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The Proof for A Convergent Integral and Another Nonzero Integral–Respectively Using the Riemann Zeta Function and the Trigonometric Sums

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Abstract

In this paper, there are the applications of the main inequalities, and show how to use the analytic properties of the Zeta function and the Laplace transform to prove the convergence of the desired integral. In addition, show how to use the trigonometric sums and the mathematical induction with the method of infinite descent to prove the non-zero value of another integral. In this way, we can obtain the important proofs concerning the Riemann Zeta function and the sum of two primes.

Keywords: Zeta function, analytic convergence, Laplace transform, sum of two primes, trigonometric sums, Diophantine equations.

MSC(2010): Primary 11M26, 11D45; Secondary 11A25, 11P32.

1. An Overview of the Key Idea

In this paper, the key elements of the innovation are as follows:

(1) In the first proof, using a kind of method concerning the Laplace transform and the analytic convergence of developing the desired integrals can complete the improvement such as the coming Main Lemma A1, which can extend and enrich the connotation of the D.J.Newman's theorem that it was initiated and used for $\Re e(z) \geq 0$ by D.J.Newman in the 1980's, and now its proof of general situation for $\Re e(z) \geq \alpha$ is improved by the author, where α is either zero or not zero of some real numbers. The novel technique is expressed in terms of the Laplace transform with the analytic convergence, and their related the integrals. I use my result of the improvement of the D.J.Newman's theorem to prove the analytic convergence about the desired Laplace transform. That is if the Laplace transform $g(z) = \int_0^\infty f(t)\, e^{-zt}\, dt$ of the function $f(t)$ extends to an analytic function for $\Re e(z) \geq 0$, then $\int_0^\infty f(t)\, dt$ exists and is equal to $g(0)$, where $f(t)$ is bounded, piecewise continuous function on the real numbers ≥ 0. Furthermore, if the Laplace transform $g(z) = \int_0^\infty f(t)\, e^{-zt}\, dt$ of the function $f(t)$ extends to an analytic function for $\Re e(z) \geq \alpha$ with some real α, then $\int_0^\infty f(t)\, e^{-\alpha t}\, dt$ exists and is equal to $g(\alpha)$, where $f(t)$ is piecewise continuous function on the real numbers ≥ 0 and $f(t) \leq Be^{\alpha t}$ for some positive constant B. Thus, we can get the limit and the convergence between the integral $\int_0^\infty f(t)\, dt$ and the integral $\int_0^\infty f(t)\, e^{-\alpha t}\, dt$ for given $f(t) \geq 0$ with $\alpha > 0$, which can guide to establish the convergence of the integral $\int_0^\infty f_3(t)\, e^{-\varepsilon t}\, dt$, where $f_3(t) = B\, e^{(\lambda - \frac{1}{2})t}$ with any $\lambda - \frac{1}{2} \geq 0$, and given any $\varepsilon > 0$ with some positive constant B. The technique of the Laplace transform is applied to the convergence of the desired two integrals

$$\int_1^\infty \frac{\psi(x) - x}{x^{\frac{3}{2}+\varepsilon}}\, dx \quad \text{and} \quad \int_1^\infty \frac{|\psi(x) - x|}{x^{\frac{3}{2}+\varepsilon}}\, dx$$

for $\forall \varepsilon > 0$ which is one of the key links of new idea, where defined the Chebyshev's function

$$\psi(x) = \sum_{p^m \leq x} \log p = \sum_{n \leq x} \Lambda(n),$$

so that the integral

$$\int_1^\infty \frac{\psi(x)-x}{x^{s+1}}\,dx$$

is convergent and analytic for $\Re e(s) > \frac{1}{2}$.

(2) In the second proof , use the properties of trigonometric sums and mathematical inductive method to derive a causal relationship. With the help of this technique, which is explained the result of the estimate that the desired integral

$$\int_0^1 \Big(\sum_{p\le 2n} e^{2\pi i p x}\Big)^2 \cdot e^{-4\pi i n x}\,dx$$

is not zero where p is a prime number, n is a positive integer ≥ 2, and $\sum_{p\le 2n}$ is extended over all the primes $p \le 2n$, which is another of the key links of new idea.

The proof of these results relies on the novel technique with the estimates for the desired integrals, which these phenomena are indicative of the intricate nature of the problems of the integrals. These in turn contribute to further analysis and correlation, so as to provide futher insight into the intrinsic and efficiently computable estimates.

2. The Proof Is Derived from Riemann's Zeta Function and Laplace Transforms

2.1 Introduction

There is the application of the main inequality

$$\int_0^\infty f_3(t)\,e^{-\varepsilon t}\,dt \ge \int_0^\infty \frac{\left|\psi(e^t)-e^t\right|}{e^{(\frac{1}{2}+\varepsilon)t}}\,dt \ge 0,$$

where $f_3(t) = Be^{(\lambda-\frac{1}{2})t}$ for some fixed λ in the inequality $\lambda - \frac{1}{2} \ge 0$ and given any $\varepsilon > 0$, with some constant $B > 0$, while one satisfies the inequality $\left|\psi(e^t) - e^t\right| \le B\,e^{\lambda t}$ concerning the form $\psi(x) = x + O(x^\lambda)$, and where the fixed λ is independent of any $\varepsilon > 0$. We can prove that the integral

$$\int_1^\infty \frac{\psi(x)-x}{x^{\frac{3}{2}+\varepsilon}}\,dx$$

converges absolutely for $\forall\varepsilon > 0$, where defined the Chebyshev's function

$$\psi(x) = \sum_{p^m\le x} \log p = \sum_{n\le x} \Lambda(n),$$

and this $\Lambda(n)$ is equal to $\log p$ if $n = p^m$, or 0 otherwise. The sum $\sum_{p^m\le x}$ is taken over those integers of the form p^m that are less than or equal to x, here p is a prime number and m is a positive integer. We can then conclude the integral

$$\int_1^\infty \frac{\psi(x)-x}{x^{s+1}}\,dx$$

is analytic convergence for $\Re e(s) > \frac{1}{2}$. Actually, this can conclude the proof of the form $\psi(x) = x + O(x^{\frac{1}{2}+\varepsilon})$ for $\forall\varepsilon > 0$.

2.2 Preliminaries

2.2.1 Some Significant Theorems

We make the precise theorems as follows.

For brevity, we know that a smooth curve γ given in $\mathbb{C}$ parametrized by $\gamma : [a, b] \to \mathbb{C}$, and f a continuous function on an open set U and suppose that γ is a curve in U, meaning that all values $\gamma(t)$ lie in U for $a \le t \le b$, we define the integral of f along γ by

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

By definition, we also know that the length of the smooth curve γ is

$$\text{length}(\gamma) = \int_a^b \left|\gamma'(t)\right| dt.$$

Arguing as we just know, it is clear that this definition is also independent of the parametrization. One has the inequality

$$\left|\int_\gamma f(z)\,dz\right| \le \sup_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma).$$

In addition, we know the well known theorems as follows.

Theorem 2.1. *A set of complex numbers is compact if and only if it is closed and bounded.*

Theorem 2.2. *Let S be a compact set of complex numbers, and let f be a continuous function on S. Then f is uniformly continuous, i.e., given ε there exists δ such that whenever $z, w \in S$ and $|z - w| < \delta$, then $|f(z) - f(w)| < \varepsilon$.*

Theorem 2.3. *Let S be a compact set of complex numbers, and let f be a continuous function on S. Then the image of f is compact.*

Theorem 2.4. *Let $\{f_n\}$ be a sequence of analytic functions on an open set U. Assume that for each compact subset K of U the sequence converges uniformly on K, and let the limit function be f, i.e., $\lim f_n = f$. Then f is analytic.*

In general, we can record a useful approximation theorem. Recall that a function on the circle is the same as a 2π-periodic function on $\mathbb{R}$. In other words, functions on $\mathbb{R}$ that 2π-periodic, and functions on an interval of length 2π that take on the same value at its end-points, are two equivalent descriptions of the same mathematical objects, namely, functions on the circle.

Theorem 2.5. *Let f be an integrable function on the circle and f is bounded by B. Then there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of continuous functions on the circle so that*

$$\sup_{x\in[0,2\pi]} |f_k(x)| \leq B \quad \textit{for all } k = 1, 2, \cdots, \quad \textit{and} \quad \int_0^{2\pi} |f(x) - f_k(x)|\, dx \to 0 \quad \textit{as } k \to \infty.$$

As is known to all, some concepts involved are that of Fourier coefficient of a function, orthogonality in a vector space equipped with an inner product, and its associated norm. We now review the definitions and summarize the results concerning the aim of the proof of the following Riemann-Lebesgue lemma. Here, if f is an integrable function given on an interval $[a, b]$ of length L (that is, $b - a = L$), then the n^{th} Fourier coefficient of f is defined by

$$\hat{f}(n) = \frac{1}{L} \int_a^b f(x)\, e^{-2\pi inx/L}\, dx, \qquad n \in \mathbb{Z}.$$

The Fourier series of f is given formally by $\sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{2\pi inx/L}$. We shall sometimes write a_n for the n^{th} Fourier coefficient of f, and use the notation

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n\, e^{2\pi inx/L}$$

to indicate that the series on the right-hand side is the Fourier series of f. For instance, if f is an integrable function on the interval $[0, 2\pi]$, then the n^{th} Fourier coefficient of f is

$$\hat{f}(n) = a_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, e^{-in\theta}\, d\theta, \qquad n \in \mathbb{Z},$$

and the Fourier series of f is $f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n\, e^{in\theta}$. Fourier series are part of a larger family called the trigonometric series which, by definition, are expressions of the form $\sum_{n=-\infty}^{\infty} c_n\, e^{2\pi inx/L}$, where $c_n \in \mathbb{Z}$. If a trigonometric series involves only finitely many non-zero terms, that is, $c_n = 0$ for all large $|n|$, it is called a trigonometric polynomial; its degree is the largest value of $|n|$ for which $c_n \neq 0$.

The N^{th} partial sum of the Fourier series of f, for N a positive integer, is a particular example of a trigonometric polynomial. It is given by

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)\, e^{2\pi inx/L}.$$

Note that by definition, the above sum is symmetric since n ranges from $-N$ to N, a choice that is natural because of the resulting decomposition of the Fourier series as sine and cosine series. As a consequence, the convergence of Fourier series will be understood as the "limit" as N tends to infinity of these symmetric sums.

We may state the following important result, even if we do not state the requirement.

Theorem 2.6. *continuous functions on the circle can be uniformly approximated by trigonometric polynomials.*

This means that if f is continuous on $[0, 2\pi]$ with $f(0) = f(2\pi)$ and $\epsilon > 0$, then there exists a trigonometric polynomial P such that $|f(x) - P(x)| < \epsilon$ for all $0 \le x \le 2\pi$.

Let $\mathcal{R}$ denote the set of complex-valued Riemann integrable functions on $[0, 2\pi]$ (or equivalently, integrable functions on the circle). This is a vector space over $\mathbb{C}$. Actually, an inner product is defined on this vector space by

$$(f, g) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\overline{g(\theta)}\,d\theta,$$

and norm $\| f \|$ defined by

$$\| f \|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2\,d\theta.$$

Two elements X and Y are "orthogonal" if $(X, Y) = 0$, and we write $X \perp Y$. The important result can be derived from this notion of orthogonality, which is the Pythagorean theorem: If X and Y are orthogonal, then $\| X + Y \|^2 = \| X \|^2 + \| Y \|^2$. Its proof of this fact is simple, it suffices to expand $(X + Y, X + Y)$ and use the assumption that $(X, Y) = (Y, X) = 0$.

For each integer n, let $e_n(\theta) = e^{in\theta}$, and observe that the family $\{e_n\}_{n\in\mathbb{Z}}$ is "orthonormal"; that is

$$(e_n, e_m) = \begin{cases} 1 & \text{if} \quad n = m; \\ 0 & \text{if} \quad n \neq m. \end{cases}$$

Let f be an integrable function on the circle, and let a_n denote its Fourier coefficients. An important observation is that these Fourier coefficients are represented by inner products of f with the elements in the orthonormal set $\{e_n\}_{n\in\mathbb{Z}}$:

$$(f, e_n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,e^{-in\theta}\,d\theta = a_n.$$

In particular, $S_N(f) = \sum_{|n|\le N} a_n e_n$. Then the orthonormal property of the family $\{e_n\}$ and the fact that this Fourier coefficient $a_n = (f, e_n)$ imply that the difference $f - \sum_{|n|\le N} a_n e_n$ is orthogonal to e_n for all $|n| \le N$. Therefore, we must have $(f - \sum_{|n|\le N} a_n e_n) \perp \sum_{|n|\le N} b_n e_n$ for any complex numbers b_n, where the orthogonal projection of the function f in the plane spanned by $\{e_{-N}, \cdots, e_0, \cdots, e_{-N}\}$ is simply $S_N(f)$. We can apply the Pythagorean theorem to the decomposition $f = f - \sum_{|n|\le N} a_n e_n + \sum_{|n|\le N} a_n e_n$, where we now choose $b_n = a_n$, to obtain

$$\| f \|^2 = \| f - \sum_{|n|\le N} a_n e_n \|^2 + \| \sum_{|n|\le N} a_n e_n \|^2 .$$

Since the orthonormal property of the family $\{e_n\}_{n\in\mathbb{Z}}$ implies that $\| \sum_{|n|\le N} a_n e_n \|^2 = \sum_{|n|\le N} |a_n|^2$, we deduce that

$$\| f \|^2 = \| f - S_N(f) \|^2 + \sum_{|n|\le N} |a_n|^2.$$

We may draw the theorem from the result

$$(f - \sum_{|n|\le N} a_n e_n) \perp \sum_{|n|\le N} b_n e_n$$

for any complex numbers b_n, with $S_N(f) = \sum_{|n|\le N} a_n e_n$.

Theorem 2.7 (Best approximation lemma). *If f is integrable on the circle with Fourier coefficients a_n, then*

$$\| f - S_N(f) \| \le \| f - \sum_{|n|\le N} c_n e_n \|$$

for any complex numbers c_n. Moreover, equality holds precisely when $c_n = a_n$ for all $|n| \le N$.

Note that the best approximation lemma and the relation $\| f \|^2 = \| f - S_N(f) \|^2 + \sum_{|n|\le N} |a_n|^2$ imply that if a_n is the n^{th} Fourier coefficient of an integrable function f, then the series $\sum_{n=-\infty}^{\infty} |a_n|^2$ converges, and in fact we have Parseval's identity $\sum_{n=-\infty}^{\infty} |a_n|^2 = \| f \|^2$. This identity provides an important connection between the norms in the two vector spaces $\ell^2(\mathbb{Z})$ and $\mathcal{R}$. The vector space $\ell^2(\mathbb{Z})$ over $\mathbb{C}$ is the set of all (two-sided) infinite sequences of complex numbers

$$(\cdots, a_{-n}, \cdots, a_{-1}, a_0, a_1, \cdots, a_n, \cdots) \quad \text{such that} \quad \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty.$$

We summarize the following important results.

Theorem 2.8. *Let f be an integrable function on the circle with the relation $f \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$. Then we have*

(i) Mean-square convergence of the Fourier series

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 \, d\theta \to 0 \quad \text{as } N \to \infty.$$

(ii) Parseval's identity

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 \, d\theta.$$

Proof. We can now give the proof that $\| f - S_N(f) \| \to 0$ using the best approximation lemma, as well as the important fact that trigonometric polynomials are dense in the space of continuous functions on the circle.

Suppose that f is continuous on the circle. Then, given $\epsilon > 0$, there exists a trigonometric polynomial P, say of degree M, such that $|f(\theta) - P(\theta)| < \epsilon$ for all θ. In particular, taking squares and integrating this inequality yields $\| f - P \| < \epsilon$, and by the best approximation lemma we conclude that $\| f - S_N(f) \| \leq \| f - P \|$, and then $\| f - S_N(f) \| < \epsilon$ whenever $N \geq M$. This proves the theorem when f is continuous.

If f is merely integrable, by definition, a Riemann integrable function is bounded, say $|f(\theta)| \leq M$ for some positive constant M, we can no longer approximate f uniformly by trigonometric polynomials. Instead, we apply the approximation Theorem 2.5 and choose a continuous function g on the circle which satisfies

$$\sup_{\theta\in[0,2\pi]} |g(\theta)| \leq \sup_{\theta\in[0,2\pi]} |f(\theta)| = B \quad \text{and} \quad \int_0^{2\pi} |f(\theta) - g(\theta)| \, d\theta < \epsilon^2.$$

Then we get

$$\| f - g \|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta) - g(\theta)|^2 \, d\theta = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta) - g(\theta)| \cdot |f(\theta) - g(\theta)| \, d\theta \leq \frac{2B}{2\pi}\int_0^{2\pi} |f(\theta) - g(\theta)| \, d\theta \leq C\epsilon^2.$$

Therefore, $\| f - g \| \leq \sqrt{C}\epsilon$ where C is a positive constant. Now we may approximate g by a trigonometric polynomial P so that the relation $\| g - P \| < \epsilon$. Then $\| f - P \| \leq \| f - g \| + \| g - P \| < C_1\epsilon$ for some positive constant C_1, and we may again conclude by applying the best the approximation lemma. This completes the proof that the partial sums of the Fourier series of f converge to f in the mean square norm $\| \cdot \|$. This proves the theorem. □

Since the terms a_n of the converging series $\sum_{n=-\infty}^{\infty} |a_n|^2$ tend to 0 as $|n|$ tends to ∞, we deduce from Parseval's identity the following result.

Theorem 2.9 (Riemann-Lebesgue lemma)**.** *If f is integrable on the circle, then $\hat{f}(n) \to 0$ as $|n| \to \infty$. An equivalent reformulation of this proposition is that if f is integrable on the interval $[0, 2\pi]$, for N a positive integer, then*

$$\lim_{N\to\infty}\int_0^{2\pi} f(\theta)\sin(N\theta) \, d\theta = \lim_{N\to\infty}\int_0^{2\pi} f(\theta)\cos(N\theta) \, d\theta = 0,$$

and

$$\lim_{T\to\infty}\int_a^b f(x)\sin(Tx) \, dx = \lim_{T\to\infty}\int_a^b f(x)\cos(Tx) \, dx = 0$$

for $f(x)$ on a finite interval $[a, b]$ of real numbers, whenever the real number $T \neq 0$.

2.2.2 Abel's Identity and Basic Properties of the Riemann Zeta Function

Sometimes, sums involving step functions of the type can be expressed as integrals by means of the following Abel's identity. For any arithmetical function $a(n)$ let $A(x) = \sum_{n\leq x} a(n)$, where $A(x) = 0$ if $x < 1$. Assume f has a continuous derivative on the interval $[y, x]$, where $0 < y < x$. Then we have

$$\sum_{y<n\leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t) \, dt.$$

Indeed, for an arbitrary function $f \in C^1([y,x])$, the Abel's identity

$$\sum_{y<n\le x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\,dt$$

can be verified directly by using integration by parts. Since $A(x)$ is a step function with jump $f(n)$ at each integer n, the sum $\sum_{y<n\le x} a(n)f(n)$ can be expressed as a Riemann-Stieltjes integral $\sum_{y<n\le x} a(n)f(n) = \int_y^x f(t)\,dA(t)$, using the definition of the Riemann-Stieltjes integral, then integration by parts gives us the result.

Note. Since $A(t) = 0$ if $t < 1$, when $y < 1$ this Abel's identity takes the form

$$\sum_{1\le n\le x} a(n)f(n) = A(x)f(x) - \int_1^x A(t)f'(t)\,dt,$$

where f has a continuous derivative on the interval $[1,x]$. We may then get

$$\sum_{1\le n\le x} a_n f(n) = \sum_{1\le n\le x} a_n \left\{ f(x) - \int_n^x f'(t)\,dt \right\} = f(x)\sum_{1\le n\le x} a_n - \int_1^x f'(t)\Big(\sum_{1\le n\le t} a_n\Big)dt$$

for an arbitrary function $f \in C^1([1,x])$ and $a_n \in \mathbb{C}$.

For $\Re e(s) > 1$, we know that the Zeta function defined by the series $\sum_{n=1}^\infty \frac{1}{n^s}$, and the Euler product $\prod_p (1-p^{-s})^{-1}$, namely $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_p (1-p^{-s})^{-1}$, where the product is over all prime numbers p. The Euler product shows that $\zeta(s) \neq 0$ for $\Re e(s) > 1$. The series and the Euler product converge absolutely and uniformly for $\Re e(s) \ge 1+\delta$, with any $\delta > 0$. It was initiated and used by Euler to prove that $\sum_p 1/p$ diverges, and the behavior of $\zeta(s)$ for real $s > 1$ with s tending to 1. While there is no difficulty in seeing that $\zeta(s)$ is well-defined when $\Re e(s) > 1$, it was Riemann who realized that the further study of prime numbers was bound up with the analytic continuation of ζ into the rest of the complex plane. In the half plane $\Re e(s) = \sigma > 1$, the ζ-function is given explicitly by the series $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$, writing $s = \sigma + it$ where σ and t are real, and it is therefore subject to the estimate $|\zeta(s)| \le \zeta(\sigma)$. Riemann recognized that there is a rather simple relationship between $\zeta(s)$ and $\zeta(1-s)$. As a consequence, one has good control of the behavior of the ζ-function also in the half $\sigma < 0$.

Moreover, the basic structural property of the Zeta function with respect to the Gamma function, which essentially characterizes it: $1/\Gamma(s)$ is an entire function which has simple zeros at exactly $s = 0, -1, -2, -3, \cdots$.

Indeed, the Gamma function $\Gamma(s)$ can be defined for $s \in \mathbb{C}$ with $\Re e(s) > 0$ by the integral

$$\Gamma(s) = \int_0^\infty e^{-t}\,t^s\,\frac{dt}{t}.$$

The integral converges absolutely for $\Re e(s) > 0$. On replacing t by nt in the integral $\Gamma(s) = \int_0^\infty e^{-t}\,t^s\,\frac{dt}{t}$ for $\Re e(s) > 1$, which leaves the integral invariant, we obtain

$$n^{-s}\,\Gamma(s) = \int_0^\infty e^{-nt}\,t^{s-1}\,dt,$$

and summation with respect to n leads to

$$\sum_{n=1}^\infty n^{-s}\,\Gamma(s) = \int_0^\infty \sum_{n=1}^\infty e^{-nt}\,t^{s-1}\,dt.$$

Because $\Re e(s) > 1$ the integral is absolutely convergent at both ends, and this justifies the interchange of integration and summation, so that

$$\zeta(s)\,\Gamma(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1}\,dt \quad \text{for } \Re e(s) > 1.$$

We shall present a more elementary approach to the properties of the Zeta function, which easily leads to its extension in the half-plane $\Re e(s) > 0$.

Proposition 2.10. *There is a sequence of entire functions $\{\delta_n(s)\}_{n=1}^\infty$ that these functions satisfy the estimate $|\delta_n(s)| \le |s|/n^{\sigma+1}$, where $s = \sigma + it$, and such that*

$$\sum_{1\le n<N} \frac{1}{n^s} - \int_1^N \frac{1}{x^s}\,dx = \sum_{1\le n<N} \delta_n(s),$$

whenever N is an integer > 1.

Proof. We compare $\sum_{1\le n<N} n^{-s}$ with $\sum_{1\le n<N}\int_n^{n+1} x^{-s}\,dx$, and set

$$\delta_n(s) = \int_n^{n+1}\left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx.$$

The integral mean-value theorem applied to $f(u) = u^{-s}$ and $f'(u) = (-s)u^{-s-1}$ such that

$$\frac{1}{n^s} - \frac{1}{x^s} = \int_n^x \frac{s}{u^{s+1}}\,du$$

yields

$$\left|\frac{1}{n^s} - \frac{1}{x^s}\right| \le \frac{|s|}{n^{\sigma+1}} \quad \text{whenever } n \le x \le n+1.$$

Therefore, $|\delta_n(s)| \le |s|/n^{\sigma+1}$, and since $\int_1^N \frac{1}{x^s}\,dx = \sum_{1\le n<N}\int_n^{n+1}\frac{1}{x^s}\,dx$, the proposition is proved. □

The proposition 2.10 has the following consequence.

Corollary 2.11. *For $\Re e(s) > 0$ we have $\zeta(s) - \frac{1}{s-1} = H(s)$, where the function $H(s) = \sum_{n=1}^{\infty}\delta_n(s)$ is analytic in the half-plane $\Re e(s) > 0$.*

Proof. Following the proposition 2.10 and turning to this idea, we assume first that $\Re e(s) > 1$. We let N tend to infinity in the formula

$$\sum_{1\le n<N}\frac{1}{n^s} - \int_1^N \frac{1}{x^s}\,dx = \sum_{1\le n<N}\delta_n(s), \quad \text{where} \quad \int_1^\infty \frac{1}{x^s}\,dx = \frac{1}{s-1},$$

and observe that by the estimate $|\delta_n(s)| \le |s|/n^{\sigma+1}$, we have the uniform convergence of the series $\sum\delta_n(s)$ which is in any half-plane $\Re e(s) > \sigma$ with $\sigma > 0$. Since $\Re e(s) > 1$, the series $\sum n^{-s}$ converges to $\zeta(s)$, and this proves the assertion when $\Re e(s) > 1$. By Theorem 2.4, the uniform convergence also shows that $\sum\delta_n(s)$ is analytic when $\Re e(s) > 0$, and thus shows that $\zeta(s)$ is extendable to the half-plane, and that the identity continues to hold there. □

Let us state the following Theorem 2.12 and Theorem 2.14.

Theorem 2.12. *For $\Re e(s) > 1$,*

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^z - 1}\,dz$$

where C beginning and ending near the positive real axis, and $(-z)^{s-1}$ is defined on the complement of positive real axis as $e^{(s-1)\log(-z)}$ with $-\pi \le \Im m\log(-z) \le \pi$.

The importance of the formula $\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^z-1}\,dz$ for $\Re e(s) > 1$, which lies in the fact that the right-hand side is defined and meromorphic for all values of s, so the formula can be used to extend $\zeta(s)$ to a meromorphic function in the whole plane. It is indeed quite obvious that the integral $\int_C \frac{(-z)^{s-1}}{e^z-1}\,dz$ is an entire function of s in the formula, while $\Gamma(1-s)$ is meromorphic with poles at $s = 1, 2, 3, \cdots$. Because $\zeta(s)$ is already known to be analytic for $\Re e(s) > 1$, the poles of $\Gamma(1-s)$ at the integers $n \ge 2$ must cancel against zeros of the integral $\int_C \frac{(-z)^{s-1}}{e^z-1}\,dz$. At $s = 1$, the function $-\Gamma(1-s)$ has a simple pole with the residue 1, as seen for instance by the explicit representation $\Gamma(s) = \frac{e^{-\gamma s}}{s}\prod_{n=1}^{\infty}\left(1 + \frac{s}{n}\right)^{-1} e^{\frac{s}{n}}$. On the other hand, by residues we obtain $\frac{1}{2\pi i}\int_C \frac{1}{e^z-1}\,dz = 1$. As a result, $\zeta(s)$ has the residue 1 at $s = 1$. We formulate the result as a corollary.

Corollary 2.13. *The $\zeta(s)$-function can be extended to a meromorphic function in the whole plane whose only pole is a simple pole at $s = 1$ with the residue 1.*

We shall reproduce a standard of the Zeta functional equation, as it is commonly called.

Theorem 2.14. *The ζ-function is given a rather simple relationship between $\zeta(s)$ and $\zeta(1-s)$, i.e., for all $s \in \mathbb{C}$ we have*

$$\zeta(s) = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\,\Gamma(1-s)\,\zeta(1-s). \tag{2.1}$$

There are equivalent forms of the functional equation. For instance, it implies $\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\,\Gamma(s)\,\zeta(s)$ for all $s \in \mathbb{C}$. In addition, the identity $\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$ reveals the symmetry of Γ about the line $\Re e(s) = 1/2$, and the functional equation (2.1) for $\zeta(s)$ such that the nontrivial zeros of $\zeta(s)$ has a certain symmetry about the critical line $\Re e(s) = 1/2$. We know the functional equation (2.1) for $\zeta(s)$ shows that all the nontrivial zeros must lie in the strip $0 < \Re e(s) < 1$, the so-called "critical strip". Moreover, note that the nontrivial zeros of $\zeta(s)$ has a certain symmetry about the real axis, namely $\zeta(\overline{s}) = \overline{\zeta(s)}$, this relation is immediate from the Euler product and Corollary 2.13 as well as the series expansion of the expression $\zeta(s) - \frac{1}{s-1} = H(s)$ in Corollary 2.11, which lead to its extension in the half-plane $\Re e(s) > 0$, with the idea described above can be developed step by step to yield the analytic continuation of ζ into the entire complex plane. These are easy to show that the nontrivial zeros are symmetrically located about the two lines. It follows that if s_0 is a complex number where $\zeta(s)$ has a zero of order m, then the complex conjugate $\overline{s_0}$ is a complex number where $\zeta(s)$ has a zero of the same order m, then $1-s_0$ is also a complex number where $\zeta(s)$ has a zero of the same order m, so is the complex conjugate $\overline{1-s_0}$, and therefore these orders m are the same (which may be a pole, in which case m is negative, but $\zeta(s)$ has no a pole in the critical strip $0 < \Re e(s) < 1$).

In fact, by Abel's identity, we often obtain the following identities, valid for $\Re e(s) > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} s\int_n^{\infty} \frac{dx}{x^{s+1}} = s\int_1^{\infty}\Big(\sum_{n\le x} 1\Big)\frac{dx}{x^{s+1}} = s\int_1^{\infty} \frac{[x]}{x^{s+1}}\,dx = \frac{s}{s-1} - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}}\,dx.$$

The symbol $[x]$ denotes the greatest integer $\le x$, it is called the integral part of x, the number $\{x\} = x - [x]$ is called the fractional part of x, it satisfies the inequalities $0 \le \{x\} < 1$, with $\{x\} = 0$ if and only if x is an integer. Moreover, we know the integral $\int_1^{\infty} \frac{\{x\}}{x^{s+1}}\,dx$ converges absolutely, and uniformly for $\Re e(s) \ge \delta$ with any $\delta > 0$.

Also, we define the so-called Chebyshev's function

$$\psi(x) = \sum_{p^m \le x} \log p = \sum_{1\le n\le x} \Lambda(n),$$

where p is a prime number and m is a positive integer. The sum is taken over those integers of the form p^m that are less than or equal to x. This $\Lambda(n)$ is equal to $\log p$ if $n = p^m$, or 0 otherwise. Note that crude estimates give

$$\psi(x) = \sum_{p\le x}\left[\frac{\log x}{\log p}\right]\log p \le \sum_{p\le x}\frac{\log x}{\log p}\log p = \pi(x)\log x.$$

We say that a step function is a piecewise constant function having only finitely many pieces with each given as the finite sum, and we observe that the formula $\psi(x) = \sum_{n=1}^{N} \Lambda(n) f_n(x)$, where $f_n(x) = 1$ if $n \le x$ and $f_n(x) = 0$ otherwise. In particular, it is obvious to see that $\psi(x)$ is a step function which begins at 0 and has a jump of $\log(p^m)\cdot(1/m) = \log p$ at each prime power p^m, whereas the positive integer $m \ge 1$. In fact, $\psi(x)$ is locally constant: there is no change in ψ between prime numbers, which the function is locally constant at a point if there exists an open set containing this point such that the function is constant on the open set. As a matter of fact, note that a piecewise function is continuous on a given interval if the conditions are met: it is defined throughout that interval, its constituent functions are continuous on that interval, there is no discontinuity at each endpoint of the subdomains within that interval. Moreover, a constant function is a trivial example of a step function, and a piecewise constant function is piecewise continuous, whereas a function is said to be piecewise constant if it is locally constant in connected regions separated by a possibly infinite number of lower-dimensional boundaries.

We know that for $\Re e(s) > 1$ taking the logarithm of the Euler product formula $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_p (1-p^{-s})^{-1}$, and using the power series expansion for the logarithm $\log\big(\frac{1}{1-x}\big) = \sum_{m=1}^{\infty}\frac{x^m}{m}$, which holds for $0 \le x < 1$, while we have the formulas $\frac{1}{1-x} = \sum_{m=1}^{\infty} x^{m-1}$ and $\int \frac{1}{1-x}\,dx = \log\big(\frac{1}{1-x}\big) + C$ for $0 \le x < 1$ with some constant C, and then we find that

$$\log\zeta(s) = \log\prod_p (1-p^{-s})^{-1} = \sum_p \log\left(\frac{1}{1-p^{-s}}\right) = \sum_{m,p}\frac{p^{-ms}}{m}.$$

Differentiating this expression gives

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{m,p}(\log p)\,p^{-ms} = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}.$$

In view of the fact that $\psi(x)$ is a monotonically increasing function on a finite interval, the sum $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$ can be expressed as a Riemann-Stieltjes integral, using the definition of the Riemann-Stieltjes integral, we have

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \int_1^{\infty} x^{-s}\, d\psi(x).$$

For $\Re e(s) > 1$, by Abel's identity, we also know the identity (∗!):

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = s\int_1^{\infty} \frac{\psi(x)}{x^{s+1}}\, dx = \frac{s}{s-1} + s\int_1^{\infty} \frac{\psi(x)-x}{x^{s+1}}\, dx,$$

and by the Euler product we get $-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s-1}$ for $\Re e(s) > 1$, where the sum $\sum_p \frac{\log p}{p^s-1}$ is extended over all primes. Moreover, we have already seen that $\zeta(1+it) \neq 0$ for any real number t so that the Zeta functional equation gives $\zeta(it) \neq 0$, while the Euler product gives $\zeta(s) \neq 0$ for $\Re e(s) > 1$. If the function $\zeta'/\zeta(s)$ has no poles on the region $1 > \Re e(s) > \frac{1}{2}$, then which implies that the function $\zeta(s)$ has no zeros on the region $1 > \Re e(s) > \frac{1}{2}$.

In a way, we need go through the basic results. We recall the notation: $f = O(g)$ or $f \ll g$ means that f, g are two functions of a variable x, defined for all x sufficiently large, and g is positive, there exists a constant $C > 0$ such that $|f(x)| \leq Cg(x)$ for all $|x|$ sufficiently large. We also recall the results, let ρ be all nontrivial zeros of $\zeta(s)$, we know $\psi(x) = x + O(x^{\sup \Re e(\rho)+\varepsilon})$ for $\forall\varepsilon > 0$. Intimately, we can get $\lambda \geq \sup \Re e(\rho) + \varepsilon$ by the result that if $\zeta(s)$ has no zero on the region $\Re e(s) > \lambda$. It is a well-known fact that the Riemann Zeta function $\zeta(s)$ has infinitely many zeros in the critical strip $0 < \Re e(s) < 1$. One can combine ideas from the functional equation $\zeta(s) = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\,\Gamma(1-s)\,\zeta(1-s)$ for $\zeta(s)$. As a result, $\zeta(s)$ has infinitely many zeros in the region $\Re e(s) \geq \frac{1}{2}$, from which the nontrivial zeros are symmetrically located about the critical line $\Re e(s) = \frac{1}{2}$ in the critical strip.

As a consequence, we may state the following proposition.

Proposition 2.15. *If $\psi(x) = x + O(x^{\lambda})$ for $0 < \lambda < 1$, then the function $-\frac{\zeta'(s)}{\zeta(s)}$ is meromorphic on the region $\Re e(s) > \lambda$, and has a pole at $s = 1$, but no other poles in this region. Furthermore, if one has the form $\psi(x) = x + O(x^{\lambda})$, then one has the consequence $\lambda \geq \frac{1}{2}$, and the function $-\frac{\zeta'(s)}{\zeta(s)} - \zeta(s)$ has no poles for $\Re e(s) > \lambda$.*

Proof. First note that $-\frac{\zeta'(s)}{\zeta(s)} - \zeta(s) = \sum_{n=1}^{\infty} (\Lambda(n) - 1)n^{-s}$ for $\Re e(s) > 1$.

For $\Re e(s) > 1$, we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = s\int_1^{\infty} \frac{\psi(x)}{x^{s+1}}\, dx = \frac{s}{s-1} + s\int_1^{\infty} \frac{\psi(x)-x}{x^{s+1}}\, dx$$

and

$$-\frac{\zeta'(s)}{\zeta(s)} - \zeta(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^s} = s\int_1^{\infty} \frac{\psi(x)-[x]}{x^{s+1}}\, dx = s\int_1^{\infty} \frac{\psi(x)-x}{x^{s+1}}\, dx + s\int_1^{\infty} \frac{\{x\}}{x^{s+1}}\, dx.$$

By our assumption that $\psi(x) = x + O(x^{\lambda})$ for $0 < \lambda < 1$, we obtain the integral $\int_1^{\infty} \frac{\psi(x)-x}{x^{s+1}}\, dx$ converges absolutely and uniformly for $\Re e(s) \geq \delta > \lambda$ with any $\delta > \lambda$. Since the integral $\int_1^{\infty} \frac{\{x\}}{x^{s+1}}\, dx$ converges absolutely and uniformly for $\Re e(s) \geq \delta > 0$ with any $\delta > 0$. By the differentiation lemma (see below), it suffices to prove that the integrals $\int_1^{\infty} \frac{\psi(x)-x}{x^{s+1}}\, dx$ and $\int_1^{\infty} \frac{\{x\}}{x^{s+1}}\, dx$ are analytic for $\Re e(s) > \lambda$, so that the function $-\frac{\zeta'(s)}{\zeta(s)} - \zeta(s)$ has an analytic continuation to the region $\Re e(s) > \lambda$, and that $-\frac{\zeta'(s)}{\zeta(s)}$ is meromorphic on the region $\Re e(s) > \lambda$, and has a pole at $s = 1$, but no other poles in this region. In particular, yet still note that the Zeta function $\zeta(s)$ has infinitely many zeros in the critical strip $0 < \Re e(s) < 1$, as a result that $\zeta(s)$ has infinitely many zeros in the region $\Re e(s) \geq \frac{1}{2}$, with the symmetry of the nontrivial zeros about the line $\Re e(s) = \frac{1}{2}$. Because if one has $\lambda < \frac{1}{2}$ in the form $\psi(x) = x + O(x^{\lambda})$ then one has no zeros in the region $\Re e(s) \geq \frac{1}{2}$, which is a contradiction, and that as a consequence, one surely has $\lambda \geq \frac{1}{2}$ in the form $\psi(x) = x + O(x^{\lambda})$. This proves the proposition. □

In a nutshell, we have $\lambda \geq \frac{1}{2}$ for $\psi(x) = x + O(x^{\lambda})$ and $\psi(x) = O(x)$. By the way, the part of the proposition that if one has the form $\psi(x) = x + O(x^{\lambda})$ then one has the consequence $\lambda \geq \frac{1}{2}$, which shall imply the proof of the form $\psi(x) = x + O(x^{\frac{1}{2}+\epsilon})$ for every $\epsilon > 0$.

2.3 Main Lemmas

Lemma 2.16 (The differentiation lemma)**.** *Let I be an interval of real numbers, possibly infinite. Let U be an open set of complex numbers. Let $f = f(t,z)$ be a continuous function on $I \times U$. Assume:*

(i) For each compact subset K of U, the integral $\int_I f(t,z)\,dt$ is uniformly convergent for $z \in K$.

(ii) For each t the function $z \mapsto f(t,z)$ is analytic.

Let $F(z) = \int_I f(t,z)\,dt$, then F is analytic on U, $D_2 f(t,z)$ satisfies the same assumptions as f, and

$$F'(z) = \int_I D_2 f(t,z)\,dt.$$

In addition, let f be a piecewise continuous function on the real numbers ≥ 0 and assume that there is constants A, B such that $|f(t)| \leq Ae^{\beta t}$ for $t \geq 0$. However, just assume for simplicity that if f is bounded, piecewise continuous, then we take the form $|f(t)| \leq M$ for some finite number $M > 0$. What we prove will hold under much less restrictive conditions: instead of piecewise continuous, it would suffice to assume that either the integral $\int_a^b |f(t)|\,dt$ exists for every pair of numbers $a, b \geq 0$ or the integral $\int_0^\infty \left|f(t)\,e^{-zt}\right|\,dt$ exists for $\Re e(z) \geq \beta$, where β is some real constant. We shall associate to f the Laplace transform defined by

$$g(z) = \int_0^\infty f(t)\,e^{-zt}\,dt \quad \text{for } \Re e(z) > 0.$$

We can then apply the differentiation lemma, whose proof applies to a function $f(t)$ satisfying our conditions (bounded and piecewise continuous), and then we easily conclude that g is analytic for $\Re e(z) > 0$. Furthermore, we can prove that the function

$$g(z) = \int_0^\infty f(t)\,e^{-zt}\,dt \quad \text{for } \Re e(z) > \beta$$

converges absolutely, and the function is analytic on the region $\Re e(z) > \beta$ for some real number β.

We shall now state special cases of the following lemmas concerning differentiation under the integral sign which are sufficient for our applications.

Lemma 2.17. *Let $f(t)$ be piecewise continuous function on the real numbers ≥ 0, and assume that there exist some real number constants B, β such that $|f(t)| \leq B\,e^{\beta t}$ for all t sufficiently large. Let f the Laplace transform g defined by*

$$g(z) = \int_0^\infty f(t)\,e^{-zt}\,dt \qquad \textit{for } \Re e(z) > \beta.$$

Then $g(z)$ converges absolutely for $\Re e(z) > \beta$, and it converges uniformly in the region $\Re e(z) \geq \alpha > \beta$, and then $g(z)$ is analytic for $\Re e(z) > \beta$.

Proof. Since $\Re e(z) > \beta$, there necessarily exists a positive number $\delta > 0$ such that $\Re e(z) > \beta + \delta$. Moreover, we have the condition $|f(t)| \leq B\,e^{\beta t}$ for all t sufficiently large with some constants B, β.
Hence

$$\int_0^\infty \left|f(t)\,e^{-zt}\right|\,dt \leq \int_0^\infty B\,e^{(\beta+\delta)t} \cdot e^{-\Re e(z)\,t}\,dt = \frac{B}{\Re e(z) - (\beta+\delta)}$$

for $\Re e(z) > \beta + \delta$. Similarly, we can also obtain $\Re e(z) \geq \alpha > \beta + \delta$ when $\Re e(z) \geq \alpha > \beta$ with some constants B, β, δ, α.
Thus

$$\int_0^\infty \left|f(t)\,e^{-zt}\right|\,dt \leq \int_0^\infty B\,e^{(\beta+\delta)t} \cdot e^{-\Re e(z)\,t}\,dt \leq \frac{B}{\alpha - (\beta+\delta)},$$

which gives that the Laplace transform $g(z)$ converges absolutely for $\Re e(z) > \beta$ and $g(z)$ converges uniformly for $\Re e(z) \geq \alpha > \beta$, and then $g(z)$ is analytic for $\Re e(z) > \beta$ as follows at once by the differentiation lemma. Therefore this proves the lemma. □

Lemma 2.18 (the so-called "D. J. Newman Theorem")**.** *Let $f(t)$ be bounded, piecewise continuous function on the real numbers ≥ 0. Let $f(t)$ the Laplace transform $g(z)$ defined by $g(z) = \int_0^\infty f(t)\,e^{-zt}\,dt$ for $\Re e(z) > 0$, then g is analytic in the region $\Re e(z) > 0$. In fact, the integral converges absolutely for $\Re e(z) > 0$. If g extends to an analytic function for $\Re e(z) \geq 0$, then $\int_0^\infty f(t)\,dt$ exists and is equal to $g(0)$. (Indeed, its proof can be contained in the proof of the next Lemma 2.19 when we let the following real number α be equal to 0.)*

Lemma 2.19 (Main Lemma A1, it contains the "D. J. Newman Theorem", which has been improved)**.** *Let $f(t)$ be piecewise continuous function on the real numbers ≥ 0, and assume that there exist some finite real numbers B, α such that $|f(t)| \leq B\,e^{\alpha t}$ for all t in the interval $0 \leq t \leq \infty$, for any $T > 0$ the interval $0 \leq t \leq T$ can be divided into finite segments, and the function $f(t)$ is certainly continuous in each finite segment. Let $f(t)$ the Laplace transform $g(z)$ defined by*

$$g(z) = \int_0^\infty f(t)\,e^{-zt}\,dt \qquad \text{for } \Re e(z) > \alpha,$$

then g is analytic in the region $\Re e(z) > \alpha$. In fact, the integral converges absolutely for $\Re e(z) > \alpha$. If g extends to an analytic function for $\Re e(z) \geq \alpha$, then

$$\int_0^\infty f(t)\,e^{-\alpha t}\,dt \quad \text{exists and is equal to } g(\alpha).$$

In particular, if the integral

$$\int_0^\infty f(t)\,e^{-\alpha t}\,dt \quad \text{converges for } f(t) \geq 0 \text{ with } \alpha > 0,$$

then

$$\lim_{\alpha\to 0}\int_0^\infty f(t)\,e^{-\alpha t}\,dt = \int_0^\infty f(t)\,dt,$$

and then this $\int_0^\infty f(t)\,dt$ also converges from which $\lim_{\alpha\to 0}\int_0^\infty f(t)\,e^{-\alpha t}\,dt$ exists as well as this limit is equal to $\int_0^\infty f(t)\,dt$.

Proof. Since $|f(t)| \leq B\,e^{\alpha t}$ for all $t \geq 0$ with some real constants B and α, then we can apply Lemma 2.17 and the differentiation lemma to conclude that g is analytic in the region $\Re e(z) > \alpha$.

For $T > 0$ define

$$g_T(z) = \int_0^T f(t)e^{-zt}\,dt.$$

Then g_T is an entire function, as follows at once by the differentiation lemma.

We have to show that

$$\lim_{T\to\infty} g_T(\alpha) = g(\alpha).$$

Let $\delta > 0$ and let C be the path consisting of the line segment $\Re e(z) = \alpha - \delta$ and the arc of circle $|z - \alpha| = R$ with $\Re e(z) \geq \alpha$, and $\Re e(z) \geq \alpha - \delta$, where α is some real number.

By our assumption that g extends to an analytic function for $\Re e(z) \geq \alpha$, where α is some real number, we can take δ small enough so that g is analytic on the region bounded by C, and on its boundary. Then by Cauchy's integral formula, we have

$$g(\alpha) - g_T(\alpha) = \frac{1}{2\pi i}\int_C (g(z) - g_T(z))\,e^{T(z-\alpha)}\Big(1 + \frac{(z-\alpha)^2}{R^2}\Big)\frac{dz}{z-\alpha} = \frac{1}{2\pi i}\int_C H_T(z)\,dz,$$

where $H_T(z)$ abbreviates the expression under the integral sign with some real number α, $|f(t)| \leq B\,e^{\alpha t}$ for all $t \geq 0$ and some real constants B with α.

Let C^+ be the semicircle $|z - \alpha| = R$, and $\Re e(z) \geq \alpha$, where α is some real numbers. Then we claim that

$$\left|\frac{1}{2\pi i}\int_{C^+} H_T(z)\,dz\right| \leq \frac{2B}{R}. \tag{2.2}$$

First note that for $\Re e(z) > \alpha$, we have

$$|g(z) - g_T(z)| = \left|\int_T^\infty f(t)e^{-zt}\,dt\right| \leq B\int_T^\infty \left|e^{(\alpha-z)t}\right|\,dt = \frac{B}{\left|\alpha - \Re e(z)\right|}\,e^{(\alpha-\Re e(z))T};$$

and for $|z - \alpha| = R$, we obtain

$$\left|e^{T(z-\alpha)}\Big(1 + \frac{(z-\alpha)^2}{R^2}\Big)\frac{1}{z-\alpha}\right| = e^{(\Re e(z)-\alpha)T}\left|\frac{R}{z-\alpha} + \frac{z-\alpha}{R}\right|\frac{1}{R} = e^{(\Re e(z)-\alpha)T}\cdot\frac{2\left|\Re e(z)-\alpha\right|}{R^2},$$

where α is some real number, and $(z-\alpha)\cdot\overline{z-\alpha} = |z-\alpha|^2 = R^2$ leads to $\frac{R}{z-\alpha} = \frac{\overline{z-\alpha}}{R}$, with $\Re e(z-\alpha) = \Re e(z)-\alpha$. Taking the product of the last two estimates and multiplying by the length of the semicircle gives a bound for the integral over the semicircle, and this proves the claim (2.2).

Let C^- be the part of the path C with $\Re e(z) < \alpha$, where α is some real number. We wish to estimate

$$\frac{1}{2\pi i}\int_{C^-}\Big(g(z)-g_T(z)\Big)e^{T(z-\alpha)}\left(1+\frac{(z-\alpha)^2}{R^2}\right)\frac{dz}{z-\alpha}.$$

Now we estimate separately the expression under the integral with g and g_T. We have

$$\left|\frac{1}{2\pi i}\int_{C^-} g_T(z)\,e^{T(z-\alpha)}\left(1+\frac{(z-\alpha)^2}{R^2}\right)\frac{dz}{z-\alpha}\right| \le \frac{B}{R}. \tag{2.3}$$

Let S^- be the semicircle with the circle $|z-\alpha| = R$ and $\Re e(z) < \alpha$, where α is some real number. Since g_T is entire, we can replace C^- by S^- in the integral without changing the value of the integral, because the integrand has no poles to the left of the line $\Re e(z) = \alpha$. Now we estimate the expression under the integral sign on S^-. We have

$$|g_T(z)| = \left|\int_0^T f(t)e^{-zt}\,dt\right| \le B\int_{-\infty}^T \left|e^{(\alpha-z)t}\right|\,dt = \frac{B\,e^{(\alpha-\Re e(z))T}}{\left|\alpha-\Re e(z)\right|}.$$

For the other factor we use the same estimate as previously, we take the product of the two estimates, and multiply by the length of the semicircle to give the desired bound in (2.3).

Third, we claim that

$$\int_{C^-} g(z)\,e^{T(z-\alpha)}\Big(1+\frac{(z-\alpha)^2}{R^2}\Big)\frac{dz}{z-\alpha} \to 0 \quad \text{as } T\to\infty. \tag{2.4}$$

We can write the expression under the integral sign as

$$g(z)\,e^{T(z-\alpha)}\left(1+\frac{(z-\alpha)^2}{R^2}\right)\frac{1}{z-\alpha} = h(z)\,e^{T(z-\alpha)},$$

where $h(z)$ is independent of T, and α is some real number with $|z-\alpha| = R$.

Given any compact subset K of the region defined by $\Re e(z) < \alpha$, we note that

$$e^{T(z-\alpha)} \to 0 \quad \text{rapidly uniformly for } z\in K, \text{ as } T\to\infty, \text{ where } T>0.$$

The word "rapidly" means that the expression divided by any power T^N also tends to 0 uniformly for z in K, as $T\to\infty$, where $T > 0$. Recall that we can take $\delta > 0$ small enough so that g is analytic on the region bounded by the path C and on its boundary, including on the path part C^- consisting of the line segment $\Re e(z) = \alpha-\delta$ and the arc of circle $|z-\alpha| = R$ with $\Re e(z) < \alpha$. The path part C^- is certainly a piecewise smooth curve and its any finite piecewise smooth part curve γ is compact in K. According to Theorem 2.3, the image of every finite piecewise smooth part curve γ of C^- in K is also compact. Let's write $\int_{C^-} = \sum_\gamma \int_\gamma$, we have

$$\int_{C^-} h(z)\,e^{T(z-\alpha)}\,dz = \sum_\gamma \int_\gamma h(z)\,e^{T(z-\alpha)}\,dz$$

and

$$\int_\gamma h(z)\,e^{T(z-\alpha)}\,dz = \int_a^b h(\gamma(t))\,\gamma'(t)\,e^{T(\gamma(t)-\alpha)}\,dt = \int_a^b \Big(u(t)+iv(t)\Big)e^{T\left(\Re e(z)-\alpha+\Im m(z)\right)}\,dt,$$

where $h(z)$ is compact for z in K. In general, a smooth curve γ given in $\mathbb{C}$ parametrized by $\gamma(t) : [a,b] \to K$, and write $h(z) = h(\gamma(t))\,\gamma'(t) = u(t)+iv(t)$, where u and v are real-valued functions on the finite interval $[a,b]$, we can apply the argument according to Theorem 2.1 to the real and imaginary parts separately so that u and v are bounded, from which $h(z)$ is compact for z in K. By Theorem 2.9 (Riemann-Lebesgue lemma), we obtain

$$\lim_{T\to\infty}\int_a^b u(t)\,e^{T\Im m(z)}\,dt = \lim_{T\to\infty}\int_a^b v(t)\,e^{T\Im m(z)}\,dt = 0.$$

Since

$$e^x = 1 + x + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \cdots ,$$

this shows that

$$x^{-n}\, e^x > \frac{x}{(n+1)!} \quad \text{for positive } x \text{ and for all } n = 0, 1, 2, 3, \cdots .$$

On replacing x by Bx with the constant $B > 0$ in the inequality $x^{-n}\, e^x > \frac{x}{(n+1)!}$, we get

$$(Bx)^{-n}\, e^{Bx} > \frac{Bx}{(n+1)!}, \quad \text{where the constant } B > 0.$$

For any fixed n, as $x \to \infty$, it follows that e^x grows faster than any fixed power of x. We can write $x^n = o(e^x)$ to mean $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for all n, namely $\log x = o(x^\delta)$ for $\delta > 0$ with $x > 0$. Furthermore, we obtain

$$\lim_{x\to\infty} \frac{(Bx)^n}{e^{Bx}} = \lim_{x\to\infty} \frac{x^n}{e^{Bx}} = \lim_{x\to\infty} \frac{B^n}{e^{Bx}} = \lim_{x\to\infty} \frac{1}{e^{Bx}} = 0 \quad \text{for any fixed } n,$$

where B is a positive constant, it is also independent of the positive number x. Of course, for any constant M with a positive constant B, and x is a positive number, we obtain

$$\lim_{x\to\infty} M e^{-Bx} = 0. \tag{2.5}$$

We compare $|b-a|\, M_1\, e^{-T \inf |\Re e(z)-\alpha|}$ with

$$\int_a^b |u(t)|\, e^{-T|\Re e(z)-\alpha|}\, dt,$$

and compare $|b-a|\, M_2\, e^{-T \inf |\Re e(z)-\alpha|}$ with

$$\int_a^b |v(t)|\, e^{-T|\Re e(z)-\alpha|}\, dt,$$

where $u(t)$ is bounded by some positive constant M_1, and $v(t)$ is bounded by some positive constant M_2, the positive number $|\Re e(z) - \alpha|$ is independent of T, with $T > 0$. The integral mean-value theorem applied to yield

$$0 \le \int_a^b |u(t)|\, e^{-T|\Re e(z)-\alpha|}\, dt \le |b-a|\, M_1\, e^{-T \inf |\Re e(z)-\alpha|} \quad \text{for } T > 0$$

and

$$0 \le \int_a^b |v(t)|\, e^{-T|\Re e(z)-\alpha|}\, dt \le |b-a|\, M_2\, e^{-T \inf |\Re e(z)-\alpha|} \quad \text{for } T > 0.$$

As a consequence of the formula (2.5) and the criteria of the limit existence, we have

$$\lim_{T\to\infty} \int_a^b \left| u(t)\, e^{T\left(\Re e(z)-\alpha\right)} \right| dt = \lim_{T\to\infty} \int_a^b |u(t)|\, e^{-T|\Re e(z)-\alpha|}\, dt = 0$$

and

$$\lim_{T\to\infty} \int_a^b \left| v(t)\, e^{T\left(\Re e(z)-\alpha\right)} \right| dt = \lim_{T\to\infty} \int_a^b |v(t)|\, e^{-T|\Re e(z)-\alpha|}\, dt = 0$$

for $T > 0$ with $\Re e(z) < \alpha$, where $u(t)$ is bounded by some positive constant M_1, and $v(t)$ is bounded by some positive constant M_2, the positive number $|\Re e(z) - \alpha|$ for z in K is independent of T. Moreover, by the triangle inequality we have

$$\begin{aligned} 0 &\le \left| \int_\gamma h(z)\, e^{T(z-\alpha)}\, dz \right| \\ &\le \left| \int_a^b u(t)\, e^{T\Im m(z)}\, dt \right| + \left| \int_a^b v(t)\, e^{T\Im m(z)}\, dt \right| \\ &\qquad + \int_a^b |u(t)|\, e^{-T|\Re e(z)-\alpha|}\, dt + \int_a^b |v(t)|\, e^{-T|\Re e(z)-\alpha|}\, dt. \end{aligned}$$

Therefore,

$$\lim_{T\to\infty}\left|\int_{\gamma} h(z)\, e^{T(z-\alpha)}\, dz\right| = 0 \quad \text{for } T > 0, \quad \text{and} \quad \sum_{\gamma}\lim_{T\to\infty}\left|\int_{\gamma} h(z)\, e^{T(z-\alpha)}\, dz\right| = 0 \quad \text{for } T > 0.$$

These conclude that

$$\lim_{T\to\infty}\left|\int_{C^-} h(z)\, e^{T(z-\alpha)}\, dz\right| = 0 \quad \text{for } T > 0.$$

Then our claim (2.4) follows.

We may now prove this lemma. We have

$$\int_0^\infty f(t)\, e^{-\alpha t}\, dt = \lim_{T\to\infty} g_T(\alpha), \quad \text{if this limit exists.}$$

But given ε, pick R so large that $2B/R < \varepsilon$. Then by (2.4), pick T so large that

$$\left|\int_{C^-} g(z)\, e^{T(z-\alpha)}\left(1 + \frac{(z-\alpha)^2}{R^2}\right)\frac{dz}{z-\alpha}\right| < \varepsilon.$$

Then by these (2.2), (2.3) and (2.4), we get $|g(\alpha) - g_T(\alpha)| < 3\varepsilon$. This shows that $\lim_{T\to\infty} g_T(\alpha)$ exists, which can be argued as the property of Cauchy sequence and the Cauchy convergence test. Since we choose

$$0 \le T_0 < T = T_1 < T_2 < \cdots < T_m < \cdots < T_n < +\infty$$

and have the relation $|g(\alpha) - g_T(\alpha)| < 3\varepsilon$ for selecting $T = T_1$ so large with given $\varepsilon > 0$. Under the same condition on T_m and T_n, we also obtain the relations $\left|g(\alpha) - g_{T_m}(\alpha)\right| < 3\varepsilon$ and $\left|g(\alpha) - g_{T_n}(\alpha)\right| < 3\varepsilon$. Then applying the triangle inequality we have $\left|g_{T_m}(\alpha) - g_{T_n}(\alpha)\right| < 6\varepsilon$ for every pair of infinite natural numbers m, n with every pair of infinite numbers T_m, T_n. We can see the sequence $\{g_{T_n}(\alpha)\}$, i.e., $g_{T_1}(\alpha), g_{T_2}(\alpha), g_{T_3}(\alpha), \cdots, g_{T_m}(\alpha), \cdots, g_{T_n}(\alpha)\cdots$ of real numbers is a Cauchy sequence, which converges uniformly. Because every positive real number ϵ, there is a positive integer N such that for all natural numbers $m, n > N$ existing $\left|g_{T_m}(\alpha) - g_{T_n}(\alpha)\right| < \epsilon$. Cauchy sequence formulated such a condition by requiring $g_{T_m}(\alpha) - g_{T_n}(\alpha)$ to be infinitesimal for every pair of infinite m, n with every pair of infinite numbers T_m, T_n. Thus, the limit

$$\lim_{T\to\infty} g_T(\alpha) \quad \text{exists.}$$

Therefore,

$$\int_0^\infty f(t)\, e^{-\alpha t}\, dt \quad \text{exists and is equal to } g(\alpha),$$

where α is some real number.

In particular, if the integral $\int_0^\infty f(t)\, e^{-\alpha t}\, dt$ converges for $f(t) \ge 0$ with $\alpha > 0$, then $\int_0^T f(t)\, dt$ increases with the increase of T. Because of this $\int_0^\infty f(t)\, dt$ is either a finite number or ∞. However, if this $\int_0^\infty f(t)\, dt$ is just equal to a limit value, then it is only a finite number.

Also,

$$\int_0^\infty f(t)\, e^{-\alpha t}\, dt \le \int_0^\infty f(t)\, dt \quad \text{for } f(t) \ge 0 \text{ with } \alpha > 0.$$

So, we have

$$\overline{\lim_{\alpha\to 0}}\int_0^\infty f(t)\, e^{-\alpha t}\, dt \le \int_0^\infty f(t)\, dt.$$

In addition,

$$\int_0^\infty f(t)\, e^{-\alpha t}\, dt \ge \int_0^T f(t)\, e^{-\alpha t}\, dt \ge e^{-\alpha T}\int_0^T f(t)\, dt \text{ for } f(t) \ge 0 \text{ with } \alpha > 0.$$

Hence

$$\underline{\lim_{\alpha\to 0}}\int_0^\infty f(t)\, e^{-\alpha t}\, dt \ge \int_0^T f(t)\, dt.$$

Making $T \to \infty$ in the inequalities

$$\int_0^T f(t)\, e^{-\alpha t}\, dt \ge e^{-\alpha T}\int_0^T f(t)\, dt \quad \text{and} \quad \underline{\lim_{\alpha\to 0}}\int_0^\infty f(t)\, e^{-\alpha t}\, dt \ge \int_0^T f(t)\, dt$$

for $f(t) \geq 0$ with $\alpha > 0$, we can immediately obtain

$$\underline{\lim}_{\alpha\to 0} \int_0^\infty f(t)\, e^{-\alpha t}\, dt \geq \int_0^\infty f(t)\, dt.$$

Therefore,

$$\lim_{\alpha\to 0} \int_0^\infty f(t)\, e^{-\alpha t}\, dt = \int_0^\infty f(t)\, dt \quad \text{for } f(t) \geq 0 \text{ with } \alpha > 0,$$

and then this $\int_0^\infty f(t)\, dt$ also converges from which $\lim_{\alpha\to 0} \int_0^\infty f(t)\, e^{-\alpha t}\, dt$ exists as well as this limit is equal to $\int_0^\infty f(t)\, dt$. This proves Main Lemma A1. □

We claim that Main Lemma A1 can also conclude that

$$\int_1^\infty \frac{\psi(x) - x}{x^{\frac{3}{2}+\varepsilon}} dx$$

converges absolutely for $\forall\varepsilon > 0$, where defined the Chebyshev's function

$$\psi(x) = \sum_{p^m \leq x} \log p = \sum_{n\leq x} \Lambda(n),$$

and this $\Lambda(n)$ is equal to $\log p$ if $n = p^m$, or 0 otherwise. The sum $\sum_{p^m\leq x}$ is taken over those integers of the form p^m that are less than or equal to x, here p is a prime number and m is a positive integer. Observe that the function ψ is piecewise continuous. In fact, it is locally constant: there is no change in ψ between prime numbers. The application of Main Lemma A1 is to prove:

Lemma 2.20 (Main Lemma A2)**.** *The pair of integrals*

$$\int_1^\infty \frac{\psi(x) - x}{x^{\frac{3}{2}+\varepsilon}}\, dx \qquad \text{and} \qquad \int_1^\infty \frac{|\psi(x) - x|}{x^{\frac{3}{2}+\varepsilon}}\, dx$$

converge for $\forall\varepsilon > 0$, *where defined the Chebyshev's function*

$$\psi(x) = \sum_{p^m \leq x} \log p = \sum_{n\leq x} \Lambda(n),$$

and this $\Lambda(n)$ *is equal to* $\log p$ *if* $n = p^m$, *or* 0 *otherwise.*

Proof. We have the fact that if $\psi(x) = x + O(x^\lambda)$ then $\lambda \geq \frac{1}{2}$.

Let

$$f_1(t) = \frac{\psi(e^t) - e^t}{e^{\frac{1}{2}t}} \quad \text{and} \quad f_2(t) = \frac{\left|\psi(e^t) - e^t\right|}{e^{\frac{1}{2}t}} \leq B\, e^{(\lambda-\frac{1}{2})t} = f_3(t).$$

Let B be a bound for $\frac{\psi(e^t)-e^t}{e^{\lambda t}}$ that it is bounded by the form $\psi(x) = x + O(x^\lambda)$, which one has the consequence $\lambda \geq \frac{1}{2}$, where $t \geq 0$ with some real number $\beta = \lambda - \frac{1}{2}$. Then $f_1(t)$, $f_2(t)$, and $f_3(t)$ are certainly piecewise continuous, and one has the result $|f_1(t)| = f_2(t) \leq f_3(t)$.

We first estimate the Laplace transform

$$g_3(z) = \int_0^\infty f_3(t)\, e^{-zt}\, dt$$

of the function $f_3(t) = B\, e^{\beta t}$, where B and β are some constant. By using Lemma 2.17, we can obtain the Laplace transform $g_3(z) = \int_0^\infty f_3(t)\, e^{-zt}\, dt$ converges absolutely and uniformly for $\Re e(z) \geq \alpha > \beta \geq 0$. By Lemma 2.17 and using the differentiation lemma, it suffices to prove that the Laplace transform $g_3(z)$ of $f_3(t)$ is analytic for $\Re e(z) > \beta$.

So, we have to compute the Laplace transform

$$g_4(z) = \int_0^\infty f_4(t)\, e^{-zt}\, dt = \int_0^\infty f_3(t)\, e^{-\varepsilon t}\, e^{-zt}\, dt$$

of the function $f_4(t) = f_3(t)\, e^{-\varepsilon t}$ for given any $\varepsilon > 0$ with $\alpha > \beta \geq 0$. Note that $t \geq 0$, $B > 0$, and $0 < f_4(t) \leq f_3(t) = B\, e^{\beta t}$ for some constant B, β. By Lemma 2.17, we can obtain the Laplace transform $g_4(z) = \int_0^\infty f_4(t)\, e^{-zt}\, dt$ converges absolutely and uniformly for $\Re e(z) \geq \alpha > \beta \geq 0$, where the function $f_4(t) = f_3(t)\, e^{-\varepsilon t}$ for given any $\varepsilon > 0$ with $\alpha > \beta \geq 0$, which α and β are independent of ε. By Lemma 2.17 and using the differentiation lemma, it suffices to prove that the Laplace transform $g_4(z)$ of $f_4(t)$ is analytic for $\Re e(z) \geq \alpha > \beta$. Moreover, by Main Lemma A1, we can obtain

$$\int_0^\infty f_4(t)\, e^{-\alpha t}\, dt = \int_0^\infty f_3(t)\, e^{-\varepsilon t}\, e^{-\alpha t}\, dt \quad \text{exists and is equal to } g_4(\alpha),$$

where $\alpha > \beta = \lambda - \frac{1}{2}$ with any $\lambda - \frac{1}{2} \geq 0$, and then

$$\int_0^\infty f_4(t)\, e^{-\alpha t}\, dt = \int_0^\infty f_3(t)\, e^{-\varepsilon t}\, e^{-\alpha t}\, dt$$

converges for given $\alpha > 0$ and $\varepsilon > 0$, whereas some real number α is independent of any $\varepsilon > 0$. From this and by Main Lemma A1, we obtain

$$\lim_{\alpha \to 0} \int_0^\infty f_3(t)\, e^{-\varepsilon t}\, e^{-\alpha t}\, dt = \int_0^\infty f_3(t)\, e^{-\varepsilon t}\, dt, \tag{2.6}$$

which the value of the right-hand side of (2.6) converges for given $\alpha > 0$ and $\varepsilon > 0$. In other words, we actually obtain the integral $\int_0^\infty f_3(t)\, e^{-\varepsilon t}\, dt$ converges for any $\beta = \lambda - \frac{1}{2}$ with any $\lambda - \frac{1}{2} \geq 0$ in the equational expression $f_3(t) = Be^{(\lambda - \frac{1}{2})t}$, meanwhile given $\alpha > 0$ and $\varepsilon > 0$ with some positive constant B.

According to the condition $f_2(t) = \frac{|\psi(e^t) - e^t|}{e^{\frac{1}{2}t}} \leq B\, e^{(\lambda - \frac{1}{2})t} = f_3(t)$ using the form $\psi(x) = x + O(x^\lambda)$ that one surely has the consequence$\lambda \geq \frac{1}{2}$, we see that there exists an integral $\int_0^\infty f_3(t)\, e^{-\varepsilon t}\, dt$ such that

$$\int_0^\infty f_3(t)\, e^{-\varepsilon t}\, dt \geq \int_0^\infty \frac{\left|\psi(e^t) - e^t\right|}{e^{(\frac{1}{2} + \varepsilon)t}}\, dt \geq 0,$$

where $f_4(t) = f_3(t)\, e^{-\varepsilon t}$ for given any $\varepsilon > 0$, and $f_3(t) = Be^{(\lambda - \frac{1}{2})t}$ for some fixed λ in the inequality $\lambda - \frac{1}{2} \geq 0$, with some constant $B > 0$, while one satisfies the inequality $\left|\psi(e^t) - e^t\right| \leq B\, e^{\lambda t}$ concerning the form $\psi(x) = x + O(x^\lambda)$, and where the fixed λ is independent of any $\varepsilon > 0$.

Since the integral $\int_0^\infty f_3(t)\, e^{-\varepsilon t}\, dt$ converges for $f_3(t) = B\, e^{(\lambda - \frac{1}{2})t}$ with any $\lambda - \frac{1}{2} \geq 0$, where given any $\varepsilon > 0$ with some positive constant B.

Making the substitution $x = e^t$ in the desired integrals, $dx = e^t dt$, where e^t is not less than 1, we can obtain

$$\int_1^\infty \frac{|\psi(x) - x|}{x^{\frac{3}{2} + \varepsilon}}\, dx = \int_0^\infty \frac{\left|\psi(e^t) - e^t\right|}{e^{(\frac{1}{2} + \varepsilon)t}}\, dt$$

converges for every positive real number $\varepsilon > 0$.

Hence

$$\int_1^\infty \frac{\psi(x) - x}{x^{\frac{3}{2} + \varepsilon}}\, dx \quad \text{and} \quad \int_1^\infty \frac{|\psi(x) - x|}{x^{\frac{3}{2} + \varepsilon}}\, dx$$

converge for $\forall \varepsilon > 0$. This proves the lemma. This proves the lemma. □

2.4 Conclusions

Using Main Lemma A2 concludes that the integral

$$\int_1^\infty \frac{\psi(x) - x}{x^{s+1}}\, dx$$

converges absolutely and uniformly for $\Re e(s) \geq \frac{1}{2} + \delta > \frac{1}{2}$ with any $\delta > 0$, and by the differentiation lemma concludes the integral

$$\int_1^\infty \frac{\psi(x) - x}{x^{s+1}}\, dx$$

is analytic for $\Re e(s) > \frac{1}{2}$. Actually, this concludes the proof of the form $\psi\,(x) = x + O(x^{\frac{1}{2} + \varepsilon})$ for $\forall \varepsilon > 0$, which immediately follows that the function $\zeta'/\zeta(s)$ has no poles on the region $1 > \Re e(s) > \frac{1}{2}$ from the formula $(*!)$, and it implies that the Riemann Zeta function $\zeta(s)$ has no zeros on the region $1 > \Re e(s) > \frac{1}{2}$.

3. The Proof Is Derived from Trigonometric Sums and Mathematical Induction

3.1 Introduction

We show how to get the main inequality

$$\int_0^1 \Big(\sum_{p \le 2n} e^{2\pi i p x} \Big)^2 \cdot e^{-4\pi i n x}\, dx \ne 0,$$

i.e., the integral expression $\int_0^1 \big(\sum_{p\le 2n} e^{2\pi ipx}\big)^2 \cdot e^{-4\pi inx}\, dx$ is a positive integer not zero, where p is a prime number, n is a positive integer ≥ 2, and $\sum_{p\le 2n}$ is extended over all the primes $p \le 2n$, which the expression is not zero that it can prove the binary linear Diophantine equation $p_x + p_y - 2n = 0$ having the solutions of a couple of prime numbers $(p_x,\ p_y)$, where all of the couple of prime numbers satisfying $p_x \le 2n$, $p_y \le 2n$, and the trigonometric sums $\sum_{p\le 2n} e^{2\pi ipx}$ is extended over all the primes $p \le 2n$. Actually, this can conclude that every even number not less than four can be expressed as the sum of two primes.

3.2 Preliminaries

We make the precise theorems which the proofs given as follows.

Theorem 3.1. *Let α be an integer. Then*

$$\int_0^1 e^{2\pi i\alpha x}\, dx = \begin{cases} \int_0^1 \cos(2\pi\alpha x)\, dx = 1, & if \quad \alpha = 0\ ; \\ \int_0^1 \cos(2\pi\alpha x)\, dx = 0, & if \quad \alpha \ne 0\ . \end{cases}$$

Proof. If $\alpha = 0$, then $e^{2\pi i\alpha x} = \cos(2\pi\alpha x) = 1$, we obtain

$$\int_0^1 e^{2\pi i\alpha x}\, dx = \int_0^1 \cos(2\pi\alpha x)\, dx = 1;$$

If $\alpha \ne 0$, then we have

$$\begin{aligned} \int_0^1 e^{2\pi i\alpha x}\, dx &= \int_0^1 \cos(2\pi\alpha x)\, dx + i\int_0^1 \sin(2\pi\alpha x)\, dx \\ &= \left[\frac{\sin 2\pi\alpha x}{2\pi\alpha}\right]_0^1 + i\left[\frac{-\cos 2\pi\alpha x}{2\pi\alpha}\right]_0^1 = 0 + 0\,i = 0. \end{aligned}$$

This proves the theorem. □

Theorem 3.2. *Let $f(x_1, \cdots, x_n) = N$ be a linear Diophantine equation with $a_v \le x_v \le b_v$, $v = 1, 2, \cdots, n$. Then the number of the integer solutions $(x_1, \cdots, x_n)$ are counted k as the expression*

$$k = \sum_{a_1\le x_1\le b_1} \cdots \sum_{a_n\le x_n\le b_n} \int_0^1 e^{2\pi i(f(x_1,\cdots,x_n)-N)y}\, dy = \sum_{a_1\le x_1\le b_1} \cdots \sum_{a_n\le x_n\le b_n} \int_0^1 \cos 2\pi(f(x_1,\cdots,\ x_n) - N)y\, dy,$$

where N is an integer and the sum $\sum_{a_v\le x_v\le b_v}$ is extended over all the integers x_v in the interval $[a_v, b_v]$, $v = 1, 2, \cdots, n$.

Proof. Apply Theorem 3.1, we have

$$\int_0^1 e^{2\pi i(f(x_1,\cdots,x_n)-N)y}\, dy = \begin{cases} 1, & \text{if } f(x_1, \cdots,\ x_n) - N = 0\ ; \\ 0, & \text{if } f(x_1, \cdots,\ x_n) - N \ne 0\ . \end{cases}$$

Thus, the linear Diophantine equation $f(x_1, \cdots,\ x_n) = N$ with the number of the integer solutions $(x_1, \cdots,\ x_n)$ are counted k as the expression

$$k = \sum_{a_1\le x_1\le b_1} \cdots \sum_{a_n\le x_n\le b_n} \int_0^1 e^{2\pi i(f(x_1,\cdots,x_n)-N)y}\, dy = \sum_{a_1\le x_1\le b_1} \cdots \sum_{a_n\le x_n\le b_n} \int_0^1 \cos 2\pi(f(x_1,\cdots,\ x_n) - N)y\, dy.$$

This proves the theorem. □

Remark 3.3. Actually, using Theorem 3.2 can conclude that the number of the prime solutions (p_x, p_y) of the equation $p_x + p_y - 2n = 0$ are counted k_n as the expression $\int_0^1 \big(\sum_{p\le 2n} e^{2\pi ipx}\big)^2 \cdot e^{-4\pi inx}\, dx$. Because we have

$$k_n = \sum_{p_x\le 2n}\sum_{p_y\le 2n}\int_0^1 e^{2\pi i\left(p_x+p_y-2n\right)x}\, dx = \int_0^1 \sum_{p_x\le 2n}\sum_{p_y\le 2n} e^{2\pi i\left(p_x+p_y-2n\right)x}\, dx$$
$$= \int_0^1 \sum_{p_x,\, p_y\le 2n} e^{2\pi i\left(p_x+p_y-2n\right)x}\, dx = \int_0^1 \Big(\sum_{p\le 2n} e^{2\pi ipx}\Big)^2 \cdot e^{-4\pi inx}\, dx,$$

where the sums $\sum_{p_x\le 2n}$, $\sum_{p_y\le 2n}$ and $\sum_{p\le 2n}$ are extended over all the primes $p \le 2n$.

Theorem 3.4. *Let $\{n\}$ be a sequence of all the positive integers greater than* 1*, i.e., a set of all natural numbers greater than* 1 *denoted by*

$$\{n\} = \{2, 3, 4, 5, \cdots, N-1, N, N+1, \cdots, \}.$$

If

$$\int_0^1 \Big(\sum_{p\le 2n} e^{2\pi ipx}\Big)^2 \cdot e^{-4\pi inx}\, dx \ne 0$$

for integers $n = 2, 3, 4, 5, \cdots, N-1, N$, where p is a prime number and $\sum_{p\le 2n}$ is extended over all the primes $p \le 2n$, which select N so sufficiently large.
Then, we have the main conclusion

$$\int_0^1 \Big(\sum_{p\le 2(N+1)} e^{2\pi ipx}\Big)^2 \cdot e^{-4\pi i(N+1)x}\, dx \ne 0,$$

and then

$$\int_0^1 \Big(\sum_{p\le 2m} e^{2\pi ipx}\Big)^2 \cdot e^{-4\pi inx}\, dx \ne 0$$

for integers $n = 2, 3, 4, 5, \cdots, N-1, N, N+1$, where m is an integer in the interval $[2, N+1]$ *satisfying $m \ge n$.*

Proof. We assume the main conclusion false and it would lead to a contradiction. Suppose there exists the minimal element $n = N + 1$ in the set $\{n\} = S$ such that whenever the natural numbers $n = 2, 3, 4, 5, \cdots, N-1, N, N+1 \in S$ meet the form

$$\int_0^1 \Big(\sum_{p\le 2n} e^{2\pi ipx}\Big)^2 \cdot e^{-4\pi inx}\, dx = 0 \quad \text{when} \quad n = N+1 \in S,$$

or rather its consequence $\int_0^1 \big(\sum_{p\le 2n} e^{2\pi ipx}\big)^2 \cdot e^{-4\pi i(N+1)x}\, dx = 0$ for integers $n = 2, 3, 4, 5, \cdots, N-1, N, N+1$, where p is a prime number and $\sum_{p\le 2n}$ is extended over all the primes $p \le 2n$, and which the value zero of the form should be the assumption except for the conditions $\int_0^1 \big(\sum_{p\le 2n} e^{2\pi ipx}\big)^2 \cdot e^{-4\pi inx}\, dx \ne 0$ for $n = 2, 3, 4, 5, \cdots, N-1, N \in S$. In accordance with the conditions, we can obtain the same situation for the natural numbers $n = N, N-1, N-2, N-3, N-4, \cdots, 5, 4, 3, 2$ except for the natural number $N + 1$, which can respectively go with setting the same form as follows:

$$\int_0^1 \Big(\sum_{p\le 2n} e^{2\pi ipx}\Big)^2 \cdot e^{-4\pi inx}\, dx = \int_0^1 \Big(\sum_{p\le 2n} e^{2\pi i(p-n)x}\Big)^2 dx = k_n \ne 0 \tag{3.1}$$

for integers $n = N, N-1, N-2, N-3, N-4, \cdots, 5, 4, 3, 2$.

In particular, we can obtain the following compatible formulas for the sum $\sum_{p\le 2n}$. Whether the integer N is an even or odd number does not affect the compatibility, which can hold under the conditions, then we obtain

$$\int_0^1 \Big(\sum_{p\le 2n} e^{2\pi i\left[p-(N-v)\right]x}\Big)^2 dx = \int_0^1 \Big(\sum_{p\le 2n} e^{2\pi ipx}\Big)^2 \cdot e^{-4(N-v)\pi ix}\, dx = k_{x_0} \ne 0, \tag{3.2}$$

and

$$\int_0^1 \Big(\sum_{p\le 2n} e^{2\pi i(p-N+v)x}\Big)^2 \cdot e^{4(v+1)\pi ix}\, dx = \int_0^1 \Big(\sum_{p\le 2n} e^{2\pi ipx}\Big)^2 \cdot e^{-4(N-2v-1)\pi ix}\, dx = k_x \ne 0, \tag{3.3}$$

where $n = 2, 3, 4, 5, \cdots, N-1, N, N+1$; and v is an integer ≥ 0 satisfying $2 \leq N - v \leq n$ with $2 \leq N-2v-1 \leq n$ regarding the sum $\sum_p \leq 2n$.

For assumed the form $\int_0^1 (\sum_{p\leq 2(N+1)} e^{2\pi i p x})^2 \cdot e^{-4\pi i(N+1)x}\, dx = 0$, by the assumption and the method of infinite descent with Theorem 3.2, we could obtain

$$\begin{aligned}
&\int_0^1 \Big(\sum_{p\leq 2(N+1)} e^{2\pi i p x}\Big)^2 \cdot e^{-4\pi i(N+1)x}\, dx = \int_0^1 \Big(\sum_{p\leq 2(N+1)} e^{2\pi i(p-N-1)x}\Big)^2 dx \\
&= \int_0^1 \Big(\sum_{p\leq 2n} e^{2\pi i p x}\Big)^2 \cdot e^{-4\pi i(N+1)x}\, dx = \int_0^1 \Big(\sum_{p\leq 2n} e^{2\pi i(p-N+v)x}\Big)^2 \cdot e^{-4(v+1)\pi i x}\, dx \\
&= \int_0^1 \Big(\sum_{p\leq 2n} e^{2\pi i(p-N-1)x}\Big)^2 dx = 0,
\end{aligned} \tag{3.4}$$

where $n = 2, 3, 4, 5, \cdots, N-1, N, N+1$; and v is an integer ≥ 0 satisfying $2 \leq N - v \leq n$ with $2 \leq N - 2v - 1 \leq n$; where $p, p_1, p_2, \cdots, p_i, p_j$ all are prime numbers with $p_1 \neq p_2, \cdots, p_i \neq p_j$, which the sums $\sum_{p\leq 2n}$ and $\sum_{p_i,p_j\leq 2n}$ are extended over the primes $\leq 2n$.

Then by (3.4) we could get

$$\int_0^1 \Big(\sum_{p\leq 2(N-1)} e^{2\pi i(p-N)x}\Big)^2 \cdot \cos 4\pi x\, dx = i \cdot \int_0^1 \Big(\sum_{p\leq 2(N-1)} e^{2\pi i(p-N)x}\Big)^2 \cdot \sin 4\pi x\, dx, \tag{3.5}$$

and

$$\int_0^1 \Big(\sum_{p\leq 2n} e^{2\pi i(p-N+v)x}\Big)^2 \cdot \cos 4(v+1)\pi x\, dx = i \cdot \int_0^1 \Big(\sum_{p\leq 2n} e^{2\pi i(p-N+v)x}\Big)^2 \cdot \sin 4(v+1)\pi x\, dx, \tag{3.6}$$

where $p, p_1, p_2, \cdots, p_i, p_j$ all are prime numbers with $p_1 \neq p_2, \cdots, p_i \neq p_j$, and v is an integer ≥ 0 satisfying $2 \leq N - v \leq n$ with $2 \leq N - 2v - 1 \leq n$ for $n = N+1, N, N-1, N-2, N-3, \cdots, 5, 4, 3, 2$; which the sums $\sum_{p\leq 2n}$ and $\sum_{p_i,p_j\leq 2n}$ are extended over the primes $\leq 2n$.

Since the decomposition

$$\begin{aligned}
&i \cdot \int_0^1 \Big(\sum_{p\leq 2n} e^{2\pi i(p-N+v)x}\Big)^2 \cdot \sin 4(v+1)\pi x\, dx \\
=&\Big(i \int_0^1 e^{4\pi i(p_1-N+v)x} \cdot \sin 4(v+1)\pi x\, dx \\
&\quad + 2\, i \int_0^1 e^{2\pi i(p_1+p_2-2N+2v)x} \cdot \sin 4(v+1)\pi x\, dx \\
&\quad + i \int_0^1 e^{4\pi i(p_2-N+v)x} \cdot \sin 4(v+1)\pi x\, dx\Big) + \cdots + \\
&\quad \cdots + \Big(i \int_0^1 e^{4\pi i(p_i-N+v)x} \cdot \sin 4(v+1)\pi x\, dx \\
&\quad + 2\, i \int_0^1 e^{2\pi i(p_i+p_j-2N+2v)x} \cdot \sin 4(v+1)\pi x\, dx \\
&\quad + i \int_0^1 e^{4\pi i(p_j-N+v)x} \cdot \sin 4(v+1)\pi x\, dx\Big) + \cdots \\
=& i \cdot \sum_{p\leq 2n} \int_0^1 e^{4\pi i(p-N+v)x} \cdot \sin 4(v+1)\pi x\, dx \\
&\quad + 2\, i \cdot \sum_{p_i,p_j\leq 2n} \int_0^1 e^{2\pi i(p_i+p_j-2N+2v)x} \cdot \sin 4(v+1)\pi x\, dx,
\end{aligned} \tag{3.7}$$

where $p, p_1, p_2, \cdots, p_i, p_j$ all are prime numbers with $p_1 \neq p_2, \cdots, p_i \neq p_j$, and v is an integer ≥ 0 satisfying $2 \leq N - v \leq n$ with $2 \leq N - 2v - 1 \leq n$ for integers $n = N+1, N, N-1, N-2, N-3, \cdots, 5, 4, 3, 2$; which the sums $\sum_{p\leq 2n}$ and $\sum_{p_i,p_j\leq 2n}$ are extended over the primes $\leq 2n$.

Also, applying trigonometric identities we have

$$\begin{aligned}
&\int_0^1 e^{4\pi i(p-N+v)x} \cdot \sin 4(v+1)\pi x\, dx \\
&= \int_0^1 \cos 4\pi(p-N+v)x \cdot \sin 4(v+1)\pi x\, dx \\
&\qquad + i\int_0^1 \sin 4\pi(p-N+v)x \cdot \sin 4(v+1)\pi x\, dx \\
&= \int_0^1 \frac{1}{2}\left[\sin 4\pi(J)x - \sin 4\pi(p-N-1)x\right] dx \\
&\qquad + i\int_0^1 (-\frac{1}{2})\left[\cos 4\pi(J)x - \cos 4\pi(p-N-1)x\right] dx
\end{aligned}$$

with setting $p - N + 2v + 1 = J$, where p, p_1, p_2, $\cdots$, p_i, p_j all are prime numbers with $p_1 \neq p_2, \cdots, p_i \neq p_j$, and v is an integer ≥ 0 satisfying $2 \leq N - v \leq n$ with the inequality $2 \leq N - 2v - 1 \leq n$ for integers $n = N + 1, N, N - 1, N - 2, N - 3, \cdots , 5, 4, 3, 2$; which the sums $\sum_{p\leq 2n}$ and $\sum_{p_i,p_j\leq 2n}$ are extended over the primes $\leq 2n$.

By the assumption, we have $2p - 2(N + 1) \neq 0$ for which the sum $\sum_{p\leq 2(N+1)}$ is extended over the primes $\leq 2(N + 1)$ except that $N + 1$ is a prime number, in which the case of the sum $\sum_{p\leq 2n}$ yet still has the result $2p - 2(N + 1) \neq 0$ that the sum is extended over the primes $\leq 2n$ for integers $n = N + 1, N, N - 1, N - 2, N - 3, \cdots , 5, 4, 3, 2$, and applying trigonometric identities we get

$$\begin{aligned}
&\int_0^1 e^{2\pi i(p_i+p_j-2N+2v)x} \cdot \sin 4(v+1)\pi x\, dx \\
&= \int_0^1 \cos 2\pi(p_i+p_j-2N+2v)x \cdot \sin 4(v+1)\pi x\, dx \\
&\qquad + i\int_0^1 \sin 2\pi(p_i+p_j-2N+2v)x \cdot \sin 4(v+1)\pi x\, dx \\
&= \int_0^1 \frac{1}{2}\Big[\sin 2\pi(K)x - \sin 2\pi(p_i+p_j-2N-2)x\Big] dx \\
&\qquad + i\int_0^1 (-\frac{1}{2})\Big[\cos 2\pi(K)x - \cos 2\pi(p_i+p_j-2N-2)x\Big] dx
\end{aligned}$$

with setting $p_i + p_j - 2N + 4v + 2 = K$, where p, p_1, p_2, $\cdots$, p_i, p_j all are prime numbers with $p_1 \neq p_2, \cdots, p_i \neq p_j$, and v is an integer ≥ 0 satisfying $2 \leq N - v \leq n$ with the inequality $2 \leq N - 2v - 1 \leq n$ for integers $n = N + 1, N, N - 1, N - 2, N - 3, \cdots , 5, 4, 3, 2$; which the sums $\sum_{p\leq 2n}$ and $\sum_{p_i,p_j\leq 2n}$ are extended over the primes $\leq 2n$.

By the assumption, we have $p_i + p_j - 2(N + 1) \neq 0$ with a couple of prime numbers (p_i, p_j) satisfying $p_i \neq p_j$ for which the sum $\sum_{p_i,p_j\leq 2(N+1)}$ is extended over the primes $p \leq 2(N + 1)$, in which the case of the sum $\sum_{p_i,p_j\leq 2n}$ yet still has the result $p_i + p_j - 2(N + 1) \neq 0$ that the sum is extended over the primes $\leq 2n$ for integers $n = N + 1, N, N - 1, N - 2, N - 3, \cdots , 5, 4, 3, 2$.

Using the formulas (3.3), (3.4), (3.6) and (3.7) or rather their consequence as above, we can enumerate the limited cases of the formula (3.7). Now to compute this formula (3.7) as follows:

(1) If an integer $N - 2v - 1$ is a composite number and which satisfies $p_i + p_j - 2(N - 2v - 1) = 0$, also assuming for the moment that the inequation $p_i + p_j - 2(N + 1) \neq 0$ and $N + 1$ is not a prime number, then we can obtain the first case of the formula (3.7), which is the value:

$$\begin{aligned}
&i\cdot\sum_{p\leq 2n}\int_0^1 (-\frac{1}{2})\left[\cos 2\pi(2p-2N+4v+2)x\right] dx \\
&\qquad + 2\,i\cdot\sum_{p_i,p_j\leq 2n}\int_0^1 (-\frac{1}{2})\left[\cos 2\pi(K)x\right] dx \\
&= i\cdot k_u\cdot\int_0^1 (-\frac{1}{2})\cos 0\, dx + 2\,i\cdot k_v\cdot\int_0^1 (-\frac{1}{2})\cos 0\, dx \\
&= -\left(\frac{k_u}{2} + k_v\right) i
\end{aligned}$$

with setting $p_i + p_j - 2N + 4v + 2 = K$, where

$$k_u = \sum_{p\le 2n} \int_0^1 \left[\cos 2\pi(2p - 2N + 4v + 2)x\right] dx,$$

$$k_v = \sum_{p_i,p_j\le 2n} \int_0^1 \left[\cos 2\pi(p_i + p_j - 2N + 4v + 2)x\right] dx,$$

and where $p, p_1, p_2, \cdots, p_i, p_j$ all are prime numbers with $p_1 \neq p_2, \cdots, p_i \neq p_j$, and v is an integer ≥ 0 satisfying the inequality $2 \leq N - v \leq n$ with $2 \leq N - 2v - 1 \leq n$ for integers $n = N + 1, N, N - 1, N - 2, N - 3, \cdots, 5, 4, 3, 2$, which the sums $\sum_{p\le 2n}$ and $\sum_{p_i,p_j\le 2n}$ are extended over the primes $\leq 2n$, with being observed that $k_u \geq 0$, $k_v \geq 1$.

Then, in this case, by the formulas (3.3) and (3.6), we could obtain

$$\int_0^1 \Big(\sum_{p\le 2n} e^{2\pi i p x}\Big)^2 \cdot e^{-4\pi i(N-2v-1)x}\, dx = \int_0^1 \Big(\sum_{p\le 2n} e^{2\pi i(p-N+v)x}\Big)^2 \cdot e^{4(v+1)\pi i x}\, dx = k_x = 2\left(\frac{k_u}{2} + k_v\right).$$

But, indeed, by Theorem 3.2, we see that $k_x = k_u + k_v$. Therefore,

$$2\left(\frac{k_u}{2} + k_v\right) = k_u + k_v, \quad \text{which is a contradiction.}$$

Hence, we can smooth away this case (1) in the discussion.

(2) If $N-2v-1$ is a prime number and $p_i+p_j-2N+4v+2 \neq 0$, also assuming for the moment that $p_i+p_j-2(N+1) \neq 0$ and $N + 1$ is not a prime number, then we can obtain the second case of the formula (3.7), which is the value:

$$i \cdot \sum_{p\le 2n} \int_0^1 (-\frac{1}{2}) \left[\cos 2\pi(2p - 2N + 4v + 2)x\right] dx = i \cdot k_u \cdot \int_0^1 (-\frac{1}{2}) \cos 0\, dx = -\frac{i}{2},$$

where

$$k_u = \sum_{p\le 2n} \int_0^1 \left[\cos 2\pi(2p - 2N + 4v + 2)x\right] dx,$$

and where $p, p_1, p_2, \cdots, p_i, p_j$ all are prime numbers with $p_1 \neq p_2, \cdots, p_i \neq p_j$, and v is an integer ≥ 0 satisfying the inequality $2 \leq N - v \leq n$ with $2 \leq N - 2v - 1 \leq n$ for integers $n = N + 1, N, N - 1, N - 2, N - 3, \cdots, 5, 4, 3, 2$; which the sums $\sum_{p\le 2n}$ and $\sum_{p_i,p_j\le 2n}$ are extended over the primes $\leq 2n$, with being observed that $k_u = 1$, $k_v = 0$.

(3) If $N-2v-1$ is a prime number and $p_i+p_j-2N+4v+2 = 0$, also assuming for the moment that $p_i+p_j-2(N+1) \neq 0$ and $N + 1$ is not a prime number, then we can obtain the third case of the formula (3.7), which is the value:

$$\begin{aligned}
& i \cdot \sum_{p\le 2n} \int_0^1 (-\frac{1}{2}) \left[\cos 2\pi(2p - 2N + 4v + 2)x\right] dx \\
& \qquad + 2\, i \cdot \sum_{p_i,p_j\le 2n} \int_0^1 (-\frac{1}{2}) \left[\cos 2\pi(K)x\right] dx \\
&= i \cdot k_u \cdot \int_0^1 (-\frac{1}{2}) \cos 0\, dx + 2\, i \cdot k_v \cdot \int_0^1 (-\frac{1}{2}) \cos 0\, dx \\
&= -\left(\frac{k_u}{2} + k_v\right) i
\end{aligned}$$

with setting $p_i + p_j - 2N + 4v + 2 = K$, where

$$k_u = \sum_{p\le 2n} \int_0^1 \left[\cos 2\pi(2p - 2N + 4v + 2)x\right] dx,$$

$$k_v = \sum_{p_i,p_j\le 2n} \int_0^1 \left[\cos 2\pi(p_i + p_j - 2N + 4v + 2)x\right] dx,$$

and where p, p_1, p_2, $\cdots$, p_i, p_j all are prime numbers with $p_1 \neq p_2, \cdots, p_i \neq p_j$, and v is an integer ≥ 0 satisfying the inequality $2 \leq N - v \leq n$ with $2 \leq N - 2v - 1 \leq n$ for integers $n = N + 1, N, N - 1, N - 2, N - 3, \cdots, 5, 4, 3, 2$; which the sums $\sum_{p \leq 2n}$ and $\sum_{p_i, p_j \leq 2n}$ are extended over the primes $\leq 2n$, with being observed that $k_u = 1$, $k_v \geq 1$.

Then, in this case, by the formulas (3.3) and (3.6) we can obtain

$$\int_0^1 \Big(\sum_{p \leq 2n} e^{2\pi i p x} \Big)^2 \cdot e^{-4\pi i (N-2v-1)x} \, dx = \int_0^1 \Big(\sum_{p \leq 2n} e^{2\pi i (p-N+v)x} \Big)^2 \cdot e^{4(v+1)\pi i x} \, dx = k_x = 2 \left(\frac{k_u}{2} + k_v \right).$$

But, indeed, by Theorem 3.2, we see that $k_x = k_u + k_v$. Therefore,

$$2 \left(\frac{k_u}{2} + k_v \right) = k_u + k_v, \quad \text{which is a contradiction.}$$

Hence, we can smooth away this case (3) in the discussion.

(4) If an integer $N - 2v - 1$ is a composite number and which it satisfies the inequation $p_i + p_j - 2N + 4v + 2 \neq 0$, also assuming for the moment that $p_i + p_j - 2(N + 1) \neq 0$ and $N + 1$ is not a prime number, then we can obtain the last case of the formula (3.7), which is the value:

$$i \cdot \sum_{p \leq 2n} \int_0^1 (-\frac{1}{2}) \left[\cos 2\pi (2p - 2N + 4v + 2)x \right] \, dx = i \cdot k_u \cdot \int_0^1 (-\frac{1}{2}) \cos 0 \, dx = -\frac{i}{2},$$

where

$$k_u = \sum_{p \leq 2n} \int_0^1 \left[\cos 2\pi (2p - 2N + 4v + 2)x \right] \, dx,$$

and where p, p_1, p_2, $\cdots$, p_i, p_j all are prime numbers with $p_1 \neq p_2, \cdots, p_i \neq p_j$, and this v is an integer ≥ 0 satisfying the inequality $2 \leq N - v \leq n$ with $2 \leq N - 2v - 1 \leq n$ for integers $n = N + 1, N, N - 1, N - 2, N - 3, \cdots$, 5, 4, 3, 2; which the sums $\sum_{p \leq 2n}$ and $\sum_{p_i, p_j \leq 2n}$ are extended over the primes $\leq 2n$, being observed that $k_u = 1$ by the formula (3.3) $= k_x$ with $k_x \neq 0$, and this case including $p_i + p_j - 2N + 4v + 2 \neq 0$ such that $k_v = 0$.

In other words, our assertion is that if $\int_0^1 (\sum_{p \leq 2n} e^{2\pi i p x})^2 \cdot e^{-4\pi i n x} \, dx \neq 0$ for $n = 2, 3, 4, 5, \cdots, N - 1, N \in S$, then

$$\int_0^1 \Big(\sum_{p \leq 2n} e^{2\pi i p x} \Big)^2 \cdot e^{-4\pi i n x} \, dx \neq 0 \quad \text{for } n = N + 1 \in S,$$

where p is a prime number and the sum $\sum_{p \leq 2n}$ is extended over all the primes $\leq 2n$. If this assertion were not true, we could obtain the assumption

$$\int_0^1 \Big(\sum_{p \leq 2(N+1)} e^{2\pi i p x} \Big)^2 \cdot e^{-4\pi i (N+1)x} \, dx = 0,$$

and then we could find the cases (1) and (3) are uninteresting. Furthermore, in the cases (2) and (4) we could find the formula

$$i \cdot \int_0^1 \Big(\sum_{p \leq 2n} e^{2\pi i (p-N+v)x} \Big)^2 \cdot \sin 4(v + 1)\pi x \, dx = \frac{1}{2},$$

i.e.,

$$\int_0^1 \Big(\sum_{p \leq 2n} e^{2\pi i (p-N+v)x} \Big)^2 \cdot \cos 4(v + 1)\pi x \, dx = \frac{1}{2}$$

from the formula (3.6), where p, p_1, p_2, $\cdots$, p_i, p_j all are prime numbers with $p_1 \neq p_2, \cdots, p_i \neq p_j$, and v is an integer ≥ 0 satisfying $2 \leq N - v \leq n$ with $2 \leq N - 2v - 1 \leq n$ for integers $n = N + 1, N, N - 1, N - 2, N - 3, \cdots, \cdots, 5, 4, 3, 2$; which the sums $\sum_{p \leq 2n}$ and $\sum_{p_i, p_j \leq 2n}$ are extended over the primes $\leq 2n$.

By (3.1), (3.3) and (3.6) with the detailed discussion of all above cases, we have $k_x = 1$, and they take the form

$$\begin{cases} \int_0^1 \left(\sum_{p \leq 2n} e^{2\pi i (p-N+v)x} \right)^2 \cdot e^{4(v+1)\pi i x} \, dx = k_x = 1 \neq 0, \\ \\ \int_0^1 \left(\sum_{p \leq 2n} e^{2\pi i (p-N+v)x} \right)^2 \cdot e^{-4(v+1)\pi i x} \, dx \\ = \int_0^1 \left(\sum_{p \leq 2n} e^{2\pi i (p-N-1)x} \right)^2 \, dx = 0, \end{cases} \tag{3.8}$$

where $p,\ p_1,\ p_2,\ \cdots,\ p_i,\ p_j$ all are prime numbers with $p_1 \neq p_2, \cdots, p_i \neq p_j$, and v is an integer ≥ 0 satisfying $2 \leq N - v \leq n$ with the inequality $2 \leq N - 2v - 1 \leq n$ for integers $n = N+1, N, N-1, N-2, N-3, \cdots, 5, 4, 3, 2$; which the sums $\sum_{p\leq 2n}$ and $\sum_{p_i, p_j \leq 2n}$ are extended over the primes $\leq 2n$.

For a more detailed discussion of all above cases, we may consider how to use the consequence and their compatibility between the formula (3.2) and the formula (3.3). Indeed, we merely apply the formula (3.3) or rather its consequence under the conditions

$$\int_0^1 \Big(\sum_{p\leq 2n} e^{2\pi i p x}\Big)^2 \cdot e^{-4\pi i n x}\, dx \neq 0$$

for $n = 2, 3, 4, 5, \cdots, N-1, N \in S$ and the assumption

$$\int_0^1 \Big(\sum_{p\leq 2n} e^{2\pi i p x}\Big)^2 \cdot e^{-4\pi i n x}\, dx = 0 \quad \text{for} \quad n = N+1 \in S,$$

where p is a prime number and the sum $\sum_{p\leq 2n}$ is extended over the primes $\leq 2n$.

By repeated applications of the same way, or rather its consequence the form (3.8) of all the above processing, it is clear that the form (3.8) has the following consequence with a suggestive compatibility:

(a) If start with a positive integer N is an odd number, then we can choose an integer $n = N - 2v - 1 = 8$ and we could write

$$\begin{cases} \int_0^1 \left(\sum_{p\leq 2n} e^{2\pi i p x}\right)^2 \cdot e^{-4(N-2v-1)\pi i x}\, dx = k_x = 1 \neq 0, \\ \int_0^1 \left(\sum_{p\leq 2n} e^{2\pi i (p-N+v) x}\right)^2 \cdot e^{-4(v+1)\pi i x}\, dx = 0; \end{cases}$$

where $p,\ p_1,\ p_2,\ \cdots,\ p_i,\ p_j$ are prime numbers with the case $p_1 \neq p_2, \cdots, p_i \neq p_j$, and this v is an integer ≥ 0 satisfying the inequality $2 \leq N - v \leq n$ with $2 \leq N - 2v - 1 \leq n$ for integers $n = N+1, N, N-1, N-2, N-3, \cdots, 5, 4, 3, 2$; which the sums $\sum_{p\leq 2n}$ and $\sum_{p_i, p_j\leq 2n}$ are extended over the primes $\leq 2n$.

(b) If start with a positive integer N is an even number, then we can choose an integer $n = N - 2v - 1 = 7$ and we could write

$$\begin{cases} \int_0^1 \left(\sum_{p\leq 2n} e^{2\pi i p x}\right)^2 \cdot e^{-4(N-2v-1)\pi i x}\, dx = k_x = 1 \neq 0, \\ \int_0^1 \left(\sum_{p\leq 2n} e^{2\pi i (p-N+v) x}\right)^2 \cdot e^{-4(v+1)\pi i x}\, dx = 0; \end{cases}$$

where $p,\ p_1,\ p_2,\ \cdots,\ p_i,\ p_j$ are prime numbers with the case $p_1 \neq p_2, \cdots, p_i \neq p_j$, and this v is an integer ≥ 0 satisfying the inequality $2 \leq N - v \leq n$ with $2 \leq N - 2v - 1 \leq n$ for integers $n = N+1, N, N-1, N-2, N-3, \cdots, 5, 4, 3, 2$; which the sums $\sum_{p\leq 2n}$ and $\sum_{p_i, p_j\leq 2n}$ are extended over the primes $\leq 2n$.

We could conclude that $k_x = 1$ by the conditions and the assumption, or rather its consequence the form (3.8).

We can choose and formulate the result as the following consequence, which the recursive scheme and the same way as previously are defined can be written in the cases (a) and (b), whether the integer N is an even or odd number does not affect the compatibility they can hold. There exists an infinite descent for the cases. However, there cannot be an infinity of ever-smaller natural numbers, and therefore by mathematical induction or rather its consequence $k_x = 1$.

But, in fact we have the identity $16 = 2 \times 8 = 3 + 13 = 5 + 11$ and the identity $14 = 2 \times 7 = 3 + 11 = 7 + 7$, we know that

$$\int_0^1 \Big(\sum_{p\leq 2\times 8} e^{2\pi i p x}\Big)^2 \cdot e^{-4\pi i\cdot 8x}\, dx = \int_0^1 \Big(\sum_{p\leq 2\times 8} e^{2\pi i (p-8) x}\Big)^2 dx = 2,$$

where p is a prime number, and $\sum_{p\leq 2\times 8}$ is extended over all the primes $\leq 2\times 8$; and

$$\int_0^1 \Big(\sum_{p\leq 2\times 7} e^{2\pi i p x}\Big)^2 \cdot e^{-4\pi i\cdot 7x}\, dx = \int_0^1 \Big(\sum_{p\leq 2\times 7} e^{2\pi i (p-7) x}\Big)^2 dx = 2,$$

where p is a prime number, and $\sum_{p\leq 2\times 7}$ is extended over all the primes $\leq 2\times 7$.

Finally, it is pointed out that $k_x = 1$ there exists the recursive scheme with an infinite descent in the presence of the form (3.8) starting with the integer N its intrinsic nature either an even or odd number, which are concluded by the conditions and the assumption, or rather its consequence more counting the value $k_x = 1$, such as the last can lead to

$$\int_0^1 \Big(\sum_{p \le 2\times 8} e^{2\pi ipx} \Big)^2 \cdot e^{-4\pi i\cdot 8x}\, dx = 1 \quad \text{or} \quad \int_0^1 \Big(\sum_{p \le 2\times 7} e^{2\pi ipx} \Big)^2 \cdot e^{-4\pi i\cdot 7x}\, dx = 1,$$

which is a contradiction. Therefore, this proves the theorem. □

3.3 Using Mathematical Induction for the Assertion

Indeed, the assertion of Theorem 3.4 expresses a fine path by taking some appropriate elements, which we can use the mathematical induction to prove that $\int_0^1 (\sum_{p\le 2n} e^{2\pi ipx})^2 \cdot e^{-4\pi inx}\, dx \neq 0$, where p is a prime number, n is any positive integer ≥ 2, and $\sum_{p\le 2n}$ is extended over all the primes $\le 2n$.

Let's recall. Mathematical induction: It is a form of direct proof, and it is done in two steps. The first step, known as the base case, is to prove the given statement for the first natural number. The second step, known as the inductive step, is to prove that the given statement for any one natural number implies the given statement for the next natural number. From these two steps, mathematical induction is the rule from which we infer that the given statement is established for all natural numbers. Mathematical induction is an inference rule used in proofs. If we want to prove a statement not for all natural numbers but only for all numbers greater than or equal to a certain number b then the proof by induction consists of two steps:

(i) The basis step: Showing that the statement holds when $n = b$.

(ii) The inductive step: Showing that if the statement holds for $n = m \ge b$ then the same statement also holds for $n = m + 1$.

Infinite descent: It might begin by showing that if a statement is true for a natural number n it must also be true for some smaller natural number m ($m < n$). Using mathematical induction (implicitly) with the inductive hypothesis being that the statement is false for all natural numbers less than or equal to m, we can conclude that the statement cannot be true for any natural number n. Although this particular form of infinite-descent proof is clearly a mathematical induction, whether one holds all proofs "by infinite descent" to be mathematical inductions depends on how one defines the term "proof by infinite descent." In mathematics, a proof by infinite descent is a particular kind of proof by contradiction which relies on the facts that the natural numbers are well ordered and that there are only a finite number of them that are smaller than any given one. However, there cannot be an infinity of ever-smaller natural numbers, and therefore by mathematical induction (repeating the same step) the original premise that any solution exists must be incorrect. It is disproven because its logical outcome would require a contradiction. An alternative way to express this is to assume one or more solutions or examples exists. Then there must be a smallest solution or example a minimal counterexample. We then prove that if a smallest solution exists, it must imply the existence of a smaller solution (in some sense) which again proves that the existence of any solution would lead to a contradiction. The method of infinite descent was developed by Fermat, who often used it for Diophantine equations.

Indeed, the proof of Theorem 3.4 in the sense is closely related to recursion and the method of infinite descent. Being Theorem 3.4 holds, we can return applications of the proof by induction consists of two steps:

(i) Showing that the statement holds when $n = b = 2, 3, 4, 5, 6, 7, 8$, which the formula $\int_0^1 (\sum_{p\le 2n} e^{2\pi ipx})^2 \cdot e^{-4\pi inx}\, dx$ is not zero in fact;

(ii) Showing that if the statement holds for all $n = 2, 3, 4, 5, 6, 7, 8, \cdots, N$ with N may be so sufficiently large then the same statement also holds for $n = N + 1$, which the formula $\int_0^1 (\sum_{p\le 2n} e^{2\pi ipx})^2 \cdot e^{-4\pi inx}\, dx$ is not zero by Theorem 3.4,

which it is the typical mathematical induction, or rather its consequence a variant as like infinite descent.

So far, the assertion is shown to be true that the proof of Theorem 3.4 and the proof of the non-zero integral

$$\int_0^1 \Big(\sum_{p\le 2n} e^{2\pi ipx} \Big)^2 \cdot e^{-4\pi inx}\, dx,$$

which are concluded by the typical mathematical induction, or rather its consequence a variant as like the method of infinite descent.

3.4 Conclusions

Using Theorem 3.4 and the mathematical induction conclude that

$$\int_0^1 \Big(\sum_{p \le 2n} e^{2\pi i p x} \Big)^2 \cdot e^{-4\pi i n x}\, dx \neq 0$$

for any positive integer $n \geq 2$, where $\sum_{p \le 2n}$ is extended over all the primes $p \leq 2n$.

4. The Overall Conclusion

Actually, the arguments can prove that the Riemann Zeta function $\zeta(s)$ has no zeros on the region $1 > \Re e(s) > \frac{1}{2}$ and every even number not less than four can be expressed as the sum of two primes.

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