Mutually algebraic structures and expansions by predicates

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Abstract

We introduce the notions of a mutually algebraic structures and theories and prove many equivalents. A theory T is mutually algebraic if and only if it is weakly minimal and trivial if and only if no model M of T has an expansion (M, A) by a unary predicate with the finite cover property. We show that every structure has a maximal mutually algebraic reduct, and give a strong structure theorem for the class of elementary extensions of a fixed mutually algebraic structure.

1 Introduction

This paper is written with two objectives in mind. On one hand, it is a continuation of [5], where a strong quantifier elimination theorem was proved for elementary diagrams of models of a weakly minimal, trivial theory. Here, we show that the crucial notion of mutual algebraicity of a formula (see Definition 2.2) has meaning in arbitrary structures, and in fact describes a specific reduct of any structure. As well, Theorem 3.3 reverses the argument in [5]. The quantifier elimination result described there can only occur as the elementary diagram of a weakly minimal, trivial theory.

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On the other hand, there has been a large body of research about whether an expansion (M, A) of a given stable structure M by a unary predicate A remains stable. Sufficient conditions abound, but the general question remains open. Here, also with Theorem 3.3, we characterize those structures M with the property that every unary expansion (M, A) satisfies the non-finite cover property (nfcp), which is a strengthening of stability.

The motivation for this came from the author's reading [1], where Baldwin and Baizhanov showed that a non-trivial, strongly minimal structure M has an unstable expansion (M, A). Thanks are due to John Baldwin for a careful reading of this paper, and for pointing out that an alternate treatment of a portion of Section 4 appears in Section 6 of [2].

2 The mutually algebraic reduct of a structure

We begin by recalling the definition of a mutually algebraic formula. This notion was introduced by Dolich, Raichev, and the author in [4] and further developed in [5]. However, in both of those papers, the ambient theory was assumed to have the non-finite cover property (nfcp). Here, we define the notions without any ambient assumptions. We begin by formulating the notion of a mutually algebraic set.

Definition 2.1 Given an arbitrary set A and an integer $n \geq 1$, a *proper partition of* n is a partition $X \sqcup Y = \{1, \ldots, n\}$ where X, Y are disjoint and each is non-empty. Given such a partition, π_Y denotes the projection of A^n onto the coordinates in Y.

A subset $B \subseteq A^n$ is *mutually algebraic* if there is a number K so that for any proper partition of the coordinates $X \sqcup Y = \{1, \ldots, n\}$, the projection π_Y restricted to B is at most K-to-1. That is, $|\pi_Y^{-1}(\bar{b}_Y) \cap B| \leq K$ for any $\bar{b}_Y \in \pi_Y(B)$.

As special cases, note that if either A is finite or B is empty, then B is mutually algebraic. Furthermore, for any set A, every subset $B \subseteq A^1$ is mutually algebraic as there are no proper partitions of a one element set.

Definition 2.2 Let M denote any L-structure. An $L(M)$-formula $\varphi(\bar{z})$ is *mutually algebraic* if $\varphi(M) := \{\bar{a} \in M^{\lg(\bar{z})} : M \models \varphi(\bar{a})\}$ is a mutually algebraic subset of $M^{\lg(\bar{z})}$. We let $\mathcal{MA}(M)$ denote the set of all mutually algebraic $L(M)$-formulas. When M is understood, we simply write $\mathcal{MA}$.

To clarify this concept and to set notation, given a formula $\varphi(\bar{z})$, a *proper partition of* $\bar{z}$ has the form $\bar{z} = \bar{x}\hat{\ }\bar{y}$, where $\bar{x}, \bar{y}$ are disjoint and $\lg(\bar{x}), \lg(\bar{y}) \geq 1$. We do not require $\bar{x}$ be an initial segment of $\bar{z}$ but to simplify notation, we write it as if it were. Then, for any L-structure M, an $L(M)$-formula $\varphi(\bar{z})$ is mutually algebraic if and only if there is an integer K so that $M \models \forall\bar{y}\exists^{\leq K}\bar{x}\varphi(\bar{x}, \bar{y})$ for every proper partition $\bar{x}\hat{\ }\bar{y}$ of $\bar{z}$.

The reader is cautioned that whether a formula $\varphi(\bar{z})$ is mutually algebraic or not depends on the choice of free variables. In particular, mutual algebraicity is **not** preserved under adjunction of dummy variables. The special cases mentioned above imply that if M is finite, then every $L(M)$-formula is in $\mathcal{MA}(M)$, and for an arbitrary M, every inconsistent formula and every $L(M)$-formula $\varphi(z)$ with exactly one free variable symbol is mutually algebraic. Our first easy Lemma gives a semantic interpretation to this notion when $\lg(\bar{z}) \geq 2$:

Lemma 2.3 *Let M be any L-structure. The following are equivalent for any $L(M)$-formula $\varphi(\bar{z})$ with $\lg(\bar{z}) \geq 2$:*

1. *$\varphi(\bar{z}) \in \mathcal{MA}(M)$;*
2. *There is an integer K so that $M \models \forall x\exists^{\leq K}\bar{y}\varphi(x, \bar{y})$ for all partitions $\bar{z} = x\hat{\ }\bar{y}$ with $\lg(x) = 1$;*
3. *For all $N \succeq M$, for all $\bar{e} \in N^{\lg(\bar{z})}$ realizing φ, and for all $e \in \bar{e}$, $\bar{e} \subseteq \mathrm{acl}(M \cup \{e\})$ (i.e, every $e' \in \bar{e}$ is in $\mathrm{acl}(M \cup \{e\})$).*

Proof. (1) $\Rightarrow$ (2) is immediate.

(2) $\Rightarrow$ (3) Fix any $N \succeq M$ and assume $N \models \varphi(\bar{e})$. Fix any variable symbol $x \in \bar{z}$ and let e be the corresponding element of $\bar{e}$. By elementarity, $N \models \exists^{\leq K}\bar{y}\varphi(e, \bar{y})$, so $\bar{e} \subseteq \mathrm{acl}(M \cup \{e\})$.

(3) $\Rightarrow$ (1) If (1) fails, then for some proper partition $\bar{z} = \bar{x}\hat{\ }\bar{y}$ we have $M \models \exists\bar{y}\exists^{\geq r}\bar{x}\varphi(\bar{x}, \bar{y})$. Thus, by compactness, there is $N \succeq M$ and $\bar{b}$ from N such that $N \models \exists^{\geq r}\bar{x}\varphi(\bar{x}, \bar{b})$ for each $r \in \omega$. By compactness again, there is $N^* \succeq N$ and $\bar{a} \in (N^*)^{\lg(\bar{x})}$ such that $\bar{a} \not\subseteq \mathrm{acl}(M \cup \bar{b})$, contradicting (3).

The following Lemma indicates some of the closure properties of the set $\mathcal{MA}$. In what follows, when we write $\varphi(\bar{x},\bar{y}) \in \mathcal{MA}$, we mean that $\bar{x}$ and $\bar{y}$ are disjoint sets of variable symbols and $\varphi(\bar{z}) \in \mathcal{MA}$ where $\bar{z} = {}^\frown\bar{y}$, but that we are concentrating on a specific proper partition of $\varphi(\bar{z})$.

Lemma 2.4 *Let M be any structure in any language L.*

1. *If $\varphi(\bar{z}) \in \mathcal{MA}$, then $\varphi(\sigma(\bar{z})) \in \mathcal{MA}$ for any permutation σ of the variable symbols;*
2. *If $\varphi(\bar{x},\bar{y}) \in \mathcal{MA}$ and $\bar{a} \in M^{\mathrm{lg}(\bar{y})}$, then both $\exists\bar{y}\varphi(\bar{x},\bar{y})$ and $\varphi(\bar{x},\bar{a}) \in \mathcal{MA}$;*
3. *If $\varphi(\bar{z}) \vdash \psi(\bar{z})$ and $\psi(\bar{z}) \in \mathcal{MA}$, then $\varphi(\bar{z}) \in \mathcal{MA}$;*
4. *If $\{\varphi_i(\bar{z}_i) : i < k\} \subseteq \mathcal{MA}$, and there is some variable x common to every $\bar{z}_i$, then $\psi(\overline{w}) := \bigwedge_{i<k} \varphi_i(\bar{z}_i) \in \mathcal{MA}$, where $\overline{w} = \bigcup_{i<k} \bar{z}_i$;*
5. *If $\varphi(\bar{x},\bar{y}) \in \mathcal{MA}$ and $r \in \omega$, then $\theta_r(\bar{y}) := \exists^{\geq r}\bar{x}\varphi(\bar{x},\bar{y}) \in \mathcal{MA}$.*

Proof. The verification of (1), (2), and (3) are immediate. Concerning (4), we apply Lemma 2.3. Fix $N \succeq M$ and $\bar{e}$ such that $N \models \psi(\bar{e})$. Let x denote a variable symbol that appears in every $\bar{z}_i$ and let e_x denote the element of $\bar{e}$ corresponding to x. Similarly, for each $i < k$ let $\bar{e}_i$ be the subsequence corresponding to $\bar{z}_i$. As each $\varphi_i(\bar{z}_i) \in \mathcal{MA}$, $e_x \in \mathrm{acl}(M \cup \{e\})$ for every $e \in \bar{e}_i$, so $e_x \in \mathrm{acl}(M \cup \{e\})$ for every $e \in \bar{e}$. But also, $e \in \mathrm{acl}(M \cup \{e_x\})$ for every $e \in \bar{e}$. Thus, by the transitivity of algebraic closure, $e \in \mathrm{acl}(M \cup \{e'\})$ for all pairs $e, e' \in \bar{e}$. So $\psi(\overline{w}) \in \mathcal{MA}$ by Lemma 2.3.

To establish (5), let $\{\bar{x}_i : i < r\}$ be disjoint sequences of variable symbols, each disjoint from $\bar{y}$. Then $\theta_r(\bar{y})$ is equivalent to

$$\exists\bar{x}_0\exists\bar{x}_1\ldots\exists\bar{x}_{r-1}\left(\bigwedge_{i<r}\varphi(\bar{x}_i,\bar{y}) \wedge \bigwedge_{i<j<r}\bar{x}_i \neq \bar{x}_j\right)$$

That this formula is in $\mathcal{MA}$ follows by successively applying Clauses (4), (3), and (2).

Definition 2.5 For any L-structure M, let M_M denote the canonical expansion of M to an $L(M)$-structure formed by adding a constant symbol c_a for each $a \in M$. We let $\mathcal{MA}^*(M)$ denote the set of all $L(M)$-formulas that are $Th(M_M)$-equivalent to a boolean combination of formulas from $\mathcal{MA}(M)$. When M is understood, we simply write $\mathcal{MA}^*$.

Whereas the definition of $\mathcal{MA}$ was rather fussy, membership in $\mathcal{MA}^*$ is more relaxed, mostly owing to the fact that $\mathcal{MA}^*$ is closed under adjunction of dummy variables. Indeed, we will see with Proposition 2.7 below, for any structure M, $\mathcal{MA}^*(M)$ specifies a reduct of the canonical expansion M_M.

Lemma 2.6 *Let M denote any L-structure.*

1. *$\mathcal{MA}^*$ is closed under boolean combinations;*
2. *$\mathcal{MA}^*$ is closed under adjunction of dummy variables, i.e., if $\varphi(\bar{z}) \in \mathcal{MA}^*$ then $\varphi(x, \bar{z}) \in \mathcal{MA}^*$;*
3. *For each $k \geq 1$, if $\{\varphi_i(x, \bar{y}_i) : i < k\} \subseteq \mathcal{MA}$ and $r \in \omega$, then each of $\exists^{=r} x \bigvee_{i<k} \varphi_i(x, \bar{y}_i)$, $\exists^{\leq r} x \bigvee_{i<k} \varphi_i(x, \bar{y}_i)$, and $\exists^{\geq r} x \bigvee_{i<k} \varphi_i(x, \bar{y}_i)$ are in $\mathcal{MA}^*$.*

Proof. The proof of (1) is immediate. For (2), note that $\psi(x) :=$ '$x = x$' is in $\mathcal{MA}$, hence in $\mathcal{MA}^*$, but $\varphi(x, \bar{z})$ is equivalent to $\varphi(\bar{z}) \wedge \psi(x)$. The verification of (3) is more substantial. We argue by induction on k that for every $r \in \omega$, $\exists^{=r} x \bigvee_{i<k} \varphi_i(x, \bar{y}_i) \in \mathcal{MA}^*$ for every k-element subset $\{\varphi_i(x, \bar{y}_i) : i < k\}$ from $\mathcal{MA}$. This suffices, as $\mathcal{MA}^*$ is closed under boolean combinations and the trivial facts that $\exists^{\leq r} x\theta$ is equivalent to $\bigvee_{s \leq r} \exists^{=s} x\theta$ and $\exists^{\geq r} x\theta$ is equivalent to $\neg\exists^{\leq r-1} x\theta$.

To handle the case when $k = 1$, fix any $\varphi(x, \bar{y}) \in \mathcal{MA}$ and any $r \in \omega$. By Lemma 2.4(5), both $\exists^{\geq r} x\varphi(x, \bar{y}) \in \mathcal{MA}$ and $\exists^{\geq r+1} x\varphi(x, \bar{y}) \in \mathcal{MA}$ and $\exists^{=r} x\varphi(x, \bar{y})$ is a boolean combination of these.

Next, inductively assume that for every $r \in \omega$, $\exists^{=r} x \bigvee_{i<k} \varphi_i(x, \bar{y}_i) \in \mathcal{MA}^*$ for every k-element subset $\{\varphi_i(x, \bar{y}_i) : i < k\}$ from $\mathcal{MA}$. Choose any $(k+1)$-element subset $\{\varphi_i(x, \bar{y}_i) : i \leq k\}$ from $\mathcal{MA}$ and choose any $r \in \omega$. As notation, let $\psi(x, \overline{w}) := \bigvee_{i<k} \varphi_i(x, \bar{y}_i)$. By the inclusion/exclusion

principle of integers, the formula $\exists^{=r}x\bigvee_{i\leq k}\varphi(x,\bar{y}_i)$, which is equivalent to $\exists^{=r}x(\psi(x,\overline{w})\vee\varphi_k(x,\bar{y}_k))$, is equivalent to

$$\bigvee_{\substack{a,b\leq r\\ a+b-c=r}}\Big(\exists^{=a}x\psi(x,\overline{w})\ \wedge\ \exists^{=b}x\varphi_k(x,\bar{y}_k)\ \wedge\ \exists^{=c}x[\psi(x,\overline{w})\wedge\varphi_k(x,\bar{y}_k)]\Big)$$

By the inductive hypothesis $\exists^{=a}x\psi(x,\overline{w})\in\mathcal{MA}^*$ and $\exists^{=b}x\varphi_k(x,\bar{y}_k)\in\mathcal{MA}^*$ by the case $k=1$. Also, note that $\psi(x,\overline{w})\wedge\varphi_k(x,\bar{y}_k)$ is equivalent to $\bigvee_{i<k}\delta_i(x,\bar{y}_i,\bar{y}_k)$, where each $\delta_i(x,\bar{y}_i,\bar{y}_k):=\varphi_i(x,\bar{y}_i)\wedge\varphi_k(x,\bar{y}_k)$ is in $\mathcal{MA}$ by Lemma 2.4(4). Thus, by applying the inductive hypothesis to this k-element subset from $\mathcal{MA}$, we conclude that $\exists^{=c}x(\psi(x,\overline{w})\wedge\varphi_k(x,\bar{y}_k))\in\mathcal{MA}^*$, completing the proof.

Proposition 2.7 *For any structure M, the set $\mathcal{MA}^*(M)$ is closed under existential quantification. Thus, the structure with universe M, together with the definable sets $MA^*(M)$, is a reduct of the canonical expansion M_M.*

Proof. The second sentence follows from the first, since $\mathcal{MA}^*$ is a set of $L(M)$-formulas closed under boolean combinations. To establish the first sentence, there are two cases. First, if the structure M is finite, then every $L(M)$-formula $\varphi(\bar{z})\in\mathcal{MA}$, so $\mathcal{MA}^*$ is precisely the elementary diagram of M and there is nothing to prove. So assume that M is infinite.

Choose $\varphi(x,\bar{y})\in\mathcal{MA}^*$ and we argue that $\exists x\varphi(x,\bar{y})$ is equivalent to a formula in $\mathcal{MA}^*$. By writing φ in Disjunctive Normal Form and noting that disjunction commutes with existential quantification, we may assume that $\varphi(x,\bar{y})$ has the form

$$\bigwedge_{i<k}\beta_i(x,\bar{y}_i)\wedge\bigwedge_{j<m}\neg\gamma_j(x,\bar{y}_j)$$

where each β_i and γ_j are in $\mathcal{MA}$ and the variable x occurs in each of these subformulas. By Lemma 2.4(4), if $k\geq 1$, then $\bigwedge_{i<k}\beta_i(x,\bar{y}_i)\in\mathcal{MA}$, so we may assume there is at most one β. If there is no β, then since the model M is infinite, then for any choice of $\bar{y}$, $\exists x\varphi(x,\bar{y})$ always holds. Thus, we assume that there is exactly one β, i.e., that $\varphi(x,\bar{y})$ has the form $\beta(x,\bar{y}^*)\wedge\bigwedge_{j<m}\neg\gamma_j(x,\bar{y}_j)$, where $\bar{y}^*$ and each $\bar{y}_j$ are subsequences of $\bar{y}$, and both β and each γ_j are from $\mathcal{MA}$.

We first consider the case where $\bar{y}^*$ is empty. In this case, we may additionally assume that no $\bar{y}_j$ is empty, since we could replace $\beta(x)$ by $\beta(x) \wedge \neg\gamma_j(x)$. Thus, for any choice of $\bar{y}$, the solution set of $\bigwedge_{j<m} \neg\gamma_j(x, \bar{y}_j)$ is a cofinite subset of M. We have two subcases: On one hand, if $\beta(x)$ were algebraic, then every solution to β lies in M, hence $\varphi(x, \bar{y})$ would be equivalent to $\bigvee_{m\in\beta(M)} \varphi(m, \bar{y})$, which would be in $\mathcal{MA}^*$ by Lemma 2.4(2). On the other hand, if $\beta(x)$ were non-algebraic, then $\beta(x)$ would have infinitely many solutions in M, so $\varphi(x, \bar{y})$ would have a solution in M for any choice of $\bar{y}$. Thus, $\exists x\varphi(x, \bar{y})$ would always hold.

Finally, assume that $\bar{y}^* \neq \emptyset$. By the definition of mutual algebraicity, there is an integer K so that $M \models \forall\bar{y}^*\exists^{\leq K} x\beta(x, \bar{y}^*)$. For each $j < m$, let $\theta_j(x, \bar{y}^*, \bar{y}_j) := \beta(x, \bar{y}^*) \wedge \gamma_j(x, \bar{y}_j)$. By Lemma 2.4(4), each $\theta_j(x, \bar{y}^*, \bar{y}_j) \in \mathcal{MA}$. Thus, the formula $\exists x\varphi(x, \bar{y})$ is equivalent to

$$\bigvee_{r\leq K} \left(\exists^{=r} x\beta(x, \bar{y}^*) \wedge \exists^{<r} x \bigvee_{j<m} \theta_j(x, \bar{y}^*, \bar{y}_j) \right)$$

which is in $\mathcal{MA}^*$ by Lemma 2.6.

The previous Proposition inspires the following two definitions:

Definition 2.8 A structure M is *mutually algebraic* if every $L(M)$-formula is in $\mathcal{MA}^*(M)$.

Definition 2.9 Let M be any structure. The *mutually algebraic reduct of* M_M is the structure with the same universe as M, and whose definable sets are precisely $\mathcal{MA}^*(M)$.

Proposition 2.7 immediately implies that the mutually algebraic reduct of a structure M is a mutually algebraic structure.

Lemma 2.10 *Mutual algebraicity of structures is preserved under elementary equivalence.*

Proof. Suppose that M is a mutually algebraic structure and that N is elementarily equivalent to M. It suffices to show that $\varphi(\bar{x}, \bar{h}) \in \mathcal{MA}^*(N)$ for any L-formula $\varphi(\bar{x}, \bar{y})$ (with $\bar{x}$ and $\bar{y}$ disjoint and there are no hidden parameters) and any $\overline{h} \in N^{\lg(\bar{y})}$. Given this data, let $\bar{z} = \bar{x}\hat{\ }\bar{y}$ and consider the L-formula $\varphi(\bar{z})$. As M is mutually algebraic, $\varphi(\bar{z}) \in \mathcal{MA}^*(M)$, so there are (finitely many) L-formulas $\delta_i(\bar{z}, \overline{w}_i)$ and $\bar{e}_i$ from M so that (1) $\varphi(\bar{z})$ is $Th(M_M)$-equivalent to a boolean combination $\theta(\bar{z}, \bar{e}^*)$ of the $\delta_i(\bar{z}, \bar{e}_i)$ ($\bar{e}^*$ denotes the concatenation of the $\bar{e}_i$'s); and (2) There is a number K so that each of the formulas $\delta_i(\bar{z}, \bar{e}_i)$ satisfy $M \models \forall \bar{y}' \exists^{\leq K} \bar{x}' \delta_i(\bar{x}', \bar{y}', \bar{e}_i)$ for every proper partition $\bar{z} = \bar{x}'\hat{\ }\bar{y}'$. Thus, by quantifying out the $\bar{e}^*$, there is an L-sentence σ asserting that

$$\exists \overline{w}^* \Big(\forall \bar{z} [\varphi(\bar{z}) \leftrightarrow \theta(\bar{z}, \overline{w}^*)] \ \wedge \ \text{`each } \delta_i(\bar{z}, \overline{w}_i) \text{ is } K\text{-mutually algebraic'} \Big)$$

As $M \models \sigma$, so does N. Choose $\bar{c}^*$ from N so that $\varphi(\bar{z})$ is $Th(N_N)$-equivalent to $\theta(\bar{z}, \bar{c}^*)$ and $\theta(\bar{z}, \bar{c}^*)$ is equivalent to a boolean combination of $\delta_i(\bar{z}, \bar{c}_i) \in \mathcal{MA}(N)$, where each $\bar{c}_i$ is the corresponding subsequence of $\bar{c}^*$. Finally, rewrite $\bar{z}$ as $(\bar{x}, \bar{y})$ and substitute $\overline{h}$ for $\bar{y}$. By Lemma 2.4(2), each of the formulas $\delta_i(\bar{x}, \overline{h}, \bar{c}_i) \in \mathcal{MA}(N)$ and $\varphi(\bar{x}, \overline{h})$ is $Th(N_N)$-equivalent to the boolean combination $\theta(\bar{x}, \overline{h}, \bar{c}^*)$. Thus, $\varphi(\bar{x}, \overline{h}) \in \mathcal{MA}^*(N)$, as required.

The following Lemma is folklore, but a proof is included for the convenience of the reader. Recall that a partitioned formula $\varphi(\bar{x}, \bar{y})$ does not have the finite cover property (i.e., has nfcp) with respect to a theory T if there is a number k so that for all sets $\{\bar{c}_i : i \in I\}$, the type $\Gamma := \{\varphi(\bar{x}, \bar{c}_i) : i \in I\}$ is consistent with T whenever every k-element subset of Γ is consistent with T.

Lemma 2.11 *Let M be any structure, and let $\varphi(\bar{x}, \bar{y})$ be any partitioned $L(M)$-formula. If, for some integer K, either $M \models \forall \bar{y} \exists^{<K} \bar{x} \varphi(\bar{x}, \bar{y})$, or $M \models \forall \bar{y} \exists^{<K} \bar{x} \neg \varphi(\bar{x}, \bar{y})$, then $\varphi(\bar{x}, \bar{y})$ does not have the finite cover property with respect to $Th(M_M)$.*

Proof. If M is finite, then every partitioned formula $\varphi(\bar{x}, \bar{y})$ has nfcp for trivial reasons, so assume that M is infinite. First, assume that $M \models \forall \bar{y} \exists^{<K} \bar{x} \varphi(\bar{x}, \bar{y})$. Choose tuples $\{\bar{c}_i : i \in I\}$ from some elementary extension of M and assume that the type $\Gamma := \{\varphi(\bar{x}, \bar{c}_i) : i \in I\}$ is inconsistent. It suffices to find a subtype of at most K elements that is inconsistent as well. Choose a maximal sequence $\langle i_j : j \leq n \rangle$ from I such that $i_0 \in I$ is arbitrary

and for each $1 \leq m \leq n$,

$$\models \exists \bar{x} \left(\bigwedge_{j<m} \varphi(\bar{x}, \bar{c}_{i_j}) \wedge \neg\varphi(\bar{x}, \bar{c}_{i_m}) \right)$$

By our hypotheses on $\varphi(\bar{x}, \bar{c}_{i_0})$, $n \leq K$. But now, if $\bigwedge_{j\leq n} \varphi(\bar{x}, \bar{c}_{i_j})$ were consistent but Γ were not, we would contradict the maximality of the sequence.

In the other case, as M is infinite, every partial type of the form $\{\varphi(\bar{x}, \bar{c}_i) : i \in I\}$ is consistent, so the nfcp of $\varphi(\bar{x}, \bar{y})$ is vacuously true.

Proposition 2.12 *For any structure M, the theory of the mutually algebraic reduct of M has nfcp.*

Proof. By the equivalence of (1) and $\forall m(2)_m$ in Theorem II 4.4 of [6] (whose proof does not use stability) it suffices to show that no partitioned formula of the form $\varphi(x, \bar{y}) \in \mathcal{MA}^*$ with $\lg(x) = 1$ has the finite cover property.

Consider any formula $\theta(x, \bar{y})$ of the form

$$\bigwedge_{i<k} \beta_i(\bar{z}_i) \wedge \bigwedge_{j<m} \neg\gamma_j(\bar{z}_j)$$

with each β_i and γ_j from $\mathcal{MA}$. First, if the variable x occurs in any β_i, then it follows that there is a number K so that $M \models \forall\bar{y}\exists^{<K}x\theta(x, \bar{y})$. Second, if x does not occur in any β_i, then there is a number K so that there is a number K so that $M \models \forall\bar{y}\exists^{<K}x\neg\theta(x, \bar{y})$. But, any formula $\varphi(x, \bar{y}) \in \mathcal{MA}^*$ is a finite disjunction of formulas $\theta(x, \bar{y})$ described above. It follows that for some K, either $M \models \forall\bar{y}\exists^{<K}x\varphi(x, \bar{y})$ or there is a number K so that $M \models \forall\bar{y}\exists^{<K}x\varphi(x, \bar{y})$ holds. Thus, $\varphi(x, \bar{y})$ has the nfcp by Lemma 2.11.

3 Characterizing theories of mutually algebraic structures

We begin with two definitions indicating that the forking behavior of 1-types (types with a single free variable) is particularly simple.

Definition 3.1 A complete, stable theory with an infinite model is *weakly minimal* if every forking extension of a 1-type is algebraic (equivalently if $R^\infty(x = x) = 1$) and is *trivial* if there do not exist a set D and three elements $\{a, b, c\}$ that are dependent, but pairwise independent over D. A type $p \in S(D)$ is trivial if there do not exist a set $\{a, b, c\}$ of realizations of p that are dependent, but pairwise independent over D.

It is well known that a weakly minimal theory is trivial if and only if every minimal type is trivial. The following Lemma generalizes the analogous result for non-trivial, strongly minimal theories that was proved by Baldwin and Baizhanov in [1].

Lemma 3.2 *If T is weakly minimal and non-trivial, then there is a model M of T and a subset $A \subseteq M$ such that (M, A) is unstable.*

Proof. Among all minimal types $p \in S(D)$ and formulas $\varphi(z, xy)$ over D that contain a dependent, but pairwise independent triple $\{a, b, c\}$ of realizations of p, with the dependency witnessed by the algebraic formula $\varphi(z, ab) \in \mathrm{tp}(c/Dab)$, choose one with the multiplicity of $\varphi(z, ab)$ as small as possible. It follows from this multiplicity condition that $\mathrm{acl}(D \cup \{a\}) \cup \mathrm{acl}(D \cup \{b\})$ does not contain any realizations of $\varphi(z, ab)$.

Fix $p \in S(D)$ and $\varphi(z, xy)$ as above, and let M be a sufficiently saturated model containing D. To ease notation, we may assume $D = \emptyset$. Let $\langle (a_i, b_i) : i \in \omega \rangle$ be a Morley sequence in $p^{(2)}$. That is, $\{a_i : i \in \omega\} \cup \{b_j : j \in \omega\}$ is an independent set of realizations of p. For each pair $(i, j) \in \omega^2$, choose $c_{i,j} \in p(M)$ realizing $\varphi(z, a_i b_j)$. Let $A = \{c_{i,j} : i \leq j < \omega\}$. We argue that the L_P-formula $\Phi(x, y) := \exists z(P(z) \wedge \varphi(z, xy))$ has the order property in (M, A).

To see this, it is clear that the element $c_{i,j}$ witnesses $\Phi(a_i, b_j)$ whenever $i \leq j$. On the other hand, suppose some $c_{k,\ell}$ witnessed $\varphi(x, a_i, b_j)$. We argue that we must have $k = i$ and $\ell = j$: If neither equality held, then we would have $c_{k,\ell}$ forking with both sets $\{a_i, b_j\}$ and $\{a_k, b_\ell\}$. This is impossible, as the doubletons are independent from each other and the type p is minimal, hence regular, hence of weight one. Similarly, suppose that $k = i$ but $\ell \neq j$. Then, working over a_i, $c_{i,\ell}$ is not algebraic over a_i, so $\mathrm{tp}(c_{i,\ell}/a_i)$ is parallel to p, hence is also regular, so of weight one. But, working over a_i, $c_{i,\ell}$ forks with each of b_j and b_ℓ, which are independent over a_i. The case where $j = \ell$ is symmetric, completing the proof.

In what follows, a *mutually algebraic expansion* of a structure M is an expansion formed by adding arbitrarily many new relation symbols R_i, whose interpretation is a mutually algebraic subset of $M^{\mathrm{arity}(R_i)}$ (see Definition 2.1). In the Theorem that follows, we do not require that the theory T be complete.

Theorem 3.3 *The following are equivalent for any theory T:*

1. *Every model of T is a mutually algebraic structure;*

2. *Every mutually algebraic expansion of every model of T is a mutually algebraic structure;*

3. *$Th((M, A))$ has the nfcp for every $M \models T$ and every expansion (M, A) by a unary predicate;*

4. *Every complete extension of T having an infinite model is weakly minimal and trivial.*

Proof. (1) $\Rightarrow$ (2) Fix $M \models T$ and let $\overline{M} = (M, R_i)_{i\in I}$ be any expansion of M, where each R_i is a $k(i)$-ary relation symbol whose interpretation in $\overline{M}$ is a mutually algebraic subset $B_i \subseteq M^{k(i)}$. By definition, the $\overline{M}$-definable subsets are the smallest class of subsets of M^ℓ for various ℓ that contain every M-definable set and every B_i and are closed under boolean combinations and projections. As M is mutually algebraic, every M-definable set is a boolean combination of mutually algebraic sets. So $\mathcal{MA}^*(\overline{M})$ contains every M-definable set and each of the sets B_i. Additionally, $\mathcal{MA}^*(\overline{M})$ is closed under boolean combinations and projections. Thus, every $\overline{M}$-definable set is in $\mathcal{MA}^*(\overline{M})$, so $\overline{M}$ is a mutually algebraic structure.

(2) $\Rightarrow$ (3) Fix any $M \models T$ and any expansion $\overline{M} = (M, A)$ by a unary predicate. As every subset of M^1 is mutually algebraic, it follows from (2) that $\overline{M}$ is a mutually algebraic structure, i.e., every $\overline{M}$-definable set is in $\mathcal{MA}^*(\overline{M})$. Thus, every partitioned $\overline{M}$-definable formula $\varphi(\bar{x}, \bar{y})$ has nfcp by Proposition 2.12. That is, the elementary diagram of $\overline{M}$ and hence the theory of $\overline{M}$ has nfcp.

(3) $\Rightarrow$ (4) Suppose T satisfies (3). Fix any complete extension T' of T with an infinite model. As the nfcp implies stability, T' must be stable. Fix a sufficiently saturated model M of T'. As T' is stable, if it were not weakly minimal then we could choose an element a and a tuple $\bar{b}$ from M such that

$\mathrm{tp}(a/\bar{b})$ forks over the empty set, but a is not algebraic over $\bar{b}$. Let $\varphi(x, \bar{y})$ be chosen so that $\varphi(x, \bar{b}) \in \mathrm{tp}(a/\bar{b})$ witnesses the forking. As M is sufficiently saturated, we can find a Morley sequence $\langle \bar{b}_i : i \in \omega \rangle$ in $\mathrm{stp}(\bar{b})$ inside M. As T' is stable, $\{\bar{b}_i : i \in \omega\}$ is an indiscernible set and there is a number k so that every element $a^* \in M$ is contained in at most k of the sets $D_i := \varphi(M, \bar{b}_i)$. As each D_i is infinite, we can construct a subset A of M such that each $c \in A$ is contained in exactly one of the sets D_i, and for each i, $|A \cap D_i| = i$. Then the theory of the expansion (M, A), where the new unary predicate symbol P is interpreted as A, has the finite cover property as witnessed by the L_P-formula $\Psi(x, \bar{y}z) := P(x) \wedge \varphi(x, \bar{y}) \wedge x \neq z$. Thus, T must be weakly minimal. That T' must be trivial as well follows from Lemma 3.2 and the fact that instability implies an instance of the finite cover property.

(4) $\Rightarrow$ (1) This is the content of Theorem 4.2 of [5]. In fact, there it is shown that every M-definable formula is a boolean combination of mutually algebraic formulas of a very special form.

Corollary 3.4 *Let M be any infinite structure. The mutually algebraic reduct of M described in Definition 2.9 is the maximal weakly minimal, trivial reduct of M.*

Proof. The mutually algebraic reduct of M is a mutually algebraic structure, so it has a weakly minimal, trivial theory. Conversely, if any reduct of M has a weakly minimal, trivial theory, then it is a mutually algebraic structure, hence all of its definable sets are contained in $\mathcal{MA}^*(M)$.

4 Mutually algebraic structures

Suppose that M is a mutually algebraic structure in a language L. We study models of the elementary diagram of M, or equivalently the class of elementary extensions of M. Note that if M is finite, then there are no proper elementary extensions of M, which will render all of the results that follow vacuous. Because of this, **throughout this section we additionally assume that M is infinite.** Thus, we may assume that M is elementarily embedded in a much larger, saturated 'monster model' $\mathfrak{C}$.

By Theorem 3.3, $Th(M)$ is weakly minimal and trivial, so the quantifier elimination offered in [5] applies. Specifically, let

$$\mathcal{A}(M) := \{\text{all quantifier-free mutually algebraic } L(M)\text{-formulas } \alpha(\bar{z})\} \text{ and}$$

$$\mathcal{E}(M) = \{\text{all } L(M)\text{-formulas of the form } \exists \bar{x}\alpha(\bar{x},\bar{y}), \text{ where } \alpha(\bar{x},\bar{y}) \in \mathcal{A}(M)\}$$

and let $\mathcal{A}^*(M)$ (respectively $\mathcal{E}^*(M)$) denote the closure of $\mathcal{A}(M)$ (respectively $\mathcal{E}(M)$) under boolean combinations. Proposition 4.1 of [5] states that every quantifier-free $L(M)$-formula is equivalent to a formula in $\mathcal{A}^*(M)$, while Theorem 4.2 states that every $L(M)$-formula is equivalent to a formula in $\mathcal{E}^*(M)$.

As $Th(M)$ is weakly minimal, the relation '$a \in \mathrm{acl}_M(B)$' satisfies the axioms of a pre-geometry, where $\mathrm{acl}_M(B)$ abbreviates $\mathrm{acl}(M \cup B)$. (Algebraic closures are always computed with respect to satisfaction in $\mathfrak{C}$.) Thus, the binary relation $a \approx b$ on $\mathfrak{C} \setminus M$ defined by $a \in \mathrm{acl}_M(\{b\})$ is an equivalence relation. The following easy Lemma is folklore.

Lemma 4.1 *Suppose $Th(M)$ is weakly minimal, $M \preceq \mathfrak{C}$, and A is any algebraically closed set satisfying $M \subseteq A \subseteq \mathfrak{C}$. Then A is the universe of an elementary submodel of $\mathfrak{C}$.*

Proof. The interpretation of any constant symbol is contained in M, and the fact that A is algebraically closed implies that it is closed under every function symbol in the language. Thus, A is the universe of a substructure of $\mathfrak{C}$. To see that this substructure is elementary, by the Tarski-Vaught criterion it suffices to show that for any L-formula $\varphi(x,\bar{y})$ and for any $\bar{a}$ from A, if $\mathfrak{C} \models \exists x\varphi(x,\bar{a})$, then there is $b \in A$ such that $\mathfrak{C} \models \varphi(b,\bar{a})$. So fix any $\varphi(x,\bar{a})$ and $b \in \mathfrak{C}$ such that $\mathfrak{C} \models \varphi(b,\bar{a})$. If $b \in A$, then we are done, so assume $b \notin A$. As A is algebraically closed, this means that $\mathrm{tp}(b/M\bar{a})$ is not algebraic. As $Th(M)$ is weakly minimal and b is a singleton, this implies that $\mathrm{tp}(b/M\bar{a})$ does not fork over M. But then, by symmetry and finite satisfiability of non-forking over models, there is $b^* \in M \subseteq A$ such that $\mathfrak{C} \models \varphi(b^*,\bar{a})$.

Recall that when combined with weak minimality, triviality implies that for any set $B \subseteq \mathfrak{C}$, $\mathrm{acl}_M(B) = \bigcup_{b\in B} \mathrm{acl}_M(\{b\})$.

Proposition 4.2 *Let M be any mutually algebraic L-structure.*

1. *If $M \subseteq A \subseteq \mathfrak{C}$ and A is an arbitrary union of $\approx$-classes, then A is an L-structure and $M \preceq A \preceq \mathfrak{C}$; and*

2. *Conversely, if $M \preceq N \preceq \mathfrak{C}$ and $B \subseteq N \backslash M$ is a set of $\approx$-representatives, then N is the disjoint union of the sets M and $\{\mathrm{acl}_M(\{b\}) \backslash M : b \in B\}$.*

Proof. (1) $Th(M)$ is weakly minimal and trivial by Theorem 3.3. By triviality, A must be algebraically closed, so $A \preceq \mathfrak{C}$ by Lemma 4.1. That $M \preceq A$ follows immediately from this.

(2) That the sets are disjoint follows by triviality. If there were an element $d \in N$ that was not in any of these sets, then d would be $\approx$-inequivalent to every element of B, contradicting the maximality of B.

In light of the previous Proposition, it is natural to refer to the sets $\mathrm{acl}_M(\{b\}) \setminus M$ as the *components* of a given $N \succeq M$. Each component has size bounded by the number of $L(M)$-formulas, and one can speak of the type of a fixed enumeration of a component over M. The notion of a component map records this amount of data.

Definition 4.3 Suppose that M is a mutually algebraic structure and N_1, N_2 are both elementary extensions of M. A *component map* $f : N_1 \to N_2$ is a bijection such that $f|_M = id$ and for each $b \in N_1 \setminus M$,

- f restricted to $\mathrm{acl}_M(\{b\})$ is elementary and
- $f(\mathrm{acl}_M(\{b\})) = \mathrm{acl}_M(\{f(b)\})$ setwise.

Proposition 4.4 *Suppose that M is mutually algebraic and $N_1, N_2 \succeq M$. Then every component map $f : N_1 \to N_2$ is an isomorphism. Conversely, every isomorphism $f : N_1 \to N_2$ that is the identity on M is a component map.*

Proof. As every quantifier-free $L(M)$-formula is equivalent to a formula in $\mathcal{A}^*(M)$, it suffices to show that f preserves every formula $\alpha(\bar{z}) \in \mathcal{A}(M)$. Choose any $\bar{a}$ from N_1. Without loss, by Lemma 2.4(2) and the fact that f fixes M pointwise, we may assume $\bar{a}$ is disjoint from M. There are now two cases. First, if $\bar{a} \subseteq \mathrm{acl}_M(\{b\})$ for some element b, then $f(\bar{a}) \subseteq \mathrm{acl}_M(\{b\})$ and $N_1 \models \alpha(\bar{a})$ if and only if $N_2 \models \alpha(f(\bar{a}))$ by the elementarity of f restricted to $\mathrm{acl}_M(\{b\})$. Second, if $\bar{a}$ intersects at least two components, then $N_1 \models \neg\alpha(\bar{a})$

automatically. Furthermore, since f maps components onto components, $f(\bar{a})$ would intersect at least two components of N_2, so $N_2 \models \neg\alpha(\bar{a})$. Thus, α is preserved in both cases, so f is an isomorphism. The converse is clear since elementary maps preserve algebraic closure.

We close with two examples of how the analog of Proposition 4.4 can fail if we work over $\mathrm{acl}(\emptyset)$ instead of a model.

Example 4.5 *Let $L = \{R, S, E\}$, and let T be the theory asserting that E is an equivalence relation with exactly two classes, both infinite, and R is a binary 'mating relation' i.e., R is symmetric, irreflexive, and $\forall x \exists^{=1} y R(x, y)$. We further require that $R(x, y) \to \neg E(x, y)$. Take S to be a 4-ary relation such that $S(x, y, z, w)$ holds if and only if the four elements are distinct, and each of the relations $R(x, y)$, $R(z, w)$, and $E(x, z)$ hold. Then T is complete, mutually algebraic, and $\mathrm{acl}(\emptyset) = \emptyset$. For any model N of T, the decomposition of N into R-mated pairs is a decomposition of N into two-element '$\emptyset$-components' i.e., sets A satisfying $\mathrm{acl}_\emptyset(A) = A$. However, in contrast to Proposition 4.4, there are '$\emptyset$-component maps' $f : N \to N$, i.e., bijections $f : N \to N$ whose restriction to each two-element $\emptyset$-component is elementary, that are not automorphisms.*

The second example is from [3]. There, Baldwin, Shelah, and the author exhibit two models M, N of the theory of infinitely many, binary splitting equivalence relations that are not isomorphic in the set-theoretic universe V, but there is a c.c.c. extension $V[G]$ of V and $M \cong N$ in $V[G]$. This theory is also weakly minimal and trivial with $\mathrm{acl}(\emptyset) = \emptyset$. In fact, this theory has a prime model and every 'component' is a singleton. The complexity exploited by this example involves which strong types over the empty set are realized in the models M and N.

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