SMOOTH COMPACTIFICATIONS OF CERTAIN NORMIC BUNDLES

ANTHONY VÁRILLY-ALVARADO AND BIANCA VIRAY

Abstract. For a finite cyclic Galois extension of fields K/k of degree n and a separable polynomial of degree dn or $dn-1$, we construct an explicit smooth compactification $X \to \mathbb{P}^1_k$ of the affine normic bundle X_0 given by

$$\mathrm{N}_{K/k}(\vec{z}) = P(x) \neq 0,$$

extending the map $X_0 \to \mathbb{A}^1_k$, where $(\vec{z}, x) \mapsto x$. The construction makes no assumption of the characteristic of k, making it a suitable departure point for studying the arithmetic of smooth compactifications of X_0 over global fields of positive characteristic.

1. Introduction

For a finite extension K/k of fields and a polynomial $P(x) \in k[x]$, the affine norm hypersurface $X_0 \subset \mathbb{A}^{n+1}_k$ given by

$$\mathrm{N}_{K/k}(\vec{z}) = P(x) \neq 0 \tag{1.1}$$

parametrizes the values of $P(x)$ that are norms for K/k.

Suppose that k is a number field. The classical Hasse norm theorem states that if K/k is a cyclic Galois extension and if $P(x)$ is a nonzero constant, then X_0 satisfies the Hasse principle. Although both the Hasse principle and weak approximation fail for more general X_0, Colliot-Thélène has conjectured that the Brauer-Manin obstruction controls failures of weak approximation on any smooth proper model of X_0. See [DSW14, §1] for a summary of progress towards this conjecture.

The existence of a smooth proper model X of X_0 extending the projection

$$X_0 \to \mathbb{A}^1, \qquad (\vec{z}, x) \mapsto x$$

to a map $X \to \mathbb{P}^1_k$ is especially useful for proving arithmetic results in the direction of Colliot-Thélène's conjecture, because the map $X \to \mathbb{P}^1_k$ affords some control over the Brauer group of X. This map can also be used to prove that certain subsets of the number field k are diophantine [Poo09, VAV12, CTvG].

The known constructions of $X \to \mathbb{P}^1$ proceed in two steps. First, one constructs a partial compactification $X' \to \mathbb{A}^1_k$ (e.g. [CTHS03, §2] or [CTvG, Proposition 2.2(i)]). Second, one extends $X' \to \mathbb{A}^1_k$ to a map $X \to \mathbb{P}^1$ via Hironaka's theorem. This second step limits the scope of the construction to fields of characteristic 0.

Our goal in this note is to give an explicit construction of a compactification $X \to \mathbb{P}^1_k$ convenient for arithmetic applications, under some hypotheses. (For example, the Picard group and Brauer group of such a compactification X are easily computable; see the proofs of [VAV12, Proposition 3.1 and Theorem 3.2].) The construction of X does not impose a restriction on the characteristic of k; it therefore serves as a starting point for studying the arithmetic of smooth compactifications of X_0 over global fields of positive characteristic.

2000 *Mathematics Subject Classification.* Primary 11 G35; Secondary 14 G05.

Theorem 1.1. *Let K/k be a cyclic Galois extension of fields of degree n, and let $P(x) \in k[x]$ be a separable polynomial of degree dn or $dn-1$. There exists a smooth proper compactification X of X_0, fibered over $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$, such that $X \to \mathbb{P}^1_k$ extends the map $X_0 \to \mathbb{A}^1_k$. Furthermore, the generic fiber of $X \to \mathbb{P}^1_k$ is a Severi-Brauer variety, and the degenerate fibers lie over $V(P(x_0/x_1)x_1^{dn})$, and consist of the union of n rational varieties all conjugate under $\operatorname{Gal}(K/k)$.*

1.1. **Outline.** Our construction of a smooth compactification takes a cue from work of Kang: the generic fiber of our construction is the embedded Severi-Brauer variety in [Kan90].

In §3.2, we construct a partial compactification $Y_a \to \operatorname{Spec} R$ of the variety $z_1 \cdots z_n = a \neq 0$ for any k-algebra R with no zero-divisors and any $a \in R \setminus 0$. In §3.3, we give an explicit open covering of Y_a, which we use in §3.4 to prove that Y is smooth if and only if $V(a) \subset \operatorname{Spec}(R)$ is smooth. We describe the geometry of the degenerate fibers of $Y_a \to \operatorname{Spec} R$ in §3.5.

In §4, we construct a K/k-twist of Y_a, $X^0_{K,a} \to \operatorname{Spec} R$. Finally, in §5, we restrict to the case $R = k[x]$ and $a = P(x)$, give a full compactification $X \to \mathbb{P}^1_k$, and prove Theorem 1.1.

Remark 1.2. Artin [Art82] gives a construction of a Severi-Brauer bundle $\tilde{X} \to \mathbb{A}^1_k$ associated to a maximal $k[x]$-order in a central simple $k(x)$-algebra. This can be translated into a partial compactification of $X_0 \to \mathbb{A}^1_k$, proper over $\mathbb{A}^1_k$, whose generic fiber is a Severi-Brauer variety, and whose degenerate fibers consist of n rational varieties conjugate under $\operatorname{Gal}(K/k)$.

Acknowledgements. We also thank Jean-Louis Colliot-Thélène and the anonymous referee for comments improving the exposition.

2. Preliminaries on vectors

Throughout, we fix an integer $n > 1$. By the weight of a vector $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathbb{Z}^n$, we mean the sum $\sum_{m=0}^{n-1} v_m$; we also say $\mathbf{v}$ has length n. Let $\mathcal{V}_n$ denote the set of nonnegative integer vectors of weight n and length n. Write $\sigma\colon \mathcal{V}_n \to \mathcal{V}_n$ for the shift operator

$$\mathbf{v} = (v_0, v_1, \ldots, v_{n-1}) \mapsto (v_1, v_2, \ldots, v_{n-1}, v_0) =: \sigma(\mathbf{v}).$$

For any $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathcal{V}_n$ we define two nonnegative integers:

$$\mu(\mathbf{v}) := \max_{i \in (0,n]} (i - v_0 - \cdots - v_{i-1}), \qquad \lambda(\mathbf{v}) := \mu(\mathbf{v}) + \mu(\sigma(\mathbf{v})) + \cdots + \mu(\sigma^{n-1}(\mathbf{v})),$$

and for any integers i, j with $i \not\equiv j \bmod n$ and $v_j > 0$, we let $\mathbf{v}^{i,j} := (v_m + \delta_{i \bmod n, m} - \delta_{j \bmod n, m})_{m=0}^{n-1}$; note that $\mathbf{v}^{i,j} \in \mathcal{V}_n$, because $v_j > 0$. We collect a few straightforward relations used frequently below.

Lemma 2.1. *We have the following relations.*

1. $\mu(\sigma^s(\mathbf{v})) = \mu(\mathbf{v}) + v_0 + v_1 + \cdots + v_{s-1} - s$ *for any* $\mathbf{v} \in \mathcal{V}_n$ *and for any* $s \in (0, n]$.
2. *For any integer r and any vectors $\mathbf{v}_i, \mathbf{w}_i \in \mathcal{V}_n$ with $1 \leq i \leq r$, such that $\sum_{i=1}^r \mathbf{w}_i = \sum_{i=1}^r \mathbf{v}_i$, we have $\sum_{i=1}^r \lambda(\mathbf{w}_i) - \sum_{i=1}^r \lambda(\mathbf{v}_i) = n\left(\sum_{i=1}^r \mu(\mathbf{w}_i) - \sum_{i=1}^r \mu(\mathbf{v}_i)\right)$.*
3. *Fix integers $0 \leq r < s < n$ and fix $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathcal{V}_n$ such that $v_r, v_s > 0$. Then: $0 \leq \mu(\mathbf{v}^{r,s}) + \mu(\mathbf{v}^{s,r}) - 2\mu(\mathbf{v}) \leq 1$, and the first inequality is strict if and only if $\mu(\mathbf{v}) = i - \sum_{m=0}^{i-1} v_m = j - \sum_{m=0}^{j-1} v_m$ for some $i \in (r, s]$ and $j \in (0, r] \cup (s, n]$.*

Proof. Let $\mathbf{w} := \sigma^s(\mathbf{v})$, so that $w_j = v_{j+s}$ if $j < n-s$ and $w_j = v_{j+s-n}$ if $j \geq n-s$. Then $i - w_0 - w_1 - \cdots - w_{i-1}$ equals

$$\begin{cases} ((i+s) - v_0 - \cdots - v_{i+s-1}) + v_0 + \cdots + v_{s-1} - s & \text{if } i+s \leq n \\ ((i+s-n) - v_0 - \cdots - v_{i+s-n-1}) + (n-s) - v_s - \cdots - v_{n-1} & \text{otherwise.} \end{cases}$$

To conclude (1), note that since $\mathbf{v}$ has weight n, we have $v_0 + v_1 + \cdots + v_{s-1} - s = (n-s) - v_s - v_{s+1} - \cdots - v_{n-1}$.

By (1), for any vector $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathcal{V}_n$ we have

$$\lambda(\mathbf{v}) = n\mu(\mathbf{v}) + \sum_{m=0}^{n-1} ((n-1-m)v_m - m)\,.$$

Using the assumption that $\sum_{i=1}^r \mathbf{w}_i = \sum_{i=1}^r \mathbf{v}_i$, the proof of (2) is now a simple manipulation.

It remains to prove (3). Let $\mathbf{w}^- := \mathbf{v}^{r,s}$ and $\mathbf{w}^+ := \mathbf{v}^{s,r}$. Since $r < s$, by the definition of μ we have

$$\mu(\mathbf{v}) - 1 \leq \mu(\mathbf{w}^-) \leq \mu(\mathbf{v}) \quad \text{and} \quad \mu(\mathbf{v}) \leq \mu(\mathbf{w}^+) \leq \mu(\mathbf{v}) + 1.$$

Furthermore, $\mu(\mathbf{w}^-) = \mu(\mathbf{v}) - 1$ if and only if the maximum of $\{i - v_0 - \cdots - v_{i-1}\}_{i \in (0,n]}$ is *only* achieved for $i \in (r,s]$. Similarly, $\mu(\mathbf{v}) = \mu(\mathbf{w}^+)$ if and only if the maximum of $\{i - v_0 - \cdots - v_{i-1}\}_{i \in (0,n]}$ is *only* achieved for $i \in (0,r] \cup (s,n]$. □

The following notion is the fundamental book-keeping device in the construction of $X \to \mathbb{P}^1_k$.

Definition 2.2. *A vector $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1}) \in \mathcal{V}_n$ is well-spaced if*

$$v_i > 0 \Rightarrow v_{i+v_i} > 0 \text{ and } v_{i+j} = 0 \text{ for all } j \in [1, v_i - 1]$$

for all $i \in [0,n)$. Here indices are considered modulo n.

For example, $\mathbf{v} = (0,3,0,0,2,0,4,0,0)$ is well-spaced whereas $\mathbf{w} = (0,3,0,0,2,4,0,0,0)$ is not. Note that $\sigma(\mathbf{v})$ is well-spaced if and only if $\mathbf{v}$ is well-spaced.

Remark 2.3. We are unaware if well-spaced vectors arise naturally in other fields. It would be interesting to have a conceptual understanding of why these vectors yield useful affine coverings of the varieties under consideration (see §3).

Lemma 2.4. *Let $\mathbf{v} \in \mathcal{V}_n$ be a well-spaced vector with $\ell + 1$ nonzero entries indexed by $i_0 < \cdots < i_\ell$. Set $i_{\ell+1} := n + i_0$. Then $\mu(\mathbf{v}) = i_0$, and for any $r, s \in [0,\ell]$ and $j \in (i_r, i_{r+1})$, we have*

$$\mu(\mathbf{v}^{j,i_r}) = \mu(\mathbf{v}) - \left\lfloor \frac{j}{n} \right\rfloor, \qquad \mu(\sigma^{i_s}(\mathbf{v}^{j,i_r})) = 0,$$

$$\text{and if } \ell \neq 0, \qquad \mu(\mathbf{v}^{i_r,i_{r+1}}) = \mu(\mathbf{v}) + \left\lfloor \frac{i_{r+1}}{n} \right\rfloor, \quad \mu(\sigma^{i_s}(\mathbf{v}^{i_r,i_{r+1}})) = \delta_{s,r+1 \bmod \ell+1}.$$

Proof. For any vector $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathcal{V}_n$, the maximum of $\{i - v_0 - \cdots - v_{i-1}\}_{i \in (0,n]}$ is never achieved at an $i = j$ where $v_j = 0$. Additionally, since $\mathbf{v}$ is well-spaced, $v_{i_r} = i_{r+1} - i_r$ for all $r \in [0,\ell]$. Hence

$$\mu(\mathbf{v}) = \max_{r \in [0,\ell]} \left(i_r - v_{i_0} - \cdots - v_{i_{r-1}}\right) = \max_{r \in [0,\ell]} \left(i_r - (i_1 - i_0) - \cdots - (i_r - i_{r-1})\right) = i_0$$

and the maximum of $\{i - v_0 - \cdots - v_{i-1}\}_{i\in(0,n]}$ is achieved at $i = i_r$ for all $r \in [0, \ell]$.

Thus, the formulas for $\mu(\mathbf{v}^{j,i_r})$ and $\mu(\mathbf{v}^{i_r,i_{r+1}})$ follow from the same argument as in Lemma 2.1 (3). Let $\mathbf{w} := \mathbf{v}^{j,i_r}$. Then

$$\begin{aligned}
\mu(\sigma^{i_s}(\mathbf{w})) &= \mu(\mathbf{w}) + w_0 + \cdots + w_{i_s-1} - i_s && \text{by Lemma 2.1 (1),}\\
&= \mu(\mathbf{v}) - \left\lfloor \frac{j}{n} \right\rfloor + w_0 + \cdots + w_{i_s-1} - i_s && \text{by the formula for } \mu(\mathbf{v}^{j,i_r}),\\
&= \mu(\mathbf{v}) - \left\lfloor \frac{j}{n} \right\rfloor + v_0 + \cdots + v_{i_s-1} - i_s + \left\lfloor \frac{j}{n} \right\rfloor && \text{by the definition of } \mathbf{v}^{j,i_r},\\
&= \mu(\mathbf{v}) - i_0 && \text{since } v_{i_r} = i_{r+1} - i_r \text{ for } r \in [0, \ell].
\end{aligned}$$

Similarly, if $\mathbf{w} := \mathbf{v}^{i_r,i_{r+1}}$, we have

$$\begin{aligned}
\mu(\sigma^{i_s}(\mathbf{w})) &= \mu(\mathbf{w}) + w_0 + \cdots + w_{i_s-1} - i_s\\
&= \mu(\mathbf{v}) + \left\lfloor \frac{i_{r+1}}{n} \right\rfloor + w_0 + \cdots + w_{i_s-1} - i_s\\
&= \mu(\mathbf{v}) + \left\lfloor \frac{i_{r+1}}{n} \right\rfloor + v_0 + \cdots + v_{i_s-1} - i_s + \delta_{s,r+1 \bmod \ell+1} - \left\lfloor \frac{i_{r+1}}{n} \right\rfloor\\
&= \mu(\mathbf{v}) - i_0 + \delta_{s,r+1 \bmod \ell+1}.
\end{aligned}$$

□

3. The auxiliary bundle $Y \to \operatorname{Spec} R$

3.1. **Notation.** Let R be a k-algebra with no zero-divisors. Given a nonzero element $a \in R$, we use the standard notation $D(a)$ to denote the open affine subscheme of $\operatorname{Spec} R$ given by $\operatorname{Spec} R_a$; if R is graded, we let $D_+(a)$ denote the open affine subscheme of $\operatorname{Proj} R$ given by $\operatorname{Spec}(R_a)_0$.

Let $N = \binom{2n-1}{n} - 1$, and fix coordinates on $\mathbb{P}^N_k = \operatorname{Proj} k\,[\{y_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}]$. We set $\mathbb{P}^N_R := \mathbb{P}^N_k \times_{\operatorname{Spec} k} \operatorname{Spec} R$.

3.2. **Construction of Y_a.** For any nonzero $a \in R$, we consider the embedding

$$\iota_a \colon \operatorname{Proj} R[t_0, \ldots, t_{n-1}] \cap D(a) \hookrightarrow \mathbb{P}^N_R \times_{\operatorname{Spec} R} D(a)$$

induced by the map $y_{\mathbf{v}} \mapsto a^{\mu(\mathbf{v})} t_0^{v_0} t_1^{v_1} \cdots t_{n-1}^{v_{n-1}}$.(This is easily seen to be an embedding since it is the composition of the (n)-uple embedding with a scaling of the coordinates by an appropriate power of a.) The image of ι_a is cut out by the equations (see [Har92, Example 2.6]):

$$a^{\sum_{i=1}^r \mu(\mathbf{w}_i)} \prod_{i=1}^r y_{\mathbf{v}_i} = a^{\sum_{i=1}^r \mu(\mathbf{v}_i)} \prod_{i=1}^r y_{\mathbf{w}_i} \tag{3.1}$$

for all integers r and all sets of vectors $\mathbf{w}_i, \mathbf{v}_i$, with $1 \le i \le r$, such that $\sum_{i=1}^r \mathbf{w}_i = \sum_{i=1}^r \mathbf{v}_i$. Let Y_a be the closure in $\mathbb{P}^N_R$ of the image of ι_a.

Lemma 3.1. *The order n automorphism $\phi \colon \mathbb{P}^N_R \to \mathbb{P}^N_R$, $y_{\mathbf{v}} \mapsto y_{\sigma(\mathbf{v})}$ preserves Y_a.*

Proof. Fix an integer r and vectors $\mathbf{v}_i, \mathbf{w}_i$, with $1 \le i \le r$, such that $\sum_{i=1}^r \mathbf{v}_i = \sum_{i=1}^r \mathbf{w}_i$. It is clear that $\sum_{i=1}^r \sigma(\mathbf{v}_i) = \sum_{i=1}^r \sigma(\mathbf{w}_i)$. Moreover, by Lemma 2.1(1),

$$\sum_{i=1}^r \mu(\sigma(\mathbf{v}_i)) - \sum_{i=1}^r \mu(\sigma(\mathbf{w}_i)) = \sum_{i=1}^r \mu(\mathbf{v}_i) - \sum_{i=1}^r \mu(\mathbf{w}_i).$$

Therefore, ϕ preserves the relations (3.1). $\square$

3.3. **An open covering.**

Proposition 3.2. *The open subvarieties* $\{D_+(y_{\mathbf{v}}) \subseteq \mathbb{P}^N_R : \mathbf{v} \in \mathcal{V}_n \text{ well-spaced}\}$ *cover* Y_a.

Proof. Let $\mathbf{w} = (w_0, w_1, \dots, w_{n-1}) \in \mathcal{V}_n$ be a vector that is not well-spaced. We will show that $D_+(y_{\mathbf{w}}) \cap Y_a \subset D_+(y_{\mathbf{v}})$ for some well-spaced vector $\mathbf{v} \in \mathcal{V}_n$. Let $i_0 < \cdots < i_\ell$ be the indices such that $w_{i_j} > 0$, and set $i_{\ell+1} = i_0 + n$.

If $\mu(\mathbf{w}) > i_0 \geq 0$, then there exists an $r \in [0, \ell)$ such that $\mu(\mathbf{w}) = i_{r+1} - w_{i_0} - w_{i_1} - \cdots - w_{i_r}$. Fix the smallest such r; then $i_{r+1} - i_r - w_{i_r} > 0$. Since $\mathbf{w}$ has length n and weight n, there exists an $r < s \leq \ell$ such that $(i_{s+1} - i_s - w_{i_s}) < 0$; fix the largest such s. Then by our choice of r and s, if $\mu(\mathbf{w}) = j - v_0 - \cdots - v_{j-1}$, we must have $j \in (i_r, i_s]$. Therefore by Lemma 2.1(3), the defining equations for Y_a include the relation

$$y_{\mathbf{w}}^2 = y_{\mathbf{w}^{i_r,i_s}} y_{\mathbf{w}^{i_s,i_r}},$$

so $D_+(y_{\mathbf{w}}) \subset D_+(y_{\mathbf{w}^{i_r,i_s}})$. After possibly repeating the argument we may assume that $\mu(\mathbf{w}) = i_0$.

If $i_{r+1} - i_r - w_{i_r} = 0$ for all r, then $\mathbf{w}$ is well-spaced. Otherwise, fix the smallest integer r such that $|i_{r+1} - i_r - w_{i_r}| > 0$; since $\mu(\mathbf{w}) = i_0$, we must have $i_{r+1} - i_r - w_{i_r} < 0$. Since $\mathbf{w}$ has length n and weight n, there exists an $r < s \leq \ell$ such that $(i_{s+1} - i_s - w_{i_s}) > 0$; fix the smallest such s. Now by our choice of r and s, if $\mu(\mathbf{w}) = j - v_0 - \cdots - v_{j-1}$, we must have $j \in [0, i_r] \cup (i_s, n)$. Then by the same argument as above, the defining equations for Y_a include the relation

$$y_{\mathbf{w}}^2 = y_{\mathbf{w}^{i_r,i_s}} y_{\mathbf{w}^{i_s,i_r}}.$$

By replacing $\mathbf{w}$ with $\mathbf{w}^{i_s,i_r}$, we reduce the value of $|i_{r+1} - i_r - w_{i_r}|$. Repeating this process we will arrive at a well-spaced vector $\mathbf{v}$ in finitely many steps. $\square$

3.4. **Smoothness of Y_a.**

Proposition 3.3. *Let* $\mathbb{A}^n_R = \operatorname{Spec} R[Z_0, \dots, Z_{n-1}]$. *Let* $\mathbf{v} \in \mathcal{V}_n$ *be a well-spaced vector with* $\ell + 1$ *nonzero entries indexed by* $i_0 < \cdots < i_\ell$ *and set* $i_{\ell+1} := i_0 + n$. *Then the map*

$$\frac{y_{\mathbf{w}}}{y_{\mathbf{v}}} \mapsto \left(\prod_{j \in [0,n),\, v_j = 0} Z_j^{w_j} \right) \times \left(\prod_{r=0}^{\ell} Z_{i_r}^{\mu(\sigma^{i_r+1}(\mathbf{w}))} \right)$$

yields an isomorphism $Y_a \cap D_+(y_{\mathbf{v}}) \cong V(Z_{i_0} \cdots Z_{i_\ell} - a) \subset \mathbb{A}^n_R$. *In particular,* Y_a *is a compactification of the variety in* $\mathbb{A}^n_R$ *given by* $Z_0 Z_1 \cdots Z_{n-1} = a$.

Corollary 3.4. *The variety* Y_a *is smooth if and only if* $V(a)$ *is smooth in* $\operatorname{Spec} R$.

Proof. This follows from Propositions 3.2 and 3.3, and the Jacobian criterion. $\square$

Proof of Proposition 3.3. Set $i_{-1} = i_\ell - n$. The proof of the proposition differs slightly in the case when $\ell = 0$. To give a unified presentation, if $\ell = 0$ then we set $y_{\mathbf{v}^{i_0,i_{\ell+1}}} := a y_{\mathbf{v}}$. Consider the following functions on $Y_a \cap D_+(y_{\mathbf{v}})$ for $j = i_0, i_0 + 1, \dots, i_0 + n - 1$:

$$g_j := \begin{cases} y_{\mathbf{v}^{j,i_r}} y_{\mathbf{v}}^{-1} & \text{if } i_r < j < i_{r+1} \text{ for some } 0 \leq r \leq \ell, \\ y_{\mathbf{v}^{i_r,i_{r+1}}} y_{\mathbf{v}}^{-1} & \text{if } j = i_r, 0 \leq r \leq \ell. \end{cases} \tag{3.2}$$

Lemma 2.4 shows that the map sends $g_j \to Z_{j \bmod n}$. In addition, Lemma 2.4 together with the relations (3.1) shows that $g_{i_0} \dots g_{i_\ell} = a$. Thus, to prove the map is a well-defined isomorphism, we will show that

$$y_{\mathbf{w}} y_{\mathbf{v}}^m = \left(\prod_{j \in [0,n),\, v_j = 0} y_{\mathbf{v}^{j,i_r}}^{w_j} \right) \times \left(\prod_{r=0}^{\ell} y_{\mathbf{v}^{i_r, i_{r+1}}}^{\mu(\sigma^{i_r+1}(\mathbf{w}))} \right)$$

where $m = -1 + \sum_{j, v_j = 0} w_j + \sum_{r=0}^{\ell} \mu(\sigma^{i_{r+1}}(\mathbf{w}))$. By Lemmas 2.4 and 2.1(1), we have

$$\sum_{\substack{j \in [0,n), \\ v_j = 0}} w_j \cdot \mu(\mathbf{v}^{j,i_r}) + \sum_{r=0}^{\ell} \mu(\sigma^{i_{r+1}}(\mathbf{w})) \mu(\mathbf{v}^{i_r, i_{r+1}}) = (m+1)\mu(\mathbf{v}) - \sum_{j=0}^{i_0 - 1} w_j + \mu(\sigma^{i_{\ell+1}}(\mathbf{w}))$$

$$= (m+1)\mu(\mathbf{v}) + \mu(\mathbf{w}) - i_0 = m\mu(\mathbf{v}) + \mu(\mathbf{w}).$$

Hence, by (3.1), it suffices to prove that $\mathbf{w} + m\mathbf{v}$ is equal to

$$\mathbf{w}' := \sum_{j \in [0,n),\, v_j = 0} w_j \mathbf{v}^{j,i_r} + \sum_{r=0}^{\ell} \mu(\sigma^{i_{r+1}}(\mathbf{w})) \mathbf{v}^{i_r, i_{r+1}}.$$

For j such that $v_j = 0$, it is evident that $w'_j = w_j + m v_j$. Further,

$$\begin{aligned} w'_{i_s} &= (m+1) v_{i_s} - \sum_{j = i_s + 1}^{i_{s+1} - 1} w_j + \mu(\sigma^{i_{s+1}}(\mathbf{w})) - \mu(\sigma^{i_s}(\mathbf{w})) \\ &= (m+1) v_{i_s} - \sum_{j = i_s + 1}^{i_{s+1} - 1} w_j + \sum_{j = i_s}^{i_{s+1} - 1} w_j - i_{s+1} + i_s \qquad \text{by Lemma 2.1(1)} \\ &= m v_{i_s} + w_{i_s} \qquad \text{since } v_{i_s} = i_{s+1} - i_s. \end{aligned}$$

□

3.5. The degenerate fibers of $Y_a \to \operatorname{Spec} R$.

Proposition 3.5. *Let $Q \in V(a) \in R$ be a closed point. The fiber $Y_{a,Q}$ consists of n rational $(n-1)$-dimensional irreducible components which are permuted cyclically by the automorphism ϕ of Lemma 3.1.*

Proof. For $i = 0, \dots, n-1$, we define $S_i := Y_{a,Q} \cap V(\langle y_{\mathbf{w}} : \mu(\sigma^{i+1}(\mathbf{w})) > 0 \rangle)$. From the definition, it follows that ϕ acts on the set $\{S_i : 0 \le i \le n-1\}$ via the permutation

$$S_0 \mapsto S_{n-1} \mapsto S_{n-2} \mapsto \dots \mapsto S_1 \mapsto S_0.$$

We claim that $Y_{a,Q} = S_0 \cup S_1 \cup \dots \cup S_{n-1}$ and that each S_i is an irreducible rational $(n-1)$-dimensional variety.

Let $\mathbf{v} \in \mathcal{V}_n$ be a well-spaced vector and let $i_0 < \dots < i_\ell$ be the indices of the nonzero entries of $\mathbf{v}$. By Proposition 3.3, $Y_{a,Q} \cap D_+(y_{\mathbf{v}})$ is isomorphic to a union of $\ell + 1$ hyperplanes in $\mathbb{A}^n_{\mathbf{k}(Q)} = \operatorname{Spec} \mathbf{k}(Q)[Z_0, \dots, Z_{n-1}]$. Furthermore, the hyperplane $Z_{i_r} = 0$ is isomorphic to the subvariety

$$V\left(\left\langle \frac{y_{\mathbf{w}}}{y_{\mathbf{v}}} : \mu(\sigma^{i_r+1}(\mathbf{w})) > 0 \right\rangle \right) \subset Y_{a,Q} \cap D_+(y_{\mathbf{v}}).$$

Hence, $Y_{a,Q} \cap D_+(y_{\mathbf{v}})$ is a dense open subset of $S_{i_0} \cup S_{i_1} \cup \dots \cup S_{i_\ell}$. Since the open subvarieties $\{D_+(y_{\mathbf{v}}) \cap Y_{a,Q} : \mathbf{v} \text{ well-spaced}\}$ cover $Y_{a,Q}$, this completes the proof. □

4. A K/k twist of Y_a

Let K/k be a cyclic Galois extension of degree n, and let $R_K := R \otimes_k K$. Fix a basis $\{\alpha_0, \ldots, \alpha_{n-1}\}$ of K as a k-vector space, as well as a generator τ of $\mathrm{Gal}(K/k)$.

Let T be a set of representatives for the orbits of $\mathcal{V}_n$under the action of the shift operator σ. Consider the K-isomorphism $\psi\colon \operatorname{Proj} K[\{z_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] \to \operatorname{Proj} K[\{y_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}]$ determined by

$$y_{\mathbf{v}} \mapsto \alpha_0 z_{\mathbf{v}} + \alpha_1 z_{\sigma(\mathbf{v})} + \cdots + \alpha_{n-1} z_{\sigma^{n-1}(\mathbf{v})} \quad \text{for } \mathbf{v} \in T,$$

$$y_{\sigma^i(\mathbf{v})} \mapsto \sum_{j=0}^{n-1} \tau^i(\alpha_j) z_{\sigma^j(\mathbf{v})} \quad \text{for } \mathbf{v} \in T \text{ and } i = 1, \ldots, n-1.$$

Define $X^0_{K,a} := \psi^{-1}_{R_K}(Y_a)$. Abusing notation, we write τ for the endomorphism of $\operatorname{Proj} R[\{z_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] \times_R R_K$ given by $\mathrm{id} \times \tau$. Let $\phi\colon \mathbb{P}^N_R \to \mathbb{P}^N_R$ be the automorphism of Lemma 3.1. The following diagram commutes

$$\begin{array}{ccc} \operatorname{Proj} R_K[\{z_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] & \xrightarrow{\psi_{R_K}} & \operatorname{Proj} R_K[\{y_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \phi_{R_K}} \\ \operatorname{Proj} R_K[\{z_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] & \xrightarrow{\psi_{R_K}} & \operatorname{Proj} R_K[\{y_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] \end{array} \tag{4.1}$$

By Lemma 3.1, the map ϕ preserves Y_a. Together with the commutativity of the above diagram, this implies that $X^0_{K,a}$ descends to a R-scheme.

5. Proof of Theorem 1.1

Let K/k be a cyclic Galois extension of degree n and let $P(x) \in k[x]$ be a separable polynomial of degree dn or $dn-1$ for some d.

Lemma 5.1. *There exists a smooth projective variety $X = X_{K/k,P(x)} \to \mathbb{P}^1_k$ such that $X_{\mathbb{A}^1} \cong X^0_{K,P(x)}$ and that $X_{\mathbb{P}^1 \setminus \{0\}} \cong X^0_{K,P(1/x')x'^{dn}}$, where $x' = 1/x$.*

Proof. We will construct X by glueing $Y_{P(x)}$ and $Y_{P(1/x')x'^{dn}}$ over $\operatorname{Spec} k[x^{\pm 1}]$ and $\operatorname{Spec} k[x'^{\pm 1}]$, in a way which is compatible with the map ψ from §4. Let $y_{\mathbf{v}}$ denote the coordinates on $Y_{P(x)}$ and let $y'_{\mathbf{v}}$ denote the coordinates on $Y_{P(1/x')x'^{dn}}$. By Lemma 2.1(2), the morphism

$$Y_{P(1/x')x'^{dn}} \times_{\mathbb{A}^1} \operatorname{Spec} k[x', x'^{-1}] \to Y_{P(x)} \times_{\mathbb{A}^1} \operatorname{Spec} k[x, x^{-1}]$$

where $y_{\mathbf{v}} \mapsto (x')^{d\lambda(\mathbf{v})} y'_{\mathbf{v}}$ and $x \mapsto 1/x'$ is well-defined and is an isomorphism. Since $\lambda(\mathbf{v}) = \lambda(\sigma(\mathbf{v}))$, this morphism is compatible with ψ and thus gives a glueing of $X^0_{K,P(x)}$ and $X^0_{K,P(1/x')x'^{dn}}$. □

Proposition 5.2. *The variety X is a smooth proper compactification of X_0, the generic fiber of $X \to \mathbb{P}^1$ is a Severi-Brauer variety, and the degenerate fibers of $X \to \mathbb{P}^1$ lie over $V(P(x_0/x_1)x_1^{dn})$ and consist of the union of n rational varieties all conjugate under* $\mathrm{Gal}(K/k)$.

Proof. The compatibility (4.1) together with Proposition 3.3 and Corollary 3.4 implies that

$$(X \times_{\mathbb{P}^1} \mathbb{A}^1) \cap D_+(z_{(1,1,\ldots,1)}) \cong X_0,$$

which gives the first claim. The second claim is immediate from the construction of X, and the third claim follows from Proposition 3.5 and the compatibility (4.1). □

Proposition 5.2 completes the proof of Theorem 1.1. □

References

[Art82] M. Artin, *Left ideals in maximal orders*, Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981), Lecture Notes in Math., vol. 917, Springer, Berlin, 1982, pp. 182–193. MR657429 (83j:16009) ↑1.2

[CTvG] Jean-Louis Colliot-Thélène and Jan van Geel, *Le complémentaire des puissances n-ièmes dans un corps de nombres est un ensemble diophantien.* Preprint. arXiv:1401.0915. ↑1

[CTHS03] Jean-Louis Colliot-Thélène, David Harari, and Alexei Skorobogatov, *Valeurs d'un polynôme à une variable représentées par une norme*, Number theory and algebraic geometry, London Math. Soc. Lecture Note Ser., vol. 303, Cambridge Univ. Press, Cambridge, 2003, pp. 69–89 (French, with English summary). MR2053456 (2005d:11095) ↑1

[DSW14] Ulrich Derenthal, Arne Smeets, and Dasheng Wei, *Universal torsors and values of quadratic polynomials represented by norms*, Math. Ann., posted on 2014, 1-22, DOI 10.1007/s00208-014-1106-7, (to appear in print). ↑1

[Har92] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1992. A first course. MR1182558 (93j:14001) ↑3.2

[Kan90] Ming Chang Kang, *Constructions of Brauer-Severi varieties and norm hypersurfaces*, Canad. J. Math. **42** (1990), no. 2, 230–238, DOI 10.4153/CJM-1990-013-7. MR1051727 (91h:12006) ↑1.1

[Poo09] Bjorn Poonen, *The set of nonsquares in a number field is Diophantine*, Math. Res. Lett. **16** (2009), no. 1, 165–170, DOI 10.4310/MRL.2009.v16.n1.a16. MR2480570 (2010c:14025) ↑1

[VAV12] Anthony Várilly-Alvarado and Bianca Viray, *Higher-dimensional analogs of Châtelet surfaces*, Bull. Lond. Math. Soc. **44** (2012), no. 1, 125–135, DOI 10.1112/blms/bdr075. MR2881330 ↑1

Department of Mathematics MS 136, Rice University, Houston, TX 77005, USA
E-mail address: varilly@rice.edu
URL: http://www.math.rice.edu/~av15

University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98195, USA
E-mail address: bviray@math.washington.edu
URL: http://math.washington.edu/~bviray