Chapter 25

Zeta and Related Functions

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Notation

25.1 Special Notation

(For other notation see pp. xiv and 873.)

k, m, n	nonnegative integers.
p	prime number.
x	real variable.
a	real or complex parameter.
$s = \sigma + it$	complex variable.
$z = x + iy$	complex variable.
γ	Euler's constant (§5.2(ii)).
$\psi(x)$	digamma function $\Gamma'(x)/\Gamma(x)$ except in §25.16. See §5.2(i).
$B_n, B_n(x)$	Bernoulli number and polynomial (§24.2(i)).
$\widetilde{B}_n(x)$	periodic Bernoulli function $B_n(x - \lfloor x \rfloor)$.
$m \mid n$	m divides n.
primes	on function symbols: derivatives with respect to argument.

The main function treated in this chapter is the Riemann zeta function $\zeta(s)$. This notation was introduced in Riemann (1859).

The main related functions are the Hurwitz zeta function $\zeta(s,a)$, the dilogarithm $\mathrm{Li}_2(z)$, the polylogarithm $\mathrm{Li}_s(z)$ (also known as Jonquière's function $\phi(z,s)$), Lerch's transcendent $\Phi(z,s,a)$, and the Dirichlet L-functions $L(s,\chi)$.

Riemann Zeta Function

25.2 Definition and Expansions

25.2(i) Definition

When $\Re s > 1$,

25.2.1
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Elsewhere $\zeta(s)$ is defined by analytic continuation. It is a meromorphic function whose only singularity in $\mathbb{C}$ is a simple pole at $s = 1$, with residue 1.

25.2(ii) Other Infinite Series

25.2.2
$$\zeta(s) = \frac{1}{1-2^{-s}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}, \qquad \Re s > 1.$$

25.2.3
$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \qquad \Re s > 0.$$

25.2.4
$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n, \quad \Re s > 0,$$

where

25.2.5
$$\gamma_n = \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{(\ln k)^n}{k} - \frac{(\ln m)^{n+1}}{n+1} \right).$$

25.2.6
$$\zeta'(s) = -\sum_{n=2}^{\infty} (\ln n) n^{-s}, \qquad \Re s > 1.$$

25.2.7
$$\zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{\infty} (\ln n)^k n^{-s}, \quad \Re s > 1,\ k = 1,2,3,\dots.$$

For further expansions of functions similar to (25.2.1) (Dirichlet series) see §27.4. This includes, for example, $1/\zeta(s)$.

25.2(iii) Representations by the Euler–Maclaurin Formula

25.2.8
$$\zeta(s) = \sum_{k=1}^{N} \frac{1}{k^s} + \frac{N^{1-s}}{s-1} - s \int_N^{\infty} \frac{x - \lfloor x \rfloor}{x^{s+1}}\, dx,$$
$$\Re s > 0,\ N = 1,2,3,\dots.$$

25.2.9
$$\zeta(s) = \sum_{k=1}^{N} \frac{1}{k^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2} N^{-s} + \sum_{k=1}^{n} \binom{s+2k-2}{2k-1} \frac{B_{2k}}{2k} N^{1-s-2k} - \binom{s+2n}{2n+1} \int_N^{\infty} \frac{\widetilde{B}_{2n+1}(x)}{x^{s+2n+1}}\, dx,$$
$$\Re s > -2n;\ n, N = 1,2,3,\dots.$$

25.2.10
$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=1}^{n} \binom{s+2k-2}{2k-1} \frac{B_{2k}}{2k} - \binom{s+2n}{2n+1} \int_1^{\infty} \frac{\widetilde{B}_{2n+1}(x)}{x^{s+2n+1}}\, dx,$$
$$\Re s > -2n,\ n = 1,2,3,\dots.$$

For B_{2k} see §24.2(i), and for $\widetilde{B}_n(x)$ see §24.2(iii).

25.2(iv) Infinite Products

25.2.11
$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \qquad \Re s > 1,$$

product over all primes p.

25.2.12
$$\zeta(s) = \frac{(2\pi)^s e^{-s-(\gamma s/2)}}{2(s-1)\,\Gamma\left(\frac{1}{2}s+1\right)} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

product over zeros ρ of ζ with $\Re \rho > 0$ (see §25.10(i)); γ is Euler's constant (§5.2(ii)).

25.3 Graphics

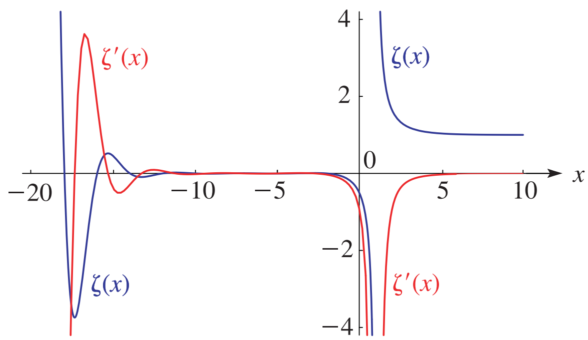

Figure 25.3.1: Riemann zeta function $\zeta(x)$ and its derivative $\zeta'(x)$, $-20 \le x \le 10$.

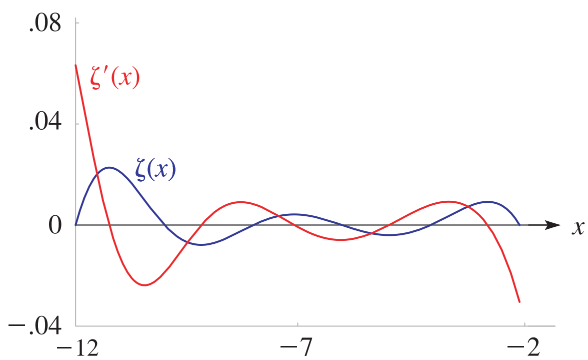

Figure 25.3.2: Riemann zeta function $\zeta(x)$ and its derivative $\zeta'(x)$, $-12 \le x \le -2$.

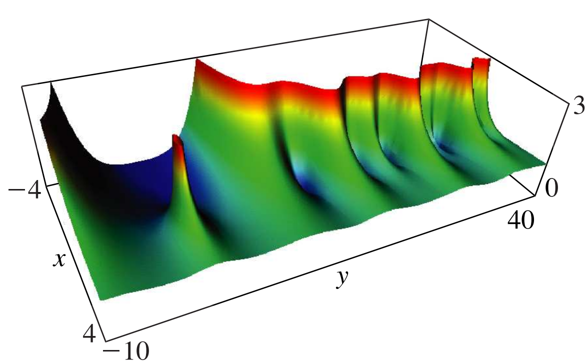

Figure 25.3.3: Modulus of the Riemann zeta function $|\zeta(x+iy)|$, $-4 \le x \le 4$, $-10 \le y \le 40$.

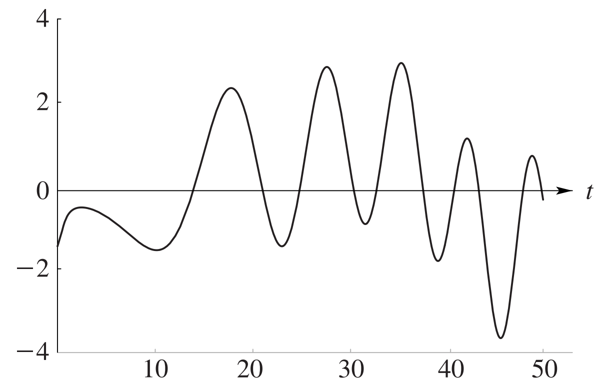

Figure 25.3.4: $Z(t)$, $0 \le t \le 50$. $Z(t)$ and $\zeta\left(\frac{1}{2}+it\right)$ have the same zeros. See §25.10(i).

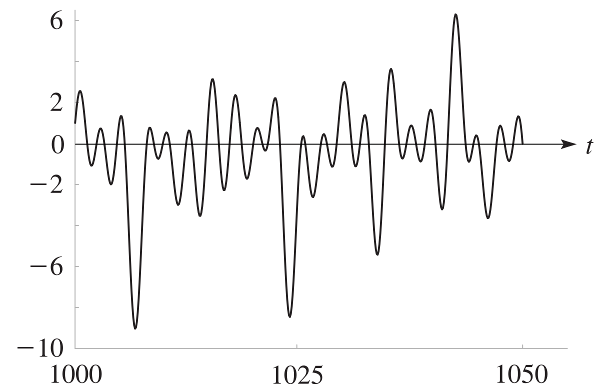

Figure 25.3.5: $Z(t)$, $1000 \le t \le 1050$.

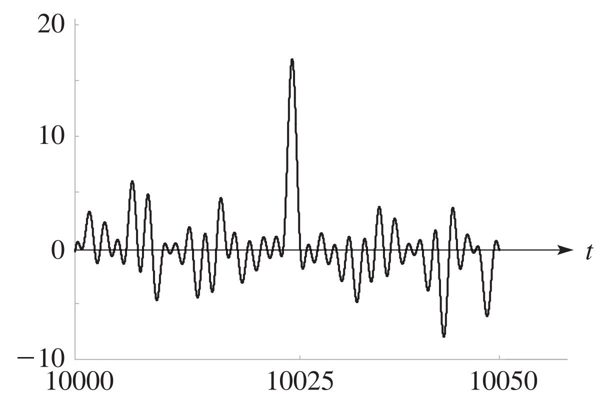

Figure 25.3.6: $Z(t)$, $10000 \le t \le 10050$.

25.4 Reflection Formulas

For $s \ne 0, 1$,

25.4.1 $$\zeta(1-s) = 2(2\pi)^{-s}\cos\left(\tfrac{1}{2}\pi s\right)\Gamma(s)\,\zeta(s),$$

25.4.2 $$\zeta(s) = 2(2\pi)^{s-1}\sin\left(\tfrac{1}{2}\pi s\right)\Gamma(1-s)\,\zeta(1-s).$$

Equivalently,

25.4.3 $$\xi(s) = \xi(1-s),$$

where $\xi(s)$ is *Riemann's ξ-function*, defined by:

25.4.4 $$\xi(s) = \tfrac{1}{2}s(s-1)\,\Gamma\left(\tfrac{1}{2}s\right)\pi^{-s/2}\,\zeta(s).$$

For $s \neq 0, 1$ and $k = 1, 2, 3, \dots$,

25.4.5
$$(-1)^k\,\zeta^{(k)}(1-s) = \frac{2}{(2\pi)^s}\sum_{m=0}^{k}\sum_{r=0}^{m}\binom{k}{m}\binom{m}{r}\left(\Re(c^{k-m})\cos\left(\tfrac{1}{2}\pi s\right) + \Im(c^{k-m})\sin\left(\tfrac{1}{2}\pi s\right)\right)\Gamma^{(r)}(s)\,\zeta^{(m-r)}(s),$$

where

25.4.6 $$c = -\ln(2\pi) - \tfrac{1}{2}\pi i.$$

25.5 Integral Representations

25.5(i) In Terms of Elementary Functions

Throughout this subsection $s \neq 1$.

25.5.1 $$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx, \qquad \Re s > 1.$$

25.5.2 $$\zeta(s) = \frac{1}{\Gamma(s+1)}\int_0^\infty \frac{e^x x^s}{(e^x-1)^2}\,dx, \qquad \Re s > 1.$$

25.5.3 $$\zeta(s) = \frac{1}{(1-2^{1-s})\,\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x+1}\,dx, \qquad \Re s > 0.$$

25.5.4 $$\zeta(s) = \frac{1}{(1-2^{1-s})\,\Gamma(s+1)}\int_0^\infty \frac{e^x x^s}{(e^x+1)^2}\,dx, \qquad \Re s > 0.$$

25.5.5 $$\zeta(s) = -s\int_0^\infty \frac{x - \lfloor x \rfloor - \frac{1}{2}}{x^{s+1}}\,dx, \qquad -1 < \Re s < 0.$$

25.5.6
$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \frac{1}{\Gamma(s)}\int_0^\infty \left(\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2}\right)\frac{x^{s-1}}{e^x}\,dx, \qquad \Re s > -1.$$

25.5.7
$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \sum_{m=1}^{n}\frac{B_{2m}}{(2m)!}\frac{\Gamma(s+2m-1)}{\Gamma(s)} + \frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2} - \sum_{m=1}^{n}\frac{B_{2m}}{(2m)!}x^{2m-1}\right)\frac{x^{s-1}}{e^x}\,dx,$$
$$\Re s > -(2n+1),\ n = 1, 2, 3, \dots.$$

25.5.8 $$\zeta(s) = \frac{1}{2(1-2^{-s})\,\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{\sinh x}\,dx, \qquad \Re s > 1.$$

25.5.9 $$\zeta(s) = \frac{2^{s-1}}{\Gamma(s+1)}\int_0^\infty \frac{x^s}{(\sinh x)^2}\,dx, \qquad \Re s > 1.$$

25.5.10 $$\zeta(s) = \frac{2^{s-1}}{1-2^{1-s}}\int_0^\infty \frac{\cos(s\arctan x)}{(1+x^2)^{s/2}\cosh\left(\frac{1}{2}\pi x\right)}\,dx.$$

25.5.11
$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + 2\int_0^\infty \frac{\sin(s\arctan x)}{(1+x^2)^{s/2}(e^{2\pi x}-1)}\,dx.$$

25.5.12 $$\zeta(s) = \frac{2^{s-1}}{s-1} - 2^s\int_0^\infty \frac{\sin(s\arctan x)}{(1+x^2)^{s/2}(e^{\pi x}+1)}\,dx.$$

25.5(ii) In Terms of Other Functions

25.5.13
$$\zeta(s) = \frac{\pi^{s/2}}{s(s-1)\,\Gamma\left(\frac{1}{2}s\right)} + \frac{\pi^{s/2}}{\Gamma\left(\frac{1}{2}s\right)}\int_1^\infty \left(x^{s/2} + x^{(1-s)/2}\right)\frac{\omega(x)}{x}\,dx, \qquad s \neq 1,$$

where

25.5.14 $$\omega(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x} = \frac{1}{2}\left(\theta_3(0|ix) - 1\right).$$

For θ_3 see §20.2(i). For similar representations involving other theta functions see Erdélyi *et al.* (1954a, p. 339).

In (25.5.15)–(25.5.19), $0 < \Re s < 1$, $\psi(x)$ is the digamma function, and γ is Euler's constant (§5.2). (25.5.16) is also valid for $0 < \Re s < 2$, $s \neq 1$.

25.5.15
$$\zeta(s) = \frac{1}{s-1} + \frac{\sin(\pi s)}{\pi}\times\int_0^\infty (\ln(1+x) - \psi(1+x))x^{-s}\,dx,$$

25.5.16
$$\zeta(s) = \frac{1}{s-1} + \frac{\sin(\pi s)}{\pi(s-1)}\times\int_0^\infty\left(\frac{1}{1+x} - \psi'(1+x)\right)x^{1-s}\,dx,$$

25.5.17 $$\zeta(1+s) = \frac{\sin(\pi s)}{\pi}\int_0^\infty (\gamma + \psi(1+x))\,x^{-s-1}\,dx,$$

25.5.18 $$\zeta(1+s) = \frac{\sin(\pi s)}{\pi s}\int_0^\infty \psi'(1+x)x^{-s}\,dx,$$

25.5.19
$$\zeta(m+s) = (-1)^{m-1}\frac{\Gamma(s)\sin(\pi s)}{\pi\,\Gamma(m+s)}\times\int_0^\infty \psi^{(m)}(1+x)x^{-s}\,dx, \qquad m = 1, 2, 3, \dots.$$

25.5(iii) Contour Integrals

25.5.20

$$\zeta(s)=\frac{\Gamma(1-s)}{2\pi i}\int_{-\infty}^{(0+)}\frac{z^{s-1}}{e^{-z}-1}\,dz,\quad s\neq 1,2,\dots,$$

where the integration contour is a loop around the negative real axis; it starts at $-\infty$, encircles the origin once in the positive direction without enclosing any of the points $z=\pm 2\pi i$, $\pm 4\pi i$, …, and returns to $-\infty$. Equivalently,

25.5.21

$$\zeta(s)=\frac{\Gamma(1-s)}{2\pi i(1-2^{1-s})}\int_{-\infty}^{(0+)}\frac{z^{s-1}}{e^{-z}+1}\,dz,\quad s\neq 1,2,\dots.$$

The contour here is any loop that encircles the origin in the positive direction not enclosing any of the points $\pm\pi i$, $\pm 3\pi i$, ….

25.6 Integer Arguments

25.6(i) Function Values

25.6.1

$$\zeta(0)=-\frac{1}{2},\quad \zeta(2)=\frac{\pi^2}{6},\quad \zeta(4)=\frac{\pi^4}{90},\quad \zeta(6)=\frac{\pi^6}{945}.$$

25.6.2 $$\zeta(2n)=\frac{(2\pi)^{2n}}{2(2n)!}\left|B_{2n}\right|,\qquad n=1,2,3,\dots.$$

25.6.3 $$\zeta(-n)=-\frac{B_{n+1}}{n+1},\qquad n=1,2,3,\dots.$$

25.6.4 $$\zeta(-2n)=0,\qquad n=1,2,3,\dots.$$

25.6.5

$$\zeta(k+1)=\frac{1}{k!}\sum_{n_1=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{1}{n_1\cdots n_k(n_1+\cdots+n_k)},\qquad k=1,2,3,\dots.$$

25.6.6

$$\zeta(2k+1)=\frac{(-1)^{k+1}(2\pi)^{2k+1}}{2(2k+1)!}\int_0^1 B_{2k+1}(t)\cot(\pi t)\,dt,\qquad k=1,2,3,\dots.$$

25.6.7 $$\zeta(2)=\int_0^1\int_0^1\frac{1}{1-xy}\,dx\,dy.$$

25.6.8 $$\zeta(2)=3\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}.$$

25.6.9 $$\zeta(3)=\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}.$$

25.6.10 $$\zeta(4)=\frac{36}{17}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}.$$

25.6(ii) Derivative Values

25.6.11 $$\zeta'(0)=-\tfrac{1}{2}\ln(2\pi).$$

25.6.12 $$\zeta''(0)=-\tfrac{1}{2}(\ln(2\pi))^2+\tfrac{1}{2}\gamma^2-\tfrac{1}{24}\pi^2+\gamma_1,$$

where γ_1 is given by (25.2.5).

With c defined by (25.4.6) and $n=1,2,3,\dots$,

25.6.13 $$(-1)^k\,\zeta^{(k)}(-2n)=\frac{2(-1)^n}{(2\pi)^{2n+1}}\sum_{m=0}^{k}\sum_{r=0}^{m}\binom{k}{m}\binom{m}{r}\Im(c^{k-m})\,\Gamma^{(r)}(2n+1)\,\zeta^{(m-r)}(2n+1),$$

25.6.14 $$(-1)^k\,\zeta^{(k)}(1-2n)=\frac{2(-1)^n}{(2\pi)^{2n}}\sum_{m=0}^{k}\sum_{r=0}^{m}\binom{k}{m}\binom{m}{r}\Re(c^{k-m})\,\Gamma^{(r)}(2n)\,\zeta^{(m-r)}(2n),$$

25.6.15 $$\zeta'(2n)=\frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!}\left(2n\,\zeta'(1-2n)-(\psi(2n)-\ln(2\pi))\,B_{2n}\right).$$

25.6(iii) Recursion Formulas

25.6.16 $$\left(n+\tfrac{1}{2}\right)\zeta(2n)=\sum_{k=1}^{n-1}\zeta(2k)\,\zeta(2n-2k),\quad n\geq 2.$$

25.6.17

$$\left(n+\tfrac{3}{4}\right)\zeta(4n+2)=\sum_{k=1}^{n}\zeta(2k)\,\zeta(4n+2-2k),\quad n\geq 1.$$

25.6.18

$$\left(n+\tfrac{1}{4}\right)\zeta(4n)+\tfrac{1}{2}(\zeta(2n))^2=\sum_{k=1}^{n}\zeta(2k)\,\zeta(4n-2k),\qquad n\geq 1.$$

25.6.19

$$\begin{aligned}&\left(m+n+\tfrac{3}{2}\right)\zeta(2m+2n+2)\\&\quad=\left(\sum_{k=1}^{m}+\sum_{k=1}^{n}\right)\zeta(2k)\,\zeta(2m+2n+2-2k),\end{aligned}$$
$$m\geq 0,\ n\geq 0,\ m+n\geq 1.$$

25.6.20
$$\tfrac{1}{2}(2^{2n}-1)\,\zeta(2n)=\sum_{k=1}^{n-1}(2^{2n-2k}-1)\,\zeta(2n-2k)\,\zeta(2k),\qquad n\geq 2.$$

For related results see Basu and Apostol (2000).

25.7 Integrals

For definite integrals of the Riemann zeta function see Prudnikov *et al.* (1986b, §2.4), Prudnikov *et al.* (1992a, §3.2), and Prudnikov *et al.* (1992b, §3.2).

25.8 Sums

25.8.1
$$\sum_{k=2}^{\infty}(\zeta(k)-1)=1.$$

25.8.2
$$\sum_{k=0}^{\infty}\frac{\Gamma(s+k)}{(k+1)!}\,(\zeta(s+k)-1)=\Gamma(s-1),\qquad s\neq 1,0,-1,-2,\dots.$$

25.8.3
$$\sum_{k=0}^{\infty}\frac{\Gamma(s+k)\,\zeta(s+k)}{k!\,\Gamma(s)2^{s+k}}=(1-2^{-s})\,\zeta(s),\qquad s\neq 1.$$

25.8.4
$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}(\zeta(nk)-1)=\ln\left(\prod_{j=0}^{n-1}\Gamma\left(2-e^{(2j+1)\pi i/n}\right)\right),\qquad n=2,3,4,\dots.$$

25.8.5
$$\sum_{k=2}^{\infty}\zeta(k)z^k=-\gamma z-z\,\psi(1-z),\qquad |z|<1.$$

25.8.6
$$\sum_{k=0}^{\infty}\zeta(2k)z^{2k}=-\tfrac{1}{2}\pi z\cot(\pi z),\qquad |z|<1.$$

25.8.7
$$\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}z^k=-\gamma z+\ln\Gamma(1-z),\qquad |z|<1.$$

25.8.8
$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}z^{2k}=\ln\left(\frac{\pi z}{\sin(\pi z)}\right),\qquad |z|<1.$$

25.8.9
$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(2k+1)2^{2k}}=\frac{1}{2}-\frac{1}{2}\ln 2.$$

25.8.10
$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}}=\frac{1}{4}-\frac{7}{4\pi^2}\,\zeta(3).$$

For other sums see Prudnikov *et al.* (1986b, pp. 648–649), Hansen (1975, pp. 355–357), Ogreid and Osland (1998), and Srivastava and Choi (2001, Chapter 3).

25.9 Asymptotic Approximations

If $x\geq 1$, $y\geq 1$, $2\pi xy=t$, and $0\leq\sigma\leq 1$, then as $t\to\infty$ with σ fixed,

25.9.1
$$\zeta(\sigma+it)=\sum_{1\leq n\leq x}\frac{1}{n^s}+\chi(s)\sum_{1\leq n\leq y}\frac{1}{n^{1-s}}+O(x^{-\sigma})+O\left(y^{\sigma-1}t^{\frac{1}{2}-\sigma}\right),$$

where $s=\sigma+it$ and

25.9.2
$$\chi(s)=\pi^{s-\frac{1}{2}}\,\Gamma\left(\tfrac{1}{2}-\tfrac{1}{2}s\right)/\,\Gamma\left(\tfrac{1}{2}s\right).$$

If $\sigma=\frac{1}{2}$, $x=y=\sqrt{t/(2\pi)}$, and $m=\lfloor x\rfloor$, then (25.9.1) becomes

25.9.3
$$\zeta\left(\tfrac{1}{2}+it\right)=\sum_{n=1}^{m}\frac{1}{n^{\frac{1}{2}+it}}+\chi\left(\tfrac{1}{2}+it\right)\sum_{n=1}^{m}\frac{1}{n^{\frac{1}{2}-it}}+O\left(t^{-1/4}\right).$$

For other asymptotic approximations see Berry and Keating (1992), Paris and Cang (1997); see also Paris and Kaminski (2001, pp. 380–389).

25.10 Zeros

25.10(i) Distribution

The product representation (25.2.11) implies $\zeta(s)\neq 0$ for $\Re s>1$. Also, $\zeta(s)\neq 0$ for $\Re s=1$, a property first established in Hadamard (1896) and de la Vallée Poussin (1896a,b) in the proof of the prime number theorem (25.16.3). The functional equation (25.4.1) implies $\zeta(-2n)=0$ for $n=1,2,3,\dots$. These are called the *trivial zeros*. Except for the trivial zeros, $\zeta(s)\neq 0$ for $\Re s\leq 0$. In the region $0<\Re s<1$, called the *critical strip*, $\zeta(s)$ has infinitely many zeros, distributed symmetrically about the real axis and about the *critical line* $\Re s=\frac{1}{2}$. The *Riemann hypothesis* states that all nontrivial zeros lie on this line.

Calculations relating to the zeros on the critical line make use of the real-valued function

25.10.1
$$Z(t)=\exp(i\vartheta(t))\,\zeta\left(\tfrac{1}{2}+it\right),$$

where

25.10.2
$$\vartheta(t)\equiv\operatorname{ph}\Gamma\left(\tfrac{1}{4}+\tfrac{1}{2}it\right)-\tfrac{1}{2}t\ln\pi$$

is chosen to make $Z(t)$ real, and $\operatorname{ph}\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)$ assumes its principal value. Because $|Z(t)|=\left|\zeta\left(\frac{1}{2}+it\right)\right|$, $Z(t)$ vanishes at the zeros of $\zeta\left(\frac{1}{2}+it\right)$, which can be separated by observing sign changes of $Z(t)$. Because $Z(t)$ changes sign infinitely often, $\zeta\left(\frac{1}{2}+it\right)$ has infinitely many zeros with t real.

25.10(ii) Riemann–Siegel Formula

Riemann developed a method for counting the total number $N(T)$ of zeros of $\zeta(s)$ in that portion of the critical strip with $0 < t < T$. By comparing $N(T)$ with the number of sign changes of $Z(t)$ we can decide whether $\zeta(s)$ has any zeros off the line in this region. Sign changes of $Z(t)$ are determined by multiplying (25.9.3) by $\exp(i\vartheta(t))$ to obtain the *Riemann–Siegel formula*:

25.10.3 $$Z(t) = 2\sum_{n=1}^{m} \frac{\cos(\vartheta(t) - t\ln n)}{n^{1/2}} + R(t),$$

where $R(t) = O\left(t^{-1/4}\right)$ as $t \to \infty$.

The error term $R(t)$ can be expressed as an asymptotic series that begins

25.10.4

$$R(t) = (-1)^{m-1}\left(\frac{2\pi}{t}\right)^{1/4} \frac{\cos\left(t - (2m+1)\sqrt{2\pi t} - \frac{1}{8}\pi\right)}{\cos\left(\sqrt{2\pi t}\right)} + O\left(t^{-3/4}\right).$$

Riemann also developed a technique for determining further terms. Calculations based on the Riemann–Siegel formula reveal that the first ten billion zeros of $\zeta(s)$ in the critical strip are on the critical line (van de Lune *et al.* (1986)). More than one-third of all the zeros in the critical strip lie on the critical line (Levinson (1974)).

For further information on the Riemann–Siegel expansion see Berry (1995).

Related Functions

25.11 Hurwitz Zeta Function

25.11(i) Definition

The function $\zeta(s,a)$ was introduced in Hurwitz (1882) and defined by the series expansion

25.11.1

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \Re s > 1,\ a \neq 0, -1, -2, \dots.$$

$\zeta(s,a)$ has a meromorphic continuation in the s-plane, its only singularity in $\mathbb{C}$ being a simple pole at $s = 1$ with residue 1. As a function of a, with s $(\neq 1)$ fixed, $\zeta(s,a)$ is analytic in the half-plane $\Re a > 0$. The Riemann zeta function is a special case:

25.11.2 $$\zeta(s,1) = \zeta(s).$$

For most purposes it suffices to restrict $0 < \Re a \leq 1$ because of the following straightforward consequences of (25.11.1):

25.11.3 $$\zeta(s,a) = \zeta(s,a+1) + a^{-s},$$

25.11.4

$$\zeta(s,a) = \zeta(s,a+m) + \sum_{n=0}^{m-1} \frac{1}{(n+a)^s}, \quad m = 1, 2, 3, \dots.$$

Most references treat real a with $0 < a \leq 1$.

25.11(ii) Graphics

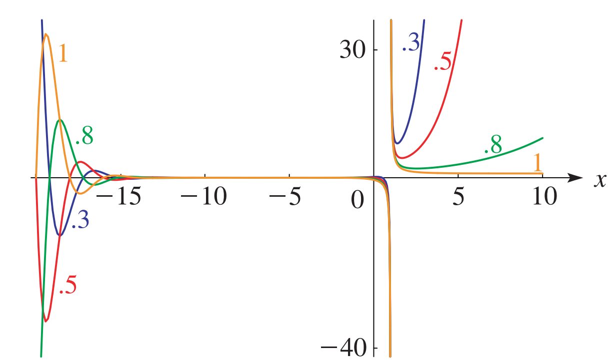

Figure 25.11.1: Hurwitz zeta function $\zeta(x,a)$, $a = 0.3$, 0.5, 0.8, 1, $-20 \leq x \leq 10$. The curves are almost indistinguishable for $-14 < x < -1$, approximately.

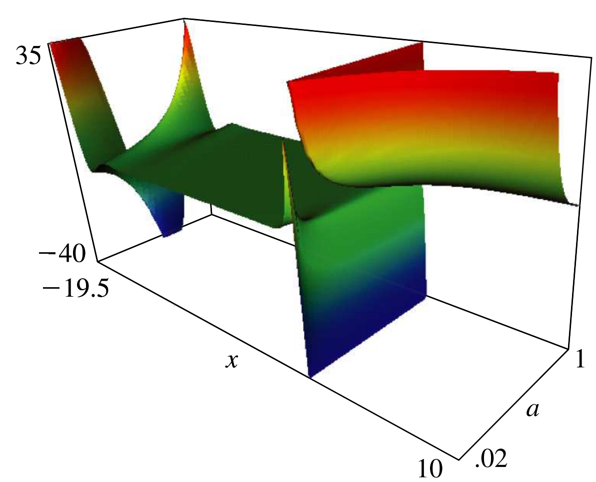

Figure 25.11.2: Hurwitz zeta function $\zeta(x,a)$, $-19.5 \leq x \leq 10$, $0.02 \leq a \leq 1$.

25.11(iii) Representations by the Euler–Maclaurin Formula

25.11.5
$$\zeta(s,a)=\sum_{n=0}^{N}\frac{1}{(n+a)^s}+\frac{(N+a)^{1-s}}{s-1}-s\int_N^\infty\frac{x-\lfloor x\rfloor}{(x+a)^{s+1}}\,dx,\quad s\neq 1,\ \Re s>0,\ a>0,\ N=0,1,2,3,\dots.$$

25.11.6
$$\zeta(s,a)=\frac{1}{a^s}\left(\frac{1}{2}+\frac{a}{s-1}\right)-s(s+1)\int_0^\infty\frac{\widetilde{B}_2(x)}{(x+a)^{s+2}}\,dx,\qquad s\neq 1,\ \Re s>-1,\ a>0.$$

25.11.7
$$\zeta(s,a)=\frac{1}{a^s}+\frac{1}{(1+a)^s}\left(\frac{1}{2}+\frac{1+a}{s-1}\right)+\sum_{k=1}^{n}\binom{s+2k-2}{2k-1}\frac{B_{2k}}{2k}\frac{1}{(1+a)^{s+2k-1}}-\binom{s+2n}{2n+1}\int_1^\infty\frac{\widetilde{B}_{2n+1}(x)}{(x+a)^{s+2n+1}}\,dx,$$
$$s\neq 1,\ a>0,\ n=1,2,3,\dots,\ \Re s>-2n.$$

For $\widetilde{B}_n(x)$ see §24.2(iii).

25.11(iv) Series Representations

25.11.8
$$\zeta(s,\tfrac12 a)=\zeta(s,\tfrac12 a+\tfrac12)+2^s\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+a)^s},\quad \Re s>0,\ s\neq 1,\ 0<a\leq 1.$$

25.11.9
$$\zeta(1-s,a)=\frac{2\,\Gamma(s)}{(2\pi)^s}\sum_{n=1}^{\infty}\frac{1}{n^s}\cos\left(\tfrac12\pi s-2n\pi a\right),\quad \Re s>1,\ 0<a\leq 1.$$

25.11.10
$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{\Gamma(n+s)}{n!\,\Gamma(s)}\zeta(n+s)(1-a)^n,\quad s\neq 1,\ |a-1|<1.$$

When $a=\frac12$, (25.11.10) reduces to (25.8.3); compare (25.11.11).

25.11(v) Special Values

Throughout this subsection $\Re a>0$.

25.11.11
$$\zeta(s,\tfrac12)=(2^s-1)\,\zeta(s),\qquad s\neq 1.$$

25.11.12
$$\zeta(n+1,a)=\frac{(-1)^{n+1}\,\psi^{(n)}(a)}{n!},\quad n=1,2,3,\dots.$$

25.11.13
$$\zeta(0,a)=\tfrac12-a.$$

25.11.14
$$\zeta(-n,a)=-\frac{B_{n+1}(a)}{n+1},\quad n=0,1,2,\dots.$$

25.11.15
$$\zeta(s,ka)=k^{-s}\sum_{n=0}^{k-1}\zeta\left(s,a+\frac{n}{k}\right),\quad s\neq 1,\ k=1,2,3,\dots.$$

25.11.16
$$\zeta\left(1-s,\frac{h}{k}\right)=\frac{2\,\Gamma(s)}{(2\pi k)^s}\sum_{r=1}^{k}\cos\left(\frac{\pi s}{2}-\frac{2\pi rh}{k}\right)\zeta\left(s,\frac{r}{k}\right),$$
$s\neq 0,1$; h,k integers, $1\leq h\leq k$.

25.11(vi) Derivatives

a-Derivative

25.11.17
$$\frac{\partial}{\partial a}\zeta(s,a)=-s\,\zeta(s+1,a),\quad s\neq 0,1;\ \Re a>0.$$

s-Derivatives

In (25.11.18)–(25.11.24) primes on ζ denote derivatives with respect to s. Similarly in §§25.11(viii) and 25.11(xii).

25.11.18
$$\zeta'(0,a)=\ln\Gamma(a)-\tfrac12\ln(2\pi),\qquad a>0.$$

25.11.19
$$\zeta'(s,a)=-\frac{\ln a}{a^s}\left(\frac12+\frac{a}{s-1}\right)-\frac{a^{1-s}}{(s-1)^2}+s(s+1)\int_0^\infty\frac{\widetilde{B}_2(x)\ln(x+a)}{(x+a)^{s+2}}\,dx-(2s+1)\int_0^\infty\frac{\widetilde{B}_2(x)}{(x+a)^{s+2}}\,dx,$$
$$\Re s>-1,\ s\neq 1,\ a>0.$$

25.11.20
$$(-1)^k\zeta^{(k)}(s,a)=\frac{(\ln a)^k}{a^s}\left(\frac12+\frac{a}{s-1}\right)+k!a^{1-s}\sum_{r=0}^{k-1}\frac{(\ln a)^r}{r!(s-1)^{k-r+1}}-s(s+1)\int_0^\infty\frac{\widetilde{B}_2(x)(\ln(x+a))^k}{(x+a)^{s+2}}\,dx$$
$$+k(2s+1)\int_0^\infty\frac{\widetilde{B}_2(x)(\ln(x+a))^{k-1}}{(x+a)^{s+2}}\,dx-k(k-1)\int_0^\infty\frac{\widetilde{B}_2(x)(\ln(x+a))^{k-2}}{(x+a)^{s+2}}\,dx,$$
$$\Re s>-1,\ s\neq 1,\ a>0.$$

25.11.21

$$\zeta'\left(1-2n,\frac{h}{k}\right)=\frac{(\psi(2n)-\ln(2\pi k))\,B_{2n}(h/k)}{2n}-\frac{(\psi(2n)-\ln(2\pi))\,B_{2n}}{2nk^{2n}}+\frac{(-1)^{n+1}\pi}{(2\pi k)^{2n}}\sum_{r=1}^{k-1}\sin\left(\frac{2\pi rh}{k}\right)\psi^{(2n-1)}\left(\frac{r}{k}\right)+\frac{(-1)^{n+1}2\cdot(2n-1)!}{(2\pi k)^{2n}}\sum_{r=1}^{k-1}\cos\left(\frac{2\pi rh}{k}\right)\zeta'\left(2n,\frac{r}{k}\right)+\frac{\zeta'(1-2n)}{k^{2n}},$$

where h,k are integers with $1\le h\le k$ and $n=1,2,3,\dots$.

25.11.22
$$\zeta'\left(1-2n,\tfrac{1}{2}\right)=-\frac{B_{2n}\ln 2}{n\cdot 4^n}-\frac{\left(2^{2n-1}-1\right)\zeta'(1-2n)}{2^{2n-1}},\qquad n=1,2,3,\dots.$$

25.11.23
$$\zeta'\left(1-2n,\tfrac{1}{3}\right)=-\frac{\pi(9^n-1)\,B_{2n}}{8n\sqrt{3}(3^{2n-1}-1)}-\frac{B_{2n}\ln 3}{4n\cdot 3^{2n-1}}-\frac{(-1)^n\,\psi^{(2n-1)}\left(\frac{1}{3}\right)}{2\sqrt{3}(6\pi)^{2n-1}}-\frac{\left(3^{2n-1}-1\right)\zeta'(1-2n)}{2\cdot 3^{2n-1}},\quad n=1,2,3,\dots.$$

25.11.24
$$\sum_{r=1}^{k-1}\zeta'\left(s,\frac{r}{k}\right)=(k^s-1)\,\zeta'(s)+k^s\,\zeta(s)\ln k,\qquad s\ne 1,\ k=1,2,3,\dots.$$

25.11(vii) Integral Representations

25.11.25
$$\zeta(s,a)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{x^{s-1}e^{-ax}}{1-e^{-x}}\,dx,\qquad \Re s>1,\ \Re a>0.$$

25.11.26
$$\zeta(s,a)=-s\int_{-a}^\infty\frac{x-\lfloor x\rfloor-\frac{1}{2}}{(x+a)^{s+1}}\,dx,\qquad -1<\Re s<0,\ 0<a\le 1.$$

25.11.27
$$\zeta(s,a)=\frac{1}{2}a^{-s}+\frac{a^{1-s}}{s-1}+\frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{1}{e^x-1}-\frac{1}{x}+\frac{1}{2}\right)\frac{x^{s-1}}{e^{ax}}\,dx,\quad \Re s>-1,\ s\ne 1,\ \Re a>0.$$

25.11.28
$$\zeta(s,a)=\frac{1}{2}a^{-s}+\frac{a^{1-s}}{s-1}+\sum_{k=1}^{n}\frac{\Gamma(s+2k-1)}{\Gamma(s)}\frac{B_{2k}}{(2k)!}a^{-2k-s+1}+\frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{1}{e^x-1}-\frac{1}{x}+\frac{1}{2}-\sum_{k=1}^{n}\frac{B_{2k}}{(2k)!}x^{2k-1}\right)x^{s-1}e^{-ax}\,dx,\qquad \Re s>-(2n+1),\ s\ne 1,\ \Re a>0.$$

25.11.29
$$\zeta(s,a)=\frac{1}{2}a^{-s}+\frac{a^{1-s}}{s-1}+2\int_0^\infty\frac{\sin(s\arctan(x/a))}{(a^2+x^2)^{s/2}(e^{2\pi x}-1)}\,dx,\qquad s\ne 1,\ \Re a>0.$$

25.11.30
$$\zeta(s,a)=\frac{\Gamma(1-s)}{2\pi i}\int_{-\infty}^{(0+)}\frac{e^{az}z^{s-1}}{1-e^z}\,dz,\qquad s\ne 1,\ \Re a>0,$$

where the integration contour is a loop around the negative real axis as described for (25.5.20).

25.11(viii) Further Integral Representations

25.11.31
$$\frac{1}{\Gamma(s)}\int_0^\infty\frac{x^{s-1}e^{-ax}}{2\cosh x}\,dx=4^{-s}\left(\zeta\left(s,\tfrac{1}{4}+\tfrac{1}{4}a\right)-\zeta\left(s,\tfrac{3}{4}+\tfrac{1}{4}a\right)\right),\qquad \Re s>0,\ \Re a>-1.$$

25.11.32
$$\int_0^a x^n\,\psi(x)\,dx=(-1)^{n-1}\,\zeta'(-n)+(-1)^n h(n)\frac{B_{n+1}}{n+1}-\sum_{k=0}^{n}(-1)^k\binom{n}{k}h(k)\frac{B_{k+1}(a)}{k+1}a^{n-k}+\sum_{k=0}^{n}(-1)^k\binom{n}{k}\zeta'(-k,a)a^{n-k},\qquad n=1,2,\dots,\ \Re a>0,$$

where

25.11.33
$$h(n)=\sum_{k=1}^{n}k^{-1}.$$

25.11.34
$$n\int_0^a\zeta'(1-n,x)\,dx=\zeta'(-n,a)-\zeta'(-n)+\frac{B_{n+1}-B_{n+1}(a)}{n(n+1)},\qquad n=1,2,\dots,\ \Re a>0.$$

25.11(ix) Integrals

See Prudnikov et al. (1990, §2.3), Prudnikov et al. (1992a, §3.2), and Prudnikov et al. (1992b, §3.2).

25.11(x) Further Series Representations

25.11.35
$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+a)^s} = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}e^{-ax}}{1+e^{-x}}\,dx = 2^{-s}\left(\zeta\left(s,\tfrac{1}{2}a\right)-\zeta\left(s,\tfrac{1}{2}(1+a)\right)\right),$$
$$\Re a>0,\ \Re s>0;\text{ or }\Re a=0,\ \Im a\neq 0,\ 0<\Re s<1.$$

When $a=1$, (25.11.35) reduces to (25.2.3).

25.11.36
$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} = k^{-s}\sum_{r=1}^{k}\chi(r)\,\zeta\left(s,\frac{r}{k}\right),\qquad \Re s>1,$$

where $\chi(n)$ is a Dirichlet character $\pmod{k}$ (§27.8).

See also Srivastava and Choi (2001).

25.11(xi) Sums

25.11.37
$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\zeta(nk,a) = -n\ln\Gamma(a) + \ln\left(\prod_{j=0}^{n-1}\Gamma\left(a-e^{(2j+1)\pi i/n}\right)\right),$$
$$n=2,3,4,\dots,\ \Re a\geq 1.$$

25.11.38
$$\sum_{k=1}^{\infty}\binom{n+k}{k}\zeta(n+k+1,a)z^k = \frac{(-1)^n}{n!}\left(\psi^{(n)}(a)-\psi^{(n)}(a-z)\right),$$
$$n=1,2,3,\dots,\ \Re a>0,\ |z|<|a|.$$

25.11.39
$$\sum_{k=2}^{\infty}\frac{k}{2^k}\zeta\left(k+1,\tfrac{3}{4}\right) = 8G,$$

where G is *Catalan's constant*:

25.11.40
$$G = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = 0.91596\,55941\,772\dots.$$

For further sums see Prudnikov et al. (1990, pp. 396–397) and Hansen (1975, pp. 358–360).

25.11(xii) a-Asymptotic Behavior

As $a\to 0$ with $s\ (\neq 1)$ fixed,

25.11.41
$$\zeta(s,a+1) = \zeta(s) - s\,\zeta(s+1)a + O\left(a^2\right).$$

As $\beta\to\pm\infty$ with s fixed, $\Re s>1$,

25.11.42
$$\zeta(s,\alpha+i\beta)\to 0,$$

uniformly with respect to bounded nonnegative values of α.

As $a\to\infty$ in the sector $|\operatorname{ph} a|\leq\pi-\delta(<\pi)$, with $s(\neq 1)$ and δ fixed, we have the asymptotic expansion

25.11.43
$$\zeta(s,a) - \frac{a^{1-s}}{s-1} - \frac{1}{2}a^{-s} \sim \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\frac{\Gamma(s+2k-1)}{\Gamma(s)}a^{1-s-2k}.$$

Similarly, as $a\to\infty$ in the sector $|\operatorname{ph} a|\leq\frac{1}{2}\pi-\delta(<\frac{1}{2}\pi)$,

25.11.44
$$\zeta'(-1,a) - \frac{1}{12} + \frac{1}{4}a^2 - \left(\frac{1}{12}-\frac{1}{2}a+\frac{1}{2}a^2\right)\ln a \sim -\sum_{k=1}^{\infty}\frac{B_{2k+2}}{(2k+2)(2k+1)2k}a^{-2k},$$

and

25.11.45
$$\zeta'(-2,a) - \frac{1}{12}a + \frac{1}{9}a^3 - \left(\frac{1}{6}a-\frac{1}{2}a^2+\frac{1}{3}a^3\right)\ln a \sim \sum_{k=1}^{\infty}\frac{2B_{2k+2}}{(2k+2)(2k+1)2k(2k-1)}a^{-(2k-1)}.$$

For the more general case $\zeta'(-m,a)$, $m=1,2,\dots$, see Elizalde (1986).

For an exponentially-improved form of (25.11.43) see Paris (2005b).

25.12 Polylogarithms

25.12(i) Dilogarithms

The notation $\mathrm{Li}_2(z)$ was introduced in Lewin (1981) for a function discussed in Euler (1768) and called the *dilogarithm* in Hill (1828):

25.12.1
$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2},\qquad |z|\leq 1.$$

25.12.2
$$\mathrm{Li}_2(z) = -\int_0^z t^{-1}\ln(1-t)\,dt,\quad z\in\mathbb{C}\backslash(1,\infty).$$

Other notations and names for $\mathrm{Li}_2(z)$ include $S_2(z)$ (Kölbig et al. (1970)), Spence function $\mathrm{Sp}(z)$ ('t Hooft and Veltman (1979)), and $\mathrm{L}_2(z)$ (Maximon (2003)).

In the complex plane $\mathrm{Li}_2(z)$ has a branch point at $z=1$. The principal branch has a cut along the interval $[1,\infty)$ and agrees with (25.12.1) when $|z|\leq 1$; see also §4.2(i). The remainder of the equations in this subsection apply to principal branches.

25.12.3
$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{z}{z-1}\right) = -\frac{1}{2}(\ln(1-z))^2,\quad z\in\mathbb{C}\backslash[1,\infty).$$

25.12.4
$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) = -\frac{1}{6}\pi^2 - \frac{1}{2}(\ln(-z))^2,\quad z\in\mathbb{C}\backslash[0,\infty).$$

25.12.5
$$\mathrm{Li}_2(z^m) = m\sum_{k=0}^{m-1}\mathrm{Li}_2\left(ze^{2\pi ik/m}\right),$$
$$m=1,2,3,\dots,\ |z|<1.$$

25.12.6
$$\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\frac{1}{6}\pi^2-(\ln x)\ln(1-x),\quad 0<x<1.$$

When $z=e^{i\theta}$, $0\le\theta\le 2\pi$, (25.12.1) becomes

25.12.7
$$\mathrm{Li}_2\left(e^{i\theta}\right)=\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n^2}+i\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2}.$$

The cosine series in (25.12.7) has the elementary sum

25.12.8
$$\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n^2}=\frac{\pi^2}{6}-\frac{\pi\theta}{2}+\frac{\theta^2}{4}.$$

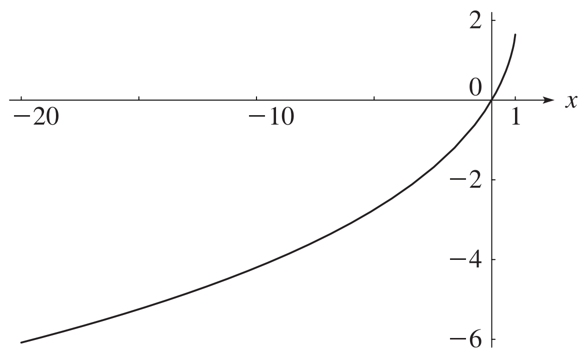

Figure 25.12.1: Dilogarithm function $\mathrm{Li}_2(x)$, $-20\le x<1$.

25.12(ii) Polylogarithms

For real or complex s and z the *polylogarithm* $\mathrm{Li}_s(z)$ is defined by

25.12.10
$$\mathrm{Li}_s(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^s}.$$

For each fixed complex s the series defines an analytic function of z for $|z|<1$. The series also converges when $|z|=1$, provided that $\Re s>1$. For other values of z, $\mathrm{Li}_s(z)$ is defined by analytic continuation.

The notation $\phi(z,s)$ was used for $\mathrm{Li}_s(z)$ in Truesdell (1945) for a series treated in Jonquière (1889), hence the alternative name *Jonquière's function*. The special case $z=1$ is the Riemann zeta function: $\zeta(s)=\mathrm{Li}_s(1)$.

Integral Representation

25.12.11
$$\mathrm{Li}_s(z)=\frac{z}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x-z}\,dx,$$

valid when $\Re s>0$ and $|\mathrm{ph}(1-z)|<\pi$, or $\Re s>1$ and $z=1$. (In the latter case (25.12.11) becomes (25.5.1)).

By (25.12.2)

25.12.9
$$\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2}=-\int_0^{\theta}\ln\left(2\sin\left(\tfrac{1}{2}x\right)\right)dx.$$

The right-hand side is called *Clausen's integral.*

For graphics see Figures 25.12.1 and 25.12.2, and for further properties see Maximon (2003), Kirillov (1995), Lewin (1981), Nielsen (1909), and Zagier (1989).

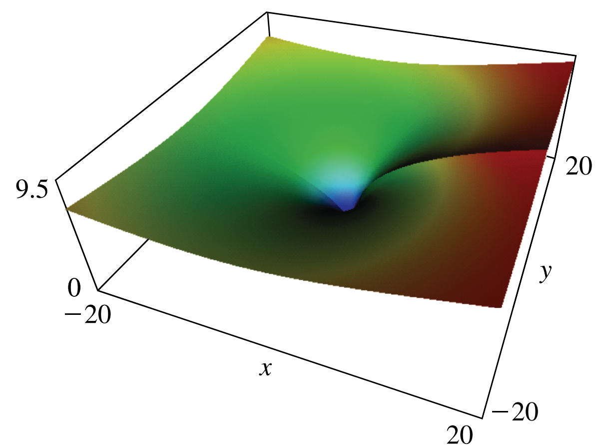

Figure 25.12.2: Absolute value of the dilogarithm function $|\mathrm{Li}_2(x+iy)|$, $-20\le x\le 20$, $-20\le y\le 20$. Principal value. There is a cut along the real axis from 1 to ∞.

Further properties include

25.12.12
$$\mathrm{Li}_s(z)=\Gamma(1-s)\left(\ln\frac{1}{z}\right)^{s-1}+\sum_{n=0}^{\infty}\zeta(s-n)\frac{(\ln z)^n}{n!},$$
$$s\ne 1,2,3,\dots,\ |\ln z|<2\pi,$$

and

25.12.13
$$\mathrm{Li}_s\left(e^{2\pi ia}\right)+e^{\pi is}\,\mathrm{Li}_s\left(e^{-2\pi ia}\right)=\frac{(2\pi)^s e^{\pi is/2}}{\Gamma(s)}\zeta(1-s,a),$$

valid when $\Re s>0$, $\Im a>0$ or $\Re s>1$, $\Im a=0$. When $s=2$ and $e^{2\pi ia}=z$, (25.12.13) becomes (25.12.4).

See also Lewin (1981), Kölbig (1986), Maximon (2003), Prudnikov *et al.* (1990, §§1.2 and 2.5), Prudnikov *et al.* (1992a, §3.3), and Prudnikov *et al.* (1992b, §3.3).

25.12(iii) Fermi–Dirac and Bose–Einstein Integrals

The Fermi–Dirac and Bose–Einstein integrals are defined by

25.12.14 $$F_s(x) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{t^s}{e^{t-x}+1}\, dt, \qquad s > -1,$$

25.12.15 $$G_s(x) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{t^s}{e^{t-x}-1}\, dt,$$
$$s > -1,\ x < 0;\ \text{or } s > 0,\ x \le 0,$$

respectively. Sometimes the factor $1/\Gamma(s+1)$ is omitted. See Cloutman (1989) and Gautschi (1993).

In terms of polylogarithms

25.12.16 $$F_s(x) = -\operatorname{Li}_{s+1}(-e^x), \quad G_s(x) = \operatorname{Li}_{s+1}(e^x).$$

For a uniform asymptotic approximation for $F_s(x)$ see Temme and Olde Daalhuis (1990).

25.13 Periodic Zeta Function

The notation $F(x,s)$ is used for the polylogarithm $\operatorname{Li}_s\left(e^{2\pi ix}\right)$ with x real:

25.13.1 $$F(x,s) = \sum_{n=1}^\infty \frac{e^{2\pi inx}}{n^s},$$

where $\Re s > 1$ if x is an integer, $\Re s > 0$ otherwise.

$F(x,s)$ is periodic in x with period 1, and equals $\zeta(s)$ when x is an integer. Also,

25.13.2 $$F(x,s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left(e^{\pi i(1-s)/2}\zeta(1-s,x) + e^{\pi i(s-1)/2}\zeta(1-s,1-x)\right),$$
$$0 < x < 1,\ \Re s > 1,$$

25.13.3 $$\zeta(1-s,x) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\pi is/2}F(x,s) + e^{\pi is/2}F(-x,s)\right),$$
$$0 < x < 1,\ \Re s > 0.$$

25.14 Lerch's Transcendent

25.14(i) Definition

25.14.1 $$\Phi(z,s,a) = \sum_{n=0}^\infty \frac{z^n}{(a+n)^s},$$
$$a \ne 0,-1,-2,\dots,|z| < 1;\ \Re s > 1, |z| = 1.$$

For other values of z, $\Phi(z,s,a)$ is defined by analytic continuation. This is the notation used in Erdélyi *et al.* (1953a, p. 27). Lerch (1887) used $\mathfrak{K}(a,x,s) = \Phi\left(e^{2\pi ix},s,a\right)$.

The Hurwitz zeta function $\zeta(s,a)$ (§25.11) and the polylogarithm $\operatorname{Li}_s(z)$ (§25.12(ii)) are special cases:

25.14.2 $$\zeta(s,a) = \Phi(1,s,a), \quad \Re s > 1,\ a \ne 0,-1,-2,\dots,$$

25.14.3 $$\operatorname{Li}_s(z) = z\,\Phi(z,s,1), \quad \Re s > 1,\ |z| \le 1.$$

25.14(ii) Properties

With the conditions of (25.14.1) and $m = 1,2,3,\dots$,

25.14.4 $$\Phi(z,s,a) = z^m\,\Phi(z,s,a+m) + \sum_{n=0}^{m-1} \frac{z^n}{(a+n)^s}.$$

25.14.5 $$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-ax}}{1-ze^{-x}}\, dx,$$
$$\Re s > 0,\ \Re a > 0,\ z \in \mathbb{C}\setminus[1,\infty).$$

25.14.6
$$\Phi(z,s,a) = \frac{1}{2}a^{-s} + \int_0^\infty \frac{z^x}{(a+x)^s}\, dx - 2\int_0^\infty \frac{\sin(x\ln z - s\arctan(x/a))}{(a^2+x^2)^{s/2}(e^{2\pi x}-1)}\, dx,$$
$$\Re s > 0 \text{ if } |z| < 1;\ \Re s > 1 \text{ if } |z| = 1, \Re a > 0.$$

For these and further properties see Erdélyi *et al.* (1953a, pp. 27–31).

25.15 Dirichlet L-functions

25.15(i) Definitions and Basic Properties

The notation $L(s,\chi)$ was introduced by Dirichlet (1837) for the meromorphic continuation of the function defined by the series

25.15.1 $$L(s,\chi) = \sum_{n=1}^\infty \frac{\chi(n)}{n^s}, \qquad \Re s > 1,$$

where $\chi(n)$ is a Dirichlet character $\pmod{k}$ (§27.8). For the principal character $\chi_1 \pmod{k}$, $L(s,\chi_1)$ is analytic everywhere except for a simple pole at $s = 1$ with residue $\phi(k)/k$, where $\phi(k)$ is Euler's totient function (§27.2). If $\chi \ne \chi_1$, then $L(s,\chi)$ is an entire function of s.

25.15.2 $$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \qquad \Re s > 1,$$

with the product taken over all primes p, beginning with $p = 2$. This implies that $L(s,\chi) \ne 0$ if $\Re s > 1$.

Equations (25.15.3) and (25.15.4) hold for all s if $\chi \ne \chi_1$, and for all s $(\ne 1)$ if $\chi = \chi_1$:

25.15.3 $$L(s,\chi) = k^{-s} \sum_{r=1}^{k-1} \chi(r)\,\zeta\left(s,\frac{r}{k}\right),$$

25.15.4 $$L(s,\chi) = L(s,\chi_0) \prod_{p|k} \left(1 - \frac{\chi_0(p)}{p^s}\right),$$

where χ_0 is a primitive character $\pmod{d}$ for some positive divisor d of k (§27.8).

When χ is a primitive character $\pmod{k}$ the L-functions satisfy the functional equation:

25.15.5
$$L(1-s,\chi) = \frac{k^{s-1}\,\Gamma(s)}{(2\pi)^s}\left(e^{-\pi is/2} + \chi(-1)e^{\pi is/2}\right) \times G(\chi)\,L(s,\overline{\chi}),$$

where $\overline{\chi}$ is the complex conjugate of χ, and

25.15.6 $$G(\chi)=\sum_{r=1}^{k}\chi(r)e^{2\pi ir/k}.$$

25.15(ii) Zeros

Since $L(s,\chi)\neq 0$ if $\Re s>1$, (25.15.5) shows that for a primitive character χ the only zeros of $L(s,\chi)$ for $\Re s<0$ (the so-called trivial zeros) are as follows:

25.15.7 $$L(-2n,\chi)=0 \text{ if } \chi(-1)=1,\quad n=0,1,2,\dots,$$

25.15.8 $$L(-2n-1,\chi)=0 \text{ if } \chi(-1)=-1,\quad n=0,1,2,\dots.$$

There are also infinitely many zeros in the critical strip $0\leq\Re s\leq 1$, located symmetrically about the critical line $\Re s=\frac{1}{2}$, but not necessarily symmetrically about the real axis.

25.15.9 $$L(1,\chi)\neq 0 \text{ if } \chi\neq\chi_1,$$

where χ_1 is the principal character $\pmod{k}$. This result plays an important role in the proof of Dirichlet's theorem on primes in arithmetic progressions (§27.11). Related results are:

25.15.10 $$L(0,\chi)=\begin{cases}-\dfrac{1}{k}\displaystyle\sum_{r=1}^{k}r\chi(r), & \chi\neq\chi_1,\\ 0, & \chi=\chi_1.\end{cases}$$

Applications

25.16 Mathematical Applications

25.16(i) Distribution of Primes

In studying the distribution of primes $p\leq x$, Chebyshev (1851) introduced a function $\psi(x)$ (not to be confused with the digamma function used elsewhere in this chapter), given by

25.16.1 $$\psi(x)=\sum_{m=1}^{\infty}\sum_{p^m\leq x}\ln p,$$

which is related to the Riemann zeta function by

25.16.2 $$\psi(x)=x-\frac{\zeta'(0)}{\zeta(0)}-\sum_{\rho}\frac{x^{\rho}}{\rho}+o(1),\qquad x\to\infty,$$

where the sum is taken over the nontrivial zeros ρ of $\zeta(s)$.

The prime number theorem (27.2.3) is equivalent to the statement

25.16.3 $$\psi(x)=x+o(x),\qquad x\to\infty.$$

The Riemann hypothesis is equivalent to the statement

25.16.4 $$\psi(x)=x+O\left(x^{\frac{1}{2}+\epsilon}\right),\qquad x\to\infty,$$

for every $\epsilon>0$.

25.16(ii) Euler Sums

Euler sums have the form

25.16.5 $$H(s)=\sum_{n=1}^{\infty}\frac{h(n)}{n^s},$$

where $h(n)$ is given by (25.11.33).

$H(s)$ is analytic for $\Re s>1$, and can be extended meromorphically into the half-plane $\Re s>-2k$ for every positive integer k by use of the relations

25.16.6 $$H(s)=-\zeta'(s)+\gamma\,\zeta(s)+\frac{1}{2}\,\zeta(s+1)+\sum_{r=1}^{k}\zeta(1-2r)\,\zeta(s+2r)+\sum_{n=1}^{\infty}\frac{1}{n^s}\int_{n}^{\infty}\frac{\widetilde{B}_{2k+1}(x)}{x^{2k+2}}\,dx,$$

25.16.7 $$H(s)=\frac{1}{2}\,\zeta(s+1)+\frac{\zeta(s)}{s-1}-\sum_{r=1}^{k}\binom{s+2r-2}{2r-1}\zeta(1-2r)\,\zeta(s+2r)-\binom{s+2k}{2k+1}\sum_{n=1}^{\infty}\frac{1}{n}\int_{n}^{\infty}\frac{\widetilde{B}_{2k+1}(x)}{x^{s+2k+1}}\,dx.$$

For integer s (≥ 2), $H(s)$ can be evaluated in terms of the zeta function:

25.16.8 $$H(2)=2\,\zeta(3),\quad H(3)=\tfrac{5}{4}\,\zeta(4),$$

25.16.9 $$H(a)=\frac{a+2}{2}\,\zeta(a+1)-\frac{1}{2}\sum_{r=1}^{a-2}\zeta(r+1)\,\zeta(a-r),\qquad a=2,3,4,\dots.$$

Also,

25.16.10 $$H(-2a)=\frac{1}{2}\,\zeta(1-2a)=-\frac{B_{2a}}{4a},\quad a=1,2,3,\dots.$$

$H(s)$ has a simple pole with residue $\zeta(1-2r)$ $(=-B_{2r}/(2r))$ at each odd negative integer $s=1-2r$, $r=1,2,3,\dots$.

$H(s)$ is the special case $H(s,1)$ of the function

25.16.11 $$H(s,z)=\sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{m=1}^{n}\frac{1}{m^z},\quad \Re(s+z)>1,$$

which satisfies the reciprocity law

25.16.12 $$H(s,z)+H(z,s)=\zeta(s)\,\zeta(z)+\zeta(s+z),$$

when both $H(s,z)$ and $H(z,s)$ are finite.

For further properties of $H(s,z)$ see Apostol and Vu (1984). Related results are:

25.16.13 $$\sum_{n=1}^{\infty}\left(\frac{h(n)}{n}\right)^2=\frac{17}{4}\,\zeta(4),$$

25.16.14 $$\sum_{r=1}^{\infty}\sum_{k=1}^{r}\frac{1}{rk(r+k)}=\frac{5}{4}\,\zeta(3),$$

25.16.15 $$\sum_{r=1}^{\infty}\sum_{k=1}^{r}\frac{1}{r^2(r+k)}=\frac{3}{4}\,\zeta(3).$$

For further generalizations, see Flajolet and Salvy (1998).

25.17 Physical Applications

Analogies exist between the distribution of the zeros of $\zeta(s)$ on the critical line and of semiclassical quantum eigenvalues. This relates to a suggestion of Hilbert and Pólya that the zeros are eigenvalues of some operator, and the Riemann hypothesis is true if that operator is Hermitian. See Armitage (1989), Berry and Keating (1998, 1999), Keating (1993, 1999), and Sarnak (1999).

The zeta function arises in the calculation of the partition function of ideal quantum gases (both Bose–Einstein and Fermi–Dirac cases), and it determines the critical gas temperature and density for the Bose–Einstein condensation phase transition in a dilute gas (Lifshitz and Pitaevskiĭ (1980)). Quantum field theory often encounters formally divergent sums that need to be evaluated by a process of regularization: for example, the energy of the electromagnetic vacuum in a confined space (*Casimir–Polder effect*). It has been found possible to perform such regularizations by equating the divergent sums to zeta functions and associated functions (Elizalde (1995)).

Computation

25.18 Methods of Computation

25.18(i) Function Values and Derivatives

The principal tools for computing $\zeta(s)$ are the expansion (25.2.9) for general values of s, and the Riemann–Siegel formula (25.10.3) (extended to higher terms) for $\zeta\left(\frac{1}{2}+it\right)$. Details are provided in Haselgrove and Miller (1960). See also Allasia and Besenghi (1989), Butzer and Hauss (1992), Kerimov (1980), and Yeremin *et al.* (1985). Calculations relating to derivatives of $\zeta(s)$ and/or $\zeta(s,a)$ can be found in Apostol (1985a), Choudhury (1995), Miller and Adamchik (1998), and Yeremin *et al.* (1988).

For the Hurwitz zeta function $\zeta(s,a)$ see Spanier and Oldham (1987, p. 653).

For dilogarithms and polylogarithms see Jacobs and Lambert (1972), Osácar *et al.* (1995), and Spanier and Oldham (1987, pp. 231–232).

For Fermi–Dirac and Bose–Einstein integrals see Cloutman (1989), Gautschi (1993), Mohankumar and Natarajan (1997), Natarajan and Mohankumar (1993), Paszkowski (1988, 1991), Pichon (1989), and Sagar (1991a,b).

25.18(ii) Zeros

Most numerical calculations of the Riemann zeta function are concerned with locating zeros of $\zeta\left(\frac{1}{2}+it\right)$ in an effort to prove or disprove the Riemann hypothesis, which states that all nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re s=\frac{1}{2}$. Calculations to date (2008) have found no nontrivial zeros off the critical line. For recent investigations see, for example, van de Lune *et al.* (1986) and Odlyzko (1987). For earlier work see Haselgrove and Miller (1960).

25.19 Tables

- Abramowitz and Stegun (1964) tabulates: $\zeta(n)$, $n=2,3,4,\dots$, 20D (p. 811); $\mathrm{Li}_2(1-x)$, $x=0(.01)0.5$, 9D (p. 1005); $f(\theta)$, $\theta=15^\circ(1^\circ)30^\circ(2^\circ)90^\circ(5^\circ)180^\circ$, $f(\theta)+\theta\ln\theta$, $\theta=0(1^\circ)15^\circ$, 6D (p. 1006). Here $f(\theta)$ denotes Clausen's integral, given by the right-hand side of (25.12.9).
- Morris (1979) tabulates $\mathrm{Li}_2(x)$ (§25.12(i)) for $\pm x=0.02(.02)1(.1)6$ to 30D.
- Cloutman (1989) tabulates $\Gamma(s+1)F_s(x)$, where $F_s(x)$ is the Fermi–Dirac integral (25.12.14), for $s=-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2}$, $x=-5(.05)25$, to 12S.
- Fletcher *et al.* (1962, §22.1) lists many sources for earlier tables of $\zeta(s)$ for both real and complex s. §22.133 gives sources for numerical values of coefficients in the Riemann–Siegel formula, §22.15 describes tables of values of $\zeta(s,a)$, and §22.17 lists tables for some Dirichlet L-functions for real characters. For tables of dilogarithms, polylogarithms, and Clausen's integral see §§22.84–22.858.

25.20 Approximations

- Cody *et al.* (1971) gives rational approximations for $\zeta(s)$ in the form of quotients of polynomials or quotients of Chebyshev series. The ranges covered are $0.5 \le s \le 5$, $5 \le s \le 11$, $11 \le s \le 25$, $25 \le s \le 55$. Precision is varied, with a maximum of 20S.

- Piessens and Branders (1972) gives the coefficients of the Chebyshev-series expansions of $s\,\zeta(s+1)$ and $\zeta(s+k)$, $k = 2, 3, 4, 5, 8$, for $0 \le s \le 1$ (23D).

- Luke (1969b, p. 306) gives coefficients in Chebyshev-series expansions that cover $\zeta(s)$ for $0 \le s \le 1$ (15D), $\zeta(s+1)$ for $0 \le s \le 1$ (20D), and $\ln \xi\left(\frac{1}{2} + ix\right)$ (§25.4) for $-1 \le x \le 1$ (20D). For errata see Piessens and Branders (1972).

- Morris (1979) gives rational approximations for $\mathrm{Li}_2(x)$ (§25.12(i)) for $0.5 \le x \le 1$. Precision is varied with a maximum of 24S.

- Antia (1993) gives minimax rational approximations for $\Gamma(s+1)F_s(x)$, where $F_s(x)$ is the Fermi–Dirac integral (25.12.14), for the intervals $-\infty < x \le 2$ and $2 \le x < \infty$, with $s = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$. For each s there are three sets of approximations, with relative maximum errors $10^{-4}, 10^{-8}, 10^{-12}$.

25.21 Software

See http://dlmf.nist.gov/25.21.

References

General References

The main references used in writing this chapter are Apostol (1976), Erdélyi *et al.* (1953a), and Titchmarsh (1986b). For additional bibliographic reading see Edwards (1974), Ivić (1985), Karatsuba and Voronin (1992).

Sources

The following list gives the references or other indications of proofs that were used in constructing the various sections of this chapter. These sources supplement the references that are quoted in the text.

§25.2 Apostol (1976, Chapter 12). For (25.2.2)–(25.2.7) see also Hardy (1912). For (25.2.8)–(25.2.10) see also Knopp (1948, p. 533). (25.2.9) follows from (25.2.8) by repeated integration by parts. For (25.2.11), (25.2.12) see also Titchmarsh (1986b, p. 30).

§25.3 These graphics were constructed at NIST.

§25.4 Apostol (1976, Chapter 12).

§25.5 Apostol (1976, Chapter 12), Erdélyi *et al.* (1953a, Chapter I). For (25.5.2) and (25.5.4) integrate (25.5.1) and (25.5.3) by parts. For (25.5.5) see Titchmarsh (1986b, p. 15). (25.5.6) comes from (25.5.1) by using the identity $e^{-x} = (1 - e^{-x})/(e^x - 1)$ in the integral $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}\,dx$ together with (5.5.1). (25.5.7) follows from (25.5.6) because $\Gamma(s+2m-1) = \int_0^\infty e^{-x}x^{s+2m-2}\,dx$. For (25.5.10) and (25.5.11) see Lindelöf (1905, p. 103). For (25.5.12) see Srivastava and Choi (2001, p. 12). For (25.5.13) see Titchmarsh (1986b, p. 22). For (25.5.14)–(25.5.19) see de Bruijn (1937). For (25.5.21) see Erdélyi *et al.* (1953a, p. 32).

§25.6 For (25.6.1)–(25.6.4) see Apostol (1976, pp. 266–268). For (25.6.5) see Mordell (1958). For (25.6.6) see Nörlund (1924, p. 66). For (25.6.7) see Apostol (1983). For (25.6.8)–(25.6.10) see van der Poorten (1980, pp. 271, 274). For (25.6.11)–(25.6.14) see Apostol (1985a). For (25.6.15) see Miller and Adamchik (1998). For (25.6.16)–(25.6.20) see Basu and Apostol (2000).

§25.8 Titchmarsh (1986b, Chapter IV), Adamchik and Srivastava (1998), Erdélyi *et al.* (1953a, pp. 45 and 51). For (25.8.2) see Landau (1953, p. 274). For (25.8.3) see Srivastava (1988). For (25.8.7), (25.8.8) divide by x in (25.8.5), (25.8.6) and integrate. For (25.8.9) see Srivastava and Choi (2001, p. 212). For (25.8.10) see Ewell (1990).

§25.9 Titchmarsh (1986b, Chapter XV), Berry (1995).

§25.10 Apostol (1976, Chapter 12), Titchmarsh (1986b, pp. 89 and 263).

§25.11 Apostol (1976, Chapter 12). Analytic properties of $\zeta(s,a)$ with respect to a follow from (25.11.30). For (25.11.5)–(25.11.6) see Apostol (1985a). For (25.11.7) take $N = 1$ in (25.11.5) and integrate by parts. For (25.11.8)–(25.11.9) see Srivastava and Choi (2001, p. 89). For (25.11.10) use Taylor's theorem (§§1.4(vi), 1.10(i)) and (25.11.17). For (25.11.11) apply (25.2.2) and (25.11.1). For (25.11.12) see Erdélyi *et al.* (1953a,

p. 45). For (25.11.13) and (25.11.14) see Apostol (1976, pp. 268, 264). For (25.11.15) use (25.11.1) and analytic continuation. For (25.11.16) see Apostol (1976, p. 263). For (25.11.17) differentiate (25.11.1). For (25.11.18) see Erdélyi *et al.* (1953a, p. 26). For (25.11.19)–(25.11.23) see Apostol (1985a, p. 231) and Miller and Adamchik (1998). For (25.11.24) use (25.11.15) with $a = 1/k$, multiply by k^s and differentiate. For (25.11.25) see Srivastava and Choi (2001, p. 89) For (25.11.26) see Berndt (1972). For (25.11.27) and (25.11.28) argue as indicated above for (25.5.6) and (25.5.7). For (25.11.29) see Lindelöf (1905, p. 106). For (25.11.30) assume $\Re s > 1$, collapse the integration path onto the real axis, apply (25.11.25) and (5.5.3) followed by analytic continuation. For (25.11.31) use (25.11.25). For (25.11.32)–(25.11.34) see Adamchik (1998). For (25.11.35) use (25.11.25) and (25.11.8). For (25.11.36) see Apostol (1976). For (25.11.37)–(25.11.40) see Adamchik and Srivastava (1998). For (25.11.41) and (25.11.42) see Apostol (1952). For (25.11.43) see Paris (2005b). For (25.11.44) and (25.11.45) see Elizalde (1986). The graphics were constructed at NIST.

§25.12 Erdélyi *et al.* (1953a, pp. 27, 29), Maximon (2003). For (25.12.13) see Erdélyi *et al.* (1953a, p. 31) with change of notation. The graphics were constructed at NIST.

§25.13 Apostol (1976, Chapter 13).

§25.15 Apostol (1976, Chapter 12), Apostol (1985b). For (25.15.9) see Apostol (1976, pp. 142, 149).

§25.16 Apostol (1976, Chapter 13). For (25.16.2) see Apostol (2000). For (25.16.4) see Ingham (1932, p. 84). For (25.16.5)–(25.16.15) see Apostol and Vu (1984) and Basu and Apostol (2000).