Local Stability Analysis for Uncertain Nonlinear Systems

Ufuk Topcu and Andrew Packard

***Abstract*—We propose a method to compute provably invariant subsets of the region-of-attraction for the asymptotically stable equilibrium points of uncertain nonlinear dynamical systems. We consider polynomial dynamics with perturbations that either obey local polynomial bounds or are described by uncertain parameters multiplying polynomial terms in the vector field. This uncertainty description is motivated by both incapabilities in modeling, as well as bilinearity and dimension of the sum-of-squares programming problems whose solutions provide invariant subsets of the region-of-attraction. We demonstrate the method on three examples from the literature and a controlled short period aircraft dynamics example.**

***Index Terms*—Region-of-attraction (ROA), uncertain systems, verification.**

I. Introduction

We consider the problem of computing invariant subsets of the region-of-attraction (ROA) for uncertain systems with polynomial nominal vector fields and local polynomial uncertainty description. Since computing the exact ROA, even for systems with known dynamics, is hard, researchers have focused on finding Lyapunov functions whose sublevel sets provide invariant subsets of the ROA [1]–[5]. Recent advances in polynomial optimization based on sum-of-squares (SOS) relaxations [6], [7] are utilized to determine invariant subsets of the ROA for systems with known polynomial and/or rational dynamics solving optimization problems with matrix inequality constraints [8]–[13]. Reference [14] provides a generalization of Zubov's method to uncertain systems and [15] investigates robustness of the ROA under time-varying perturbations and proposes an iterative algorithm that asymptotically gives the robust ROA. Parametric uncertainties are considered in [16]–[18]. The focus in [16] is on computing the largest sublevel set of a *given* Lyapunov function that can be certified to be an invariant subset of the ROA. [17], [18] propose parameter-dependent Lyapunov functions which lead to potentially less conservative results at the expense of increased computational complexity.

Similar to other problems in local analysis of dynamical systems based on Lyapunov arguments and SOS relaxations [9], [11]–[13], [17], [19], our formulation leads to optimization problems with bilinear matrix inequality (BMI) constraints. BMIs are nonconvex and bilinear semidefinite programs (SDPs) (those with BMI constraints) are generally harder than linear SDPs. Consequently, approaches for solving SDPs with BMIs are limited to local search schemes [20]–[23]).

The uncertainty description detailed in Section III contains two types of uncertainty: uncertain components in the vector field that obey local polynomial bounds and/or uncertain parameters appearing affinely and multiplying polynomial terms. Using this description, we develop an SDP with BMIs to compute robustly invariant subsets of the ROA. The number of BMIs (and consequently the number of variables) in this problem increases exponentially with the sum of the number of components of the vector field containing uncertainty with polynomial bounds and the number of uncertain parameters. One way to deal with this difficulty is first to compute a Lyapunov function for a particular system (imposing extra robustness constraints) and then determine the largest sublevel set in which the computed Lyapunov function serves as a local stability certificate for the whole family of systems. Once a Lyapunov function is determined for the system in the first step, second step involves solving smaller decoupled linear SDPs. Therefore, this two step procedure is well suited for parallel computation leading to relatively efficient numerical implementation. Moreover, recently developed methods [13], [24], which use simulation to aid in the nonconvex search for Lyapunov functions, extend easily to the robust ROA analysis using simulation data for finitely many systems from the family of possible systems (e.g., systems corresponding to the vertices of the uncertainty polytope when the uncertainty can be described by a polytope). In the examples in this technical note, we implement this generalization of the simulation based ROA analysis method from [13], [24].

The rest of the technical note is organized as follows: Section II reviews results on computing the ROA for systems with known polynomial dynamics. Section III is devoted to the discussion of the motivation for this work and the setup for the uncertain system analysis. In Section IV provides a generalization of the results from Section II to the case of dynamics with uncertainty. The methodology is demonstrated with three small examples from the literature and a five-state example in Section V.

Notation: For $x \in \mathcal{R}^n$, $x \succeq 0$ means that $x_k \geq 0$ for $k = 1, \ldots, n$. For $Q = Q^T \in \mathcal{R}^{n\times n}$, $Q \succeq 0$ ($Q \succ 0$) means that $x^TQx \geq 0$ ($\succ 0$) for all $x \in \mathcal{R}^n$. $\mathbb{R}[x]$ represents the set of polynomials in x with real coefficients. The subset $\Sigma[x] := \{\pi \in \mathbb{R}[x] : \pi = \pi_1^2 + \pi_2^2 + \cdots + \pi_m^2, \pi_1, \ldots, \pi_m \in \mathbb{R}[x]\}$ of $\mathbb{R}[x]$ is the set of SOS polynomials. For $\pi \in \mathbb{R}[x]$, $\partial(\pi)$ denotes the degree of π. For subsets $\mathcal{X}_1$ and $\mathcal{X}_2$ of a vector space $\mathcal{X}$, $\mathcal{X}_1 + \mathcal{X}_2 := \{x_1 + x_2 : x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2\}$. In several places, a relationship between an algebraic condition on some real variables and state properties of a dynamical system is claimed, and same symbol for a particular real variable in the algebraic statement as well as the state of the dynamical system is used. This could be a source of confusion, so care on the reader's part is required. ◁

II. Computation of Invariant Subsets of Region-of-Attraction

In this section, we give a characterization of invariant subsets of ROA using Lyapunov functions and formulate a bilinear optimization problem for computing these functions when they are restricted to be polynomial. These results will be modified to compute invariant subsets of the ROA for systems with uncertainty in Section IV. Now, consider the system governed by

$$\dot{x}(t) = f(x(t)) \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the state vector and $f : \mathcal{R}^n \to \mathcal{R}^n$ is such that $f(0) = 0$, i.e., the origin is an equilibrium point of (1) and f is locally Lipschitz on $\mathcal{R}^n$. Let $\varphi(t; \mathbf{x}_0)$ denote the solution to (1) with the initial condition $x(0) = \mathbf{x}_0$. If the origin is asymptotically stable but not globally attractive, one often wants to know which trajectories converge to the origin as time approaches ∞. This gives rise to the following definition of the *region-of-attraction*:

Definition 2.1: The region-of-attraction R_0 of the origin for the system (1) is

$$R_0 := \left\{\mathbf{x}_0 \in \mathcal{R}^n : \lim_{t\to\infty} \varphi(t; \mathbf{x}_0) = 0\right\}.$$

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The Authors are with the Department of Mechanical Engineering, The University of California, Berkeley, CA 94720-1740 USA (e-mail: utopcu@jagger.me.berkeley.edu; pack@jagger.me.berkeley.edu).

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For $\eta > 0$ and a function $V : \mathcal{R}^n \rightarrow \mathcal{R}$, define the η-sublevel set of V as

$$\Omega_{V,\eta} := \{x \in \mathcal{R}^n : V(x) \leq \eta\}.$$

For simplicity, $\Omega_{V,1}$ is denoted by Ω_V. Lemma 2.2 provides a characterization of invariant subsets of the ROA in terms of sublevel sets of appropriate Lyapunov functions.

Lemma 2.2: If there exists a continuously differentiable function $V : \mathcal{R}^n \rightarrow \mathcal{R}$ such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \neq 0, \quad (2)$$

$$\Omega_V \text{ is bounded, and} \quad (3)$$

$$\Omega_V \setminus \{0\} \subset \{x \in \mathcal{R}^n : \nabla V(x) f(x) < 0\} \quad (4)$$

then for all $\mathbf{x}_0 \in \Omega_V$, the solution of (1) exists, satisfies $\varphi(t; \mathbf{x}_0) \in \Omega_V$ for all $t \geq 0$, and $\lim_{t\rightarrow\infty} \varphi(t; \mathbf{x}_0) = 0$, i.e., Ω_V is an invariant subset of R_0. ◃

Lemma 2.2 is proven in [11], [12] using a similar result from [25]. If the dynamical system has an exponentially stable linearization, one can impose a stricter condition replacing (4), for $\mu \geq 0$, by

$$\Omega_V \setminus \{0\} \subset \{x \in \mathcal{R}^n : \nabla V(x) f(x) < -\mu V(x)\}. \quad (5)$$

With nonzero μ, (5) not only assures that trajectories starting in Ω_V stay in Ω_V and converge to the origin but also imposes a bound on the rate of exponential decay of V certifying the convergence and provides an implicit threshold for the level of a disturbance that could drive the system out of Ω_V. Therefore, one may consider the stability property implied by (5) with nonzero μ to be more desirable in practice. With this in mind, all subsequent derivations contain the μV term. The relaxed condition in (4) can be recovered by setting $\mu = 0$.

III. Setup and Motivation

We now introduce the uncertainty description used in the rest of the technical note and explain its usefulness in ROA analysis based on computing Lyapunov functions using SOS programming. Consider the system governed by

$$\dot{x}(t) = f(x(t)) = f_0(x(t)) + \phi(x(t)) + \psi(x(t)) \quad (6)$$

where $f_0, \phi, \psi : \mathcal{R}^n \rightarrow \mathcal{R}^n$ are locally Lipschitz. Assume that f_0 is known, $\phi \in \mathcal{D}_\phi$, and $\psi \in \mathcal{D}_\psi$, where

$$\mathcal{D}_\phi := \{\phi : \phi_l(x) \preceq \phi(x) \preceq \phi_u(x) \;\forall\, x \in \mathcal{G}\}$$
$$\mathcal{D}_\psi := \{\psi : \psi(x) = \Psi(x)\alpha \;\forall\, x \in \mathcal{G}, \alpha_l \preceq \alpha \preceq \alpha_u\}.$$

Here, $\mathcal{G}$ is a given subset of $\mathcal{R}^n$ containing the origin, ϕ_l and ϕ_u are n dimensional vectors of known polynomials satisfying $\phi_l(x) \preceq 0 \preceq \phi_u(x)$ for all $x \in \mathcal{G}$, $\alpha, \alpha_l, \alpha_u \in \mathcal{R}^N$, and Ψ is a matrix of known polynomials. Let ϕ_i, $\phi_{l,i}$, $\phi_{u,i}$, α_i, $\alpha_{l,i}$, and $\alpha_{u,i}$ denote i-th entry of ϕ, ϕ_l, ϕ_u, α, α_l, and α_u, respectively. Define $\mathcal{D} := \mathcal{D}_\phi + \mathcal{D}_\psi$. We assume that $f_0(0) = 0$, $\phi(0) = 0$ for all $\phi \in \mathcal{D}_\phi$ (i.e., $\phi_l(0) = 0$, and $\phi_u(0) = 0$), and $\psi(0) = 0$ for all $\psi \in \mathcal{D}_\psi$ (i.e., $\Psi(0) = 0$), which assure that all systems in (6) have a common equilibrium point at the origin.[1] In order to be able to use SOS programming, we restrict our attention to the case where f_0, ϕ_l, ϕ_u, and Ψ have only polynomial entries and $\mathcal{G}$ is defined as $\mathcal{G} := \{x \in \mathcal{R}^n : g(x) \succeq 0,\ g_i \in \mathbb{R}[x],\ i = 1, \ldots, m\}$. Note that entries of ϕ do not have to be polynomial but have to satisfy local polynomial bounds.

TABLE I
N_{SDP} (Left Columns) and N_{decision} (Right Columns) for Different Values of n and $2d$

n	2d = 4		6		8		10	
2	6	6	10	27	15	75	21	165
5	21	105	56	1134	126	6714	**252**	**2e4**
9	55	825	**220**	**1e4**	**715**	**2e5**	⋆	⋆
14	120	4200	**680**	⋆	⋆	⋆	⋆	⋆
16	**153**	**6936**	⋆	⋆	⋆	⋆	⋆	⋆

Motivation for this kind of system description stems from the following sources:

i) Perturbations as in (6) may be due to modeling errors, aging, disturbances, and uncertainties due to environment which may be present in any realistic problem. Prior knowledge about the system may provide local bounds on the entries of ϕ and/or bounds for the parametric uncertainties α. Moreover, uncertainties that do not change system order can always be represented as in (6) (see [27, p.339]).

ii) Analysis of dynamical systems using SOS programming is often limited to systems with polynomial or rational vector field. In [28], a procedure for re-casting non-rational vector fields into rational ones at the expense of increasing the state dimension is proposed. Another way to deal with a non-polynomial vector field is to locally approximate the vector field with a polynomial and bound the error. For practical purposes only finite number of terms can be used. Finite-term approximations are relatively accurate in a restricted region containing the origin. However, they are not exact. On the other hand, it may be possible to represent terms, for which the error between the exact vector field and its finite-term approximation obey local polynomial bounds, using ϕ in (6) (see Table I).

iii) SOS programming can be used to analyze systems with polynomial vector fields. The number of decision variables N_{decision} and the size N_{SDP} of the matrix in the SDP for checking existence of a SOS decomposition for a degree $2d$ polynomial in n variables grows polynomially with n if d is fixed and vice versa [6]. However, N_{SDP} and N_{decision} get practically intractable for the state-of-the-art SDP solvers even for moderate values of n for fixed d (see Table II, where solid lines in the table represent a fuzzy boundary between tractable and intractable SDPs). Moreover, using higher degree Lyapunov functions and/or higher degree multipliers (used in the sufficient conditions for certain set containment constraints in Section IV) as well as higher degree vector fields increases the problem size, and, in fact, the growth of the problem size with the simultaneous increase in n and d is exponential. Therefore, in order to be able to use SOS programming, one may have to simplify the dynamics by truncating higher degree terms in the vector field. In this case, ϕ_l and ϕ_u provide local bounds on the truncated terms. This is discussed further at the end of Section IV. It is also worth mentioning that bilinearity, a common feature of the optimization problems for

[1]The assumption that all possible systems in (6) have a common equilibrium point can be alleviated by generalizing the analysis based on contraction metrics and SOS programming studied in [26] to address local stability (rather than global stability as in [26]). However, this method leads to higher computational cost. Therefore, we do not pursue this direction here.

TABLE II
OPTIMAL VALUES OF β IN THE PROBLEM (15) WITH DIFFERENT VALUES OF μ AND $\partial(v) = 2$ AND 4

μ \ $\partial(V)$	2	4
0	0.623	0.771
0.01	0.603	0.763
0.05	0.494	0.742
0.1	0.404	0.720
0.15	0.277	0.698
0.2	0.137	0.676

local analysis using Lyapunov arguments (see Section IV), introduces extra complexity [29] and therefore a further necessity for simplifying the system dynamics.

In summary, representation in (6) and definitions of $\mathcal{D}_\phi$ and $\mathcal{D}_\psi$ are motivated by uncertainties introduced due to incapabilities in modeling and/or analysis.

IV. Computation of Robustly Invariant Sets

In this section, we will develop tools for computing invariant subsets of the robust ROA. The robust ROA is the intersection of the ROAs for all possible systems governed by (6) and formally defined, assuming that the origin is an asymptotically equilibrium point of (6) for all $\delta \in \mathcal{D}$, as

Definition 4.1: The robust ROA R_0^r of the origin for systems governed by (6) is

$$R_0^r := \bigcap_{\delta \in \mathcal{D}} \{\mathbf{x}_0 \in \mathcal{R}^n : \lim_{t\to\infty} \varphi(t;\mathbf{x}_0,\delta) = 0\}$$

where $\varphi(t;\mathbf{x}_0,\delta)$ denotes the solution of (6) for $\delta \in \mathcal{D}$ with the initial condition $x(0) = \mathbf{x}_0$. ◁

We focus on computing invariant subsets of the robust ROA characterized by sublevel sets of appropriate Lyapunov functions. Since the uncertainty description for ϕ and ψ holds only for $x \in \mathcal{G}$, we will also require the computed invariant set to be a subset of $\mathcal{G}$. To this end, we modify Lemma 2.2 such that condition (5) holds for (6) for all $\delta \in \mathcal{D}$ (i.e., for all $\phi \in \mathcal{D}_\phi$ and $\psi \in \mathcal{D}_\psi$).

Proposition 4.2: If there exists a continuously differentiable function $V : \mathcal{R}^n \to \mathcal{R}$ and $\mu \geq 0$ such that, for all $\delta \in \mathcal{D}$, conditions (2)–(3)

$$\Omega_V \subseteq \mathcal{G} \tag{7}$$

and

$$\Omega_V \setminus \{0\} \subset \{x \in \mathcal{R}^n : \nabla V(x)(f_0(x) + \delta(x)) < -\mu V(x)\} \tag{8}$$

hold, then for all $\mathbf{x}_0 \in \Omega_V$ and for all $\delta \in \mathcal{D}$, the solution of (6) exists, satisfies $\varphi(t;\mathbf{x}_0,\delta) \in \Omega_V$ for all $t \geq 0$, and $\lim_{t\to\infty}\varphi(t;\mathbf{x}_0,\delta) = 0$, i.e., Ω_V is an invariant subset of R_0^r. ◁

Proof: Proposition 4.2 follows from Lemma 2.2. Indeed, for any given system $\dot{x} = f_0(x) + \delta(x)$, (8) assures that (5) is satisfied. Then, for any fixed $\delta \in \mathcal{D}$ and for all $\mathbf{x}_0 \in \Omega_V$, $\varphi(t;\mathbf{x}_0,\delta)$ exists and satisfies $\varphi(t;\mathbf{x}_0,\delta) \in \Omega_V$ for all $t \geq 0$, $\lim_{t\to\infty}\varphi(t;\mathbf{x}_0,\delta) = 0$, and Ω_V is an invariant subset of $\{\mathbf{x}_0 \in \mathcal{R}^n : \varphi(t;\mathbf{x}_0,\delta) \to 0\}$. Therefore, Ω_V is an invariant subset of R_0^r. ■

Remark 4.3: In fact, Ω_V is invariant for both time-invariant and time-varying perturbations. The conclusion of Proposition 4.2 holds for time-varying $\delta(= \phi + \alpha)$ as long as $\phi_l(x) \preceq \phi(x,t) \preceq \phi_u(x)$ and $\alpha_l \preceq \alpha(t) \preceq \alpha_u$ for all $x \in \mathcal{G}$ and $t \geq 0$. Recall that, in the uncertain linear system literature, e.g., [30], the notion of quadratic stability is similar, where a single quadratic Lyapunov function proves the stability of an entire family of uncertain linear systems. ◁

Note that $\mathcal{D}$ has infinitely many elements; therefore, there are infinitely many constraints in (8). Now, define

$$\mathcal{E}_\phi := \{\phi : \phi_i \in \mathbb{R}[x] \text{ and } \phi_i \text{ is equal to } \phi_{l,i} \text{ or } \phi_{u,i}\}$$
$$\mathcal{E}_\psi := \{\psi : \psi(x) = \Psi(x)\alpha, \alpha_i \text{ is equal to } \alpha_{l,i} \text{ or } \alpha_{u,i}\}$$

and $\mathcal{E} := \mathcal{E}_\phi + \mathcal{E}_\psi$. $\mathcal{E}$, a finite subset of $\mathcal{D}$, can be used to transform condition (8) to a finite set of constraints that are more suitable for numerical verification:

Proposition 4.4: If

$$\Omega_V \setminus \{0\} \subseteq \{x \in \mathcal{R}^n : \nabla V(x)(f_0(x) + \xi(x)) < -\mu V(x)\} \tag{9}$$

holds for all $\xi \in \mathcal{E}$, then (8) holds for all $\delta \in \mathcal{D}$. ◁

Proof: Let $\tilde{x} \in \Omega_V$ be nonzero and $\delta \in \mathcal{D}$. Then, $\tilde{x} \in \mathcal{G}$ by (7); therefore, there exist $\ell_1,\ldots,\ell_n$, $k_1,\ldots,k_N$ (depending on $\tilde{x}$) with $0 \leq \ell_i \leq 1$ and $0 \leq k_i \leq 1$ such that $\delta(\tilde{x}) = L\phi_l(\tilde{x}) + (I - L)\phi_u(\tilde{x}) + \Psi(\tilde{x})(K\alpha_l + (I-K)\alpha_u)$, where L and K are diagonal with $L_{ii} = \ell_i$ and $K_{ii} = k_i$. Hence, there exist nonnegative numbers ν_ξ (determined from ℓ's and k's) for $\xi \in \mathcal{E}$ with $\sum_{\xi\in\mathcal{E}}\nu_\xi = 1$ such that $\delta(\tilde{x}) = \sum_{\xi\in\mathcal{E}}\nu_\xi\xi(\tilde{x})$. Consequently, by $\nabla V(\tilde{x})(f_0(\tilde{x}) + \delta(\tilde{x})) = \nabla V(\tilde{x})(f_0(\tilde{x}) + \sum_{\xi\in\mathcal{E}}\nu_\xi\xi(\tilde{x})) = \sum_{\xi\in\mathcal{E}}\nu_\xi\nabla V(\tilde{x})(f_0(\tilde{x}) + \xi(\tilde{x})) < -\sum_{\xi\in\mathcal{E}}\nu_\xi\mu V(\tilde{x}) = -\mu V(\tilde{x})$, (8) follows. ■

In order to enlarge the computed invariant subset of the robust ROA, we define a variable sized region $\mathcal{P}_\beta := \{x \in \mathcal{R}^n : p(x) \leq \beta\}$, where $p \in \mathbb{R}[x]$ is a fixed, positive definite, convex polynomial, and maximize β while imposing constraints (2)–(3), (7), (9), and $\mathcal{P}_\beta \subseteq \Omega_V$. This can be written as an optimization problem

$$\beta^*(\mathcal{V}) := \max_{V\in\mathcal{V},\beta>0} \beta \text{ subject to} \tag{10a}$$

$$\left.\begin{array}{c} V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \neq 0, \\ \Omega_V = \{x \in \mathcal{R}^n : V(x) \leq 1\} \text{ is bounded}, \end{array}\right\} \tag{10b}$$

$$\left.\begin{array}{c} \mathcal{P}_\beta = \{x \in \mathcal{R}^n : p(x) \leq \beta\} \subseteq \Omega_V \\ \Omega_V \subseteq \mathcal{G}, \end{array}\right\} \tag{10c}$$

$$\Omega_V \setminus \{0\} \subseteq \bigcap_{\xi\in\mathcal{E}} \{x \in \mathcal{R}^n : \nabla V(f_0(x) + \xi(x)) < -\mu V(x)\}. \tag{10d}$$

Here, $\mathcal{V}$ denotes the set of candidate Lyapunov functions over which the maximum is defined (e.g., $\mathcal{V}$ may be equal to all continuously differentiable functions).

In order to make the problem in (10) amenable to numerical optimization (specifically SOS programming), we restrict V to be a polynomial in x of fixed degree. We use the well-known sufficient condition for polynomial positivity [6]: for any $\pi \in \mathbb{R}[x]$, if $\pi \in \Sigma[x]$, then π is positive semidefinite. Using simple generalizations of the S-procedure (Lemmas 8.1 and 8.2 in the appendix), we obtain sufficient conditions for set containment constraints. Specifically, let l_1 and l_2 be a positive definite polynomials (typically $\epsilon x^T x$ with some (small) real number ϵ). Then, since l_1 is radially unbounded, the constraint

$$V - l_1 \in \Sigma[x] \tag{11}$$

and $V(0) = 0$ are sufficient conditions for the constraints in (10b). By Lemma 8.1, if $s_1 \in \Sigma[x]$ and $s_{4k} \in \Sigma[x]$ for $k = 1,\ldots,m$, then

$$-[(\beta - p)s_1 + (V - 1)] \in \Sigma[x] \tag{12}$$

$$g_k - (1 - V)s_{4k} \in \Sigma[x], \quad k = 1,\ldots,m \tag{13}$$

imply the first and second constraints in (10c), respectively. By Lemma 8.2, if $s_{2\xi}, s_{3\xi} \in \Sigma[x]$ for $\xi \in \mathcal{E}$, then

$$-[(1 - V)s_{2\xi} + (\nabla V(f_0 + \xi) + \mu V)s_{3\xi} + l_2] \in \Sigma[x] \tag{14}$$

is a sufficient condition for the feasibility of the constraint in (10d). Using these sufficient conditions, a lower bound on $\beta^*(\mathcal{V})$ can be defined as an optimization:

Proposition 4.5: Let β_B^* be defined as

$$\beta_B^*(\mathcal{V}_{poly}, \mathcal{S}) := \max_{V,\beta,s_1,s_{2\xi},s_{3\xi},s_{4k}} \beta \text{ subject to } (11)\text{–}(14) \quad (15)$$

$V(0) = 0, V \in \mathcal{V}_{poly}$, $s_1 \in \mathcal{S}_1$, $s_{2\xi} \in \mathcal{S}_{2\xi}$, $s_{3\xi} \in \mathcal{S}_{3\xi}$, $s_{4k} \in \mathcal{S}_{4k}$, and $\beta > 0$. Here, $\mathcal{V}_{poly} \subset \mathcal{V}$ and $\mathcal{S}$'s are prescribed finite-dimensional subspaces of $\mathbb{R}[x]$ and $\Sigma[x]$, respectively. Then, $\beta_B^*(\mathcal{V}_{poly}, \mathcal{S}) \leq \beta^*(\mathcal{V}_{poly})$. ◁

The optimization problem in (15) provides a recipe to compute subsets of $\mathcal{R}^n$ that are invariant under the flow of all possible systems described by (6). The number of constraints in (15) (consequently the number of decision variables since each new constraint includes new variables) increases exponentially with N and $n - \bar{n}$ where $\bar{n}$ is defined as the number of entries of the vectors ϕ_l and ϕ_u satisfying $\phi_l(x) = \phi_u(x) = 0$ for all $x \in \mathcal{G}$. Namely, there are $2^{(n-\bar{n})+N}$ SOS conditions in (15) due to the constraint in (14). Revisiting the discussion in item (iii) at the end of Section III, we note that covering the high degree vector field with low degree uncertainty reduces the dimension of the SOS constraints but increases (exponentially, depending on $n - \bar{n}$) the number of constraints. Consequently, the utility of this approach will depend on $n - \bar{n}$ and is problem dependent. Example (3) in Section V-A illustrates this technique.

This difficulty can be partially alleviated by accepting suboptimal solutions for (15) in two steps: First compute a Lyapunov function for a finite sample of systems corresponding to the finite set $\mathcal{D}_{sample} \subset \mathcal{D}$ (for example, $\mathcal{D}_{sample}$ can be taken as the singleton corresponding to the "center" of $\mathcal{D}$) solving the problem

$$\begin{aligned} &\max_{V,\beta,s_1,s_{2\delta},s_{3\delta},s_{4k}} \beta \text{ subject to} \\ &V - l_1 \in \Sigma[x] \\ &-[(\beta - p)s_1 + (V-1)] \in \Sigma[x] \\ &g_k - (1-V)s_{4k} \in \Sigma[x], \quad k = 1,\ldots,m \\ &-[(1-V)s_{2\delta} + \nabla V(f_0+\delta)s_{3\delta} + l_2] \in \Sigma[x] \end{aligned} \quad (16)$$

for $\delta \in \mathcal{D}_{sample}$, where $s_1 \in \mathcal{S}_1$, $s_{2\delta} \in \mathcal{S}_2$, $s_{3\delta} \in \mathcal{S}_3$, $s_{4k} \in \mathcal{S}_4$ are SOS, $V \in \mathcal{V}_{poly}$, $V(0) = 0$, and let $\tilde{V}$ the Lyapunov function computed by solving (16). In the second step, compute the largest sublevel set $\mathcal{P}_{\beta subopt}$ such that $\tilde{V}$ certifies $\mathcal{P}_{\beta subopt}$ to be in the ROA for every vertex system by solving several smaller decoupled affine SDPs. For $\xi \in \mathcal{E}$, define

$$\begin{aligned} \gamma_\xi := &\max_{\gamma, s_2 \in \mathcal{S}_2, s_3 \in \mathcal{S}_3} \gamma \text{ subject to } \; s_2, s_3 \in \Sigma[x] \\ &-\left[(\gamma - \tilde{V})s_2 + \nabla\tilde{V}(f_0+\xi)s_3 + l_2\right] \in \Sigma[x] \end{aligned} \quad (17)$$

and $\gamma^{subopt} := \min\{\gamma_\xi : \xi \in \mathcal{E}\}$. Then, a lower bound for β^{subopt} can be computed through

$$\begin{aligned} \beta^{subopt} := &\max_{\beta, s_1 \in \mathcal{S}_1} \beta \text{ subject to } \; s_1 \in \Sigma[x] \\ &-\left[(\beta - p)s_1 + (\tilde{V} - \gamma^{subopt})\right] \in \Sigma[x]. \end{aligned} \quad (18)$$

While the two-step procedure sacrifices optimality, it has practical computational advantages. The constraints in (14) decouple in the problem (17). In fact, for each $\xi \in \mathcal{E}_\Delta$, the problem in (17) contains only a single constraint from (14). Therefore, this decoupling enables suboptimal local stability analysis for systems with uncertainty without solving optimization problems larger than those one would have to solve in local stability analysis for systems without uncertainty. Furthermore, problems in (17) can be solved independently for different $\xi \in \mathcal{E}_\Delta$ and therefore computations can be trivially parallelized. Advantages of this decoupling may be better appreciated by noting that one of the main difficulties in solving large-scale SDPs is the memory requirements of the interior-point type algorithms [31]. Consequently, it is possible to perform some ROA analysis on systems with relatively reasonable number of states and/or uncertain parameters using the proposed suboptimal solution technique.

Finally, the following upper bound on the value of μ, for which (14) can be feasible, will be useful in Section V.

Proposition 4.6: Let $L_2 \succ 0$ and $l_2(x) := x^T L_2 x$. Then

$$\begin{aligned} &\bar{\mu} := \max_{\mu \geq 0, P = P^T \succeq 0} \mu \text{ subject to} \\ &A_\xi^T P + P A_\xi + \mu P \preceq -L_2, \text{ for all } \xi \in \mathcal{E} \end{aligned} \quad (19)$$

where $A_\xi := (\partial(f_0+\xi)/\partial x)|_{x=0}$, is an upper bound for the values of μ such that (14) can be feasible. ◁

Proof: With l_2 as defined and $s_{2\xi}, s_{3\xi} \in \Sigma[x]$, if $b_\xi(x) := -[(1-V)s_{2\xi} + (\nabla V(f_0+\xi) + \mu V)s_{3\xi} + l_2] \in \Sigma[x]$, then $s_{2\xi}$ and b_ξ cannot contain constant and linear monomials and the quadratic part of b_ξ has to be SOS and equivalently positive semidefinite. Therefore, the result follows from the fact that, for fixed $\mu \geq 0$ and positive definite quadratic l_2, the existence of $P \succeq 0$ satisfying $A_\xi^T P + P A_\xi + \mu P \preceq -L_2$ is necessary for the existence of V, $s_{2\xi}$, and $s_{3\xi}$ feasible for (14). ■

Note that the problem in (19) can be solved as a sequence of affine SDPs by a line search on μ.

V. EXAMPLES

In the following examples, $p(x) = x^T x$ (except for example (2) in Section V-A), $l_1(x) = 10^{-6} x^T x$, and $l_2(x) = 10^{-6} x^T x$. All certifying Lyapunov functions and multipliers are available at [32]. All computations use the generalization of the simulation based ROA analysis method from [13], [24]. Representative computation times on 2.0 GHz desktop PC are listed with each example.

A. Examples From the Literature

Example 1) Consider the following system from [16]: $\dot{x}_1 = x_2$ and $\dot{x}_2 = -x_2 + \alpha\left(-x_1 + x_1^3\right)$, where $\alpha \in [1, 3]$ is a parametric uncertainty. We solved problem (15) with $\partial(V) = 2$ and $\partial(V) = 4$ for $\mu = 0, 0.01, 0.05, 0.1, 0.15$, and 0.2. Note that $\bar{\mu}$ (as defined in Proposition 4.6) is 0.244. Typical computation times are 5 and 8 s for $\partial(V) = 2$ and 4, respectively.

Fig. 1 shows the invariant subset of the robust ROA reported in [16] (solid) and those computed here with $\partial(V) = 2$ (dash) and $\partial(V) = 4$ (dot) for $\mu = 0$ along with two points (stars) that are initial conditions for divergent trajectories of the system corresponding to $\alpha = 1$.Table II shows the optimal values of β in the problem (15) with $\partial(V) = 2$ and 4 for different values of μ.

Example 2) Consider the system (from [17]) of $\dot{x}_1 = -x_2 + 0.2\alpha x_2$ and $\dot{x}_2 = x_1 + (x_1^2 - 1)x_2$ where $\alpha \in [-1, 1]$. For easy comparison with the results in [17], let $p(x) = 0.378x_1^2 - 0.274x_1x_2 + 0.278x_2^2$ and $\mu = 0$. In [17], it was shown that $\mathcal{P}_{0.545}$ (with a single parameter independent quartic V), $\mathcal{P}_{0.772}$ (with pointwise maximum of two parameter independent quartic V's), $\mathcal{P}_{0.600}$ (with a single parameter dependent quartic (in state) V), $\mathcal{P}_{0.806}$ (with pointwise maximum of two parameter dependent quartic (in state) V's) are contained in the robust

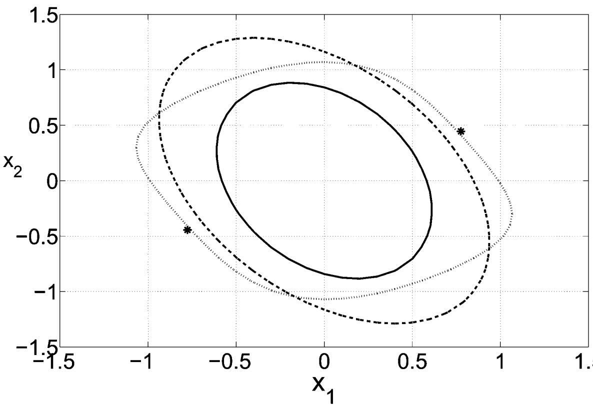

Fig. 1. Invariant subsets of ROA reported in [16] (solid) and those computed solving the problem in (15) with $\partial(V) = 2$ (dash) and $\partial(V) = 4$ (dot) along with initial conditions (stars) for some divergent trajectories of the system corresponding to $\alpha = 1$.

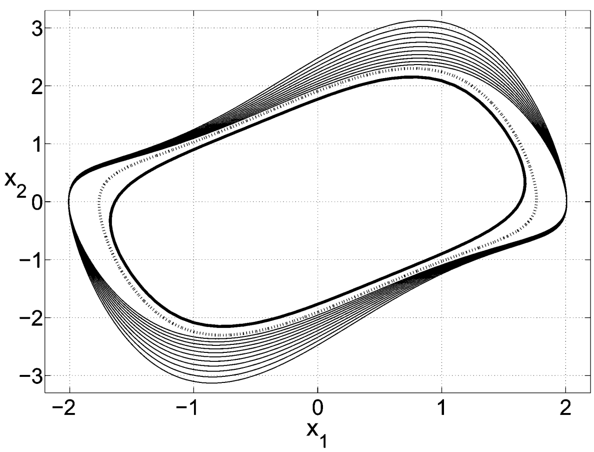

Fig. 2. Invariant subsets of ROA with $\partial(V) = 4$ (inner solid) and $\partial(V) = 6$ (dash) along with the unstable limit cycle (outer solid curves) of the system corresponding to $\alpha = -1.0, -0.8, \ldots, 0.8, 1.0$.

TABLE III
OPTIMAL VALUES OF β IN THE PROBLEM (15) WITH DIFFERENT VALUES OF μ AND $\partial(v) = 4$ AND 6

μ \ $\partial(V)$	4	6
0	0.773	0.826
0.01	0.767	0.820
0.05	0.741	0.803
0.1	0.708	0.787
0.2	0.640	0.750
0.5	0.517	0.651
0.75	0.406	0.573

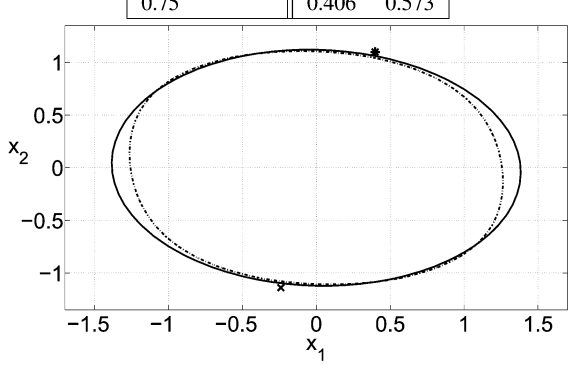

Fig. 3. Invariant subsets of ROA with $\partial(V) = 2$ (solid) and $\partial(V) = 4$ (dash) along with initial conditions for divergent trajectories ("$*$" for $\phi(x) = (0.76x_2^2, 0.19\,(x_1^2 + x_2^2))$ and "$\times$" for $\phi(x) = (-0.76x_2^2, -0.19\,(x_1^2 + x_2^2)))$.

ROA. On the other hand, the solution of problem (15) with $\partial(V) = 4$ and $\partial(V) = 6$ certifies that $\mathcal{P}_{0.773}$ and $\mathcal{P}_{0.826}$ are subsets of the robust ROA, respectively. Fig. 2 shows invariant subsets of the robust ROA computed using $\partial(V) = 4$ (inner solid) and $\partial(V) = 6$ (dash) along with the unstable limit cycle (outer solid curves) of the system corresponding to $\alpha = -1.0, -0.8, \ldots, 0.8, 1.0$. In order to demonstrate the effect of the parameter μ on the size of the invariant subsets of the robust ROA verifiable solving the optimization problem in (15), the analysis is repeated with $\mu = 0.01, 0.05, 0.1., 0.2, 0.5$, and 0.75. Note that $\bar{\mu}$ (as defined in Proposition 4.6) is 0.769. Table III shows the optimal values of β in the problem (15) with $\partial(V) = 4$ and 6 for different values of μ. Typical computation times are 19 and 24 s for $\partial(V) = 4$ and 6, respectively.

Example 3) Consider the system governed by

$$\dot{x} = \begin{bmatrix} -2x_1 + x_2 + x_1^3 + 1.58x_2^3 \\ -x_1 - x_2 + 0.13x_2^3 + 0.66x_1^2 x_2 \end{bmatrix} + \phi(x) \qquad (20)$$

where ϕ satisfies the bounds $-0.76x_2^2 \leq \phi_1(x) \leq 0.76x_2^2$ and $-0.19\left(x_1^2 + x_2^2\right) \leq \phi_2(x) \leq 0.19\left(x_1^2 + x_2^2\right)$ in the set $\mathcal{G} = \{x \in \mathcal{R}^2 : g(x) = x^T x \leq 2.1\}$. Fig. 3 shows invariant subsets of the robust ROA computed with $\partial(V) = 2$ (solid) and $\partial(V) = 4$ (dash) along with two points that are initial conditions for divergent trajectories ("$*$" for $\phi(x) = \left(0.76x_2^2, 0.19\left(x_1^2 + x_2^2\right)\right)$ and "$\times$" for $\phi(x) = \left(-0.76x_2^2, -0.19\left(x_1^2 + x_2^2\right)\right)$). Typical computation times are 13 and 35 s for $\partial(V) = 2$ and 4, respectively.

B. Controlled Short Period Aircraft Dynamics

Consider the plant dynamics

$$\begin{aligned} \dot{z} = &\begin{bmatrix} -3 & -1.35 & -0.56 \\ 0.91 & -0.64 & -0.02 \\ 1 & 0 & 0 \end{bmatrix} z \\ &+ \begin{bmatrix} 1.35 - 0.04z_2 \\ 0.4 \\ 0 \end{bmatrix} u \\ &+ \begin{bmatrix} (1+\alpha_1)(0.08z_1 z_2 + 0.44z_2^2 + 0.01z_2 z_3 + 0.22z_2^3) \\ (1+\alpha_2)(-0.05z_2^2 + 0.11z_2 z_3 - 0.05z_3^2) \\ 0 \end{bmatrix} \end{aligned} \qquad (21)$$

$y = [z_1 \quad z_3]^T$, where z_1, z_2 and, z_3 are the pitch rate, the angle of attack, and the pitch angle, respectively. The input u is the elevator deflection and determined by

$$\dot{\eta} = \begin{bmatrix} -0.60 & 0.09 \\ 0 & 0 \end{bmatrix} \eta + \begin{bmatrix} -0.06 & -0.02 \\ -0.75 & -0.28 \end{bmatrix} y \qquad (22)$$

$u = \eta_1 + 2.2\eta_2$, where η is the controller state. Here, α_1 and α_2 are two uncertain parameters introducing 10% uncertainty for the entries of the plant dynamics that are nonlinear in v, i.e., $\alpha_1 \in [-0.1, 0.1]$ and $\alpha_2 \in [-0.1, 0.1]$. Entries in the vector fields above are shown up to three significant digits. The exact vector field used for this example is available at [32]. The solution of (15) with $\partial(V) = 2$ and $\mu = 0$ verifies that $\mathcal{P}_{7.2} \subset R_0^r$ whereas it can be certified that $\mathcal{P}_{8.6}$ is a subset of the ROA for the nominal system (i.e., for $\alpha_{l,1} = \alpha_{l,2} = \alpha_{u,1} = \alpha_{u,2} = 0$). With $\partial(V) = 4$ the problem in (15) has more than 4000 decision variables. Therefore, we computed a suboptimal solution in two steps for $\mu = 0$: We first computed a Lyapunov function for the nominal system (35 min, which certifies that $\mathcal{P}_{15.2}$ is in the ROA for the nominal system) and then verified (3 min) that $\mathcal{P}_{9.6}$ is an invariant subset of the ROA for the uncertain system. To assess the suboptimality of the results, we performed extensive simulations for the uncertain system setting α_1 and α_2 to their limit values and found a diverging trajectory with the initial condition satisfying $p(z(0), \eta(0)) \approx 14$. The gap between the value of $\beta \approx 14$ for which $\mathcal{P}_\beta$ cannot be a subset of the robust ROA and the value of $\beta = 9.6$ for which $\mathcal{P}_\beta \subset R_0^r$ is verified may be due to the finite dimensional parametrization for V, the issues mentioned in Remark 4.3, the fact that we only use sufficient conditions and/or suboptimality of the two step procedure used for this example; nevertheless, it demonstrates a necessity of further study to make local system analysis based on Lyapunov functions and SOS relaxations more efficient.

VI. Conclusion

We proposed a method to compute provably invariant subsets of the region-of-attraction for the asymptotically stable equilibrium points of uncertain nonlinear dynamical systems. We considered polynomial dynamics with perturbations that either obey local polynomial bounds or are described by uncertain parameters multiplying polynomial terms in the vector field. This uncertainty description is motivated by both incapabilities in modeling, as well as bilinearity and dimension of the sum-of-squares programming problems whose solutions provide invariant subsets of the region-of-attraction. We demonstrated the method on three examples from the literature and a controlled short period aircraft dynamics example.

Appendix

Following two lemmas are simple generalizations of the S-procedure. The proof of the first one is trivial. We provide a proof for the second one.

Lemma 8.1: Given $g_0, g_1, \ldots, g_m \in \mathbb{R}[x]$, if there exist $s_1, \ldots, s_m \in \Sigma[x]$ such that $g_0 - \sum_{i=1}^m s_i g_i \in \Sigma[x]$, then $\{x \in \mathcal{R}^n : g_1(x), \ldots, g_m(x) \geq 0\} \subseteq \{x \in \mathcal{R}^n : g_0(x) \geq 0\}$. ◃

Lemma 8.2: Let $g \in \mathbb{R}[x]$ be positive definite, $h \in \mathbb{R}[x]$, $\gamma > 0$, $s_1, s_2 \in \Sigma[x]$, $l \in \mathbb{R}[x]$ be positive definite and satisfy $l(0) = 0$. Suppose that $-[(\gamma - g)s_1 + hs_2 + l] \in \Sigma[x]$ holds. Then, $\Omega_{g,\gamma}\backslash\{0\} \subset \{x \in \mathcal{R}^n : h(x) < 0 \text{ and } s_2(x) > 0\}$. ◃

Proof: Let $x \in \Omega_{g,\gamma}$ be nonzero. Then

$$0 > -l(x) - (\gamma - g(x))s_2(x) \geq h(x)s_2(x)$$

and, consequently, $s_2(x) > 0$ (since $s_2(x) \geq 0$) and $h(x) < 0$. ■

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