_2}{t_3 - t_2}\right)\left(\frac{s_2 - t_2}{t_3 - t_2}\right)}\left(0, \left(\frac{t_3 - s_2}{t_3 - t_2}\right), \left(\frac{s_2 - t_2}{t_3 - t_2}\right), 0\right)^T$$

And

$$R(I_0') = R(I_0) + \Delta R$$

This is indeed the correct solution which can be seen mathematically since one has:

$$\min_{\Delta R} |\Delta R|^2$$

$$s.t \quad A_3 \Delta R = \Delta rate_2$$

Where $A_3 = (0, \left(\frac{t_3 - s_2}{t_3 - t_2}\right), \left(\frac{s_2 - t_2}{t_3 - t_2}\right), 0)$ and coincides with the gradient. Let ΔR_{s2} be any other solution satisfying $A_3 \Delta R_{s2} = \Delta rate_2$ then

$0 = \Delta rate_2 - \Delta rate_2 = A_3 \Delta R_{s2} - A_3 \Delta R_s = A_3(\Delta R_{s2} - \Delta R_s)$ so $(\Delta R_{s2} - \Delta R_s) \in \ker(A_3)$ i.e. $(\Delta R_{s2} - \Delta R_s)$ belongs to the kernel of A_3. Define $x_k = (\Delta R_{s2} - \Delta R_s)$ then $\Delta R_s^T x_k = \alpha A_3 x_k = 0$ so $\Delta R_s \perp x_k$. Hence for any solution ΔR_{s2}:

$$|\Delta R_{s2}|^2 = |\Delta R_s + x_k|^2 = \{pythagorian\ theorem, \Delta R_s \perp x_k\} = |\Delta R_s|^2 + |x_k|^2 \geq |\Delta R_s|^2,$$

with equality if and only if $x_k = 0$. That is, the solution ΔR_s is optimal and unique.

Solving for ΔR_s, gives the numerical values (0%, 0.05%, 0.05%, 0%) and consequently $R(I_0') = (1.0\%, 1.15\%, 1.35\%, 1.6\%))$ recall that $R(I_0) = (1.0\%, 1.1\%, 1.3\%, 1.6\%)$.

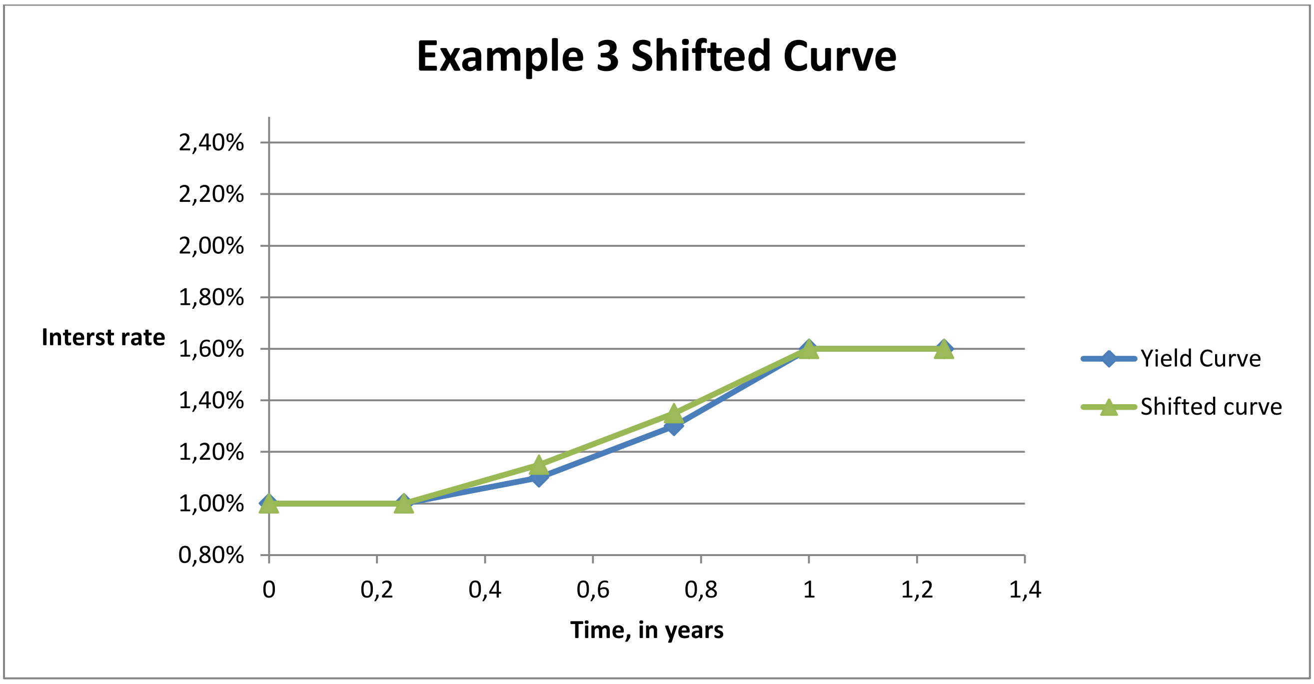

Figure 15. The two SEK spot curves plotted together the upper one in green corresponds to the scenario I_0' and the blue lower curve corresponds to the true market data, I_0.

One can see from the above derivation that the gradient of $\Delta rate_k(\Delta R_1, \Delta R_2, \Delta R_3, \Delta R_4)$ as a function of $\Delta R_1, \Delta R_2, \Delta R_3, \Delta R_4$ will determine the relative shifts in the interest rate points of I_0 . In the above case where the gradient of $\Delta rate_2(\Delta R_1, \Delta R_2, \Delta R_3, \Delta R_4)$ is $(0, \left(\frac{t_3 - s_2}{t_3 - t_2}\right), \left(\frac{s_2 - t_2}{t_3 - t_2}\right), 0)$ one can see that $r_{I_0}^1$ and $r_{I_0}^4$ will not be shifted at all relative to $r_{I_0'}^1$ and $r_{I_0'}^4$, and that $r_{I_0}^2$ and $r_{I_0}^3$ will be shifted the same amount relative to $r_{I_0'}^2$ and $r_{I_0'}^3$. This is independent of the size of $rate_2$. With $rate_2 = 1.50\%$ instead of $rate_2 = 1.25\%$ one would instead get the $R(I_0')$ shown in the figure below.

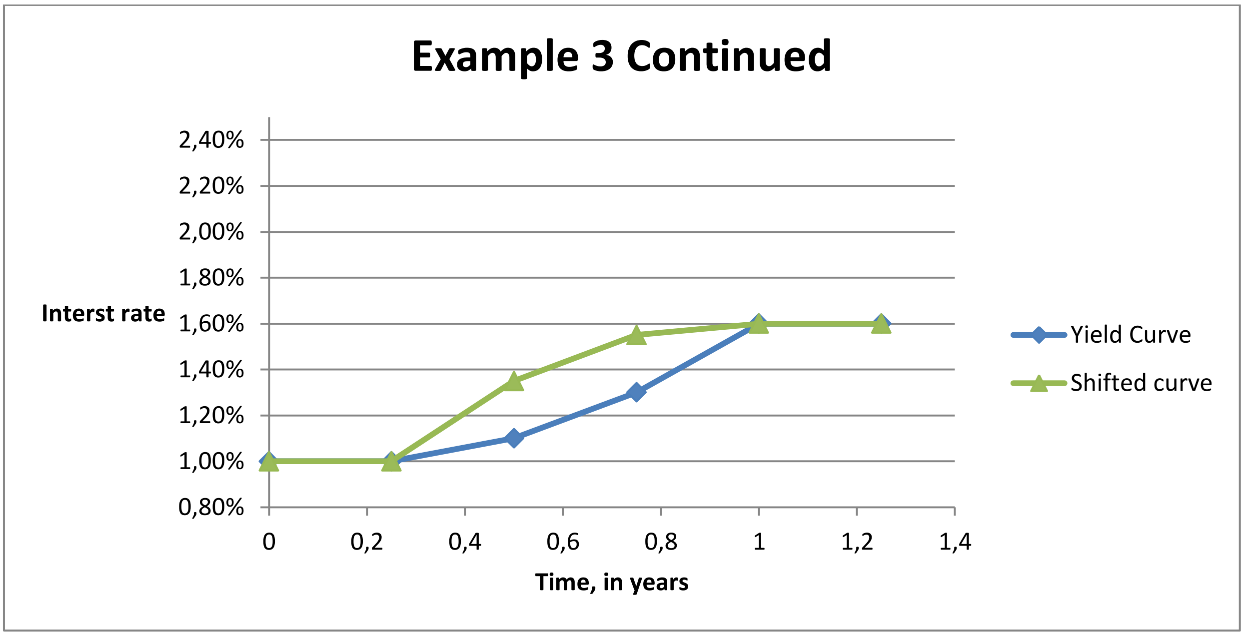

Figure 16. the plot is the same as figure 15 with the only difference that $y_{I_0}(s_2)$ was believed to rise by 25 basis points, i.e considerably more than the first scenario for g_0^2 where $y_{I_0}(s_2)$ was believed to rise with 5 basis points.

Hence g_0^2 is unrelated to I_0^1 and I_0^4, so that there is no shift in the interest rate points of I_0^1 and I_0^4 should not come as a surprise. The reason the generic benchmark instrument g_0^2 shift $r_{I_0}^2$ and $r_{I_0}^3$ to $r_{I_0'}^2$ and $r_{I_0'}^3$ by the same amount is due to the fact that the maturity of g_0^2 lies just between the maturity of I_0^2 and I_0^3 i.e. $t_3 - s_2 = s_2 - t_2$, making the corresponding gradient components equal. Thus if one had set a scenario for only g_0^3 then the interest rate points of I_0^1 and I_0^2 would have remained the same by unrelatedness and the interest rate point $r_{I_0}^3$ and $r_{I_0}^4$ had been shifted. But since $t_4 - s_3 \neq s_3 - t_3$ they would not be shifted by the same amount. In fact $0 \neq t_4 - s_3 < s_3 - t_3$ so $r_{I_0}^4$ would be shifted more than $r_{I_0}^3$ by a factor of $\frac{s_3 - t_3}{t_4 - s_3}$.

Example 4. Multiple Generic Benchmark Scenarios

This section considers a belief regarding several generic benchmark instruments of g_0. Here the belief regarding two generic benchmark instruments is considered but the example is easily extended to cover the case of more than two generic benchmark instruments.

The situation is the same as in Example 3 with four market instrument and three generic benchmark instruments but the belief is that the interest rates will rise faster than reflected in the spot curve, perhaps due to stronger economic recovery. The scenario is that interest rate of g_0^2 will increase by 10 basis points from $r_{I_0}^2 = 1.2\%$ to $r_{I_0'}^2 = rate_2 = 1.3\%$ and that the interest rate of g_0^3 will increase by 20 basis points from $r_{I_0}^3 = 1.5\%$ to $r_{I_0'}^3 = rate_3 = 1.7\%$. The corresponding price drop is from 0.9925 to 0.9919 for g_0^2 and from 0.9863 to 0.9845 for g_0^3.

As in Example 3 this is formulated as the optimization problem:

$$\min_{\mathrm{R}(I_0')} |R(I_0') - R(I_0)|^2 \tag{4}$$

$$s.t \quad \begin{cases} \beta_2(R(I_0')) = price_2 \\ \beta_3(R(I_0')) = price_3 \end{cases}$$

Just as in Example 3, putting $\Delta R = R(I_0') - R(I_0)$ and using $y_{I_0}(s_2)$, the problem can be reformulated as

$$\min_{\Delta R} |\Delta R|^2$$

$$s.t \quad \begin{cases} \left(\dfrac{t_3 - s_2}{t_3 - t_2}\right)\Delta R_2 + \left(\dfrac{s_2 - t_2}{t_3 - t_2}\right)\Delta R_3 = \Delta rate_2 \\ \left(\dfrac{t_4 - s_3}{t_4 - t_3}\right)\Delta R_3 + \left(\dfrac{s_3 - t_3}{t_4 - t_3}\right)\Delta R_4 = \Delta rate_3 \end{cases}$$

Where $\Delta rate_2 = rate_2 - \frac{(t_3-s_2)r_{I_0}^2+(s_2-t_2)r_{I_0}^3}{t_3-t_2}$ and $\Delta rate_3 = rate_3 - \frac{(t_4-s_3)r_{I_0}^3+(s_3-t_3)r_{I_0}^4}{t_4-t_3}$. Putting

$$\Delta r = (\Delta r_2, \Delta r_3)^T = (\Delta rate_2, \Delta rate_3)^T$$

and

$$B = \begin{bmatrix} 0 & \left(\frac{t_3 - s_2}{t_3 - t_2}\right) & \left(\frac{s_2 - t_2}{t_3 - t_2}\right) & 0 \\ 0 & 0 & \left(\frac{t_4 - s_3}{t_4 - t_3}\right) & \left(\frac{s_3 - t_3}{t_4 - t_3}\right) \end{bmatrix}$$

the problem can thus succinctly be stated as:

$$\begin{gathered} \min_{\Delta R} |\Delta R|^2 \\ s.t.\ \ B\Delta R = \Delta r. \end{gathered} \tag{5}$$

This formulation can be stated for an arbitrary number of the g_0, that one has formed beliefs regarding. Furthermore, the optimization problem (5) is nothing more than a convex optimization problem with equality constraints. The rows of B are linearly independent as might be suspected since g_0^3 is unrelated to I_0^2 but g_0^2 is not. Using the Lagrange multiplier, $\lambda = (\lambda_1, \lambda_2)^T$, to solve (5) gives:

$$\begin{cases} 2\Delta R + B^T\lambda = 0 \\ B\Delta R = \Delta r \end{cases}$$

The first equation gives: $\Delta R = -0.5B^T\lambda$, and inserting this into the second equation yields:

$$B(-0.5B^T\lambda) = \Delta r \Leftrightarrow -0.5BB^T\lambda = \Delta r \Leftrightarrow \lambda = -2(BB^T)^{-1}\Delta r$$

BB^T is invertible since $range(B) = range(BB^T)$ and $rank(B) = rank(B^T) = 2$. Now inserting that $\lambda = -2(BB^T)^{-1}\Delta r$ into the first equation gives that:

$$\Delta R = B^T(BB^T)^{-1}\Delta r \tag{6}$$

Recall that $s_2 = \frac{7.5}{12}, s_3 = \frac{11}{12}, t_2 = \frac{6}{12}, t_3 = \frac{9}{12}, t_4 = \frac{12}{12}$ and inserting the numerical values for B gives:

$$B = \begin{bmatrix} 0 & \frac{1.5}{3} & \frac{1.5}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix},$$

And with $r_{I_0}^2 = 1.1\%,\ r_{I_0}^3 = 1.3\%,\ r_{I_0}^4 = 1.6\%,\ rate_2 = 1.3\%,\ rate_3 = 1.7\%$

$$\Delta r = \begin{bmatrix} \Delta rate_2 \\ \Delta rate_3 \end{bmatrix} = \begin{bmatrix} rate_2 - \dfrac{(t_3 - s_2)r_{I_0}^2 + (s_2 - t_2)r_{I_0}^3}{t_3 - t_2} \\ rate_3 - \dfrac{(t_4 - s_3)r_{I_0}^3 + (s_3 - t_3)r_{I_0}^4}{t_4 - t_3} \end{bmatrix} = \begin{bmatrix} 0.1\% \\ 0.2\% \end{bmatrix} = \begin{bmatrix} 0.001 \\ 0.002 \end{bmatrix}.$$

This is no coincidence since, $y_{I_0}(s_i) = \left(\frac{t_{k+1} - s_i}{t_{k+1} - t_k}\right) r_{I_0}^k + \left(\frac{s_i - t_k}{t_{k+1} - t_k}\right) r_{I_0}^{k+1}$. Hence any $\Delta rate_k$ will simply be the change in interest rate of the generic benchmark instrument from the level given by $y_{I_0}(s_k)$. $\Delta rate_k = rate_k - y_{I_0}(s_k)$. If the scenario is stated as a new price of g_0^k then the change in interest rate needs to be calculated otherwise $rate_k$ will be explicitly given in the scenario assumption and therefore $\Delta rate_k$ is easily calculated.

The numerical values of Δr and B gives in this example ΔR equal to, using matlab to solve(6),

$$\Delta R = \begin{bmatrix} 0 \\ 0.0004 \\ 0.0016 \\ 0.0022 \end{bmatrix} = \begin{bmatrix} 0\% \\ 0.04\% \\ 0.16\% \\ 0.22\% \end{bmatrix}$$

Giving:

$$R(I_0') = R(I_0) + \Delta R = \begin{bmatrix} 1.0\% \\ 1.1\% \\ 1.3\% \\ 1.6\% \end{bmatrix} + \begin{bmatrix} 0\% \\ 0.04\% \\ 0.16\% \\ 0.22\% \end{bmatrix} = \begin{bmatrix} 1.0\% \\ 1.14\% \\ 1.46\% \\ 1.86\% \end{bmatrix}$$

The figure below illustrates this new scenario.

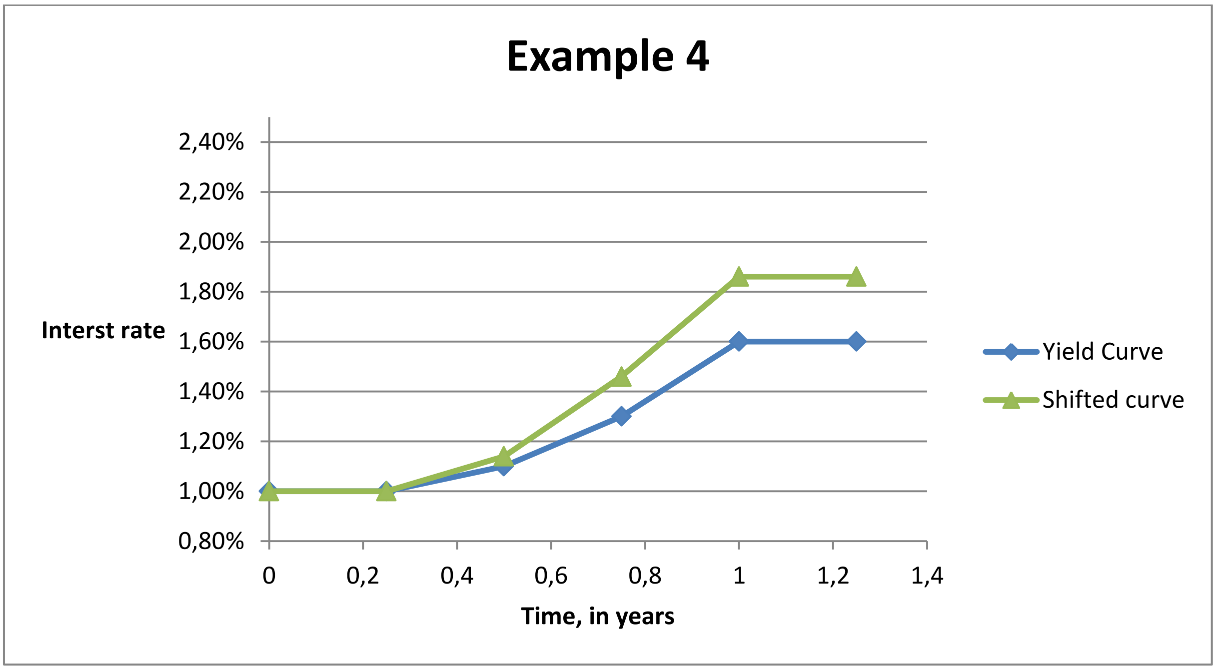

Figure 17. The upper green curve is the generated scenario, I_0', and the lower blue curve is the market curve, I_0.

Computational aspects

Equation (6) holds more generally for more than two constraints and any structure of I_0. However for large environments and with possible hierarchies of different yield curves computational effort may be a factor of greater importance than in this simple environment and with only two views of the generic benchmarks in Example 4. When using the Lagrange multiplier $(BB^T)^{-1}$ may be demanding to compute for larger environments, i.e., many columns in B and more than two views on the generic benchmarks that is more than two rows in B. Therefore, ΔR would be found more effectively by solving $B\Delta R = \Delta r$ such that $|\Delta R|^2$ is minimized. Solving (5) can be done more effectively by for example QR-factorization.

Furthermore, the problem of solving (4) can be made simpler by paying attention to the structure of the matrix B. It is no surprise that $\Delta R_1 = 0$ in Example 4, since both g_0^2 and g_0^3 are unrelated to I_0^1. The unrelatedness property implies that $\frac{\partial \beta_k\left(R(I_0')\right)}{\partial r_{I_0'}^1} = 0$ for $k = 2,3$ in Example 4 and also that ΔR_1 can not affect the price of g_0^2 or g_0^3. Therefore the first column of B is zero in Example 4. Similarly column 1 and column 4 in Example 3 are zero due to the fact that g_0^2 is unrelated to I_0^1 and I_0^4. Consequently with n generic benchmark instruments the set of instruments $E = \bigcap_{k=1}^{n} U\left(g_0^k\right)$will not change an can thus be excluded from (5) which reduces the dimension of B. Any ΔR_k such that I_0^k is in the exclusion set E will thus be equal to zero, $\Delta R_k = 0 : I_0^k \in E$.

By working in the formulation (5) with interest rate instead of prices as in (4), the $\Delta rate_k$ as a function of ΔR became linear in the arguments of ΔR meaning that the gradient of $\Delta rate_k$ is constant and independent of the particular value I_0. Implying that numerical approximations of the partial derivatives are accurate and that numerical solutions to (4) such as Newton-Raphson converges to the true solution given that $\Delta r \in range(B)$.

6. THE MINIATURE ENVIRONMENT EXAMPLE WITH CUBIC SPLINE INTERPOLATION

The examples in Section 5 will now be generalized to the situation where the more realistic interpolation technique called cubic spline is utilized. The curve instruments, I_0, are the same as in Section 5 but the interpolation technique is changed. This means that the interest rate curve is now given by:

$$y_{I_0}(s) = \begin{cases} a_0^{(k)} + a_1^{(k)}s + a_2^{(k)}s^2 + a_3^{(k)}s^3 \quad if\ t_k \leq s < t_{k+1} & : k \in \{1,2,3\} \\ a_0^{(3)} + a_1^{(3)}s + a_2^{(3)}s^2 + a_3^{(3)}s^3 \qquad if \quad s = t_4 \end{cases}$$

The extrapolation method can be chosen arbitrarily in this example since there will be no benchmarks depending on the interest rate for maturities greater than t_4 or smaller than t_1. The coefficients

$$a_0^{(k)}, a_1^{(k)}, a_2^{(k)}, a_3^{(k)} \ : k \in \{1,2,3\},$$

are chosen such that:

$$r_{I_0}^k = a_0^{(k)} + a_1^{(k)}t_k + a_2^{(k)}t_k^2 + a_3^{(k)}t_k^3 = y_{I_0}(t_k),$$
$$r_{I_0}^{k+1} = a_0^{(k)} + a_1^{(k)}t_{k+1} + a_2^{(k)}t_{k+1}^2 + a_3^{(k)}t_{k+1}^3 = y_{I_0}(t_{k+1}),$$

for $k \in \{1,2,3\}$, and such that the first and second derivative of $y_{I_0}(s)$ exists and are continuous on the interval (t_1, t_4), in accordance with the design of cubic splines. Furthermore, the second derivative at the end points are constrained to be zero, which gives the following system of linear equations:

$$Ta = Y, \tag{7}$$

where

$$T=\begin{bmatrix}1 & t_1 & t_1^2 & t_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & t_2 & t_2^2 & t_2^3 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_3 & t_3^2 & t_3^3\\ 1 & t_2 & t_2^2 & t_2^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & t_3 & t_3^2 & t_3^3 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_4 & t_4^2 & t_4^3\\ 0 & 1 & 2t_2 & 3t_2^2 & 0 & -1 & -2t_2 & -3t_2^2 & 0 & 0 & 0 & 0\\ 0 & 0 & 2 & 6t_2 & 0 & 0 & -2 & -6t_2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 2t_3 & 3t_3^2 & 0 & -1 & -2t_3 & -3t_3^2\\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6t_3 & 0 & 0 & -2 & -6t_3\\ 0 & 0 & 2 & 6t_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6t_4\end{bmatrix},$$

$$a=\left(a_0^{(1)},\ a_1^{(1)},\ a_2^{(1)},\ a_3^{(1)},\ a_0^{(2)},\ a_1^{(2)},\ a_2^{(2)},\ a_3^{(2)}, a_0^{(3)},\ a_1^{(3)},\ a_2^{(3)},\ a_3^{(3)}\right)^T,$$

$$Y=\left(r_{I_0}^1,\ r_{I_0}^2,\ r_{I_0}^3,\ r_{I_0}^2,\ r_{I_0}^3,\ r_{I_0}^4, 0,0,0,0,0,0\right)^T.$$

For subsequent use, partition $T=\begin{bmatrix}T_u\\ T_d\end{bmatrix}$, where T_u contains the first six rows, and T_d contains the last six row. Note that T has full row rank if and only if $\neg(t_1=t_2=t_3=t_4)$, which is the case in this example.

$$T_d=\begin{bmatrix}0 & 1 & 2t_2 & 3t_2^2 & 0 & -1 & -2t_2 & -3t_2^2 & 0 & 0 & 0 & 0\\ 0 & 0 & 2 & 6t_2 & 0 & 0 & -2 & -6t_2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 2t_3 & 3t_3^2 & 0 & -1 & -2t_3 & -3t_3^2\\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6t_3 & 0 & 0 & -2 & -6t_3\\ 0 & 0 & 2 & 6t_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6t_4\end{bmatrix}$$

Let t denote the matrix

$$t=\begin{bmatrix}1 & t_1 & t_1^2 & t_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & t_2 & t_2^2 & t_2^3 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_3 & t_3^2 & t_3^3\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_4 & t_4^2 & t_4^3\end{bmatrix}$$

and let s denote the matrix

$$s = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & s_2 & s_2^2 & s_2^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & s_3 & s_3^2 & s_3^3 \end{bmatrix}$$

Let´s now return to the setup of Example 4, meaning that the same I_0 and g_0 is used. Given that the same scenarios for the generic benchmark instruments as in Example 4, then the only difference is that cubic splines are utilized instead of linear interpolation. The problem is thus to find the correspondence scenario. The general formulation of the problem is again:

$$\min_{\mathrm{R}(I_0')} |R(I_0') - R(I_0)|^2 , \tag{8}$$
$$s.t. \ \begin{cases} \beta_2(R(I_0')) = price_2, \\ \beta_3(R(I_0')) = price_3. \end{cases}$$

But it will not take the form of (5) since the interpolation technique has been changed. Since $ta = R(I_0)$ and letting

$a' = \left(a_0^{(1)\prime},\, a_1^{(1)\prime},\, a_2^{(1)\prime},\, a_3^{(1)\prime},\, a_0^{(2)\prime},\, a_1^{(2)\prime},\, a_2^{(2)\prime},\, a_3^{(2)\prime}, a_0^{(3)\prime},\, a_1^{(3)\prime},\, a_2^{(3)\prime},\, a_3^{(3)\prime}\right)^T$ be the coefficient as in a but with the values from the spline interpolated curve generated by $R(I_0')$ it gives that $ta' = R(I_0')$. Similarly put $Y' = \left(r_{I_0'}^1,\, r_{I_0'}^2,\, r_{I_0'}^3,\, r_{I_0'}^2,\, r_{I_0'}^3,\, r_{I_0'}^4, 0, 0, 0, 0, 0, 0\right)^T$. The constraints of the new prices are converted to their equivalent form in the form of interest rates as in (1) and can then be written as $sa' = rate$ where $ratet = (rate_2, rate_3)^T$. Therefore one gets

$$\min_{\mathrm{R}(I_0')} |R(I_0') - R(I_0)|^2,$$
$$s.t. \ \{sa' = rate.$$

There is a one to one mapping between a' and $R(I_0')$ by (7).

But this is not enough since one needs to make sure that a' actually is a cubic spline interpolation to the interest rates, $R(I_0') = ta'$. Which means that one must have $Ta' = Y'$ where $Y' = \left(r_{I_0'}^1,\, r_{I_0'}^2,\, r_{I_0'}^3,\, r_{I_0'}^2,\, r_{I_0'}^3,\, r_{I_0'}^4, 0, 0, 0, 0, 0, 0\right)^T$. It is obvious that

$$T_u a' = \left(r_{I_0'}^1,\, r_{I_0'}^2,\, r_{I_0'}^3,\, r_{I_0'}^2,\, r_{I_0'}^3,\, r_{I_0'}^4\right)^T,$$

for any a', by definition. The constraint means that $T_d a' = \mathbf{0}$ where $\mathbf{0}$ denotes the zero vector with dimension 6. Thus the problem formulation is:

$$\min_{\mathrm{R}(I_0')} |R(I_0') - R(I_0)|^2, \tag{9}$$
$$s.t. \begin{cases} sa' = rate, \\ T_d a' = \mathbf{0}. \end{cases}$$

The formulation (9) can be simplified to ease the task of finding the solution a_s' and hence the correspondence scenario. Starting with the objective function one puts, as in Section 5:

$$\Delta R = R(I_0') - R(I_0).$$

Moreover, sa, are the interest rates of the generic benchmark instruments given by $P(I_0)$ and

$$sa' = rate \iff sa' - sa = rate - sa \iff s(a' - a) = \Delta rate,$$

where $\Delta rate = rate - sa$. $\Delta rate$ is simply the believed increase or decrease of the generic benchmark instruments for which the scenario specifies a belief.

With

$$T_d a' = 0 \iff T_d a' - T_d a + T_d a = 0 \iff T_d(a' - a) = -T_d a = \mathbf{0},$$

putting $\Delta a = (\Delta a_1, \ldots, \Delta a_{12}) = a' - a$ one has

$$\min_{\Delta R} |\Delta R|^2, \tag{10}$$
$$s.t. \begin{cases} s\Delta a = \Delta rate, \\ T_d \Delta a = \mathbf{0}. \end{cases}$$

But since there is a one to one mapping between a' and $R(I_0')$ there is also a one to one mapping between Δa and ΔR:

$$Ta' = Y' \iff T\Delta a = T(a' - a) = Ta' - Ta = Y' - Y = WR(I_0') - WR(I_0) = W\Delta R \iff$$
$$\iff \Delta a = T^{-1} W \Delta R,$$

where W is defined as the full column rank matrix:

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Inserting $\Delta a = T^{-1}W\Delta R$ into (10) gives the optimization problem:

$$\min_{\Delta R} \ |\Delta R|^2, \tag{11}$$
$$s.t. \ \ \begin{bmatrix} S \\ T_d \end{bmatrix} T^{-1}W\Delta R = \begin{bmatrix} \Delta rate \\ \mathbf{0} \end{bmatrix},$$

T and T^{-1} are square matrixes with full rank and it is easy to see that W has full column rank. However, $\begin{bmatrix} S \\ T_d \end{bmatrix}$ does not have full row rank, the first column is in fact the zero vector. This implies that $\begin{bmatrix} S \\ T_d \end{bmatrix} T^{-1}W$ does not have full column rank. In fact, $rank\left(\begin{bmatrix} S \\ T_d \end{bmatrix} T^{-1}W\right) \neq 4$, $\begin{bmatrix} \Delta rate \\ \mathbf{0} \end{bmatrix} \in range(\begin{bmatrix} S \\ T_d \end{bmatrix} T^{-1}W)$ so there are infinitely many feasible solutions to $\begin{bmatrix} S \\ T_d \end{bmatrix} T^{-1}W\Delta R = \begin{bmatrix} \Delta rate \\ \mathbf{0} \end{bmatrix}$. The least norm solution to an underdetermined system of linear equations is an easy matter. Let $A_6 = \begin{bmatrix} S \\ T_d \end{bmatrix} T^{-1}W$ and $b_6 = \begin{bmatrix} \Delta rate \\ \mathbf{0} \end{bmatrix}$, then

$$A_6\Delta R = b, \tag{12}$$

and the columns of A_6 are linearly dependent. If ΔR_p is a solution to (11) then so is ΔR_k for any $\Delta R_k \in kernel(A_6)$, since any vector ΔR_p can be decomposed into $\Delta R_p = \Delta R_{k\perp} + \Delta R_k$ where $\Delta R_k \in kernel(A_6)$ and $\Delta R_{k\perp} \in \ kernel(A_6)^{\perp}$. Here $kernel(A_6)^{\perp}$ denotes the orthogonal complement of $kernel(A_6)$. It is clear that $\Delta R_{k\perp}$ is a solution to (12) since

$$b = A_6(\Delta R_{k\perp} + \Delta R_k) = A_6\Delta R_{k\perp} + A_6\Delta R_k = A_6\Delta R_{k\perp}.$$

The solution $\Delta R_{k\perp}$ is independent of ΔR_p and is the minimum norm solution to (12). Let $\Delta R_{p'}$ be any other solution to (12) then

$$0 = b - b = A_6 \Delta R_{p'} - A_6 \Delta R_{k\perp} = A_6\left(\Delta R_{p'} - \Delta R_{k\perp}\right).$$

Hence $\Delta R_{p'} - \Delta R_{k\perp} \in kernel(A_6)$ and by defining $\Delta R_{k'} = \Delta R_{p'} - \Delta R_{k\perp}$ one gets that $\Delta R_{p'} = \Delta R_{k\perp} + \mathit{\Delta} R_{k'}$. Hence any solution $\Delta R_{p'} = \Delta R_{k\perp} + \mathit{\Delta} R_{k'}$ for some $\mathit{\Delta} R_{k'} \in kernel(A_6)$.

It follows that

$$\left|\Delta R_{p'}\right|^2 = |\Delta R_{k\perp} + \mathit{\Delta} R_{k'}|^2 = |\Delta R_{k\perp}|^2 + |\mathit{\Delta} R_{k'}|^2 \geq |\Delta R_{k\perp}|^2,$$

with equality if and only if $|\Delta R_{k'}|^2 = 0 \Leftrightarrow \Delta R_{k'} = 0$. Since $\Delta R_{k\perp} \in kernel(A_6)^{\perp}$ and $kernel(A_6)^{\perp} = Range(A_6^T)$ there exist a vector, w, such that $\Delta R_{k\perp} = A_6^T w$. The vector w may not be unique but $\Delta R_{k\perp}$ is in fact w will not be unique since A_6^T does not have full column rank. Together with (12) this gives:

$$A_6 A_6^T w = b. \qquad (13)$$

Note that $det(A_6 A_6^T) = 0$ due to linearly dependent columns of A_6^T so one cannot invert $A_6 A_6^T$ which would yield the solution $w = (A_6 A_6^T)^{-1} b \Rightarrow \Delta R_{k\perp} = A_6^T (A_6 A_6^T)^{-1} b$.

Solving (13) for any solution w_s such that $A_6 A_6^T w_s = b$ gives $\Delta R_{k\perp} = A_6^T w_s$.

The above solution gives the numerical values for $\Delta R_{k\perp}$ as

$$\Delta R_{k\perp} = (0.0009\%, \quad 0.0031\%, \quad 0.1935\%, \quad 0.1726\%)^T,$$

and

$$R(I_0') = R(I_0) + \Delta R_{k\perp} = (1.0009\%, \quad 1.1031\%, \quad 1.4935\%, \quad 1.7726\%)^T$$

The spine interpolated yield curve is illustrated below.

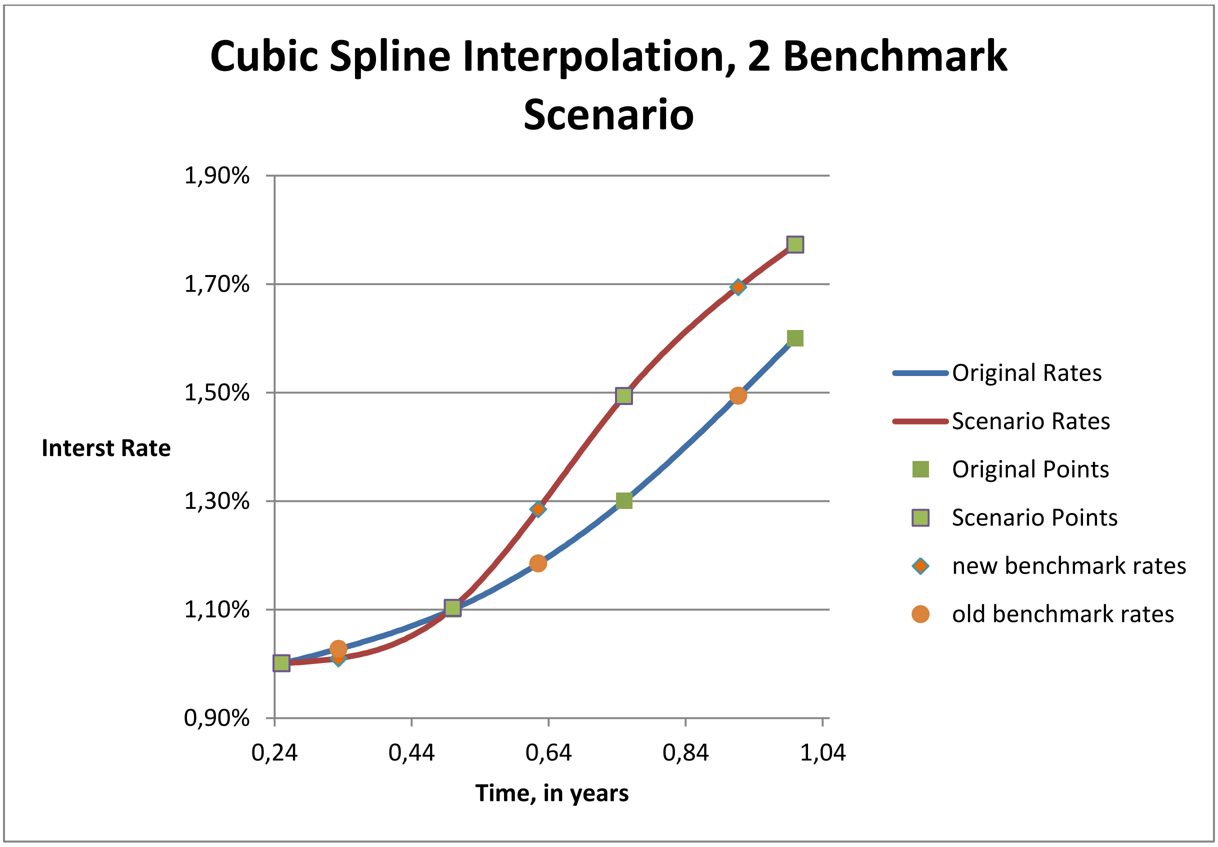

Figure 18. The new yield curve given the views of benchmark instrument 2 and 3 compared with the initial yield curve. Benchmark 2 was believed to go up 10 basis points and benchmark 3 was believed to go up 20 basis points.

One can see in the above plot that $r_{I_0}^3$ and $r_{I_0}^4$ is affected the most and that $r_{I_0}^1$ and $r_{I_0}^2$ is affected less which is hard to detect visually the magnitude is found in the vector.

Let us view another situation with cubic splines, call it scenario 2 and the above scenario scenario 1. The scenario given is that the interest rate of g_0^2 will increase by 10 basis points from 1.2% to $rate_2 = 1.30\%$ and no view is held regarding g_0^1 or g_0^3. The formulation of (11) still holds generally with the only difference that s and $\Delta rate$ will be

$$s = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad s_2 \quad s_2^2 \quad s_2^3 \quad 0 \quad 0 \quad 0 \quad 0\,],$$

$$\Delta rate = [0.1\%].$$

Numerically one gets

$$R(I_0') = R(I_0) + \Delta R_{k\perp} = (0.989\%, \quad 1.186\%, \quad 1.386\%, \quad 1.589\%)^T.$$

The new situation is illustrated below:

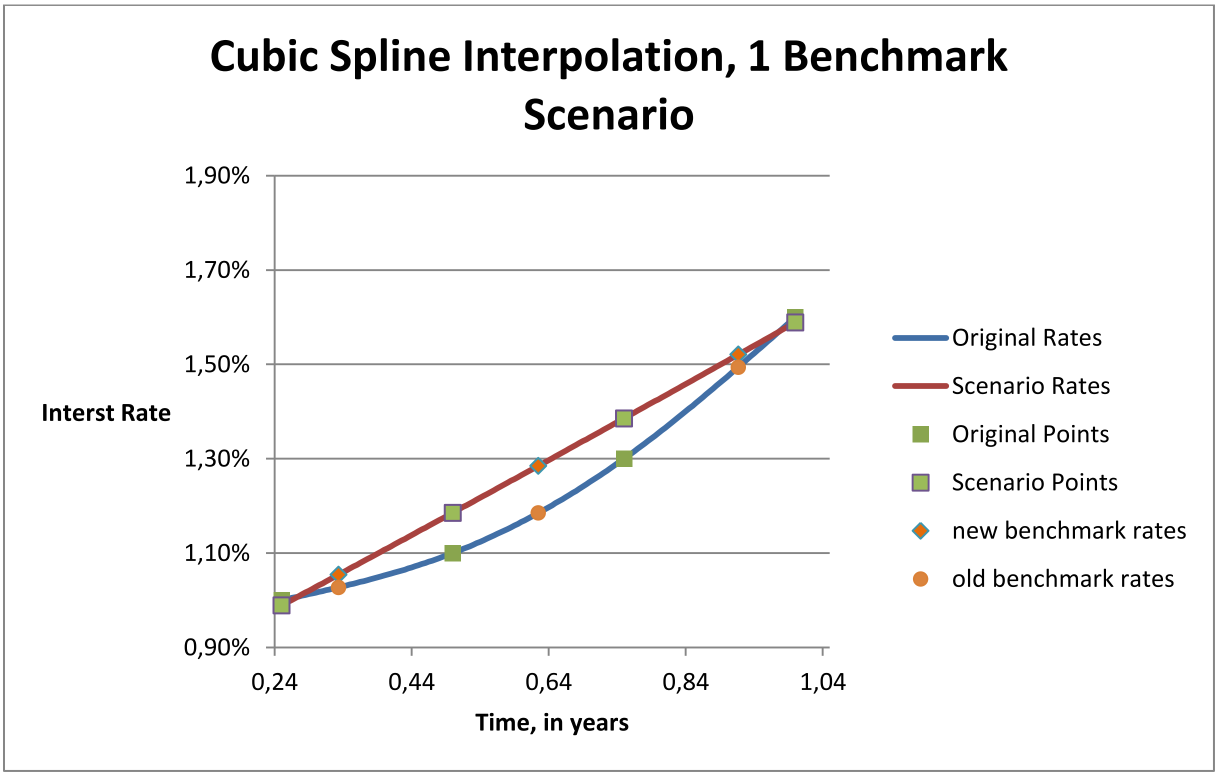

Figure 19. The scenario that g_0^2 increases by 10 basis points and no view is held regarding g_0^1 or g_0^3.

One can see in the above plot that the correspondence scenario is not the same as when a view was held regarding both g_0^2 and g_0^3 even though the view regarding g_0^2 is the same in both scenarios. This is natural since scenario 2 places no restrictions on g_0^3. Thus the correspondence scenario is generated by a total squared shift in the yield curve which is less than the squared shift of scenario 1. Furthermore one realizes by this line of thought that a scenario, call it scenario 3, where g_0^2 is believed to increase by 10 basis points and g_0^3 is believed to stay at the same interest rate is not the same as scenario 2. This can be seen from Figure 19 where the interest rate of g_0^3 has in fact gone up somewhat. However scenario 3 would force g_0^3 to stay at an interest rate of 1.5%. The plot corresponding to scenario 3 is shown below and the numerical values for the correspondence scenario is

$$R(I_0') = R(I_0) + \Delta R_{k\perp} = (0.987\%, \quad 1.198\%, \quad 1.369\%, \quad 1.560\%)^T.$$

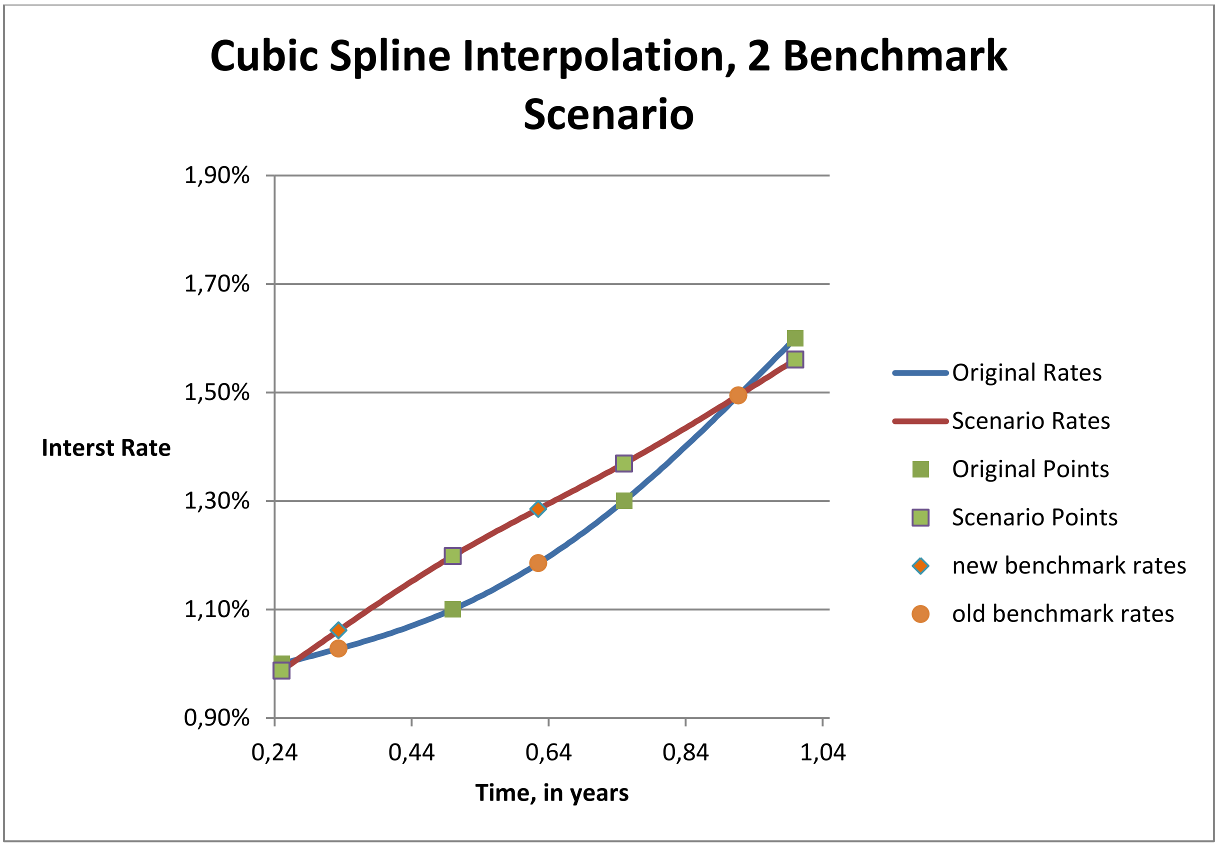

Figure 20. The scenario that g_0^2 increases by 10 basis points and that the interest rate of g_0^3 stay the same no view is held regarding g_0^1.

It is clear from the derivation of (11) that there is no pair of generic benchmarks that are unrelated to each other. This is exposed in the numeric of the above scenarios. Recall Example 3 in the section with linear interpolation. Employing the same scenario, call it scenario 4 where the interest rate of g_0^2 increases by 25 basis points, in the context of cubic spline interpolation yields the numeric:

$$R(I_0') = R(I_0) + \Delta R_{k\perp} = (0.972\%, \quad 1.314\%, \quad 1.514\%, \quad 1.572\%)^T$$

The corresponding plot is shown below, notice how the unrelatedness property have been lost and all the rates of I_0 are affected by a view regarding only g_0^2.

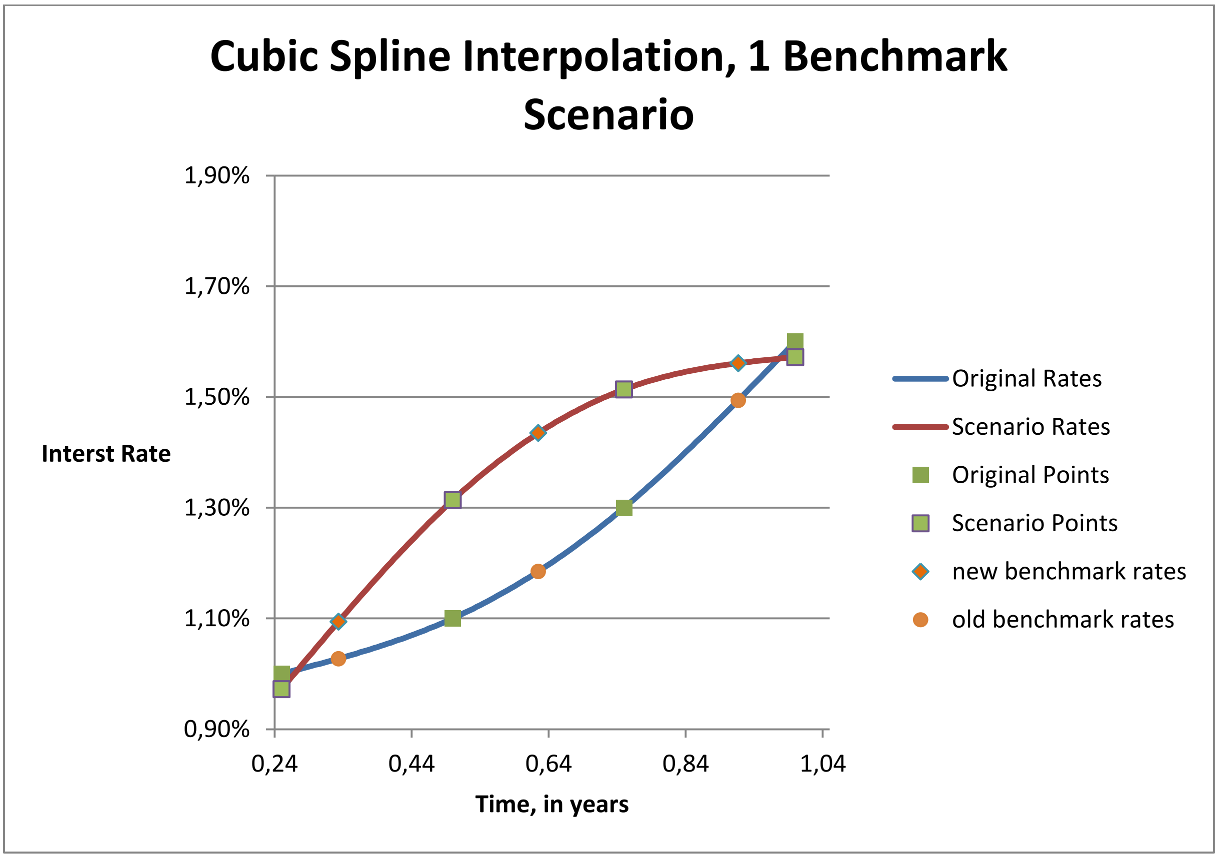

Figure 21. The scenario that g_0^2 increases by 25 basis points and no view is held regarding g_0^1 or g_0^3.

7. CONCLUSIONS & ANALYSIS

Using generic benchmarks for interest rate risk assessment and interest rate risk limiting is a viable alternative for doing so in a consistent way. However the generic benchmark instruments need to be chosen with care as to guarantee the existence of at least one correspondence scenario. Furthermore the generic benchmark should be plenty enough to capture the most central movements of the risk space.

With linear and cubic spline interpolation scheme in the yield curve, properties such as monotonicity and convexity in the optimization problems have been exhibited. This makes it safe to use numerical solutions for finding the correspondence scenario. There is thus reason to believe that a numerical treatment of a larger environment which also introduces the mentioned hierarchy and therefore some non-linearity will behave nicely. However further analysis would be required to assert such a hypothesis. In the situation with only one view, regarding one benchmark instruments the gradients have been shown to be independent of $R(I_t)$ when the interest rates of the benchmark instruments are seen as a function of $R(I_t)$. Likewise in the multivariate case with views regarding several interest rate of generic benchmarks viewed as a function of $R(I_t)$. The derivative with respect to $R(I_t)$ has been seen to be independent of $R(I_t)$.

It has been demonstrated that some properties in the generic benchmark approach will depend on which interpolation technique that is utilized. Namely the extent to which, two generic benchmarks are unrelated to each other or not. In the one yield curve situation, it is possible that generic benchmarks are unrelated, which was shown in Section 5. When that is the case computations can be made less expensive by concentrating on the instruments that is not unrelated to the benchmark for which one has a view. However, it was shown above that when a change is made to cubic interpolation scheme no generic benchmark is unrelated to market instruments. The impact of a scenario for a given benchmark instrument on a shift of the market instrument corresponding to the correspondence scenario can however be non-significant.

Last but not least the interpretability of interest rate risk measured with the use of generic benchmark is an advantageous feature. The generic benchmark approach make transparent not only the interest rate exposure but also a clear way on how to act/trade, to change the interest rate risk exposure in an investment portfolio. As opposed to simply reporting an empirical value at risk numeric, for the same portfolio of interest rate contracts, without any clear directive on how to act/trade, in order to reduce the interest rate risk.

8. FURTHER RESEARCH TOPICS & LIMITATION

Further research topics of interest, are to expand the environment to include a hierarchy structure and to determine if the correspondence can be found analytically and how a numerical approach to finding the correspondence scenario would behave. It may be the case that the convexity property is maintained in a hierarchy structure but it is not evident form the investigation of this paper.

Another generalization would be to introduce more complex interest rate contracts as generic benchmark instruments. One might make the hypotheses that the best way is to is to keep the generic benchmark instruments as simple as possible since every complex interest rate contract is a sum of such simpler contract an because of the linearity of the pricing operator, that is, $p(g_t^1 + g_t^2) = p(g_t^1) + p(g_t^2)$. This view was adopted in this paper but an analysis of this assumption might bring about further understanding to the generic benchmark approach in general.

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