

TWO-DIMENSIONAL VARIATIONAL MODE DECOMPOSITION

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ABSTRACT

In this paper we propose a method to adaptively decompose an image into few different modes of separate spectral bands, which are unknown before. A popular method for recursive one dimensional signal decomposition is the Empirical Mode Decomposition algorithm, introduced by Huang in the nineties. This algorithm as well as its 2D extension, though extensively used, suffers from a lack of exact mathematical model, interpolation choice, and sensitivity to both noise and sampling. Other state-of-the-art models include synchrosqueezing, the empirical wavelet transform, and recursive variational decomposition into smooth signals and residuals. Here, we have created an entirely non-recursive 2D variational mode decomposition (2D-VMD) model, where the modes are extracted concurrently. The model looks for a number of 2D modes and their respective center frequencies, such that the bandlimited modes reproduce the input image (exactly or in least-squares sense). Preliminary results show excellent performance on both synthetic images and real data.

Index Terms— variational image decomposition, spectral segmentation, sparse image approximation, directional filter bank, wiener filter.

1. INTRODUCTION

In this paper we are interested in decomposing images into ensembles of constituent modes (components) that have specific directional and oscillatory characteristics. Simply put, the goal is to retrieve a small number of modes, that each have a very limited bandwidth around their characteristic center frequency. These modes are called intrinsic mode functions (IMF) and can be seen as amplitude- and frequency-modulated (AM-FM) 2D signals (“plane”-waves). Such a mode can have limited spatial support, its local (instantaneous) frequency and amplitude vary smoothly, several modes can overlap in space, and together the ensemble of modes should reconstruct the given input image up to noise and singular features. Possible applications include the decomposition of images into (possibly overlapping) regions of

essentially wave-like nature, in order to make these components accessible for further analysis, such as space-frequency analysis and demodulation.

The problem is inspired by the one dimensional Empirical Mode Decomposition (EMD) algorithm [1] and its two dimensional extension [2] for recursive sifting of 2D spatial signals by means of interpolating upper and lower envelopes, median envelopes, and thus extracting image components in different “frequency” bands. This 2D-EMD, however, suffers from the same drawbacks in robustness as the original EMD in extremal point finding, interpolation of envelopes, and stopping criteria imposed.

Other methods of directional image decomposition work by mostly rigid frames, decomposing the Fourier spectrum into fixed, mostly or strictly disjoint, (quasi-)orthogonal basis elements. Examples include Gabor filters [3], wavelets [4, 5, 6], curvelets [7], shearlets [8, 9], etc. These methods are not adaptive relative to the signal, and can attribute principle components of the image to different bands, as well as contain several different image components in the same band. Adaptivity and tuned sparsity concerns have been addressed through synchrosqueezed wavelet transforms [10, 11], where unimportant wavelet coefficients are removed by thresholding based on energy content. In pursuit of the same goal, the 2D Empirical Wavelet Transform (EWT) [12, 13] decomposes an image by creating an adaptive wavelet basis.

In this paper we propose a natural two dimensional extension of the (1D) Variational Mode Decomposition (VMD) algorithm [14] in the context of image segmentation and directional decomposition. 2D-VMD is a non-recursive, fully adaptive, variational method which sparsely decomposes images in a mathematically well-founded manner with minimal parameters and no explicit interpolation.

Indeed, the variational mode decomposition in 1D is essentially based on well-established concepts such as Wiener filtering, the 1D Hilbert transform and the analytic signal, and heterodyne demodulation. The goal of 1D-VMD is to decompose an input signal into a discrete number of sub-signals (modes), where each mode has limited bandwidth in spectral domain. In other words, we require each mode k to be mostly compact around a center pulsation ω_k , which is to be determined along with the decomposition. In order to assess

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the bandwidth of a mode, we have proposed the following scheme: 1) for each mode u_k , compute the associated analytic signal by means of the Hilbert transform in order to obtain a unilateral frequency spectrum. 2) for each mode, shift the mode's frequency spectrum to "baseband", by mixing with an exponential tuned to the respective estimated center frequency. 3) The bandwidth is now estimated through the H^1 smoothness of the demodulated signal, i.e. the squared L^2 -norm of the gradient. The resulting constrained variational problem is the following:

$$\begin{aligned} \min_{u_k, \omega_k} & \left\{ \sum_k \left\| \partial_t \left[\left(\delta(t) + \frac{j}{\pi t} \right) * u_k(t) \right] e^{-j\omega_k t} \right\|_2^2 \right\} \\ \text{s.t.} & \sum_k u_k = f, \end{aligned} \quad (1)$$

where the reconstruction constraint can be addressed in different ways, such as through simple quadratic penalty or by including an augmented Lagrangian in order to render the problem unconstrained [15, 16]. The solution to the 1D-VMD problem is then found as a sequence of simple sub-optimizations [14].

Here, we propose a generalization of the 1D-VMD model to two (and possibly more) dimensions, in order to make the method applicable to images.

2. 2D VARIATIONAL MODE DECOMPOSITION

We adapt the original 1D-VMD model to dimension two, with the intent of application to images. The 2D model is relatively analogous to its 1D counterpart, minimizing the constituent sub-signals bandwidth while maintaining data fidelity.

The analogy runs into a natural impasse due to a lack of a natural unique extension of the 1D analytic signal for two dimensions. With the intent of being able to shift image frequencies to baseband through heterodyne demodulation, we need the Fourier spectra to be single sided.

2.1. 2D Analytic Signal

While in 1D the analytic signal is achieved by suppressing the negative frequencies, in 2D one half-plane of the frequency domain must be set to zero; it is chosen relative to a vector, in our case to $\vec{\omega}_k$. Thus the 2D analytic signal of interest can first be defined in the frequency domain.

$$\begin{aligned} \hat{u}_{AS,k}(\vec{\omega}) &= \begin{cases} 2\hat{u}_k(\omega), & \text{if } \vec{\omega} \cdot \vec{\omega}_k > 0 \\ \hat{u}_k(\omega), & \text{if } \vec{\omega} \cdot \vec{\omega}_k = 0 \\ 0, & \text{if } \vec{\omega} \cdot \vec{\omega}_k < 0 \end{cases} \\ &= (1 + \text{sgn}(\vec{\omega} \cdot \vec{\omega}_k))\hat{u}_k(\vec{\omega}) \end{aligned} \quad (2)$$

We can now define the 2D analytic signal with the previous Fourier property, as per a definition in [17]:

$$u_{AS,k}(\vec{x}) = u_k(\vec{x}) * \left(\delta(\langle \vec{x}, \vec{\omega}_k \rangle) + \frac{j}{\pi \langle \vec{x}, \vec{\omega}_k \rangle} \right) \delta(\langle \vec{x}, \vec{\omega}_{k,\perp} \rangle) \quad (3)$$

Where $*$ denotes convolution and the transform is separable. Here, the analytic signal is calculated line-wise along the direction of reference, $\vec{\omega}_k$. These lines are processed independently, hence this definition is intrinsically 1D, but has the desired 2D Fourier property.

2.2. 2D VMD Functional

The functional to be minimized, stemming from this definition of 2D analytic signal, is:

$$\min_{u_k, \vec{\omega}_k} \left\{ \sum_k \left\| \nabla [u_{AS,k}(\vec{x}) e^{-j\langle \vec{\omega}_k, \vec{x} \rangle}] \right\|_2^2 \right\} \quad \text{s.t.} \quad \sum_k u_k = f \quad (4)$$

The reconstruction constraint is addressed through quadratic penalty or augmented Lagrangian, and we proceed by ADMM for optimization, as in the 1D case [14, 15, 16].

2.2.1. Minimization w.r.t. the modes u_k

With this definition of the 2D analytic signal in mind, we can optimize for u_k and ω_k analogously to the 1D counterpart. Since we are dealing with L^2 -norms, the functional including the augmented Lagrangian can be written in Fourier domain:

$$\begin{aligned} \hat{u}_k^{n+1} = \arg \min_{\hat{u}_k} & \left\{ \alpha \left\| j(\vec{\omega} - \vec{\omega}_k) [(1 + \text{sgn}(\vec{\omega} \cdot \vec{\omega}_k))\hat{u}_k(\vec{\omega})] \right\|_2^2 \right. \\ & \left. + \left\| \hat{f}(\vec{\omega}) - \sum_i \hat{u}_i(\vec{\omega}) + \frac{\hat{\lambda}(\vec{\omega})}{2} \right\|_2^2 \right\}, \end{aligned} \quad (5)$$

which yields the following Wiener-filter result:

$$\begin{aligned} \hat{u}_k^{n+1}(\vec{\omega}) &= \left(\hat{f}(\vec{\omega}) - \sum_{i \neq k} \hat{u}_i(\vec{\omega}) + \frac{\hat{\lambda}(\vec{\omega})}{2} \right) \frac{1}{1 + 2\alpha |\vec{\omega} - \vec{\omega}_k|^2} \\ &\forall \vec{\omega} \in \Omega_k : \Omega_k = \{ \vec{\omega} \mid \vec{\omega} \cdot \vec{\omega}_k \geq 0 \} \end{aligned} \quad (6)$$

2.2.2. Minimization w.r.t. the center frequencies $\vec{\omega}_k$

Optimizing for $\vec{\omega}_k$ is similar to the 1D VMD case, except that now we are considering the domains to be the half-planes, and that there are two components. The update goal is

$$\vec{\omega}_k^{n+1} = \arg \min_{\vec{\omega}_k} \left\{ \sum_k \left\| \nabla [u_{AS,k}(\vec{x}) e^{-j\langle \vec{\omega}_k, \vec{x} \rangle}] \right\|_2^2 \right\} \quad (7)$$

Or, again in the Fourier domain:

$$\vec{\omega}_k^{n+1} = \arg \min_{\vec{\omega}_k} \left\{ \alpha \left\| j(\vec{\omega} - \vec{\omega}_k) [(1 + \text{sgn}(\vec{\omega} \cdot \vec{\omega}_k))\hat{u}_k(\vec{\omega})] \right\|_2^2 \right\} \quad (8)$$

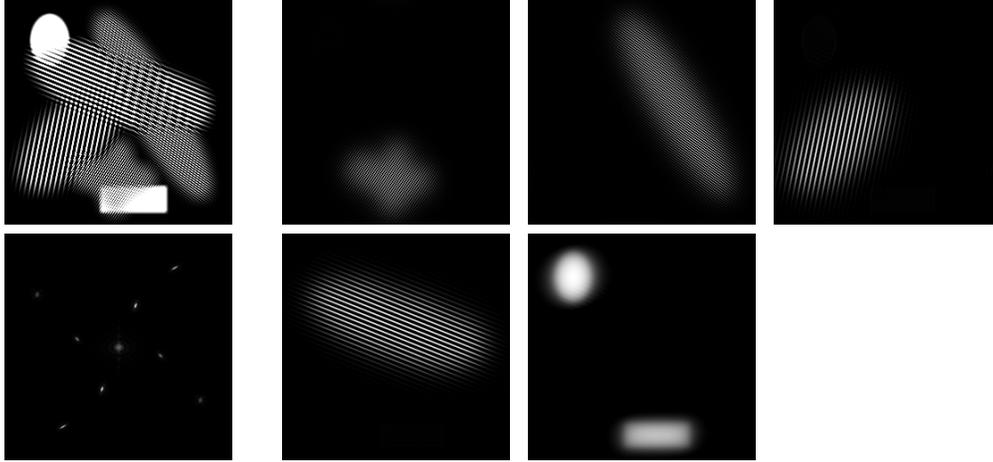


Fig. 1. Results of 2D-VMD on f_{Synth} . **Top left:** The synthetic image. **Bottom left:** Spectrum. **Right:** 5 recovered modes.

The minimization is solved by letting the first variation w.r.t. $\vec{\omega}_k$ vanish. The resulting solutions are the first moments of the mode's power spectrum $|\hat{u}_k(\vec{\omega})|^2$ on the half-plane Ω_k :

$$\vec{\omega}_k^{n+1} = \frac{\int_{\Omega_k} \vec{\omega} |\hat{u}_k(\vec{\omega})|^2 d\vec{\omega}}{\int_{\Omega_k} |\hat{u}_k(\vec{\omega})|^2 d\vec{\omega}} \quad (9)$$

3. EXPERIMENTS AND RESULTS

We have implemented the 2D-VMD method in MATLAB[®]¹, and test the algorithm on both a synthetic and a real image.

3.1. Synthetic Image

The first synthetic image is a composition of spatially overlapping basic shapes, more precisely five ellipses and a rectangle, with frequency patterns varying in both periodicity and direction. The spectrum is ideal for segmentation due to modes being deliberately both well isolated and narrowbanded. The synthetic image experiment was run with parameters $\alpha = 1000$ and $K = 5$. The algorithm has no problems in accuracy nor timeliness in segmenting the image into its five constituent sub-images. The first four decompositions in Fig. 1 show clear separation of the patterned ellipses, while the fifth is the DC component of the image - a solid ellipse and rectangle. Due to these solid pieces having sharp edges, their spectra are not bandlimited and only smoothed versions are recovered. The number of iterations and speed will be addressed below in the subsection on ω_k initialization.

3.2. Peptide β -sheet Microscopy Image

The second test image is a microscopy scan of peptide β -sheets bonding on a graphite base, courtesy of the Weiss-

group at the California NanoScience Institute (CNSI), [18]. The peptide sheets grow in regions of directional homogeneity and form natural spatial boundaries where the regions meet. It is important to scientists to have accurate segmentation for their dual interests in the homogenous regions and their boundaries. Identifying regions of homogeneity allows for the subsequent study of isolated peptide sheets of one particular bonding class. For these types of scans, manually finding the boundaries is a tedious problem that demands the attention of a skilled scientist on a rote task. In addition to speed and automation, the proposed 2D VMD is superior in accuracy to manual boundary identification due to regions potentially having very similar patterns, varying by only a few degrees, that are difficult to discern to even the trained eye. The success of the 1D VMD algorithm in tone separation is expected to carry over to its 2D counterpart in separating patterns that are very close, yet distinct, in spectrum.

As a common pre-processing step in image analysis, here we apply a band-pass filter to the image in order to remove both noise and the DC component. Subsequently, the 2D-VMD algorithm decomposes the piecewise homogeneous peptide sheet image into its five principle components with the purpose of segmentation. Fig. 2 illustrates these individual components, and then compares their superposition to the original peptide sheet with manual boundaries added. This experiment was run with parameters $\alpha = 5000$ and $K = 5$.

3.3. ω_k Initialization

An important degree of freedom in this algorithm is the initialization of the variables. While the u_k have a natural initialization of $u_k \equiv 0$, the $\vec{\omega}_k$ are somewhat more sensitive. Qualitatively, a high α leads to finer separation of constituent subsignals due to the Wiener filter being more narrowly concentrated around its center frequency. However, this same narrow filter, if centered away from a non-principle frequency,

¹Code will be made available at <http://www.math.ucla.edu/~zosso/code.html>

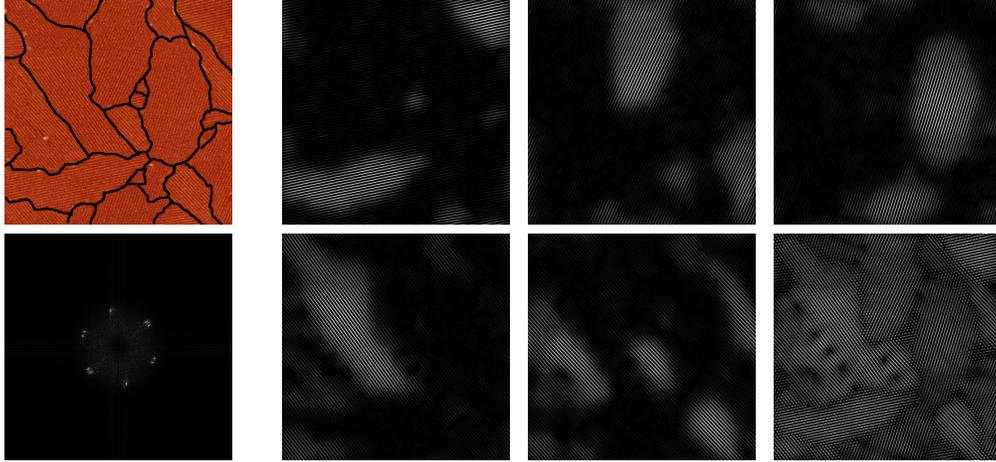


Fig. 2. Results of 2D-VMD on peptide β -sheet image. **Top left:** Peptide β -sheet (with manual boundaries), and **Bottom left:** its power spectrum. **Right:** Five recovered constituent modes, and **Bottom right:** their mode superposition.

may fail to capture the relevant principle frequencies. Conversely, a low α creates a wider filter, allowing the algorithm to “see and travel” to the correct frequencies, but yields worse separation. Given that we know the correct frequencies about which to initialize, a high α will produce accurate results. If we do not know estimates of the principle frequencies of the subcomponents *a priori*, it seems that we are forced to use a lower α , where the ω_k gains freedom of mobility to the appropriate modes at the expense of proper separation. Instead of sacrificing accuracy for mobility, we keep both at the expense of computation time in the following way:

Initialize the ω_k^0 for $k = 1, \dots, K$ randomly on any half-plane, such as $\{\vec{\omega} \mid \omega_1 \geq 0\}$. Using a high α , run the VMD algorithm and record the final values of ω_k^N . Perform this repeatedly for a number of times and create a histogram of the convergent ω_k^N , then observe the top K values. Individual iterations may converge to local minimizers, where qualitatively non-principle modes such as noise will be found, or where multiple ω_k converge to the same principle mode while others are not picked up. The silver lining is that the non-principle convergent modes will be mostly uniformly spread across the spectrum, while the principle ones will show up with much higher consistency. This histogram of convergent modes captures the location of the consistent modes, from which we may get an excellent initialization for a final “clean” iteration. In the above peptide sheet image, we used 200 such iterations to unambiguously determine a proper initialization, though as few as 25 iterations were needed for the histogram to begin to resemble the power spectrum.

4. CONCLUSIONS AND OUTLOOKS

In this paper, we have presented a 2D variational method for decomposing an image into an ensemble of band-limited intrinsic mode functions.

Our decomposition model solves the inverse problem as follows: decompose an image into a given number of modes such that each individual mode has limited bandwidth. We assess the mode’s bandwidth as the squared H^1 norm of its directional 2D analytic signal with only half-space frequencies, shifted to baseband by mixing with a complex exponential of the current center frequency estimate. The modes are updated by simple Wiener filtering, directly in Fourier domain with a filter tuned to the current center frequency. Then the center frequencies are updated as the center of gravity of the mode’s power spectrum. We apply our model to both synthetic and real image data and can show successful decomposition.

The most important limitation of the proposed 2D VMD is with boundary effects, and sudden signal onset in general. Despite the mostly successful decomposition, there is an issue of components’ boundaries being overly smooth due to the narrow-band violation caused by discontinuous envelopes in such AM-FM signals, and a quadratic functional disfavoring and overly penalizing sharp amplitude changes, as with domain boundaries and piecewise regions. Conversely, this is also reflected by implicit periodicity assumptions when optimizing in Fourier domain. Another point is the required explicit selection of the number of active modes in the decomposition. A way to handle this issue is through the histogram method mentioned above. With many random initialization iterations, the dominant modes would show up with highest frequency, and this appropriate number of bins would become obvious by looking at an ordered distribution of the convergent frequency bins, exhibiting a gap between true frequencies and random noise. Rather than relying on iterations, both of these issues are being addressed in current work on an extended mathematical model.

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