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IMMERSED LAGRANGIAN FLOER THEORY

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Abstract

Let (M,ω) be a compact symplectic manifold, and L a compact embedded Lagrangian submanifold in M. Fukaya, Oh, Ohta and Ono [**8**] construct *Lagrangian Floer cohomology*, yielding groups $HF^*(L,b;\Lambda_{\rm nov})$ for one or $HF^*\bigl((L_1,b_1),(L_2,b_2);\Lambda_{\rm nov}\bigr)$ for two Lagrangians, where b,b_1,b_2 are choices of *bounding cochains*, and exist if and only if L,L_1,L_2 have *unobstructed Floer cohomology*. These are independent of choices up to isomorphism, and have important invariance properties under Hamiltonian equivalence. Floer cohomology groups are the morphism groups in the derived Fukaya category of (M,ω), and so are an essential part of the Homological Mirror Symmetry Conjecture of Kontsevich.

The goal of this paper is to extend [**8**] to *immersed* Lagrangians $\iota:L\rightarrow M$, with transverse self-intersections. In the embedded case, Floer cohomology $HF^*(L,b;\Lambda_{\rm nov})$ is a modified, 'quantized' version of singular homology $H_{n-*}(L;\Lambda_{\rm nov})$ over the Novikov ring $\Lambda_{\rm nov}$. In our immersed case, $HF^*(L,b;\Lambda_{\rm nov})$ turns out to be a quantized version of $H_{n-*}(L;\Lambda_{\rm nov})\oplus\bigoplus_{(p_-,p_+)\in R}\Lambda_{\rm nov}\cdot(p_-,p_+)$, where $R=\bigl\{(p_-,p_+):p_-,p_+\in L,\ p_-\neq p_+,\ \iota(p_-)=\iota(p_+)\bigr\}$ is a set of two extra generators for each self-intersection point of L, and (p_-,p_+) has degree $\eta_{(p_-,p_+)}\in\mathbb{Z}$, an index depending on how L intersects itself at $\iota(p_-)=\iota(p_+)$.

The theory becomes simpler and more powerful for *graded* Lagrangians in Calabi–Yau manifolds, when we can work over a smaller Novikov ring $\Lambda_{\rm CY}$. The proofs involve associating a gapped filtered A_∞ algebra over $\Lambda_{\rm nov}^0$ or $\Lambda_{\rm CY}^0$ to $\iota:L\rightarrow M$, which is independent of nearly all choices up to canonical homotopy equivalence, and is built using a series of finite approximations called $A_{N,0}$ algebras for $N=0,1,2,\ldots$.

1. Introduction

Let (M,ω) be a compact symplectic manifold, and L a compact embedded Lagrangian submanifold in M. Fukaya, Oh, Ohta and Ono

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[**8**] have undertaken the mammoth task of rigorously constructing *Lagrangian Floer cohomology* for such M, L. In brief, to each Lagrangian L in M they associate a (*gapped filtered*) A_∞ *algebra* $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\rm nov}, \mathfrak{m})$. A *bounding cochain* $b \in \mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\rm nov}$ is a solution of $\sum_{k\geqslant 0}\mathfrak{m}_k(b,\ldots,b) = 0$ in $\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\rm nov}$. Given such b, they define the Lagrangian Floer cohomology $HF^*(L, b; \Lambda_{\rm nov})$. If L does not admit a bounding cochain, we say L has *obstructed Lagrangian Floer cohomology.* If L_1, L_2 are transversely intersecting Lagrangians in M with bounding cochains b_1, b_2, they define the Lagrangian Floer cohomology $HF^*\bigl((L_1, b_1), (L_2, b_2); \Lambda_{\rm nov}\bigr)$ of L_1, L_2. These are the morphism groups in the derived Fukaya category of (M, ω), and so are an essential part of the Homological Mirror Symmetry Conjecture of Kontsevich [**14**].

The purpose of this paper is to extend the work of Fukaya, Oh, Ohta and Ono [**8**] to *immersed* Lagrangians L in M with immersion $\iota : L \to M$, with transverse self-intersections. This was done by the first author [**1**] under the simplifying assumption that $\pi_2(M, \iota(L)) = \{1\}$, which eliminates the issues of disc bubbling, A_∞ algebras and bounding cochains. We now discuss the much more difficult general case.

Suppose $\iota : L \to M$ is a compact immersed Lagrangian in (M, ω), such that $\iota^{-1}(p)$ is at most two points for each $p \in \iota(L)$, and when $\iota^{-1}(p) = \{p_+, p_-\}$ is two points the two sheets of L intersect transversely at p, that is, $\mathrm{d}\iota(T_{p_+}L) \cap \mathrm{d}\iota(T_{p_-}L) = \{0\}$ in T_pM. We will construct a gapped filtered A_∞ algebra $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\rm nov}, \mathfrak{m})$ associated to L, independent of choices up to canonical homotopy equivalence, which generalizes both the embedded case in Fukaya et al. [**8**, §3], and the gapped filtered A_∞ category associated to finitely many embedded Lagrangian submanifolds by Fukaya [**7**]. Thus we can define *bounding cochains* b for L, and *Lagrangian Floer cohomology groups* $HF^*(L, b; \Lambda_{\rm nov})$ and $HF^*\bigl((L_1, b_1), (L_2, b_2); \Lambda_{\rm nov}\bigr)$, which are independent of choices up to canonical isomorphism.

Fukaya et al. [**8**] mainly develop two subjects: geometry and algebra. In the geometric part, they realize $A_{N,K}$ structures on some singular chains of an embedded Lagrangian submanifold L through moduli spaces of isomorphism classes of stable maps from a genus 0 prestable bordered Riemann surface with boundary attached to L. In the algebraic part, they develop the homotopy theory, or homological algebra, of $A_{N,K}$ and gapped filtered A_∞ algebras. Finally, they apply the homotopy theory to the geometric realization, and obtain a gapped filtered A_∞ algebra associated to an embedded Lagrangian submanifold.

Here we develop a generalization of their geometry, that is, we construct $A_{N,0}$ structures associated to an immersed Lagrangian submanifold with transverse self-intersections. Then we apply the homotopy theory to our generalization, and obtain a gapped filtered A_∞ algebra associated to an immersed Lagrangian submanifold.

Fukaya et al. also construct a gapped filtered A_∞ bimodule associated to a pair of transversely intersecting embedded Lagrangian submanifolds [**8**], and a gapped filtered A_∞ category associated to a finite number of transversely intersecting embedded Lagrangian submanifolds [**7**]. Regarding a finite union of embedded Lagrangians as a single immersed Lagrangian, their gapped filtered A_∞ modules and categories become part of our gapped filtered A_∞ algebras.

Here is one reason why extending Lagrangian Floer cohomology to immersed Lagrangians may be important. Using the embedded Lagrangian Floer theory of [**8**], one can define the *Fukaya category* $\mathrm{Fuk}(M,\omega)_{\mathrm{em}}$, whose objects are roughly speaking pairs (L,b) of an embedded Lagrangian and a bounding cochain b for L, and the *derived Fukaya category* $D^b\big(\mathrm{Fuk}(M,\omega)_{\mathrm{em}}\big)$. Kontsevich's Homological Mirror Symmetry Conjecture [**14**] says (roughly) that for (M,ω) a symplectic Calabi–Yau with mirror complex Calabi–Yau $(\check M,\check J)$, the derived Fukaya category $D^b\big(\mathrm{Fuk}(M,\omega)_{\mathrm{em}}\big)$ should be equivalent as a triangulated category to the derived category $D^b\big(\mathrm{coh}(\check M,\check J)\big)$ of coherent sheaves on $(\check M,\check J)$.

The theory of this paper would yield an *immersed Fukaya category* $\mathrm{Fuk}(M,\omega)_{\mathrm{im}}$ using immersed Lagrangians, and the *derived immersed Fukaya category* $D^b\big(\mathrm{Fuk}(M,\omega)_{\mathrm{im}}\big)$. We could then use $D^b\big(\mathrm{Fuk}(M,\omega)_{\mathrm{im}}\big)$ in place of $D^b\big(\mathrm{Fuk}(M,\omega)_{\mathrm{em}}\big)$ in Homological Mirror Symmetry. Actually, it seems likely that $D^b\big(\mathrm{Fuk}(M,\omega)_{\mathrm{im}}\big)$ and $D^b\big(\mathrm{Fuk}(M,\omega)_{\mathrm{em}}\big)$ are *equivalent categories*, although $D^b\big(\mathrm{Fuk}(M,\omega)_{\mathrm{im}}\big)$ has more objects.

Motivated by conjectures of Thomas and Yau [**21**] and more recent ideas of Bridgeland [**2**] and the String Theorists Douglas and Aspinwall, we can state the following (approximate) conjecture, which is an extension of the Homological Mirror Symmetry story: let (M,J,ω,Ω) be a Calabi–Yau n-fold. Then there should exist a Bridgeland stability condition $(Z,\mathcal{P})$ on $D^b\big(\mathrm{Fuk}(M,\omega)\big)$ depending on the holomorphic $(n,0)$-form Ω on M, such that each isomorphism class of stable objects in $D^b\big(\mathrm{Fuk}(M,\omega)\big)$ is represented by a unique special Lagrangian.

For this conjecture to hold, we need $D^b\big(\mathrm{Fuk}(M,\omega)\big)$ to contain as many actual geometric Lagrangians as possible. In particular, the conjecture should be *false* for $D^b\big(\mathrm{Fuk}(M,\omega)_{\mathrm{em}}\big)$ when $n>2$, since then there could exist (L,b) and (L',b') isomorphic in $D^b\big(\mathrm{Fuk}(M,\omega)_{\mathrm{im}}\big)$ with L embedded, and L' special Lagrangian and immersed but not embedded. If the conjecture were true for the embedded case, there would exist an embedded special Lagrangian L'' with (L'',b'') isomorphic to (L,b). But the uniqueness argument of Thomas and Yau [**21**, Th. 4.3] applied to our immersed case implies that there cannot be two different special Lagrangian representatives (L',b') and (L'',b'') for $[(L,b)]$, a contradiction. Thus, to make our modified Thomas–Yau conjecture true we need at least to include immersed Lagrangians in the Fukaya category, and perhaps also some classes of singular Lagrangians as well.

We begin with some background material on Kuranishi spaces, multisections, and virtual chains in §2, and on A_∞ algebras and $A_{N,K}$ algebras in §3. Section 4 introduces the moduli spaces of isomorphism classes of stable maps from a genus 0 prestable bordered Riemann surface with boundary attached to an immersed Lagrangian submanifold. They are Kuranishi spaces, with corners, whose boundaries are fibre products of other such moduli spaces. Section 5 discusses orientations of our moduli spaces.

Sections 6–11 construct gapped filtered A_∞ algebras from immersed Lagrangian submanifolds $\iota : L \to M$, and show they are independent of choices such as the almost complex structure J, up to canonical homotopy equivalence. First, in §6–§7, we construct $A_{N,0}$ algebras $(\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m})$ from $\iota : L \to M$ for all $N = 0, 1, 2, \ldots$, involving different arbitrary choices for each N. In §8–§9, we show that the $A_{N,0}$ algebras of §6–§7 are unique up to $A_{N,0}$ homotopy equivalences $\mathfrak{j} : (\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m}) \to (\mathbb{Q}\mathcal{X}'_N, \mathcal{G}, \mathfrak{m}')$, and §10 proves that these $\mathfrak{j}$ are unique up to homotopy.

Section 11 passes from $A_{N,0}$ algebras $(\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m})$ to gapped filtered A_∞ algebras $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}}, \mathfrak{m})$ by a limiting process as $N \to \infty$, and shows that $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}}, \mathfrak{m})$ is independent of choices up to canonical homotopy equivalence. Section 12 defines *graded* Lagrangians in Calabi–Yau manifolds, and explains how in the graded case we can redo §6–§11 using the smaller Novikov ring Λ^0_{CY}. Finally, §13 defines bounding cochains and Lagrangian Floer cohomology, discusses some applications, and suggests some questions and conjectures for future research.

By its very nature, this paper exists wholly in the shadow of Fukaya, Oh, Ohta and Ono's massive work [**8**]. Despite this, we have tried hard to make our paper *independent of* [**8**], in the sense that it is self-contained, requiring no more than the usual background for research papers in the area, and readers do not need to read [**8**] to understand our paper.

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2. Background material on Kuranishi spaces and multisections

We now summarize results from Fukaya, Ono et al. [**9**, §3–§6], [**8**, §A] on Kuranishi spaces, multisections and virtual chains that we will need later. Where the notation of [**9**, **8**] differs, for instance if Kuranishi neighbourhoods are (V, E, Γ, s, ψ) with V a manifold or (V, E, s, ψ) with V an orbifold, we generally follow [**8**].

2.1. Kuranishi structures on topological spaces. We define *Kuranishi spaces*, following Fukaya, Ono et al. [**9**, §5] and [**8**, §A1.1].

Definition 2.1. Let X be a compact, metrizable topological space. A *Kuranishi neighbourhood* of $p \in X$ is a quintet $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ such that:

(i) V_p is a smooth finite-dimensional manifold, which may or may not have boundary or corners;
(ii) $E_p \to V_p$ is a vector bundle over V_p;
(iii) Γ_p is a finite group which acts smoothly on V_p, and acts compatibly on E_p preserving the vector bundle structure;
(iv) $s_p : V_p \to E_p$ is a Γ_p-equivariant smooth section; and
(v) ψ_p is a homeomorphism from $s_p^{-1}(0)/\Gamma_p$ to a neighbourhood of p in X, where $s_p^{-1}(0)$ is the subset of V_p where the section s_p is zero.

We call E_p the *obstruction bundle*, and s_p the *Kuranishi map*.

Here we follow [**9**, Def. 5.1] in taking E_p to be a vector bundle, rather than a finite-dimensional vector space as in [**8**, Def. A1.1].

Definition 2.2. Let $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ and $(V_q, E_q, \Gamma_q, s_q, \psi_q)$ be Kuranishi neighbourhoods of $p \in X$ and $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$ respectively. We call a triple $(\hat{\phi}_{pq}, \phi_{pq}, h_{pq})$ a *coordinate change* if:

(a) $h_{pq} : \Gamma_q \to \Gamma_p$ is an *injective* group homomorphism;
(b) $\phi_{pq} : V_q \to V_p$ is an h_{pq}-equivariant smooth *embedding*;
(c) $(\hat{\phi}_{pq}, \phi_{pq})$ is an h_{pq}-equivariant smooth *embedding* of vector bundles $E_q \to E_p$;
(d) $\hat{\phi}_{pq} \circ s_q \equiv s_p \circ \phi_{pq}$; and
(e) $\psi_q \equiv \psi_p \circ \phi_{pq}$.

We define the notions of a *germ of a Kuranishi neighbourhood* and a *germ of a coordinate change* in the obvious way.

Definition 2.3. A *Kuranishi structure* on X assigns a germ of a Kuranishi neighbourhood $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ for each $p \in X$ and a germ of a coordinate change $(\hat{\phi}_{pq}, \phi_{pq}, h_{pq})$ for each $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$, such that the following hold:

(i) $\dim V_p - \operatorname{rank} E_p$ is independent of p; and
(ii) if $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$ and $r \in \psi_q(s_q^{-1}(0)/\Gamma_q)$ then $h_{pq} \circ h_{qr} = h_{pr}$, $\phi_{pq} \circ \phi_{qr} = \phi_{pr}$ and $\hat{\phi}_{pq} \circ \hat{\phi}_{qr} = \hat{\phi}_{pr}$.

We call $\operatorname{vdim} X = \dim V_p - \operatorname{rank} E_p$ the *virtual dimension* of the Kuranishi structure. A topological space X with a Kuranishi structure is called a *Kuranishi space*.

The point of these definitions is that in many moduli problems in geometry in which there are obstructions, the moduli spaces can be equipped with Kuranishi structures in a natural way. This holds for the moduli spaces of J-holomorphic maps from a bordered Riemann surface studied by Fukaya et al. [**8**] and Liu [**17**], as we shall explain in §4.

2.2. Boundaries, strongly smooth maps, and fibre products. We now define the *boundary* ∂X of a Kuranishi space X, which is itself a Kuranishi space of dimension $\operatorname{vdim} X - 1$. To understand the definition, recall that in Definition 2.1(i), V_p may be a manifold *with boundary*, or *with corners*. An n-manifold M with boundary is locally modelled on $[0,\epsilon)\times(-\epsilon,\epsilon)^{n-1}$, and an n-manifold M with corners is locally modelled on $[0,\epsilon)^k\times(-\epsilon,\epsilon)^{n-k}$, for small $\epsilon>0$. If x lies in a codimension k corner of M then k different $(n-1)$-dimensional boundary strata of M meet at x. The *boundary* ∂M is the set of pairs (x,B), where $x\in M$ and B is a local choice of $(n-1)$-dimensional boundary stratum of M containing x.

Thus, if x lies in a codimension k corner of M then x is represented by k *distinct points* (x,B_i) in ∂M for $i=1,\dots,k$. The point of making ∂M a set of pairs (x,B) and not points x is that this way ∂M is a manifold with corners, but if we defined ∂M as the obvious subset of M it would not be a manifold with corners near a codimension k corner of M for $k>1$.

Definition 2.4. Let X be a Kuranishi space. We shall define a Kuranishi space ∂X called the *boundary* of X. The points of ∂X are equivalence classes $[p,v,B]$ of triples (p,v,B), where $p\in X$, $(V_p,E_p,\Gamma_p,s_p,\psi_p)$ lies in the germ of Kuranishi neighbourhoods at p, $v\in V_p$ with $s_p(v)=0$ and $\psi_p(\Gamma_p v)=p$, and B is a local boundary stratum of V_p containing v.

Two triples $(p,v,B),(q,w,C)$ are *equivalent* if $p=q$ and $\gamma\cdot(v,B)=(w,C)$ for some $\gamma\in\Gamma_p$; we also have an obvious notion of equivalence for choices of different Kuranishi neighbourhoods $(V_p,E_p,\Gamma_p,s_p,\psi_p)$, $(V_p',E_p',\Gamma_p',s_p',\psi_p')$ in the germ at p. Basically, this just means that points of ∂X are $p\in X$ together with a choice of boundary stratum of the Kuranishi neighbourhoods V_p lying over p, up to the action of Γ_p.

We can then define a unique natural *topology* and *Kuranishi structure* on ∂X, with $(\partial V_p, E_p|_{\partial V_p},\Gamma_p, s_p|_{\partial V_p},\psi_p|_{\partial V_p/\Gamma_p})$ a Kuranishi neighbourhood on ∂X for each Kuranishi neighbourhood $(V_p,E_p,\Gamma_p,s_p,\psi_p)$ on X. It is easy to verify that $\operatorname{vdim}\partial X=\operatorname{vdim} X-1$, and ∂X is compact if X is compact.

Here is [**9**, Def. 6.6]. The equivalent definition in [**8**, Def. A1.13] instead uses *good coordinate systems*. Fukaya et al. [**8**, Def. A1.13] use the notation *weakly submersive* rather than strong submersion.

Definition 2.5. Let X be a Kuranishi space, and Y be a topological space. Roughly speaking, a *strongly continuous map* $\boldsymbol{f}: X\to Y$ consists of a continuous map $f_p: V_p\to Y$ with $f_p\circ\gamma\equiv f_p$ for all $\gamma\in\Gamma_p$

for each Kuranishi neighbourhood $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ in X, such that if $(\hat{\phi}_{pq}, \phi_{pq}, h_{pq})$ is a coordinate change between $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ and $(V_q, E_q, \Gamma_q, s_q, \psi_q)$, then $f_p \circ \phi_{pq} = f_q$. But because Kuranishi spaces are defined using germs of Kuranishi neighbourhoods, we define a strongly continuous map $\boldsymbol{f}$ to be a system of germs of Γ_p-invariant continuous maps $f_p : V_p \to Y$, satisfying $f_p \circ \phi_{pq} = f_q$ for germs of coordinate changes. Then $\boldsymbol{f}$ induces a continuous map $f : X \to Y$ in the obvious way. If Y is a smooth manifold and all f_p are smooth, we call $\boldsymbol{f}$ *strongly smooth*, and if all f_p are submersions, we call $\boldsymbol{f}$ a *strong submersion*.

Fukaya et al. [**8**, Def. A1.37] define *fibre products* of Kuranishi spaces.

Definition 2.6. Let X, X' be Kuranishi spaces, Y be a smooth manifold, and $\boldsymbol{f} : X \to Y$, $\boldsymbol{f}' : X' \to Y$ be strongly smooth maps, at least one of which is a strong submersion, inducing continuous maps $f : X \to Y$ and $f' : X' \to Y$. Then we can form the *fibre product* $X \times_Y X' = \{(p, p') \in X \times X' : f(p) = f'(p')\}$, a paracompact Hausdorff topological space. We also write $X \times_Y X'$ as $X \times_{\boldsymbol{f},Y,\boldsymbol{f}'} X'$ when we wish to specify $\boldsymbol{f}, \boldsymbol{f}'$.

Let $(p, p') \in X \times_Y X'$, $(V_p, E_p, \Gamma_p, s_p, \psi_p)$, $(V'_{p'}, E'_{p'}, \Gamma'_{p'}, s'_{p'}, \psi'_{p'})$ be sufficiently small Kuranishi neighbourhoods in the germs at p, p' in X, X', and $f_p : V_p \to Y$, $f'_{p'} : V'_{p'} \to Y$ be smooth maps in the germs of $\boldsymbol{f}, \boldsymbol{f}'$ at p, p' respectively. Define a Kuranishi neighbourhood in $X \times_Y X'$ by

$$\begin{aligned}(1)\qquad \big(V_p \times_{f_p,Y,f'_{p'}} V'_{p'}, &(E_p \oplus E'_{p'})|_{V_p \times_Y V'_{p'}}, \Gamma_p \times \Gamma'_{p'},\\ &(s_p \oplus s'_{p'})|_{V_p \times_Y V'_{p'}}, (\psi_p \times \psi'_{p'})|_{V_p \times_Y V'_{p'}}\big).\end{aligned}$$

Here $V_p \times_{f_p,Y,f'_{p'}} V'_{p'}$ is the fibre product of smooth manifolds, defined as at least one of $f_p, f'_{p'}$ is a submersion. It is a submanifold of $V_p \times V'_{p'}$, so we can restrict $E_p \oplus E'_{p'}$, $s_p \oplus s'_{p'}$ and $\psi_p \times \psi'_{p'}$ to it.

It is easy to verify that coordinate changes between Kuranishi neighbourhoods in X and X' induce coordinate changes between neighbourhoods (1). So the systems of germs of Kuranishi neighbourhoods and coordinate changes on X, X' induce such systems on $X \times_Y X'$. This gives a *Kuranishi structure* on $X \times_Y X'$, making it into a *Kuranishi space*. Clearly $\operatorname{vdim}(X \times_Y X') = \operatorname{vdim} X + \operatorname{vdim} X' - \dim Y$, and $X \times_Y X'$ is compact if X, X' are compact.

2.3. Tangent bundles and orientations. Here is [**9**, Def. 5.6]. The equivalent definition in [**8**, Def. A1.14] involves a choice of *good coordinate system*.

Definition 2.7. Let X be a Kuranishi space. Then X has a germ of coordinate changes $(\hat{\phi}_{pq}, \phi_{pq}, h_{pq})$ between Kuranishi neighbourhoods $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ and $(V_q, E_q, \Gamma_q, s_q, \psi_q)$. We say that X *has a tangent bundle* if associated to this germ of coordinate changes $(\hat{\phi}_{pq}, \phi_{pq}, h_{pq})$ we have a germ of Γ_q- and h_{pq}-equivariant isomorphisms of vector bundles over V_q:

$$\chi_{pq} : \frac{\phi_{pq}^*(E_p)}{\hat{\phi}_{pq}(E_q)} \longrightarrow \frac{\phi_{pq}^*(TV_p)}{(\mathrm{d}\phi_{pq})(TV_q)}, \tag{2}$$

where $\hat{\phi}_{pq} : E_q \to \phi_{pq}^*(E_p)$ and $\mathrm{d}\phi_{pq} : TV_q \to \phi_{pq}^*(TV_p)$ are morphisms of vector bundles over V_q. These must agree on triple overlaps: if $(V_p, E_p, \Gamma_p, s_p, \psi_p)$, $(V_q, E_q, \Gamma_q, s_q, \psi_q)$ and $(V_r, E_r, \Gamma_r, s_r, \psi_r)$ are Kuranishi neighbourhoods and $(\hat{\phi}_{pq}, \phi_{pq}, h_{pq}), (\hat{\phi}_{pr}, \phi_{pr}, h_{pr}), (\hat{\phi}_{qr}, \phi_{qr}, h_{qr})$ coordinate changes between them with $\hat{\phi}_{pr} = \hat{\phi}_{pq} \circ \hat{\phi}_{qr}$, $\phi_{pr} = \phi_{pq} \circ \phi_{qr}$ and $h_{pr} = h_{pq} \circ h_{qr}$, then the following diagram of vector bundles over V_r must commute:

$$\begin{tikzcd}
0 \arrow[r] & \frac{\phi_{qr}^*(E_q)}{\hat{\phi}_{qr}(E_r)} \arrow[r, "\phi_{qr}^*(\hat{\phi}_{pq})"] \arrow[d, "\chi_{qr}"] & \frac{\phi_{pr}^*(E_p)}{\hat{\phi}_{pr}(E_r)} \arrow[r, "\text{project}"] \arrow[d, "\chi_{pr}"] & \frac{\phi_{pr}^*(E_p)}{\phi_{qr}^*(\hat{\phi}_{pq}(E_q))} \arrow[r] \arrow[d, "\phi_{qr}^*(\chi_{pq})"] & 0 \\
0 \arrow[r] & \frac{\phi_{qr}^*(TV_q)}{(\mathrm{d}\phi_{qr})(TV_r)} \arrow[r, "\phi_{qr}^*(\mathrm{d}\phi_{pq})"] & \frac{\phi_{pr}^*(TV_p)}{(\mathrm{d}\phi_{pr})(TV_r)} \arrow[r, "\text{project}"] & \frac{\phi_{pr}^*(TV_p)}{\phi_{qr}^*(\mathrm{d}\phi_{pq})(\phi_{qr})^*(TV_q)} \arrow[r] & 0.
\end{tikzcd}$$

We can now discuss *orientations* of Kuranishi spaces.

Definition 2.8. Let X be a Kuranishi space with a tangent bundle. We say that the Kuranishi structure on X is *oriented* if associated to the germ of Kuranishi neighbourhoods $(V_p, E_p, \Gamma_p, s_p, \psi_p)$ on X we are given a germ of orientations of the fibres of the vector bundles $E_p \oplus TV_p$ varying continuously over V_p. These must be compatible on overlaps, in the following sense. Let $(\hat{\phi}_{pq}, \phi_{pq}, h_{pq})$ be in the germ of coordinate changes, and χ_{pq} be as in (2).

Then if $v \in V_q$ and $(e_q^1, \ldots, e_q^m)$, $(t_q^1, \ldots, t_q^n)$ are bases of $E_q|_v$ and $T_v V_q$ such that $(e_q^1, \ldots, e_q^m, t_q^1, \ldots, t_q^n)$ is an oriented basis of $(E_q \oplus TV_q)|_v$, and $\bigl(e_p^1, \ldots, e_p^k, \hat{\phi}_{pq}(e_q^1), \ldots, \hat{\phi}_{pq}(e_q^m)\bigr)$, $\bigl(t_p^1, \ldots, t_p^k, (\mathrm{d}\phi_{pq})(t_q^1), \ldots, (\mathrm{d}\phi_{pq})(t_q^n)\bigr)$ are bases of $E_p|_{\phi_{pq}(v)}, T_{\phi_{pq}(v)}V_p$ such that $\chi_{pq}\bigl(e_p^i + \hat{\phi}_{pq}(E_q|_v)\bigr) = t_p^i + (\mathrm{d}\phi_{pq})(T_v V_q)$ for $i \leqslant k$, then $\bigl(e_p^1, \ldots, e_p^k, \hat{\phi}_{pq}(e_q^1), \ldots, \hat{\phi}_{pq}(e_q^m), t_p^1, \ldots, t_p^k, (\mathrm{d}\phi_{pq})(t_q^1), \ldots, (\mathrm{d}\phi_{pq})(t_q^n)\bigr)$ is an oriented basis for $(E_p \oplus TV_p)|_{\phi_{pq}(v)}$.

2.4. Orientation conventions. Suppose X, X' are Kuranishi spaces with tangent bundles and orientations, Y is an oriented smooth manifold, and $\boldsymbol{f} : X \to Y$, $\boldsymbol{f}' : X' \to Y$ are strongly smooth maps. Then by §2.2 we have Kuranishi spaces ∂X and $X \times_Y X'$. These can also be given orientations in a natural way. We use the orientation conventions of Fukaya et al. [**8**, §8.2].

Convention 2.9. First, our conventions for smooth manifolds:

(a) Let X be an oriented smooth manifold with boundary ∂X. Then we define the orientation on ∂X such that

$$TX|_{\partial X} = \mathbb{R}_{\text{out}} \oplus T(\partial X)$$

is an isomorphism of oriented vector spaces, where $\mathbb{R}_{\text{out}}$ is oriented by an outward-pointing normal vector to ∂X.

(b) Let X, X', Y be oriented smooth manifolds, and $f : X \to Y$, $f' : X' \to Y$ be smooth submersions. Then $\mathrm{d}f : TX \to f^*(TY)$ and $\mathrm{d}f' : TX' \to (f')^*(TY)$ are surjective maps of vector bundles over X, X'. Choosing Riemannian metrics on X, X' and identifying the orthogonal complement of $\operatorname{Ker} \mathrm{d}f$ in TX with the image $f^*(TY)$ of $\mathrm{d}f$, and similarly for f', we have isomorphisms of vector bundles over X, X':

$$TX \cong \operatorname{Ker} \mathrm{d}f \oplus f^*(TY) \quad \text{and} \quad TX' \cong (f')^*(TY) \oplus \operatorname{Ker} \mathrm{d}f'. \tag{3}$$

Define orientations on the fibres of $\operatorname{Ker} \mathrm{d}f$, $\operatorname{Ker} \mathrm{d}f'$ over X, X' such that (3) are isomorphisms of oriented vector bundles, where TX TX' are oriented by the orientations on X, X', and $f^*(TY)$, $(f')^*(TY)$ by the orientation on Y. Then we define the orientation on $X \times_Y X'$ so that

$$\begin{aligned} T(X \times_Y X') &\cong \pi_X^*(\operatorname{Ker} \mathrm{d}f) \oplus (f \circ \pi_X)^*(TY) \oplus \pi_{X'}^*(\operatorname{Ker} \mathrm{d}f') \\ &\cong \pi_X^*(\operatorname{Ker} \mathrm{d}f) \oplus \pi_{X'}^*(TX') \\ &\cong \pi_X^*(TX) \oplus \pi_{X'}^*(\operatorname{Ker} \mathrm{d}f') \end{aligned} \tag{4}$$

are isomorphisms of oriented vector bundles. Here $\pi_X : X \times_Y X' \to X$ and $\pi_{X'} : X \times_Y X' \to X'$ are the natural projections, and $f \circ \pi_X \equiv f' \circ \pi_{X'}$.

Note that the second line of (4) makes sense if f is a submersion but f' is only smooth, and the third line makes sense if f' is a submersion but f is only smooth. Thus, our convention extends to fibre products $X \times_{f,Y f'} X'$ in which only one of f, f' is a submersion.

Here is how to extend (b) to X, X' Kuranishi spaces:

(c) Let X, X' be oriented Kuranishi spaces, Y an oriented smooth manifold, and $\boldsymbol{f} : X \to Y, \boldsymbol{f}' : X' \to Y$ strong submersions. Take Kuranishi neighbourhoods $(V_p, E_p, \Gamma_p, s_p, \psi_p), (V'_{p'}, E'_{p'}, \Gamma'_{p'}, s'_{p'}, \psi'_{p'})$ for X, X', respectively. First, choose orientations of V_p and $V'_{p'}$, and we have the orientation of $V_p \times_Y V'_{p'}$ by Convention 2.9(b). Secondly, the orientations of $E_p \oplus TV_p$ and $E'_{p'} \oplus TV'_{p'}$ induce the orientation of $(E_p \oplus E'_{p'})|_{V_p \times_Y V'_{p'}} \oplus T_{(p,p')}\big(V_p \times_Y V'_{p'}\big)$. Then we *define* an orientation of the Kuranishi neighbourhood (1) with the following sign correction term:

$$(-1)^{\operatorname{rank} E'_{p'}(\dim V_p - \operatorname{rank} E_p - \dim Y)}(E_p \oplus E'_{p'})|_{V_p \times_Y V'_{p'}} \oplus T_{(p,p')}(V_p \times_Y V'_{p'}),$$

where -1 means the opposite orientation. This orientation convention is independent of the choice of Kuranishi neighbourhood. It extends to only one of $\boldsymbol{f}, \boldsymbol{f}'$ a strong submersion as in (b).

If X is a Kuranishi space with tangent bundle and orientation, we will write $-X$ for the same Kuranishi space with the opposite orientation. Here is [**8**, Lem. 8.2.3], except the second line of (5), which is elementary.

Proposition 2.10. *Let $X_1, X_2, \dots$ be Kuranishi spaces with tangent bundles and orientations, $Y, Y_1, \dots$ be oriented smooth manifolds without boundary, and $\boldsymbol{f}_1 : X_1 \to Y, \dots$ be strong submersions. Then the following hold, in Kuranishi spaces with tangent bundles and orientations:*

(a) *For $\boldsymbol{f}_1 : X_1 \to Y$ and $\boldsymbol{f}_2 : X_2 \to Y$ we have*

$$\begin{aligned} &\partial(X_1 \times_Y X_2) = (\partial X_1) \times_Y X_2 \amalg (-1)^{\operatorname{vdim} X_1 + \dim Y} X_1 \times_Y (\partial X_2) \\ &\textit{and} \quad X_1 \times_Y X_2 = (-1)^{(\operatorname{vdim} X_1 - \dim Y)(\operatorname{vdim} X_2 - \dim Y)} X_2 \times_Y X_1. \end{aligned} \tag{5}$$

(b) *For $\boldsymbol{f}_1 : X_1 \to Y_1$, $\boldsymbol{f}_2 : X_2 \to Y_1 \times Y_2$ and $\boldsymbol{f}_3 : X_3 \to Y_2$, we have*

$$(X_1 \times_{Y_1} X_2) \times_{Y_2} X_3 = X_1 \times_{Y_1} (X_2 \times_{Y_2} X_3).$$

(c) *For $\boldsymbol{f}_1 : X_1 \to Y_1 \times Y_2$, $\boldsymbol{f}_2 : X_2 \to Y_1$ and $\boldsymbol{f}_3 : X_3 \to Y_2$, we have*

$$X_1 \times_{Y_1 \times Y_2} (X_2 \times X_3) = (-1)^{\dim Y_2 (\dim Y_1 + \operatorname{vdim} X_2)} (X_1 \times_{Y_1} X_2) \times_{Y_2} X_3.$$

2.5. Good coordinate systems. *Good coordinate systems* are convenient choices of finite coverings of X by Kuranishi neighbourhoods, [**9**, Def. 6.1], [**8**, Lem. A1.11].

Definition 2.11. Let X be a compact Kuranishi space. A *good coordinate system* on X consists of a finite indexing set I, an order $<$ on I, a family $\{(V^i, E^i, \Gamma^i, s^i, \psi^i) : i \in I\}$ of Kuranishi neighbourhoods on X with $X = \bigcup_{i \in I} \operatorname{Im} \psi^i$, and for all $i, j \in I$ with $j < i$ and $\operatorname{Im} \psi^i \cap \operatorname{Im} \psi^j \neq \emptyset$, a quadruple $(V^{ij}, \hat{\phi}^{ij}, \phi^{ij}, h^{ij})$, where V^{ij} is a Γ^j-invariant open neighbourhood of $(\psi^j)^{-1}(\operatorname{Im} \psi^i)$ in V^j, and $(\hat{\phi}^{ij}, \phi^{ij}, h^{ij})$ is a coordinate change from $(V^{ij}, E^j|_{V^{ij}}, \Gamma^j, s^j|_{V^{ij}}, \psi^j|_{V^{ij}})$ to $(V^i, E^i, \Gamma^i, s^i, \psi^i)$. Whenever $i, j, k \in I$ with $k < j < i$ these should satisfy $\hat{\phi}^{ij} \circ \hat{\phi}^{jk} = \hat{\phi}^{ik}$, $\phi^{ij} \circ \phi^{jk} = \phi^{ik}$ and $h^{ij} \circ h^{jk} = h^{ik}$ over $(\phi^{jk})^{-1}(V^{ij}) \cap V^{jk} \cap V^{ik}$.

Then Fukaya and Ono prove [**9**, Lem. 6.3], [**8**, Lem. A1.11]:

Proposition 2.12. *Let X be a compact Kuranishi space and $\{U_\alpha : \alpha \in A\}$ an open cover of X. Then there exists a good coordinate system on X such that for each $i \in I$ we have $\operatorname{Im} \psi^i \subseteq U_\alpha$ for some $\alpha \in A$.*

2.6. Chains and homology. Let Y be a smooth manifold. We now explain the complexes we will use to define the homology of Y. We shall work throughout with *singular homology* defined using *smooth simplicial chains* on Y, following Fukaya et al. [**8**]. Write Δ_k for the *k-simplex*

$$\big\{(x_0,\dots,x_k)\in\mathbb{R}^{k+1}: x_i\geqslant 0,\ x_0+\cdots+x_k=1\big\}. \tag{6}$$

The *singular chain complex* $\big(C_*^{\mathrm{si}}(Y;\mathbb{Q}),\partial\big)$ of Y has $C_k^{\mathrm{si}}(Y;\mathbb{Q})$ the $\mathbb{Q}$-vector space with basis smooth maps $f:\Delta_k\to Y$, and boundary operator $\partial: C_k^{\mathrm{si}}(Y;\mathbb{Q})\to C_{k-1}^{\mathrm{si}}(Y;\mathbb{Q})$ given by

$$\partial:\textstyle\sum_{a\in A}\rho_a\,f_a\longmapsto\sum_{a\in A}\sum_{j=0}^k(-1)^j\rho_a(f_a\circ F_j^k), \tag{7}$$

where for $j=0,\dots,k$ the map $F_j^k:\Delta_{k-1}\to\Delta_k$ is $F_j^k(x_0,\dots,x_{k-1})=(x_0,\dots,x_{j-1},0,x_j,\dots,x_{k-1})$. The *singular homology* $H_*^{\mathrm{si}}(Y;\mathbb{Q})$ of Y is the homology of $\big(C_*^{\mathrm{si}}(Y;\mathbb{Q}),\partial\big)$.

However, following Fukaya et al. [**8**], when we define $A_{N,0}$ algebras and A_∞ algebras below we will not use the full chain complex $\big(C_*^{\mathrm{si}}(Y;\mathbb{Q}),\partial\big)$, but certain *subcomplexes* $(\mathbb{Q}\mathcal{X},\partial)$. When we do this, we will use the following conventions:

- $\mathcal{X}$ is a countable set of smooth maps $f:\Delta_k\to Y$, ranging over different $k=0,1,\dots$, and allowing $k>\dim Y$. We generally refer to elements of $\mathcal{X}$ as f, taking the domain Δ_k of f (that is, the choice of $k=0,1,\dots$) to be implicit.
- $\mathbb{Q}\mathcal{X}$ is the graded $\mathbb{Q}$-vector subspace of $C_*^{\mathrm{si}}(Y;\mathbb{Q})$ with basis $\mathcal{X}$.
- If $f\in\mathcal{X}$ maps $\Delta_k\to Y$, then $f\circ F_j^k\in\mathcal{X}$ for $j=0,\dots,k$. Thus $\mathbb{Q}\mathcal{X}$ is closed under ∂ by (7), and $(\mathbb{Q}\mathcal{X},\partial)$ is a subcomplex of $\big(C_*^{\mathrm{si}}(Y;\mathbb{Q}),\partial\big)$. The inclusion $\mathbb{Q}\mathcal{X}\hookrightarrow C_*^{\mathrm{si}}(Y;\mathbb{Q})$ induces a morphism $H_*(\mathbb{Q}\mathcal{X},\partial)\to H_*^{\mathrm{si}}(Y;\mathbb{Q})$ from the homology of $(\mathbb{Q}\mathcal{X},\partial)$ to the singular homology of Y. We require $\mathcal{X}$ to be chosen so that this morphism is an isomorphism.
- We shall also consider (completed) tensor products $\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda_{\mathrm{nov}}^*$ with a *Novikov ring* $\Lambda_{\mathrm{nov}}^*=\Lambda_{\mathrm{nov}}^0$ or Λ_{nov}. Then $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda_{\mathrm{nov}}^*,\partial)$ is a complex of Λ_{nov}^*-modules.

The reason for using countably generated subcomplexes $(\mathbb{Q}\mathcal{X},\partial)$ is that in the construction of an A_∞ algebra for a Lagrangian submanifold, when we perturb our moduli spaces $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$ to make them transverse, just one perturbation is not enough, we need a different choice of perturbation for each k-tuple $(f_1,\dots,f_k)$ of chains $f_1,\dots,f_k$ in our chain complex $\mathbb{Q}\mathcal{X}$. To keep these choices under control, we cannot work with the full complex $C_*^{\mathrm{si}}(Y;\mathbb{Q})$, but only with countably generated subcomplexes $\mathbb{Q}\mathcal{X}$, which are constructed together with associated perturbations of $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\dots,f_k)$ for $f_1,\dots,f_k\in\mathcal{X}$ using an inductive method. The following proposition will be an important tool in constructing such $\mathcal{X}$.

Proposition 2.13. *Let Y be a compact manifold, possibly with corners. Let $\mathcal{W}$ be a countable set of smooth maps $f : \Delta_k \to Y$, ranging over different $k = 0, 1, \dots$. Then there exists a countable set $\mathcal{X}$ of smooth maps $f : \Delta_k \to Y$, ranging over different $k = 0, 1, \dots,$ with the following properties:*

(i) $\mathcal{W} \subseteq \mathcal{X}$;

(ii) *if $f : \Delta_k \to Y$ lies in $\mathcal{X}$ and $k > 0$ then $f \circ F_j^k : \Delta_{k-1} \to Y$ lies in $\mathcal{X}$ for all $j = 0, \dots, k$; and*

(iii) *part* (ii) *implies that $\mathbb{Q}\mathcal{X}$ is closed under ∂, and a subcomplex of the singular chains $C_*^{\mathrm{si}}(Y; \mathbb{Q})$. We require that the natural projection $H_*(\mathbb{Q}\mathcal{X}, \partial) \to H_*^{\mathrm{si}}(Y; \mathbb{Q})$ should be an isomorphism.*

2.7. Multisections and virtual chains. In many geometric situations, if a moduli space X is singular or does not have the expected dimension, then one can make a small perturbation to get a new moduli space X' which is smooth and of the expected dimension. The Kuranishi structure formalism allows us to make these perturbations in an abstract way. The basic idea is to choose a *good coordinate system*, as in Definition 2.11, and then perturb the sections $s^i : V^i \to E^i$ to smooth $\tilde{s}^i : V^i \to E^i$ which are *transverse*, that is, $\mathrm{d}\tilde{s}^i : T_v V^i \to E^i$ is surjective for each $v \in (\tilde{s}^i)^{-1}(0)$. Then $(\tilde{s}^i)^{-1}(0)$ is a smooth manifold of dimension $\mathrm{vdim}\, X$. The perturbations $\tilde{s}^i, \tilde{s}^j$ must be compatible on the overlaps V^{ij}.

However, it may be impossible to choose $\tilde{s}^i$ both transverse and Γ^i-equivariant. To deal with this, Fukaya and Ono [**9**, §3], [**8**, §A1] introduce *multisections*.

Definition 2.14. Let (V, E, Γ, s, ψ) be a Kuranishi neighbourhood on some space X. For each $n \geqslant 1$, write $S^n E \to V$ for the quotient of the vector bundle $E^n \to V = E \times \cdots \times E \to V$ by the symmetric group S_n. That is, the fibre of the bundle $S^n E$ over $v \in V$ is $(E|_v)^n / S_n$.

Define an *n-multisection* $\boldsymbol{s}$ of the orbibundle $E \to V$ to be a continuous, Γ-equivariant section of the bundle $S^n E \to V$. An n-multisection $\boldsymbol{s}$ is called *liftable* if there exists $\tilde{\boldsymbol{s}} = (s_1, \dots, s_n) : V \to E^n$ with each s_a continuous such that $\boldsymbol{s} = \pi \circ \tilde{\boldsymbol{s}}$, where $\pi : E^n \to S^n E$ is the projection. Note that we do *not* require the s_a for $a = 1, \dots, n$ to be Γ-equivariant. A liftable n-multisection $\boldsymbol{s}$ is called *smooth* if it has a lift $\tilde{\boldsymbol{s}} = (s_1, \dots, s_n)$ with each s_a smooth, and *transverse* if these smooth s_a are transverse, that is, $\mathrm{d}s_a : T_v V \to E$ is surjective for each $v \in s_a^{-1}(0)$. When V has corners, we also require that the restriction of each s_a to each codimension k corner of V should be transverse. This implies that $s_a^{-1}(0)$ is a submanifold of V, of dimension $\dim V - \mathrm{rank}\, E$, with corners.

For $n, m \geqslant 1$, there is an obvious map $E^n \to E^{nm}$ in which each E factor of E^n is repeated m times. This induces a map $S^n E \to S^{nm} E$. Composing with this maps an n-multisection to an nm-multisection.

An n-multisection $\boldsymbol{s}$ and an m-multisection $\boldsymbol{s}'$ are called *equivalent* if the induced nm-multisections coincide. A (smooth, or transverse) *multisection* $\mathfrak{s}$ of $E \to V$ is defined to be an equivalence class of (smooth, or transverse) n-multisections $\boldsymbol{s}$ over all n.

We now sketch the construction of *virtual chains* in Fukaya and Ono [**9**, §3 & §6], [**8**, §A1], without going into detail. Let X be a compact Kuranishi space with a tangent bundle and an orientation, which may have boundary or corners, let Y be an orbifold, and $\boldsymbol{g} : X \to Y$ a strongly smooth map. By Proposition 2.12 we may choose a *good coordinate system* $\boldsymbol{I} = \bigl(I, <, (V^i, \dots, \psi^i) : i \in I\bigr)$ for X, and smooth maps $g^i : V^i \to Y$ representing $\boldsymbol{g}$ for $i \in I$, with $g^i \circ \phi^{ij} \equiv g^j|_{V^{ij}}$ when $j < i$ in I and $\operatorname{Im}\psi^i \cap \operatorname{Im}\psi^j \neq \emptyset$. By induction on $i \in I$ in the order $<$, for each $i \in I$ Fukaya and Ono choose a sequence $(\mathfrak{s}^i_n)^\infty_{n=1}$ of smooth, transverse multisections on $(V^i, E^i, \Gamma^i, s^i, \psi^i)$, such that $\mathfrak{s}^i_n \to s^i$ in the C^0 topology as $n \to \infty$.

When $j < i$ in I and $\operatorname{Im}\psi^i \cap \operatorname{Im}\psi^j \neq \emptyset$, the $(\mathfrak{s}^i_n)^\infty_{n=1}$ and $(\mathfrak{s}^j_n)^\infty_{n=1}$ satisfy compatibility conditions: we have $\hat\phi^{ij} \circ \mathfrak{s}^j_n \equiv \mathfrak{s}^i_n \circ \phi^{ij}$ on V^{ij} for all $n = 1, 2, \dots$. Furthermore, since X has a tangent bundle we have isomorphisms χ^{ij} over V^{ij} as in (2), and Fukaya and Ono use these and $\hat\phi^{ij} \circ \mathfrak{s}^j_n$ to prescribe $\mathfrak{s}^i_n$ on an open neighbourhood of $\phi^{ij}(V^{ij})$ in V^i.

If the multisections $\mathfrak{s}^i_n$ were single-valued sections of E^i, then as they are transverse $(\mathfrak{s}^i_n)^{-1}(0)$ would be a smooth oriented Γ^i-invariant submanifold of V^i of dimension $\operatorname{vdim} X$, so $(\mathfrak{s}^i_n)^{-1}(0)/\Gamma^i$ would be a smooth orbifold. The compatibility conditions over V^{ij} mean that ϕ^{ij} induces a local diffeomorphism of $(\mathfrak{s}^i_n)^{-1}(0)/\Gamma^i$ and $(s^j_n)^{-1}(0)/\Gamma^j$ over V^{ij}/Γ^j. Gluing the $(\mathfrak{s}^i_n)^{-1}(0)/\Gamma^i$ for fixed n and all $i \in I$ together using ϕ^{ij} yields a smooth oriented orbifold $\tilde X_n$. When $n \gg 0$, so that $(\mathfrak{s}^i_n)^{-1}(0)$ is C^0 close to $(s^i)^{-1}(0)$, this $\tilde X_n$ would be both compact and Hausdorff, so we would have perturbed X to a compact, smooth, oriented orbifold $\tilde X_n$ of dimension $k = \operatorname{vdim} X$, which may have boundary or corners.

The smooth maps $g^i : V^i \to Y$ would glue together to give a smooth map $\tilde g_n : \tilde X_n \to Y$. We would then choose a *triangulation* of $\tilde X_n$ by smooth singular simplices $f_a : \Delta_k \to \tilde X_n$ for $a \in A$, a finite indexing set. The *virtual chain* for $(X, \boldsymbol{g})$ would then be $VC(X, \boldsymbol{g}) = \sum_{a\in A} \epsilon_a(\tilde g_n \circ f_a)$ in $C^{\mathrm{si}}_k(Y; \mathbb{Q})$, where ϵ_a is 1 if f_a is orientation-preserving, and -1 if f_a is orientation-reversing. If $\partial X = \emptyset$ then $\partial \tilde X_n = \emptyset$, so $\partial VC(X, \boldsymbol{g}) = 0$. Then $VC(X, \boldsymbol{g})$ is called the *virtual cycle* of $(X, \boldsymbol{g})$, and its homology class $[VC(X, \boldsymbol{g})] \in H^{\mathrm{si}}_k(Y; \mathbb{Q})$ is independent of choices of $\boldsymbol{I}, \mathfrak{s}^i_n, n, \dots$, and is called the *virtual class* of $(X, \boldsymbol{g})$.

Although the multisections $\mathfrak{s}^i_n$ are not in general single-valued sections of E^i, we can still follow the method above, with some adaptations. Represent $\mathfrak{s}^i_n$ by a liftable m-multisection on V^i with lift $(s^i_{n,1}, \dots, s^i_{n,m})$. Then each $(s^i_{n,b})^{-1}(0)$ is an oriented submanifold of V^i, not necessarily

Γ^i-invariant. In place of $(\mathfrak{s}_n^i)^{-1}(0)$, we write $\frac{1}{m}\sum_{b=1}^m (s_{n,b}^i)^{-1}(0)$, considered as a $\mathbb{Q}$-linear combination of oriented submanifolds of V^i, and this is then Γ^i-invariant, and essentially independent of the choice of m-multisection and lift $(s_{n,1}^i,\ldots,s_{n,m}^i)$ representing $\mathfrak{s}_n^i$. Here we do not distinguish sheets of $\frac{1}{m}\sum_{b=1}^m (s_{n,b}^i)^{-1}(0)$ that lie on top of each other locally, but regard them as a single sheet and add up the multiplicities $\frac{1}{m}$. So we regard $\bigl(\frac{1}{m}\sum_{b=1}^m (s_{n,b}^i)^{-1}(0)\bigr)/\Gamma^i$ as a kind of *non-Hausdorff suborbifold of V^i/Γ^i, with multiplicity in $\mathbb{Q}$*.

With this convention, we can glue the $\bigl(\frac{1}{m}\sum_{b=1}^m (s_{n,b}^i)^{-1}(0)\bigr)/\Gamma^i$ for all $i\in I$ using the ϕ^{ij} to get a kind of compact, oriented, non-Hausdorff orbifold $\tilde X_n$ with multiplicity in $\mathbb{Q}$, with a smooth map $\tilde g_n:\tilde X_n\to Y$. Fukaya and Ono then triangulate $\tilde X_n$ into k-simplices $f_a:\Delta_k\to\tilde X_n$, such that on the interior $f_a(\Delta_k^\circ)$ of each simplex $\tilde X_n$ is Hausdorff and the multiplicity is a constant $c_a\in\mathbb{Q}$. The *virtual chain* or *cycle* $VC(X,\boldsymbol{g})$ is then defined to be $\sum_{a\in A}(\epsilon_a c_a)(\tilde g_n\circ f_a)$ in $C_k^{\rm si}(Y;\mathbb{Q})$, as in §2.6.

Perturbation data is the choices for constructing a virtual chain.

Definition 2.15. Let X be a compact Kuranishi space with a tangent bundle and an orientation, Y an orbifold, and $\boldsymbol{g}:X\to Y$ a strongly smooth map. A set of *perturbation data* $\mathfrak{s}_X$ for $(X,\boldsymbol{g})$ consists of a good coordinate system $\boldsymbol{I}=\bigl(I,<,(V^i,\ldots,\psi^i):i\in I\bigr)$ for X, and smooth maps $g^i:V^i\to Y$ representing $\boldsymbol{g}$ for $i\in I$, with $g^i\circ\phi^{ij}\equiv g^j|_{V^{ij}}$ when $j<i$ in I and $\operatorname{Im}\psi^i\cap\operatorname{Im}\psi^j\neq\emptyset$, and smooth, transverse multisections $\mathfrak{s}^i$ on $(V^i,E^i,\Gamma^i,s^i,\psi^i)$ for $i\in I$ which are compatible on overlaps V^{ij} and near $\phi^{ij}(V^{ij})$ as above, and such that each $\mathfrak{s}^i$ is sufficiently close to s^i in C^0 that the construction of virtual chains above works; in particular, gluing the $(\mathfrak{s}^i)^{-1}(0)/\Gamma^i$ for all $i\in I$ together as above should yield a *compact* oriented non-Hausdorff manifold $\tilde X$ with corners.

The last item in a set of perturbation data is a choice of triangulation of $\tilde X$ into k-simplices $f_a:\Delta_k\to\tilde X$ for $a\in A$, where $k=\operatorname{vdim}X$ and A is a finite indexing set, such that on the interior $f_a(\Delta_k^\circ)$ of each simplex $\tilde X$ is Hausdorff and the multiplicity is a constant $c_a\in\mathbb{Q}$. We shall often use $\mathfrak{s}_X$, or similar notation, to denote this collection of data. The *virtual chain* or *cycle* $VC(X,\boldsymbol{g},\mathfrak{s}_X)$ constructed using this data $\mathfrak{s}_X$ is then defined to be $VC(X,\boldsymbol{g},\mathfrak{s}_X)=\sum_{a\in A}(\epsilon_a c_a)(\tilde g\circ f_a)$ in $C_k^{\rm si}(Y;\mathbb{Q})$, where ϵ_a is 1 if f_a is orientation-preserving, and -1 if f_a is orientation-reversing.

Remark 2.16. **(a)** Perturbation data involves not a series $(\mathfrak{s}_n^i)_{n=1}^\infty$ for each $(V^i,\ldots,\psi^i)$, but only a single choice $\mathfrak{s}^i$, which we think of as $\mathfrak{s}_n^i$ for some fixed $n\gg 0$. Because of this, we have to require the $\mathfrak{s}^i$ to be 'sufficiently close to s^i in C^0'. This is rather unsatisfactory, and will cause problems later; the reason why we have to introduce $A_{N,0}$ algebras, rather than going straight to A_∞ algebras, is roughly

speaking that we can make only finitely many choices of $\mathfrak{s}^i$ at once and still have these 'sufficiently close' conditions satisfied.
(b) When we choose perturbation data $\mathfrak{s}_X$ for $(X, \boldsymbol{g})$, we usually need $VC(X, \boldsymbol{g}, \mathfrak{s}_X)$ to lie in some chain complex $\mathbb{Q}\mathcal{X}$, as in §2.6. That is, we need $\tilde{g} \circ f_a : \Delta_k \to Y$ to lie in $\mathcal{X}$ for all $a \in A$. When this happens we will say that 'the simplices of $VC(X, \boldsymbol{g}, \mathfrak{s}_X)$ lie in $\mathcal{X}$'. Actually, we first choose more-or-less arbitrary perturbations $\mathfrak{s}_X$, and then enlarge $\mathcal{X}$ so that it contains the simplices of $VC(X, \boldsymbol{g}, \mathfrak{s}_X)$. We never try to choose $\mathfrak{s}_X$ so that the simplices of $VC(X, \boldsymbol{g}, \mathfrak{s}_X)$ lie in a fixed complex $\mathcal{X}$, as this would probably be impossible.
(c) Given perturbation data $\mathfrak{s}_X$ for $(X, \boldsymbol{g})$, we can restrict it to perturbation data $\mathfrak{s}_X|_{\partial X}$ for $(\partial X, \boldsymbol{g}|_{\partial X})$ in a natural way, and then the virtual chains satisfy $\partial VC(X, \boldsymbol{g}, \mathfrak{s}_X) = VC(\partial X, \boldsymbol{g}|_{\partial X}, \mathfrak{s}_X|_{\partial X})$. Conversely, given perturbation data $\mathfrak{s}_{\partial X}$ for $(\partial X, \boldsymbol{g}|_{\partial X})$, we often want to choose perturbation data $\mathfrak{s}_X$ for $(X, \boldsymbol{g})$ with $\mathfrak{s}_X|_{\partial X} = \mathfrak{s}_{\partial X}$, or at least, we want $\mathfrak{s}_X|_{\partial X}$ and $\mathfrak{s}_{\partial X}$ to be equivalent in some sense that implies that $VC(\partial X, \boldsymbol{g}|_{\partial X}, \mathfrak{s}_X|_{\partial X}) = VC(\partial X, \boldsymbol{g}|_{\partial X}, \mathfrak{s}_{\partial X})$. But there is a problem here, that referred to in (a) above, as the condition $\mathfrak{s}_X|_{\partial X} = \mathfrak{s}_{\partial X}$ may not be compatible with the condition that the $\mathfrak{s}^i$ in $\mathfrak{s}_X$ are 'sufficiently close to s^i in C^0'.

3. Introduction to A_∞ algebras and $A_{N,K}$ algebras

A_∞ algebras were introduced by Stasheff [**20**]. The following treatment is based on Fukaya et al. [**8**], and uses their conventions. Two survey papers by Keller [**12**, **13**] are useful introductions; note that [**13**] uses the conventions of [**8**], as we do, but [**12**] has different conventions on signs and grading. We restrict to A_∞ algebras over $\mathbb{Q}$, but one can also work over any commutative ring R.

3.1. (Weak) A_∞ algebras and morphisms.

Following [**8**, §3.2.1], we define

Definition 3.1. A *weak A_∞ algebra* $(A, \mathfrak{m})$ (over $\mathbb{Q}$) consists of:

(a) A $\mathbb{Z}$-graded $\mathbb{Q}$-vector space $A = \bigoplus_{d \in \mathbb{Z}} A^d$; and

(b) graded $\mathbb{Q}$-multilinear maps $\mathfrak{m}_k : \overbrace{A \times \cdots \times A}^{k \text{ copies}} \to A$ for $k = 0, 1, \ldots,$ of degree $+1$. That is, $\mathfrak{m}_k$ maps $A^{d_1} \times \cdots \times A^{d_k} \to A^{d_1 + \cdots + d_k + 1}$ for all $d_1, \ldots, d_k \in \mathbb{Z}$. When $k = 0$ we take $\mathfrak{m}_0 \in A^1$. Write $\mathfrak{m} = (\mathfrak{m}_k)_{k \geqslant 0}$.

These must satisfy the following condition. Call $a \in A$ *pure* if $a \in A^d \setminus \{0\}$ for some $d \in \mathbb{Z}$, and then define the *degree* of a to be $\deg a = d$. Then we require that for all $k \geqslant 0$ and all pure $a_1, \ldots, a_k$ in A we have

$$\sum_{\substack{i, k_1, k_2 : 1 \leqslant i \leqslant k_1, \\ k_2 \geqslant 0,\ k_1 + k_2 = k+1}} (-1)^{\sum_{l=1}^{i-1} \deg a_l} \mathfrak{m}_{k_1}(a_1, \ldots, a_{i-1}, \mathfrak{m}_{k_2}(a_i, \ldots, a_{i+k_2-1}), a_{i+k_2} \ldots, a_k) = 0. \tag{8}$$

We call $(A, \mathfrak{m})$ an A_∞ *algebra* if it is a weak A_∞ algebra and $\mathfrak{m}_0 = 0$. We sometimes write $A^{\times^k}$ for $\overbrace{A \times \cdots \times A}^{k \text{ copies}}$, to distinguish it from the k^{th} graded subspace A^k of $A = \bigoplus_{d \in \mathbb{Z}} A^d$.

If $(A, \mathfrak{m})$ is an A_∞ algebra, so that $\mathfrak{m}_0 = 0$, then (8) for $k = 1$ becomes $\mathfrak{m}_1 \circ \mathfrak{m}_1(a_1) = 0$. Thus $\mathfrak{m}_1 : A \to A$ is a graded linear map of degree $+1$ with $\mathfrak{m}_1 \circ \mathfrak{m}_1 = 0$, so $(A, \mathfrak{m}_1)$ is a complex, and we can form its cohomology $H^*(A)$ by

$$H^p(A) = \frac{\operatorname{Ker} \mathfrak{m}_1 : A^p \to A^{p+1}}{\operatorname{Im} \mathfrak{m}_1 : A^{p-1} \to A^p}.$$

Then $\mathfrak{m}_k$ for $k > 1$ induce various operations on $H^*(A)$. For example, (8) when $k = 2$ yields $\mathfrak{m}_2(\mathfrak{m}_1(a_1), a_2) + (-1)^{\deg a_1} \mathfrak{m}_2(a_1, \mathfrak{m}_1(a_2)) + \mathfrak{m}_1(\mathfrak{m}_2(a_1, a_2)) = 0$. This implies that the bilinear product $\bullet : H^p(A) \times H^q(A) \to H^{p+q+1}(A)$ given by

$$(a_1 + \operatorname{Im} \mathfrak{m}_1) \bullet (a_2 + \operatorname{Im} \mathfrak{m}_1) = (-1)^{(\deg a_1 + 1) \deg a_2} \mathfrak{m}_2(a_1, a_2) + \operatorname{Im} \mathfrak{m}_1$$

is well-defined. Then (8) when $k = 3$ implies that $\bullet$ is *associative.*

If $(A, \mathfrak{m})$ is only a weak A_∞ algebra, with $\mathfrak{m}_0 \neq 0$, then (8) for $k = 1$ yields

$$\mathfrak{m}_1 \circ \mathfrak{m}_1(a_1) = -\mathfrak{m}_2(\mathfrak{m}_0, a_1) - (-1)^{\deg a_1} \mathfrak{m}_2(a_1, \mathfrak{m}_0).$$

So we may no longer have $\mathfrak{m}_1 \circ \mathfrak{m}_1 = 0$, and we cannot form the cohomology $H^*(A)$. We regard $\mathfrak{m}_0$ as the *obstruction* to $(A, \mathfrak{m}_1)$ being a complex.

Equation (8) can be expressed more naturally using the *bar complex* of $(A, \mathfrak{m})$.

Definition 3.2. Let $(A, \mathfrak{m})$ be a weak A_∞ algebra. The *tensor coalgebra* $T(A)$ of A is $T(A) = \bigoplus_{n \geqslant 0} A^{\otimes^n}$, where we write $A^{\otimes^0} = \mathbb{Q}$. It is graded in the obvious way, so that $T(A)^d = \bigoplus_{d_1 + \cdots + d_n = d} A^{d_1} \otimes \cdots \otimes A^{d_n}$. It has a *coproduct* $\Delta : T(A) \to T(A) \otimes T(A)$ given by

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \textstyle\sum_{k=0}^n (a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_n),$$

taking the $k = 0$ and $k = n$ terms to be $1 \otimes (a_1 \otimes \cdots \otimes a_n)$ and $(a_1 \otimes \cdots \otimes a_n) \otimes 1$ respectively. Define a linear map $\bar{\mathfrak{m}}_k : T(A) \to T(A)$ for $k \geqslant 0$ by

$$\begin{aligned} \bar{\mathfrak{m}}_k(a_1 \otimes \cdots \otimes a_n) = \sum_{l=1}^{n-k+1} & (-1)^{\deg a_1 + \cdots + \deg a_{l-1}} a_1 \otimes \cdots \otimes a_{l-1} \otimes \\ & \mathfrak{m}_k(a_l, \ldots, a_{l+k-1}) \otimes a_{l+k} \otimes \cdots \otimes a_n, \end{aligned}$$

for all $n \geqslant 0$ and pure $a_1, \ldots, a_n$ in A. In the case $k = n = 0$ we set $\bar{\mathfrak{m}}_0(\lambda) = \lambda \mathfrak{m}_0 \in A^1$ for $\lambda \in \mathbb{Q}$. Define $\bar{\mathrm{d}} = \sum_{k=0}^\infty \bar{\mathfrak{m}}_k$. Then $\bar{\mathrm{d}} : T(A) \to T(A)$ is a graded linear map of degree $+1$, and equation (8) is equivalent to $\bar{\mathrm{d}} \circ \bar{\mathrm{d}} = 0$, so that $\big(T(A), \bar{\mathrm{d}}\big)$ is a complex, the *bar complex* of $(A, \mathfrak{m})$.

Note too that $\bar{\mathfrak{m}}_k$ and $\bar{\mathrm{d}}$ are derivations for the coproduct Δ, so that $\bigl(T(A),\Delta,\bar{\mathrm{d}}\bigr)$ is a *differential graded coalgebra.*

Here is the notion of morphism of A_∞ algebras.

Definition 3.3. Let $(A,\mathfrak{m})$ and $(B,\mathfrak{n})$ be A_∞ algebras. An A_∞ *morphism* $\mathfrak{f}:(A,\mathfrak{m})\rightarrow(B,\mathfrak{n})$ is $\mathfrak{f}=(\mathfrak{f}_k)_{k\geqslant 1}$, where $\mathfrak{f}_k:A^{\times^k}\rightarrow B$ for $k=1,2,\ldots$ are graded $\mathbb{Q}$-multilinear maps of degree 0, satisfying

$$\begin{aligned}\sum_{1\leqslant i<j\leqslant k}&(-1)^{\sum_{l=1}^{i-1}\deg a_l}\mathfrak{f}_{k-j+i+1}\bigl(a_1,\ldots,a_{i-1},\\&\qquad\mathfrak{m}_{j-i}(a_i,\ldots,a_{j-1}),a_j,\ldots,a_k\bigr)\\=\sum_{0<k_1<k_2<\cdots<k_l=k}&\mathfrak{n}_l\bigl(\mathfrak{f}_{k_1}(a_1,\ldots,a_{k_1}),\mathfrak{f}_{k_2-k_1}(a_{k_1+1},\ldots,a_{k_2}),\\&\qquad\ldots,\mathfrak{f}_{k_l-k_{l-1}}(a_{k_{l-1}+1},\ldots,a_{k_l})\bigr),\end{aligned}\tag{9}$$

for all $k\geqslant 0$ and pure $a_1,\ldots,a_k$ in A. We can rewrite (9) in terms of the bar complexes of $(A,\mathfrak{m})$ and $(B,\mathfrak{n})$: define $\bar{\mathfrak{f}}:T(A)\rightarrow T(B)$ by

$$\begin{aligned}\bar{\mathfrak{f}}(a_1\otimes\cdots\otimes a_n)=\sum_{0<k_1<\cdots<k_l=n}&\mathfrak{f}_{k_1}(a_1,\ldots,a_{k_1})\otimes\mathfrak{f}_{k_2-k_1}(a_{k_1+1},\ldots,a_{k_2})\otimes\\&\cdots\otimes\mathfrak{f}_{k_l-k_{l-1}}(a_{k_{l-1}+1},\ldots,a_{k_l}),\end{aligned}\tag{10}$$

for all $n>0$ and $a_1,\ldots,a_n$ in A. Then (9) is equivalent to $\bar{\mathrm{d}}_B\circ\bar{\mathfrak{f}}=\bar{\mathfrak{f}}\circ\bar{\mathrm{d}}_A:T(A)\rightarrow T(B)$, that is, $\bar{\mathfrak{f}}$ is a morphism of bar complexes $\bigl(T(A),\bar{\mathrm{d}}_A\bigr)\rightarrow\bigl(T(B),\bar{\mathrm{d}}_B\bigr)$. It also intertwines the coproducts Δ_A,Δ_B on $T(A),T(B)$.

We call an A_∞ morphism $\mathfrak{f}:(A,\mathfrak{m})\rightarrow(B,\mathfrak{n})$ *strict* if $\mathfrak{f}_k=0$ for $k\neq 1$, an A_∞ *isomorphism* if $\mathfrak{f}_1:A\rightarrow B$ is an isomorphism of vector spaces, and a *strict* A_∞ *isomorphism* if it is both strict and an A_∞ isomorphism. When $n=1$, equation (9) becomes $\mathfrak{f}_1\circ\mathfrak{m}_1=\mathfrak{n}_1\circ\mathfrak{f}_1:A\rightarrow B$. Thus $\mathfrak{f}_1$ is a morphism of complexes $(A,\mathfrak{m}_1)\rightarrow(B,\mathfrak{n}_1)$, and induces a morphism of cohomology groups $(\mathfrak{f}_1)_*:H^*(A)\rightarrow H^*(B)$. We call $\mathfrak{f}$ a *weak homotopy equivalence*, or *quasi-isomorphism*, if $(\mathfrak{f}_1)_*$ is an isomorphism.

If $(A,\mathfrak{m}),(B,\mathfrak{n}),(C,\mathfrak{o})$ are A_∞ algebras and $\mathfrak{f}:(A,\mathfrak{m})\rightarrow(B,\mathfrak{n})$, $\mathfrak{g}:(B,\mathfrak{n})\rightarrow(C,\mathfrak{o})$ are A_∞ morphisms, the *composition* $\mathfrak{g}\circ\mathfrak{f}:(A,\mathfrak{m})\rightarrow(C,\mathfrak{o})$ is given by

$$\begin{aligned}(\mathfrak{g}\circ\mathfrak{f})_n(a_1,\ldots,a_n)=\sum_{0<k_1<\cdots<k_l=n}&\mathfrak{g}_l\bigl(\mathfrak{f}_{k_1}(a_1,\ldots,a_{k_1}),\mathfrak{f}_{k_2-k_1}(a_{k_1+1},\ldots,a_{k_2}),\\&\ldots,\mathfrak{f}_{k_l-k_{l-1}}(a_{k_{l-1}+1},\ldots,a_{k_l})\bigr).\end{aligned}\tag{11}$$

On bar complexes this implies that $\overline{(\mathfrak{g}\circ\mathfrak{f})}=\bar{\mathfrak{g}}\circ\bar{\mathfrak{f}}$. Composition is associative.

This definition of A_∞ morphism also makes sense for *weak* A_∞ algebras, allowing $n\geqslant 0$ and $i\leqslant j$ in (9). In the weak case it would look more natural to take $\mathfrak{f}=(\mathfrak{f}_k)_{k\geqslant 0}$, and include $\mathfrak{f}_0$ terms in (9) and (10). However, both (9) and (10) would then become *infinite* sums, for instance, (10) when $n=0$ would be $\mathfrak{f}_1(\mathfrak{m}_0)=\sum_{l\geqslant 0}\mathfrak{n}_l(\mathfrak{f}_0,\ldots,\mathfrak{f}_0)$. So we would need an appropriate notion of convergence of series in A,B. But the definition of weak homotopy equivalence does *not* make sense for weak A_∞ algebras, since $H^*(A), H^*(B)$ are not defined.

3.2. Homotopy between A_∞ morphisms and algebras. Now let $(A,\mathfrak{m}),(B,\mathfrak{n})$ be A_∞ algebras, and $\mathfrak{f},\mathfrak{g}:(A,\mathfrak{m})\to(B,\mathfrak{n})$ be A_∞ morphisms. We will define the notion of *homotopy* $\mathfrak{H}$ from $\mathfrak{f}$ to $\mathfrak{g}$. Our definition is based on Keller [**12**, §3.7]. Fukaya et al. [**8**, §4.2.1–§4.2.2] use a different, more complicated definition, involving 'models of $[0,1]\times B$', but they show in [**8**, Prop. 4.2.40] that the two definitions yield the same notion of whether $\mathfrak{f},\mathfrak{g}$ are homotopic.

Definition 3.4. Let $(A,\mathfrak{m}),(B,\mathfrak{n})$ be A_∞ algebras, and $\mathfrak{f},\mathfrak{g}:(A,\mathfrak{m})\to(B,\mathfrak{n})$ be A_∞ morphisms. A *homotopy* from $\mathfrak{f}$ to $\mathfrak{g}$ is $\mathfrak{H}=(\mathfrak{H}_k)_{k\geqslant1}$, where $\mathfrak{H}_k:A^{\times^k}\to B$ for $k=1,2,\ldots$ are graded $\mathbb{Q}$-multilinear maps of degree -1, satisfying

$$
\begin{aligned}
&\mathfrak{f}_n(a_1,\ldots,a_n)-\mathfrak{g}_n(a_1,\ldots,a_n)= \qquad (12)\\
&\sum_{\substack{0<j_1<j_2<\cdots<j_l<\\ k_1<k_2<\cdots<k_m=n}}\mathfrak{n}_{l+m+1}\bigl(\mathfrak{f}_{j_1}(a_1,\ldots,a_{j_1}),\mathfrak{f}_{j_2-j_1}(a_{j_1+1},\ldots,a_{j_2}),\ldots,\\
&\qquad\mathfrak{f}_{j_l-j_{l-1}}(a_{j_{l-1}+1},\ldots,a_{j_l}),\mathfrak{H}_{k_1-j_l}(a_{j_l+1},\ldots,a_{k_1}),\\
&\qquad\mathfrak{g}_{k_2-k_1}(a_{k_1+1},\ldots,a_{k_2}),\ldots,\mathfrak{g}_{k_m-k_{m-1}}(a_{k_{m-1}+1},\ldots,a_{k_m})\bigr)\\
&+\sum_{0\leqslant i<j\leqslant n}(-1)^{\sum_{l=1}^{i}\deg a_l}\mathfrak{H}_{n-j+i+1}\bigl(a_1,\ldots,a_i,\mathfrak{m}_{j-i}(a_{i+1},\ldots,a_j),a_{j+1},\ldots,a_n\bigr),
\end{aligned}
$$

for all $n\geqslant0$ and pure $a_1,\ldots,a_n$ in A. We can rewrite (12) in terms of the bar complexes of $(A,\mathfrak{m})$ and $(B,\mathfrak{n})$: define $\bar{\mathfrak{H}}:T(A)\to T(B)$ by

$$
\begin{aligned}
\bar{\mathfrak{H}}(a_1\otimes\cdots\otimes a_n)=\sum_{\substack{0<j_1<j_2<\cdots<j_l<\\ k_1<k_2<\cdots<k_m=n}}&\mathfrak{f}_{j_1}(a_1,\ldots,a_{j_1})\otimes\mathfrak{f}_{j_2-j_1}(a_{j_1+1},\ldots,a_{j_2})\otimes\cdots\otimes\\
&\mathfrak{f}_{j_l-j_{l-1}}(a_{j_{l-1}+1},\ldots,a_{j_l})\otimes\mathfrak{H}_{k_1-j_l}(a_{j_l+1},\ldots,a_{k_1})\\
&\otimes\mathfrak{g}_{k_2-k_1}(a_{k_1+1},\ldots,a_{k_2})\otimes\cdots\otimes\mathfrak{g}_{k_m-k_{m-1}}(a_{k_{m-1}+1},\ldots,a_{k_m}),
\end{aligned}
$$

for all $n\geqslant0$ and $a_1,\ldots,a_n$ in A. Then $\bar{\mathfrak{H}}$ satisfies $\Delta_B\circ\bar{\mathfrak{H}}=(\bar{\mathfrak{f}}\otimes\bar{\mathfrak{H}}+\bar{\mathfrak{H}}\otimes\bar{\mathfrak{g}})\circ\Delta_A$, and (12) is equivalent to $\bar{\mathfrak{f}}-\bar{\mathfrak{g}}=\bar{\mathrm{d}}_B\circ\bar{\mathfrak{H}}+\bar{\mathfrak{H}}\circ\bar{\mathrm{d}}_A$, so that $\bar{\mathfrak{f}}$ and $\bar{\mathfrak{g}}$ are homotopic as morphisms of chain complexes in the usual sense.

A_∞ algebras form a 2-*category*, with A_∞ morphisms as 1-morphisms, and homotopies as 2-morphisms. We will sometimes write a homotopy $\mathfrak{H}$ from $\mathfrak{f}$ to $\mathfrak{g}$ as $\mathfrak{H}:\mathfrak{f}\Rightarrow\mathfrak{g}$, using 2-category notation. There are various notions of composition between homotopies and A_∞-morphisms: given $\mathfrak{f},\mathfrak{g},\mathfrak{h}:(A,\mathfrak{m})\to(B,\mathfrak{n})$ and $\mathfrak{H}:\mathfrak{f}\Rightarrow\mathfrak{g}$, $\mathfrak{I}:\mathfrak{g}\Rightarrow\mathfrak{h}$, we can define $\mathfrak{I}\circ\mathfrak{H}:\mathfrak{f}\Rightarrow\mathfrak{h}$. Given $\mathfrak{f},\mathfrak{g}:(A,\mathfrak{m})\to(B,\mathfrak{n})$, $\mathfrak{h}:(B,\mathfrak{n})\to(C,\mathfrak{o})$ and $\mathfrak{H}:\mathfrak{f}\Rightarrow\mathfrak{g}$, we can define $\mathfrak{h}\circ\mathfrak{H}:(\mathfrak{h}\circ\mathfrak{f})\Rightarrow(\mathfrak{h}\circ\mathfrak{g})$. Given $\mathfrak{f}:(A,\mathfrak{m})\to(B,\mathfrak{n})$, $\mathfrak{g},\mathfrak{h}:(B,\mathfrak{n})\to(C,\mathfrak{o})$ and $\mathfrak{I}:\mathfrak{g}\Rightarrow\mathfrak{h}$, we can define $\mathfrak{I}\circ\mathfrak{f}:(\mathfrak{g}\circ\mathfrak{f})\Rightarrow(\mathfrak{h}\circ\mathfrak{f})$. The definitions, as compositions of maps $\mathfrak{f}_k,\mathfrak{g}_k,\mathfrak{h}_k,\mathfrak{H}_k,\mathfrak{I}_k$, are straightforward. They satisfy the usual 2-category associativity properties.

Definition 3.5. Let $(A,\mathfrak{m}),(B,\mathfrak{n})$ be A_∞ algebras, and $\mathfrak{f}:(A,\mathfrak{m})\to(B,\mathfrak{n})$ an A_∞ morphism. A *homotopy inverse* for $\mathfrak{f}$ is an A_∞ morphism $\mathfrak{g}:(B,\mathfrak{n})\to(A,\mathfrak{m})$ such that $\mathfrak{g}\circ\mathfrak{f}:(A,\mathfrak{m})\to(A,\mathfrak{m})$ is homotopic to $\mathrm{id}_A:(A,\mathfrak{m})\to(A,\mathfrak{m})$, and $\mathfrak{f}\circ\mathfrak{g}:(B,\mathfrak{n})\to(B,\mathfrak{n})$ is homotopic to $\mathrm{id}_B:(B,\mathfrak{n})\to(B,\mathfrak{n})$. If $\mathfrak{f}$ has a homotopy inverse, we call $\mathfrak{f}$ a *homotopy equivalence*, and we call $(A,\mathfrak{m}),(B,\mathfrak{n})$ *homotopic.*

The following theorem is proved by Fukaya et al. [**8**, Cor. 4.2.44, Th. 4.2.45(1)]; see also Keller [**12**, §3.7], who cites the thesis of Prouté (Paris, 1984).

Theorem 3.6. *Let $(A,\mathfrak{m}),(B,\mathfrak{n})$ be A_∞ algebras. Then:*

(a) *Homotopy is an equivalence relation on A_∞ morphisms $\mathfrak{f}:(A,\mathfrak{m})\to(B,\mathfrak{n})$.*
(b) *Homotopy is an equivalence relation on A_∞ algebras.*
(c) *An A_∞ morphism $\mathfrak{f}:(A,\mathfrak{m})\to(B,\mathfrak{n})$ is a homotopy equivalence if and only if it is a weak homotopy equivalence.*

In practice, homotopy is a more useful notion of when two A_∞ algebras are 'the same' than either A_∞ isomorphism or strict A_∞ isomorphism. We are interested in properties of A_∞ algebras which are invariant under homotopy. Constructions of A_∞ algebras generally depend on some arbitrary choices (such as the almost complex structure J below), and different choices yield homotopic but not (strictly) isomorphic A_∞ algebras.

3.3. Minimal models, and sums over planar trees. An A_∞ algebra $(B,\mathfrak{n})$ is called *minimal* if $\mathfrak{n}_1=0$, so that $H^*(B)=B$. If $(A,\mathfrak{m})$ is an A_∞ algebra, then one can make $H^*(A)$ into a minimal A_∞ algebra $\bigl(H^*(A),\mathfrak{n}\bigr)$, such that there is an A_∞-morphism $\boldsymbol{\pi}:(A,\mathfrak{m})\to\bigl(H^*(A),\mathfrak{n}\bigr)$ inducing the identity in cohomology. Thus $\bigl(H^*(A),\mathfrak{n}\bigr)$ is homotopic to $(A,\mathfrak{m})$. We call $\bigl(H^*(A),\mathfrak{n}\bigr)$ a *minimal model* or *canonical model* for $(A,\mathfrak{m})$. It is unique up to A_∞ isomorphism. We will explain a proof of this using the method of sums over 'planar rooted trees' due to Kontsevich and Soibelman [**15**, §6.4]; see also Markl [**18**] and Keller [**13**, Th. 2.3].

Definition 3.7. A *planar rooted tree* is a finite, connected, simply-connected graph T in the plane $\mathbb{R}^2$, whose vertices are divided into $k+1$ *external vertices* numbered $0,1,\ldots,k$, and at least one *internal vertices.* Each external vertex must be connected to exactly one edge, and the external vertices should be *cyclically ordered*, in the sense that if we embed T into the unit disc $\{x^2+y^2\leqslant 1\}$ such that $T\cap\{x^2+y^2=1\}$ is vertices $0,1,\ldots,k$, then the external vertices appear in the cyclic order $0,1,\ldots,k$ anticlockwise around the circle.

Here when we say T is a *graph in the plane*, we mean that T is embedded in $\mathbb{R}^2$ up to continuous deformations. Since T is simply-connected, such an embedding class of T is equivalent to prescribing the cyclic order of the edges at each vertex.

We call vertex 0 the *root* of T, and vertices $1, \ldots, k$ the *leaves* of T. Define a unique *orientation* on T such that each edge is oriented in the direction of the minimal path to the root vertex. Then every vertex except the root has exactly one outgoing edge, and the rest incoming edges. We call an edge the *root edge* if it is connected to the root vertex, a *leaf edge* if it is connected to a leaf vertex, and an *internal edge* if it is connected to no distinguished vertices. (See Figure 3.1(a).)

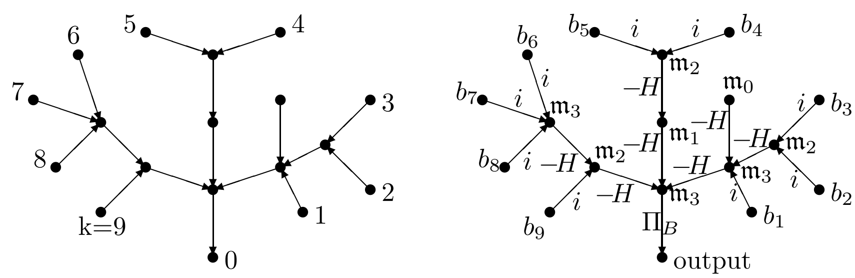


Figure 3.1. (a) A planar rooted tree (b) operators assigned to it

Definition 3.8. Let $(A, \mathfrak{m})$ be an A_∞ algebra. Then $(A, \mathfrak{m}_1)$ is a complex. Let B be a graded vector subspace of A closed under $\mathfrak{m}_1$, such that the inclusion $i : B \hookrightarrow A$ induces an isomorphism $i_* : H^*(B, \mathfrak{m}_1|_B) \to H^*(A, \mathfrak{m}_1)$. We will construct $\mathfrak{n} = (\mathfrak{n}_k)_{k \geqslant 1}$ making $(B, \mathfrak{n})$ into an A_∞ algebra homotopic to $(A, \mathfrak{m})$.

Since i_* is an isomorphism, we can choose a graded vector subspace C of A such that $C \cap \operatorname{Ker} \mathfrak{m}_1 = \{0\}$ and $A = B \oplus C \oplus \mathfrak{m}_1(C)$. Then $\mathfrak{m}_1 : C \to \mathfrak{m}_1(C)$ is invertible, so there is a unique graded linear map $H : A \to A$ of degree -1 with $H(b) = H(c) = 0$ and $H \circ \mathfrak{m}_1(c) = c$ for all $b \in B$ and $c \in C$. Let $\Pi_B : A \to B$ be the projection, with kernel $C \oplus \mathfrak{m}_1(C)$. Then $\mathrm{id}_A - \Pi_B = \mathfrak{m}_1 \circ H + H \circ \mathfrak{m}_1$ on A.

For each planar rooted tree T with k leaves, define a graded multilinear operator $\mathfrak{n}_{k,T} : B^{\times^k} \to B$ of degree $+1$, as follows. To define $\mathfrak{n}_{k,T}(b_1, \ldots, b_k)$, assign objects and operators to the vertices and edges of T:

- assign $b_1, \ldots, b_k$ to the leaf vertices $1, \ldots, k$ respectively.
- for each internal vertex with 1 outgoing edge and n incoming edges, assign $\mathfrak{m}_n$.
- assign i to each leaf edge.
- assign Π_B to the root edge.

- assign $-H$ to each internal edge.

This is illustrated in Figure 3.1(b). Then we define $\mathfrak{n}_{k,T}(b_1,\ldots,b_k)$ to be the composition of all these objects and morphisms, where we follow the orientations of the edges, and at each interior vertex with 1 outgoing edge and n incoming edges, we apply $\mathfrak{m}_n$ to the n inputs from the n incoming edges in the order counting anticlockwise from the outgoing edge. In the example of Figure 3.1, this yields

$$\begin{aligned}\mathfrak{n}_{9,T}(b_1,\ldots,b_9) = \Pi\circ\mathfrak{m}_3\bigl(&-H\circ\mathfrak{m}_3(i(b_1),-H\circ\mathfrak{m}_2(i(b_2),i(b_3)),-H(\mathfrak{m}_0)),\\ &-H\circ\mathfrak{m}_1(-H\circ\mathfrak{m}_2(i(b_4),i(b_5))),\\ &-H\circ\mathfrak{m}_2(-H\circ\mathfrak{m}_3(i(b_6),i(b_7),i(b_8)),i(b_9))\bigr).\end{aligned}$$

This includes an $\mathfrak{m}_0$ term, and so is zero in the A_∞ algebra case.

Define $\mathfrak{n}_1 = \mathfrak{m}_1|_B$, and for $k \geqslant 2$ define $\mathfrak{n}_k = \sum_T \mathfrak{n}_{k,T}$, where the sum is over all planar rooted trees T with k leaves, such that *every internal vertex has at least three edges*. (This excludes Figure 3.1. For filtered A_∞ algebras we will also allow internal vertices with one or two edges.) This condition implies that T has at most $2k$ vertices and $2k-1$ edges, so there are only finite many such trees T, and $\mathfrak{n}_k = \sum_T \mathfrak{n}_{k,T}$ is a finite sum.

In a similar way, for each planar rooted tree T with k leaves, define a graded multilinear operator $\mathfrak{i}_{k,T} : B^{\times^k} \to A$ of degree 0, as follows. Assign objects and operators to the vertices and edges of T:

- assign $b_1,\ldots,b_k$ to the leaf vertices $1,\ldots,k$ respectively.
- for each internal vertex with 1 outgoing edge and n incoming edges, assign $\mathfrak{m}_n$.
- assign i to each leaf edge.
- assign $-H$ to the root edge and to each internal edge.

Define $\mathfrak{i}_{k,T}(b_1,\ldots,b_k)$ to be the composition of all these objects and morphisms. Define $\mathfrak{i}_1 : B \to A$ by $\mathfrak{i}_1 = i$, and for $k \geqslant 2$ define $\mathfrak{i}_k = \sum_T \mathfrak{i}_{k,T}$, where the sum is over all rooted planar trees T with k leaves, such that every internal vertex has at least three edges. Then Markl [**18**] proves:

Theorem 3.9. *In Definition* 3.8, $(B,\mathfrak{n})$ *is an* A_∞ *algebra, and* $\mathfrak{i} : (B,\mathfrak{n}) \to (A,\mathfrak{m})$ *is an* A_∞ *morphism, and a homotopy equivalence. If we choose* $B \cong H^*(A)$ *to be a subspace representing* $H^*(A)$, *so that* $\mathfrak{n}_1 = \mathfrak{m}_1|_B = 0$, *then* $(B,\mathfrak{n})$ *is a minimal model for* $(A,\mathfrak{m})$.

Markl [**18**] also gives much more complicated explicit formulae for a homotopy inverse $\mathfrak{j} : (A,\mathfrak{m}) \to (B,\mathfrak{n})$ for $\mathfrak{i}$ and a homotopy $\mathfrak{H}$ from $\mathfrak{i}\circ\mathfrak{j}$ to id_A. Later we will need a special case of this construction.

Definition 3.10. Let $(A,\mathfrak{m})$ and $(D,\mathfrak{o})$ be A_∞ algebras, and $\mathfrak{p} : (A,\mathfrak{m}) \to (D,\mathfrak{o})$ a strict, surjective A_∞ morphism which is a weak homotopy equivalence. That is, $\mathfrak{p}_k = 0$ for $k \neq 1$, and $\mathfrak{p}_1 : A \to D$ is surjective and induces an isomorphism $(\mathfrak{p}_1)_* : H^*(A,\mathfrak{m}_1) \to H^*(D,\mathfrak{o}_1)$. In Definition 3.8, choose the subspaces B,C of A such that $C \oplus \mathfrak{m}_1(C) = \operatorname{Ker}\mathfrak{p}_1$, and $\mathfrak{p}_1|B : B \to D$ is an isomorphism. This is possible as $\mathfrak{p}_1$ is surjective and $(\mathfrak{p}_1)_*$ is an isomorphism.

As $\mathfrak{p}$ is a strict A_∞ morphism, we have $\mathfrak{p}_1 \circ \mathfrak{m}_j = \mathfrak{o}_j \circ (\mathfrak{p}_1 \times \cdots \times \mathfrak{p}_1)$ for all $j = 1,2,\ldots$. Since $\operatorname{Ker}\Pi_B = \operatorname{Ker}\mathfrak{p}_1$ and $\operatorname{Im} H \subseteq \operatorname{Ker}\mathfrak{p}_1$, this implies that $\Pi_B \circ \mathfrak{m}_j(a_1,\ldots,a_{i-1},-H(a_i),a_{i+1},\ldots,a_j) = 0$ for all $a_1,\ldots,a_j \in A$ and $i = 1,\ldots,j$. Applying this to the root vertex of T, we see that $\mathfrak{n}_{k,T} = 0$ in Definition 3.8 whenever T has an internal edge. Thus, the only nonzero $\mathfrak{n}_{k,T}$ is the unique T with one internal vertex and k leaves, and we have $\mathfrak{n}_k = \Pi_B \circ \mathfrak{m}_k \circ (i \times \cdots \times i)$ for all $k = 1,2,\ldots$. Comparing this with $\mathfrak{p}_1 \circ \mathfrak{m}_k = \mathfrak{o}_k \circ (\mathfrak{p}_1 \times \cdots \times \mathfrak{p}_1)$ and noting that $\mathfrak{p}_1|B : B \to D$ is an isomorphism, we see that $\mathfrak{p}_1|B : B \to D$ identifies $\mathfrak{m}_k$ and $\mathfrak{o}_k$ for $k = 1,2,\ldots$. Hence, $\mathfrak{p}_1|_B$ induces a *strict* A_∞ *isomorphism* $(B,\mathfrak{n}) \to (D,\mathfrak{o})$.

Now define a graded multilinear operator $\mathfrak{q}_k : D^{\times^k} \to A$ of degree 0 by $\mathfrak{q}_k = \mathfrak{i}_k \circ \big((\mathfrak{p}_1|B)^{-1} \times \cdots \times (\mathfrak{p}_1|B)^{-1}\big)$, and write $\mathfrak{q} = (\mathfrak{q}_k)_{k\geqslant 1}$. Then Theorem 3.9 implies that $\mathfrak{q} : (D,\mathfrak{o}) \to (A,\mathfrak{m})$ is an A_∞ *morphism*, and a *homotopy equivalence*. It is easy to check that $\mathfrak{p}\circ\mathfrak{q} : (D,\mathfrak{o}) \to (D,\mathfrak{o})$ is the identity on $(D,\mathfrak{o})$, so $\mathfrak{q}$ is a *homotopy inverse* for $\mathfrak{p} : (A,\mathfrak{m}) \to (D,\mathfrak{o})$. We have proved:

Corollary 3.11. *Let $\mathfrak{p} : (A,\mathfrak{m}) \to (D,\mathfrak{o})$ be a strict, surjective A_∞ morphism of A_∞ algebras which is a weak homotopy equivalence. Then we can construct an explicit homotopy inverse $\mathfrak{q} : (D,\mathfrak{o}) \to (A,\mathfrak{m})$ for $\mathfrak{p}$ using sums over planar trees.*

3.4. Novikov rings, and modules over them. To define Lagrangian Floer cohomology, we have to consider sums involving infinitely many terms, coming from J-holomorphic discs of larger and larger area. To ensure these sums converge, we work over a ring of formal power series known as a *Novikov ring*, as in Fukaya et al. [**8**, Conv. 4–Conv. 6, §1.7]. We consider two kinds, *general Novikov rings* $\Lambda_{\rm nov},\Lambda^0_{\rm nov}$ and *Calabi–Yau Novikov rings* $\Lambda_{\rm CY},\Lambda^0_{\rm CY}$, to be used in §11, §13, and §12–§13, respectively.

The reason for having two kinds is this. In $\Lambda_{\rm nov},\Lambda^0_{\rm nov}$, terms $T^\lambda e^\mu$ keep track of J-holomorphic discs in M with boundary in L, area λ, and Maslov index 2μ. However, if M is *Calabi–Yau* and L is *graded* then all J-holomorphic curves in M with boundary in L have Maslov index 0, so the e^μ are unnecessary, and we can use the smaller rings $\Lambda_{\rm CY},\Lambda^0_{\rm CY}$. We restrict to Novikov rings over $\mathbb{Q}$.

Definition 3.12. Let T and e be formal variables, graded of degree 0 and 2, respectively. Define four *universal Novikov rings* (over $\mathbb{Q}$) by

$$\Lambda_{\rm nov}=\bigl\{\textstyle\sum_{i=0}^{\infty}a_iT^{\lambda_i}e^{\mu_i}:a_i\in\mathbb{Q},\ \lambda_i\in\mathbb{R},\ \mu_i\in\mathbb{Z},\ \lim_{i\to\infty}\lambda_i=\infty\bigr\},\tag{13}$$

$$\Lambda_{\rm nov}^0=\bigl\{\textstyle\sum_{i=0}^{\infty}a_iT^{\lambda_i}e^{\mu_i}:a_i\in\mathbb{Q},\lambda_i\in[0,\infty),\mu_i\in\mathbb{Z},\lim_{i\to\infty}\lambda_i=\infty\bigr\},\tag{14}$$

$$\Lambda_{\rm CY}=\bigl\{\textstyle\sum_{i=0}^{\infty}a_iT^{\lambda_i}:a_i\in\mathbb{Q},\ \lambda_i\in\mathbb{R},\ \lim_{i\to\infty}\lambda_i=\infty\bigr\},\tag{15}$$

$$\Lambda_{\rm CY}^0=\bigl\{\textstyle\sum_{i=0}^{\infty}a_iT^{\lambda_i}:a_i\in\mathbb{Q},\ \lambda_i\in[0,\infty),\ \lim_{i\to\infty}\lambda_i=\infty\bigr\}.\tag{16}$$

Then $\Lambda_{\rm nov}^0\subset\Lambda_{\rm nov}$ and $\Lambda_{\rm CY}^0\subset\Lambda_{\rm CY}$ are $\mathbb{Q}$-vector spaces. For brevity we shall write $\Lambda_{\rm nov}^*$ to mean either $\Lambda_{\rm nov}^0$ or $\Lambda_{\rm nov}$, and $\Lambda_{\rm CY}^*$ to mean either $\Lambda_{\rm CY}^0$ or $\Lambda_{\rm CY}$. Define multiplications '$\cdot$' by $\bigl(\sum_{i=0}^{\infty}a_iT^{\lambda_i}e^{\mu_i}\bigr)\cdot\bigl(\sum_{j=0}^{\infty}b_jT^{\nu_j}e^{\xi_j}\bigr)=\sum_{i,j=0}^{\infty}a_ib_jT^{\lambda_i+\nu_j}e^{\mu_i+\xi_j}$ on $\Lambda_{\rm nov}^*$, and similarly for $\Lambda_{\rm CY}^*$. Here since $\lambda_i,\nu_j\to\infty$ as $i,j\to\infty$, the sum over i,j can be rewritten as a sum over $k=0,1,\ldots$ such that $\lambda_{i_k}+\nu_{j_k}\to\infty$ as $k\to\infty$, and so it lies in $\Lambda_{\rm nov}^*$. With these multiplications, $\Lambda_{\rm nov}^*,\Lambda_{\rm CY}^*$ are *commutative* $\mathbb{Q}$-*algebras* with identity $1=1T^0e^0$ or $1T^0$.

The condition that $\lim_{i\to\infty}\lambda_i=\infty$ in (13)–(16) is equivalent to saying that for all $C\geqslant 0$, there are only finitely many (λ_i,μ_i) or λ_i in the sums with $\lambda_i\leqslant C$. We will often write similar conditions this way. Define *filtrations* of $\Lambda_{\rm nov}^*,\Lambda_{\rm CY}^*$ by

$$F^\lambda\Lambda_{\rm nov}^*=\bigl\{\textstyle\sum_{i=0}^{\infty}a_iT^{\lambda_i}e^{\mu_i}\in\Lambda_{\rm nov}^*:\lambda_i\geqslant\lambda\text{ for all }i=0,1,\ldots\bigr\},$$
$$F^\lambda\Lambda_{\rm CY}^*=\bigl\{\textstyle\sum_{i=0}^{\infty}a_iT^{\lambda_i}\in\Lambda_{\rm CY}^*:\lambda_i\geqslant\lambda\text{ for all }i=0,1,\ldots\bigr\},$$

for $\lambda\in\mathbb{R}$. Then $F^\lambda\Lambda_{\rm nov}^*\subseteq F^\nu\Lambda_{\rm nov}^*$ if $\lambda\geqslant\nu$, and $(F^\lambda\Lambda_{\rm nov}^*)\cdot(F^\nu\Lambda_{\rm nov}^*)=F^{\lambda+\nu}\Lambda_{\rm nov}^*$, and $\Lambda_{\rm nov}^0=F^0\Lambda_{\rm nov}$, and $F^\lambda\Lambda_{\rm nov}=T^\lambda\Lambda_{\rm nov}^0$.

These filtrations induce *topologies* on $\Lambda_{\rm nov}^*,\Lambda_{\rm CY}^*$, and notions of *convergence* for sequences and series, which have nothing to do with the topology on $\mathbb{Q}$ or convergence in $\mathbb{Q}$. An infinite sum $\sum_{k=0}^{\infty}\alpha_k$ in $\Lambda_{\rm nov}^*$ converges in $\Lambda_{\rm nov}^*$ if and only if for all $\lambda\in\mathbb{R}$ we have $\alpha_k\in F^\lambda\Lambda_{\rm nov}^*$ for all except finitely many $k=0,1,2,\ldots$.

As T,e are graded of degrees 0,2, we can regard $\Lambda_{\rm nov},\Lambda_{\rm nov}^0$ as *graded* rings. Write $\Lambda_{\rm nov}^{(k)},\Lambda_{\rm nov}^{0\,(k)}$ for the degree k parts of $\Lambda_{\rm nov},\Lambda_{\rm nov}^0$, for $k\in\mathbb{Z}$. Then

$$\Lambda_{\rm nov}^{(2k)}=\bigl\{\textstyle\sum_{i=0}^{\infty}a_iT^{\lambda_i}e^k:a_i\in\mathbb{Q},\ \lambda_i\in\mathbb{R},\ \lim_{i\to\infty}\lambda_i=\infty\bigr\},\quad\Lambda_{\rm nov}^{(2k+1)}=0,$$

for $k\in\mathbb{Z}$. Note that $\nu\in\Lambda_{\rm nov}$ can have nonzero components $\nu^{(2k)}\in\Lambda_{\rm nov}^{(2k)}$ for infinitely many $k\in\mathbb{Z}$, but $\nu=\sum_{k\in\mathbb{Z}}\nu^{(2k)}$ holds as a convergent sum in $\Lambda_{\rm nov}$. Identifying $e^0=1$ gives $\Lambda_{\rm CY}=\Lambda_{\rm nov}^{(0)}$ and $\Lambda_{\rm CY}^0=\Lambda_{\rm nov}^{0\,(0)}$.

We can also consider *modules* over $\Lambda_{\rm nov}^*$ and $\Lambda_{\rm CY}^*$. Let V be a graded vector space over $\mathbb{Q}$. Then $V\otimes_{\mathbb{Q}}\Lambda_{\rm nov}^*$ is a $\Lambda_{\rm nov}^*$-module, which is *graded* with grading $(V\otimes\Lambda_{\rm nov}^*)^l=\bigoplus_{j+k=l}V^j\otimes\Lambda_{\rm nov}^{*\,(k)}$, and *filtered* with filtration $F^\lambda(V\otimes\Lambda_{\rm nov}^*)=V\otimes F^\lambda\Lambda_{\rm nov}^*$ for $\lambda\in\mathbb{R}$. If V is not finite-dimensional then $V\otimes_{\mathbb{Q}}\Lambda_{\rm nov}^*$ is not complete, so we pass to the *completion* with respect to the filtration $F^\lambda(V\otimes\Lambda^*_{\rm nov})$, which we write as $V\hat\otimes_{\mathbb Q}\Lambda^*_{\rm nov}$. Similarly, we will work with $\Lambda^*_{\rm CY}$-modules $V\otimes_{\mathbb Q}\Lambda^*_{\rm CY}$ and their completions $V\hat\otimes_{\mathbb Q}\Lambda^*_{\rm CY}$.

3.5. Gapped filtered A_∞ algebras. Next we define *gapped filtered A_∞ algebras*, following Fukaya et al. [**8**, §3.2.2], and extend the material of §3.2–§3.3 to them. The rest of the section, §3.5–§3.7, can be done either over $\Lambda^0_{\rm nov}$ or $\Lambda^0_{\rm CY}$. We shall work over $\Lambda^0_{\rm nov}$, as it is more general; the changes for the $\Lambda^0_{\rm CY}$ case are obvious. For instance, in Definition 3.13(i) for $\Lambda^0_{\rm CY}$ we would take $\mathcal G\subset[0,\infty)$ closed under addition with $0\in\mathcal G$ and $\mathcal G\cap[0,C]$ finite for $C\geqslant 0$, and write $\mathfrak m_k=\sum_{\lambda\in\mathcal G}T^\lambda\mathfrak m_k^\lambda$.

Definition 3.13. A *gapped filtered A_∞ algebra* $(A\hat\otimes\Lambda^0_{\rm nov},\mathfrak m)$ is:

(a) A $\mathbb Z$-graded $\mathbb Q$-vector space $A=\bigoplus_{d\in\mathbb Z}A^d$, so that $A\hat\otimes\Lambda^0_{\rm nov}$ is a graded filtered $\Lambda^0_{\rm nov}$-module.

(b) Graded $\Lambda^0_{\rm nov}$-multilinear maps $\mathfrak m_k:(A\hat\otimes\Lambda^0_{\rm nov})^{\times^k}\to A\hat\otimes\Lambda^0_{\rm nov}$ for $k=0,1,2,\ldots$, of degree $+1$. Write $\mathfrak m=(\mathfrak m_k)_{k\geqslant 0}$.

These must satisfy the following conditions:

(i) there exists a subset $\mathcal G\subset[0,\infty)\times\mathbb Z$, closed under addition, such that $\mathcal G\cap(\{0\}\times\mathbb Z)=\{(0,0)\}$ and $\mathcal G\cap([0,C]\times\mathbb Z)$ is finite for any $C\geqslant 0$, and $\mathfrak m_k$ for $k\geqslant 0$ may be written $\mathfrak m_k=\sum_{(\lambda,\mu)\in\mathcal G}T^\lambda e^\mu\mathfrak m_k^{\lambda,\mu}$, for unique $\mathbb Q$-multilinear maps $\mathfrak m_k^{\lambda,\mu}:A^{\times^k}\to A$ graded of degree $1-2\mu$. When $k=0$, we take $\mathfrak m_0\in(A\hat\otimes\Lambda^0_{\rm nov})^{(1)}$ and $\mathfrak m_0^{\lambda,\mu}\in A^{1-2\mu}$;

(ii) $\mathfrak m_0^{0,0}=0$, in the notation of (i); and

(iii) call $a\in A\hat\otimes\Lambda^0_{\rm nov}$ *pure* if $a\in(A\hat\otimes\Lambda^0_{\rm nov})^{(d)}\setminus\{0\}$ for some $d\in\mathbb Z$, and then define the *degree* of a to be $\deg a=d$. Then we require that (8) holds for all $k\geqslant 0$ and all pure $a_1,\ldots,a_k$ in $A\hat\otimes\Lambda^0_{\rm nov}$.

There is a unique smallest choice of subset $\mathcal G$ satisfying (i). Part (iii) may be rewritten in terms of the $\mathfrak m_k^{\lambda,\mu}$ as follows: for all $k\geqslant 0$, all $(\lambda,\mu)\in\mathcal G$ and all pure $a_1,\ldots,a_k$ in A, we have

$$\sum_{\substack{i,k_1,k_2,\lambda_1,\lambda_2,\mu_1,\mu_2:\,1\leqslant i\leqslant k_1,k_2\geqslant 0,\\ k_1+k_2=k+1,\;\lambda_1+\lambda_2=\lambda,\;\mu_1+\mu_2=\mu}}(-1)^{\sum_{l=1}^{i-1}\deg a_l}\mathfrak m_{k_1}^{\lambda_1,\mu_1}(a_1,\ldots,a_{i-1},\mathfrak m_{k_2}^{\lambda_2,\mu_2}(a_i,\ldots,a_{i+k_2-1}),a_{i+k_2}\ldots,a_k)=0.\tag{17}$$

Note that a gapped filtered A_∞ algebra $(A\hat\otimes\Lambda^0_{\rm nov},\mathfrak m)$ is a *weak A_∞ algebra* in the sense of Definition 3.1, with extra structure. Also, if $(\lambda,\mu)=(0,0)$ then as $\mathcal G\cap(\{0\}\times\mathbb Z)=\{(0,0)\}$, equation (17) reduces to

$$\sum_{\substack{i,k_1,k_2:1\leqslant i\leqslant k_1,\\ k_2\geqslant 0,\;k_1+k_2=k+1}}(-1)^{\sum_{l=1}^{i-1}\deg a_l}\mathfrak m_{k_1}^{0,0}(a_1,\ldots,a_{i-1},\mathfrak m_{k_2}^{0,0}(a_i,\ldots,a_{i+k_2-1}),a_{i+k_2}\ldots,a_k)=0,$$

for all $k\geqslant 0$ and all pure $a_1,\ldots,a_k$ in A. Thus, if $(A\hat\otimes\Lambda^0_{\rm nov},\mathfrak m)$ is a gapped filtered A_∞ algebra, then $(A,\mathfrak m^{0,0})$ is an A_∞ *algebra*, where $\mathfrak m^{0,0}=(\mathfrak m_k^{0,0})_{k\geqslant 0}$. In particular, $(A,\mathfrak m_1^{0,0})$ is a complex, and we can form its cohomology $H^*(A,\mathfrak{m}_1^{0,0})$. Generalizing §3.3, we call a gapped filtered A_∞ algebra $(A\hat{\otimes}\Lambda_{\rm nov}^0,\mathfrak{m})$ *minimal* if $\mathfrak{m}_1^{0,0}=0$.

A gapped filtered A_∞ algebra $(A\hat{\otimes}\Lambda_{\rm nov}^0,\mathfrak{m})$ is called *strict* if $\mathfrak{m}_0=0$. Then (iii) implies that $\mathfrak{m}_1\circ\mathfrak{m}_1=0$, so $(A\hat{\otimes}\Lambda_{\rm nov}^0,\mathfrak{m}_1)$ is a complex of $\Lambda_{\rm nov}^0$-modules, and we can form its *cohomology* $H^*(A\hat{\otimes}\Lambda_{\rm nov}^0,\mathfrak{m}_1)$, which is a graded filtered $\Lambda_{\rm nov}^0$-module. Also, $(A\hat{\otimes}\Lambda_{\rm nov},\mathfrak{m}_1)$ is a complex of $\Lambda_{\rm nov}$-modules, whose cohomology $H^*(A\hat{\otimes}\Lambda_{\rm nov},\mathfrak{m}_1)$ is a graded filtered $\Lambda_{\rm nov}$-module. These are the kinds of cohomology we will use to define Lagrangian Floer cohomology.

The term *gapped* [**8**, Def. 3.2.26] refers to condition (i) above. This structure arises naturally in J-holomorphic curve problems, and is useful for inductive arguments. We generalize Definitions 3.3–3.5 to the gapped filtered case.

Definition 3.14. Let $(A\hat{\otimes}\Lambda_{\rm nov}^0,\mathfrak{m})$ and $(B\hat{\otimes}\Lambda_{\rm nov}^0,\mathfrak{n})$ be gapped filtered A_∞ algebras. A *gapped filtered A_∞ morphism* $\mathfrak{f}:(A\hat{\otimes}\Lambda_{\rm nov}^0,\mathfrak{m})\to(B\hat{\otimes}\Lambda_{\rm nov}^0,\mathfrak{n})$ is $\mathfrak{f}=(\mathfrak{f}_k)_{k\geqslant 0}$, where $\mathfrak{f}_k:(A\hat{\otimes}\Lambda_{\rm nov}^0)^{\times^k}\to B\hat{\otimes}\Lambda_{\rm nov}^0$ for $k=0,1,\dots$ are graded $\Lambda_{\rm nov}^0$-multilinear maps of degree 0, satisfying

- (i) there exists a subset $\mathcal{G}'\subset[0,\infty)\times\mathbb{Z}$, closed under addition, such that $\mathcal{G}'\cap(\{0\}\times\mathbb{Z})=\{(0,0)\}$ and $\mathcal{G}'\cap([0,C]\times\mathbb{Z})$ is finite for any $C\geqslant 0$, and the maps $\mathfrak{f}_k$ for $k\geqslant 0$ may be written $\mathfrak{f}_k=\sum_{(\lambda,\mu)\in\mathcal{G}'}T^\lambda e^\mu\mathfrak{f}_k^{\lambda,\mu}$, for unique $\mathbb{Q}$-multilinear maps $\mathfrak{f}_k^{\lambda,\mu}:A^{\times^k}\to B$ graded of degree -2μ. When $k=0$, we take $\mathfrak{f}_0\in(B\hat{\otimes}\Lambda_{\rm nov}^0)^{(0)}$ and $\mathfrak{f}_0^{\lambda,\mu}\in B^{-2\mu}$;
- (ii) $\mathfrak{f}_0^{0,0}=0$, in the notation of (i); and
- (iii) for all $k\geqslant 0$ and pure $a_1,\dots,a_k$ in $A\hat{\otimes}\Lambda_{\rm nov}^0$, we have

$$\begin{aligned}&\sum_{1\leqslant i\leqslant j\leqslant k}(-1)^{\sum_{l=1}^{i-1}\deg a_l}\mathfrak{f}_{k-j+i+1}\bigl(a_1,\dots,a_{i-1},\\&\qquad\qquad\mathfrak{m}_{j-i}(a_i,\dots,a_{j-1}),a_j,\dots,a_k\bigr)\\&=\sum_{0\leqslant k_1\leqslant k_2\leqslant\cdots\leqslant k_l=k}\mathfrak{n}_l\bigl(\mathfrak{f}_{k_1}(a_1,\dots,a_{k_1}),\mathfrak{f}_{k_2-k_1}(a_{k_1+1},\dots,a_{k_2}),\\&\qquad\qquad\dots,\mathfrak{f}_{k_l-k_{l-1}}(a_{k_{l-1}+1},\dots,a_{k_l})\bigr).\end{aligned}\tag{18}$$

As for (17), equation (18) may be rewritten using the $\mathfrak{f}_k^{\lambda,\mu},\mathfrak{m}_k^{\lambda,\mu},\mathfrak{n}_k^{\lambda,\mu}$ as

$$\begin{aligned}&\sum_{\substack{1\leqslant i\leqslant j\leqslant k\\ \lambda_1+\lambda_2=\lambda,\ \mu_1+\mu_2=\mu}}(-1)^{\sum_{l=1}^{i-1}\deg a_l}\mathfrak{f}_{k-j+i+1}^{\lambda_1,\mu_1}\bigl(a_1,\dots,a_{i-1},\\&\qquad\qquad\mathfrak{m}_{j-i}^{\lambda_2,\mu_2}(a_i,\dots,a_{j-1}),a_j,\dots,a_k\bigr)\\&=\sum_{\substack{0\leqslant k_1\leqslant k_2\leqslant\cdots\leqslant k_l=k\\ \lambda_0+\cdots+\lambda_l=\lambda,\ \mu_0+\cdots+\mu_l=\mu}}\mathfrak{n}_l^{\lambda_0,\mu_0}\bigl(\mathfrak{f}_{k_1}^{\lambda_1,\mu_1}(a_1,\dots,a_{k_1}),\mathfrak{f}_{k_2-k_1}^{\lambda_2,\mu_2}(a_{k_1+1},\dots,a_{k_2}),\\&\qquad\qquad\dots,\mathfrak{f}_{k_l-k_{l-1}}^{\lambda_l,\mu_l}(a_{k_{l-1}+1},\dots,a_{k_l})\bigr),\end{aligned}\tag{19}$$

for all $k\geqslant 0$, pure $a_1,\dots,a_k$ in A, $\lambda\geqslant 0$ and $\mu\in\mathbb{Z}$.

Note the difference between (9) and (18): as we now allow $\mathfrak{f}_0$ to be nonzero, the second line of (18) is a sum over $0\leqslant k_1\leqslant k_2\leqslant\cdots\leqslant k_l=n$ rather than over $0<k_1<k_2<\cdots<k_l=n$. Thus, the second

line of (18) is an *infinite* sum, as for instance it includes the terms $\mathfrak{n}_l(\mathfrak{f}_0,\dots,\mathfrak{f}_0,\mathfrak{f}_n(a_1,\dots,a_n))$ for all $l \geqslant 1$. We claim that the second line of (18) is a *convergent sum* in the complete filtered Λ^0_{nov}-module $B\hat{\otimes}\Lambda^0_{\mathrm{nov}}$, in the sense of §3.4. This is more-or-less equivalent to (19) being finite sums for all λ,μ.

To see this, let $\lambda_0 = \min_{(0,0)\neq(\lambda,n)\in\mathcal{G}'}\lambda$, which is well-defined and positive by (i), unless $\mathcal{G}'=\{(0,0)\}$, in which case $\mathfrak{f}_0=0$ and the result is trivial. Then $\mathfrak{f}_0\in F^{\lambda_0}(B\hat{\otimes}\Lambda^0_{\mathrm{nov}})$ by (ii). Now for any given $N\geqslant 0$, there are only finitely many terms in the second line of (18) including fewer that N $\mathfrak{f}_0$'s. Thus, there are only finitely many terms which do not lie in $F^{N\lambda_0}(B\hat{\otimes}\Lambda^0_{\mathrm{nov}})$. Since $N\lambda_0\to\infty$ as $N\to\infty$, this implies that for any $\lambda\in[0,\infty)$, all but finitely many terms in the second line of (18) lie in $F^{\lambda}(B\hat{\otimes}\Lambda^0_{\mathrm{nov}})$, so it is a convergent sum.

A gapped filtered A_∞ morphism $\mathfrak{f}:(A\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})\to(B\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{n})$ is called *strict* if $\mathfrak{f}_k=0$ for $k\neq 1$, and a *gapped filtered A_∞ isomorphism* if $\mathfrak{f}_1:A\hat{\otimes}\Lambda^0_{\mathrm{nov}}\to B\hat{\otimes}\Lambda^0_{\mathrm{nov}}$ is an isomorphism.

If $\mathfrak{f}:(A\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})\to(B\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{n})$ is a gapped filtered A_∞ morphism, then $\mathfrak{f}^{0,0}:(A,\mathfrak{m}^{0,0})\to(B,\mathfrak{n}^{0,0})$ is an A_∞ morphism, where $\mathfrak{f}^{0,0}=(\mathfrak{f}^{0,0}_k)_{k\geqslant 1}$. We call $\mathfrak{f}$ a *weak homotopy equivalence* of gapped filtered A_∞ algebras if $\mathfrak{f}^{0,0}:(A,\mathfrak{m}^{0,0})\to(B,\mathfrak{n}^{0,0})$ is a weak homotopy equivalence of A_∞ algebras in the sense of Definition 3.3, that is, if $\mathfrak{f}^{0,0}_1$ induces an isomorphism $H^*(A,\mathfrak{m}^{0,0}_1)\to H^*(B,\mathfrak{n}^{0,0}_1)$.

If $(A\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m}),(B\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{n}),(C\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{o})$ are gapped filtered A_∞ algebras and $\mathfrak{f}:(A\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})\to(B\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{n})$, $\mathfrak{g}:(B\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{n})\to(C\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{o})$ are gapped filtered A_∞ morphisms, the *composition* $\mathfrak{g}\circ\mathfrak{f}:(A\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})\to(C\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{o})$ is

$$\begin{aligned}(\mathfrak{g}\circ\mathfrak{f})_n(a_1,\dots,a_n)=\sum_{0\leqslant k_1\leqslant\cdots\leqslant k_l=n}\mathfrak{g}_l\bigl(&\mathfrak{f}_{k_1}(a_1,\dots,a_{k_1}),\mathfrak{f}_{k_2-k_1}(a_{k_1+1},\dots,a_{k_2}),\\ &\dots,\mathfrak{f}_{k_l-k_{l-1}}(a_{k_{l-1}+1},\dots,a_{k_l})\bigr),\end{aligned}\tag{20}$$

which is (11) but allowing equalities in the sum over $0<k_1<k_2<\cdots<k_l=n$. As for (18), this is an infinite, convergent sum. Composition is associative.

Let $\mathfrak{f},\mathfrak{g}:(A\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})\to(B\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{n})$ be gapped filtered A_∞ morphisms of gapped filtered A_∞ algebras. A *homotopy* from $\mathfrak{f}$ to $\mathfrak{g}$ is $\mathfrak{H}=(\mathfrak{H}_k)_{k\geqslant 0}$, where $\mathfrak{H}_k:(A\hat{\otimes}\Lambda^0_{\mathrm{nov}})^{\times^k}\to B\hat{\otimes}\Lambda^0_{\mathrm{nov}}$ for $k=0,1,\dots$ are graded Λ^0_{nov}-multilinear maps of degree -1, satisfying

(i) there exists a subset $\mathcal{G}''\subset[0,\infty)\times\mathbb{Z}$, closed under addition, such that $\mathcal{G}''\cap(\{0\}\times\mathbb{Z})=\{(0,0)\}$ and $\mathcal{G}''\cap([0,C]\times\mathbb{Z})$ is finite for any $C\geqslant 0$, and the maps $\mathfrak{H}_k$ for $k\geqslant 0$ may be written $\mathfrak{H}_k=\sum_{(\lambda,\mu)\in\mathcal{G}''}T^\lambda e^\mu\mathfrak{H}^{\lambda,\mu}_k$, for unique $\mathbb{Q}$-multilinear maps $\mathfrak{H}^{\lambda,\mu}_k:A^{\times^k}\to B$ graded of degree $-1-2\mu$. When $k=0$, we take $\mathfrak{H}_0\in(B\hat{\otimes}\Lambda^0_{\mathrm{nov}})^{(-1)}$ and $\mathfrak{H}^{\lambda,\mu}_0\in B^{-1-2\mu}$;

(ii) $\mathfrak{H}^{0,0}_0=0$, in the notation of (i); and

(iii) for all $n \geqslant 0$ and pure $a_1,\ldots,a_n$ in $A\hat{\otimes}\Lambda^0_{\rm nov}$, we have

$$
\begin{aligned}
&\mathfrak{f}_n(a_1,\ldots,a_n)-\mathfrak{g}_n(a_1,\ldots,a_n)= \\
&\sum_{\substack{0\leqslant j_1\leqslant\cdots\leqslant j_l\leqslant\\ k_1\leqslant\cdots\leqslant k_m=n}} \begin{aligned}[t]&\mathfrak{n}_{l+m+1}\bigl(\mathfrak{f}_{j_1}(a_1,\ldots,a_{j_1}),\mathfrak{f}_{j_2-j_1}(a_{j_1+1},\ldots,a_{j_2}),\ldots,\\ &\mathfrak{f}_{j_l-j_{l-1}}(a_{j_{l-1}+1},\ldots,a_{j_l}),\mathfrak{H}_{k_1-j_l}(a_{j_l+1},\ldots,a_{k_1}),\\ &\mathfrak{g}_{k_2-k_1}(a_{k_1+1},\ldots,a_{k_2}),\ldots,\mathfrak{g}_{k_m-k_{m-1}}(a_{k_{m-1}+1},\ldots,a_{k_m})\bigr)\end{aligned}\\
&+\sum_{0\leqslant i\leqslant j\leqslant n}(-1)^{\sum_{l=1}^{i}\deg a_l}\mathfrak{H}_{n-j+i+1}\bigl(a_1,\ldots,a_i,\mathfrak{m}_{j-i}(a_{i+1},\ldots,a_j),a_{j+1},\ldots,a_n\bigr),
\end{aligned}
\tag{21}
$$

which is (12), but allowing equalities in $0<j_1<\cdots<k_m=n$ and $0\leqslant i<j\leqslant n$. As for (18) and (20), (21) is a convergent infinite sum.

Let $\mathfrak{f}:(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})\to(B\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{n})$ be a gapped filtered A_∞ morphism. A *homotopy inverse* for $\mathfrak{f}$ is a gapped filtered A_∞ morphism $\mathfrak{g}:(B\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{n})\to(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})$ such that $\mathfrak{g}\circ\mathfrak{f}:(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})\to(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})$ is homotopic to $\mathrm{id}_A:(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})\to(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})$, and $\mathfrak{f}\circ\mathfrak{g}:(B\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{n})\to(B\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{n})$ is homotopic to $\mathrm{id}_B:(B\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{n})\to(B\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{n})$. If $\mathfrak{f}$ has a homotopy inverse, we call $\mathfrak{f}$ a *homotopy equivalence*, and we call $(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m}),(B\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{n})$ *homotopic*.

Here is the analogue of Theorem 3.6, in Fukaya et al. [**8**, Th. 4.2.45(2)].

Theorem 3.15. *Let $(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m}),(B\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{n})$ be gapped filtered A_∞ algebras. Then:*

(a) *Homotopy is an equivalence relation on gapped filtered A_∞ morphisms $\mathfrak{f}:(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})\to(B\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{n})$.*

(b) *Homotopy is an equivalence relation on gapped filtered A_∞ algebras.*

(c) *A gapped filtered A_∞ morphism $\mathfrak{f}:(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})\to(B\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{n})$ is a homotopy equivalence if and only if it is a weak homotopy equivalence.*

We can also generalize the ideas of §3.3 to the gapped filtered case. Here are the analogues of Definition 3.8 and Theorem 3.9.

Definition 3.16. Let $(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})$ be a gapped filtered A_∞ algebra. Then $\bigl(A,\mathfrak{m}_1^{0,0}\bigr)$ is a complex. Let B be a graded vector subspace of A closed under $\mathfrak{m}_1^{0,0}$, such that the inclusion $i:B\hookrightarrow A$ induces an isomorphism $i_*:H^*\bigl(B,\mathfrak{m}_1^{0,0}|_B\bigr)\to H^*\bigl(A,\mathfrak{m}_1^{0,0}\bigr)$. We will construct $\mathfrak{n}=(\mathfrak{n}_k)_{k\geqslant 0}$ making $(B\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{n})$ into a gapped filtered A_∞ algebra homotopic to $(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})$.

Since i_* is an isomorphism, we can choose a graded vector subspace C of A such that $C\cap\operatorname{Ker}\mathfrak{m}_1^{0,0}=\{0\}$ and $A=B\oplus C\oplus\mathfrak{m}_1^{0,0}(C)$. Then $\mathfrak{m}_1^{0,0}:C\to\mathfrak{m}_1^{0,0}(C)$ is invertible, so there is a unique graded linear map

$H : A \to A$ of degree -1 with $H(b) = H(c) = 0$ and $H \circ \mathfrak{m}_1^{0,0}(c) = c$ for all $b \in B$ and $c \in C$. Let $\Pi_B : A \to B$ be the projection, with kernel $C \oplus \mathfrak{m}_1^{0,0}(C)$. Then $\mathrm{id}_A - \Pi_B = \mathfrak{m}_1^{0,0} \circ H + H \circ \mathfrak{m}_1^{0,0}$ on A. Let $\hat{\imath} : B\hat{\otimes}\Lambda_{\mathrm{nov}}^0 \hookrightarrow A\hat{\otimes}\Lambda_{\mathrm{nov}}^0$, $\hat{H} : A\hat{\otimes}\Lambda_{\mathrm{nov}}^0 \to A\hat{\otimes}\Lambda_{\mathrm{nov}}^0$ and $\hat{\Pi}_B : A\hat{\otimes}\Lambda_{\mathrm{nov}}^0 \to B\hat{\otimes}\Lambda_{\mathrm{nov}}^0$ be the Λ_{nov}^0-linear extensions of i, H, Π_B.

For each planar rooted tree T with k leaves, define a graded multilinear operator $\mathfrak{n}_{k,T} : (B\hat{\otimes}\Lambda_{\mathrm{nov}}^0)^{\times^k} \to B\hat{\otimes}\Lambda_{\mathrm{nov}}^0$ of degree $+1$, as follows. To define $\mathfrak{n}_{k,T}(b_1, \ldots, b_k)$, assign objects and operators to the vertices and edges of T:

- assign $b_1, \ldots, b_k$ to the leaf vertices $1, \ldots, k$ respectively.
- for each internal vertex with 1 outgoing edge and n incoming edges, $n \neq 1$, assign $\mathfrak{m}_n$.
- for each internal vertex with 1 outgoing edge and 1 incoming edge, assign $\mathfrak{m}_1 - \mathfrak{m}_1^{0,0}$.
- assign $\hat{\imath}$ to each leaf edge.
- assign $\hat{\Pi}_B$ to the root edge.
- assign $-\hat{H}$ to each internal edge.

Let $\mathfrak{n}_{k,T}(b_1, \ldots, b_k)$ be the composition of all these objects and morphisms, as in Definition 3.8. Define $\mathfrak{n}_k : (B\hat{\otimes}\Lambda_{\mathrm{nov}}^0)^{\times^k} \to B\hat{\otimes}\Lambda_{\mathrm{nov}}^0$ by

$$\mathfrak{n}_k = \begin{cases} \mathfrak{m}_1^{0,0} + \sum_T \mathfrak{n}_{1,T}, & k = 1, \\ \sum_T \mathfrak{n}_{k,T}, & k = 0, 2, 3, 4, \ldots, \end{cases} \tag{22}$$

where the sums are over all planar rooted trees T with k leaves.

The sums in (22) are *infinite* sums, since such trees T can contain arbitrarily large numbers of internal vertices with 1 edge, which are weighted by $\mathfrak{m}_0$, or with 2 edges, which are weighted by $\mathfrak{m}_1 - \mathfrak{m}_1^{0,0}$. We claim they are *convergent*. To see this, let $\mathcal{G}$ be as in Definition 3.13(i), and set $\lambda_0 = \min_{(0,0)\neq(\lambda,\mu)\in\mathcal{G}} \lambda$. Then $\lambda_0 > 0$, provided $\mathcal{G} \neq \{(0,0)\}$, and $\mathfrak{m}_0 \in F^{\lambda_0}(A\hat{\otimes}\Lambda_{\mathrm{nov}}^0)$, and $\mathfrak{m}_1 - \mathfrak{m}_1^{0,0} : F^{\lambda}(A\hat{\otimes}\Lambda_{\mathrm{nov}}^0) \to F^{\lambda+\lambda_0}(A\hat{\otimes}\Lambda_{\mathrm{nov}}^0)$ for all $\lambda \in [0, \infty)$. Therefore, if T has N internal vertices with 1 or 2 edges, then $\mathfrak{n}_{k,T}$ maps to $F^{N\lambda_0}(B\hat{\otimes}\Lambda_{\mathrm{nov}}^0)$. As there are only finitely many rooted planar trees T with k leaves and fewer than N internal vertices with 1 or 2 edges, and $N\lambda_0 \to \infty$ as $N \to \infty$, it follows that (22) is convergent.

In a similar way, for each planar rooted tree T with k leaves, define a graded multilinear operator $\mathfrak{i}_{k,T} : (B\hat{\otimes}\Lambda_{\mathrm{nov}}^0)^{\times^k} \to A\hat{\otimes}\Lambda_{\mathrm{nov}}^0$ of degree 0, as follows. Assign objects and operators to the vertices and edges of T:

- assign $b_1, \ldots, b_k$ to the leaf vertices $1, \ldots, k$ respectively.
- for each internal vertex with 1 outgoing edge and n incoming edges, $n \neq 1$, assign $\mathfrak{m}_n$.
- for each internal vertex with 1 outgoing edge and 1 incoming edge, assign $\mathfrak{m}_1 - \mathfrak{m}_1^{0,0}$.
- assign $\hat{\imath}$ to each leaf edge.

- assign $-\hat{H}$ to the root edge and to each internal edge.

Define $\mathfrak{i}_{k,T}(b_1,\ldots,b_k)$ to be the composition of all these objects and morphisms. Define $\mathfrak{i}_k : (B\hat{\otimes}\Lambda^0_{\rm nov})^{\times^k} \to A\hat{\otimes}\Lambda^0_{\rm nov}$ by

$$\mathfrak{i}_k = \begin{cases} \hat{\imath} + \sum_T \mathfrak{i}_{1,T}, & k=1, \\ \sum_T \mathfrak{i}_{k,T}, & k=0,2,3,4,\ldots, \end{cases}$$

where the sums are over all planar rooted trees T with k leaves. As for (22), these are convergent infinite sums.

Theorem 3.17. *In Definition* 3.16, $(B\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{n})$ *is a gapped filtered* A_∞ *algebra, and* $\mathfrak{i} : (B\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{n}) \to (A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})$ *is a gapped filtered* A_∞ *morphism, and a homotopy equivalence. If we choose* $B \cong H^*(A,\mathfrak{m}_1^{0,0})$ *to be a subspace representing* $H^*(A,\mathfrak{m}_1^{0,0})$, *so that* $\mathfrak{n}_1^{0,0} = \mathfrak{m}_1^{0,0}|_B = 0$, *then* $(B\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{n})$ *is a minimal model for* $(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})$.

As for Corollary 3.11, we prove:

Corollary 3.18. *Let* $\mathfrak{p} : (A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m}) \to (D\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{o})$ *be a strict, surjective gapped filtered* A_∞ *morphism of gapped filtered* A_∞ *algebras which is a weak homotopy equivalence. Then we can construct an explicit homotopy inverse* $\mathfrak{q} : (D\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{o}) \to (A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})$ *for* $\mathfrak{p}$ *using sums over planar trees.*

3.6. Bounding cochains. As in Definition 3.13, to define Lagrangian Floer cohomology we need *strict* gapped filtered A_∞ algebras. *Bounding cochains* are a method of modifying gapped filtered A_∞ algebras to make them strict, introduced by Fukaya et al. [**8**, §2.4.5, §3.6].

Definition 3.19. Let $(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})$ be a gapped filtered A_∞ algebra, and suppose $b \in F^\lambda(A\hat{\otimes}\Lambda^0_{\rm nov})^{(0)}$ for some $\lambda > 0$. Define graded $\Lambda^0_{\rm nov}$-multilinear maps $\mathfrak{m}^b_k : (A\hat{\otimes}\Lambda^0_{\rm nov})^{\times^k} \to A\hat{\otimes}\Lambda^0_{\rm nov}$ for $k=0,1,2,\ldots$, of degree $+1$, by

$$\begin{aligned}\mathfrak{m}^b_k(a_1,\ldots,a_k) = \sum_{n_0,\ldots,n_k\geqslant 0} \mathfrak{m}_{k+n_0+\cdots+n_k}\bigl(&\overbrace{b,\ldots,b}^{n_0}, a_1, \overbrace{b,\ldots,b}^{n_1}, a_2, \overbrace{b,\ldots,b}^{n_2}, \\ &\ldots, \overbrace{b,\ldots,b}^{n_{k-1}}, a_k, \overbrace{b,\ldots,b}^{n_k}\bigr).\end{aligned}$$

This is an infinite sum, but converges as $b \in F^\lambda(A\hat{\otimes}\Lambda^0_{\rm nov})$ for $\lambda > 0$. Write $\mathfrak{m}^b = (\mathfrak{m}^b_k)_{k\geqslant 0}$. We call b a *bounding cochain* for $(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})$ if $\mathfrak{m}^b_0 = 0$, that is, if

$$\textstyle\sum_{k\geqslant 0} \mathfrak{m}_k(b,\ldots,b) = 0.$$

This is called the *Maurer–Cartan equation*, or *Batalin–Vilkovisky master equation.*

It is then easy to prove [**8**, Prop. 3.6.10]:

Lemma 3.20. *In Definition* 3.19, $(A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m}^b)$ *is a gapped filtered* A_∞ *algebra, which is strict if and only if* b *is a bounding cochain. Moreover,* $\mathfrak{f} : (A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m}^b) \to (A\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m})$ *defined by* $\mathfrak{f}_0 = b$, $\mathfrak{f}_1 = \mathrm{id}_{A\hat{\otimes}\Lambda^0_{\rm nov}}$

and $\mathfrak{f}_k = 0$ *for* $k \geqslant 2$ *is an* A_∞ *isomorphism. Thus* $(A\hat{\otimes}\Lambda^0_{\rm nov}, \mathfrak{m}^b)$ *is homotopy equivalent to* $(A\hat{\otimes}\Lambda^0_{\rm nov}, \mathfrak{m})$.

Thus, if b is a bounding cochain then $(A\hat{\otimes}\Lambda^0_{\rm nov}, \mathfrak{m}^b_1)$ is a complex, and we may form its cohomology $H^*(A\hat{\otimes}\Lambda^0_{\rm nov}, \mathfrak{m}^b_1)$, which is a $\Lambda^0_{\rm nov}$-module. We can also work over $\Lambda_{\rm nov}$ rather than $\Lambda^0_{\rm nov}$, so that $(A\hat{\otimes}\Lambda_{\rm nov}, \mathfrak{m}^b_1)$ is a complex, with cohomology $H^*(A\hat{\otimes}\Lambda_{\rm nov}, \mathfrak{m}^b_1)$.

3.7. $A_{N,K}$ **algebras.** Stasheff [**20**] introduced A_K *algebras* [**8**, §4.4.2], a finite approximation to A_∞ algebras. An A_K algebra $(A,\mathfrak{m})$ is as in Definition 3.1 with $\mathfrak{m}_0 = 0$, except that $\mathfrak{m} = (\mathfrak{m}_k)^K_{k=1}$ rather than $(\mathfrak{m}_k)^\infty_{k=1}$, and (8) holds for $k = 1,\ldots,K$ rather than $k = 1,\ldots,\infty$. Similarly, $A_{N,K}$ *algebras* [**8**, §7.2.6] are a finite approximation of gapped filtered A_∞ algebras. We omit the phrase 'gapped filtered' used in [**8**]. Here is the $A_{N,K}$ analogue of Definitions 3.13 and 3.14.

Definition 3.21. Let $\mathcal{G} \subset [0,\infty)\times\mathbb{Z}$ be closed under addition with $\mathcal{G}\cap(\{0\}\times\mathbb{Z}) = \{(0,0)\}$ and $\mathcal{G}\cap([0,C]\times\mathbb{Z})$ finite for any $C \geqslant 0$. Define $\|\,.\,\| : \mathcal{G}\to\mathbb{N}$ by

$$\|(\lambda,\mu)\| = \max\Bigl\{d : (\lambda,\mu) = \sum_{i=1}^d (\lambda_i,\mu_i),\ (0,0)\neq(\lambda_i,\mu_i)\in\mathcal{G}\Bigr\} + [\lambda] \tag{23}$$

for $(0,0) \neq (\lambda,\mu) \in \mathcal{G}$, where $[\lambda]$ is the greatest integer $\leqslant \lambda$, and $\|(0,0)\| = 0$. (This differs by 1 from $\|(\lambda,\mu)\|$ in [**8**, Def. 7.2.61].)

Let $N,K \geqslant 0$. An $A_{N,K}$ *algebra* $(A,\mathcal{G},\mathfrak{m})$ consists of a $\mathbb{Z}$-graded $\mathbb{Q}$-vector space $A = \bigoplus_{d\in\mathbb{Z}} A^d$, $\mathcal{G}$ as above, and a family $\mathfrak{m}$ of graded $\mathbb{Q}$-multilinear maps $\mathfrak{m}^{\lambda,\mu}_k : A^{\times^k} \to A$ of degree $1-2\mu$ for all $(\lambda,\mu)\in\mathcal{G}$ and $k \geqslant 0$ such that either (a) $\|(\lambda,\mu)\| + k - 1 < N + K$, or (b) $\|(\lambda,\mu)\| + k - 1 = N + K$ and $\|(\lambda,\mu)\| - 1 \leqslant N$, satisfying equation (17) for all $(\lambda,\mu)\in\mathcal{G}$ and $k \geqslant 0$ such that (a) or (b) hold.

Now suppose $(A,\mathcal{G},\mathfrak{m})$ and $(B,\mathcal{G},\mathfrak{n})$ are $A_{N,K}$ algebras. Modifying the first part of Definition 3.14, an $A_{N,K}$ *morphism* $\mathfrak{f} : (A,\mathcal{G},\mathfrak{m}) \to (B,\mathcal{G},\mathfrak{n})$ consists of $\mathbb{Q}$-multilinear maps $\mathfrak{f}^{\lambda,\mu}_k : A^{\times^k} \to B$ graded of degree -2μ for all $(\lambda,\mu)\in\mathcal{G}$ and $k \geqslant 0$ such that (a) or (b) hold, with $\mathfrak{f}^{0,0}_0 = 0$, satisfying equation (19) for all $(\lambda,\mu)\in\mathcal{G}$, $k \geqslant 0$ such that (a) or (b) hold and pure $a_1,\ldots,a_k \in A$. Note that we use *the same* $\mathcal{G}$ for $(A,\mathcal{G},\mathfrak{m}), (B,\mathcal{G},\mathfrak{n})$ and $\mathfrak{f}$, and we regard $\mathcal{G}$ as fixed once and for all. The issue of changing $\mathcal{G}$ will be addressed in the proof of Theorem 11.2.

Composition of $A_{N,K}$ morphisms is defined in the obvious way. If $\mathfrak{f} : (A,\mathcal{G},\mathfrak{m}) \to (B,\mathcal{G},\mathfrak{n})$ is an $A_{N,K}$ morphism then $\mathfrak{f}^{0,0}_1 : A \to B$ is a well-defined morphism of complexes $(A,\mathfrak{m}^{0,0}_1) \to (B,\mathfrak{n}^{0,0}_1)$, and induces $(\mathfrak{f}^{0,0}_1)_* : H^*(A,\mathfrak{m}^{0,0}_1) \to H^*(B,\mathfrak{n}^{0,0}_1)$. We call $\mathfrak{f}$ a *weak homotopy equivalence* if $(\mathfrak{f}^{0,0}_1)_*$ is an isomorphism. We can also define *homotopy* $\mathfrak{H} : \mathfrak{f} \Rightarrow \mathfrak{g}$ between $A_{N,K}$-morphisms $\mathfrak{f},\mathfrak{g} : (A,\mathcal{G},\mathfrak{m}) \to (B,\mathcal{G},\mathfrak{n})$ by rewriting (21) in terms of the $\mathfrak{H}^{\lambda,\mu}_k$ and only requiring it to hold for $(\lambda,\mu), k$ satisfying

(a) or (b). Thus we define *homotopy inverse* and *homotopy equivalence*.

Here is the analogue of Theorems 3.6 and 3.15, [**8**, Rem. 7.2.71].

Theorem 3.22. *Let $(A,\mathcal{G},\mathfrak{m}),(B,\mathcal{G},\mathfrak{n})$ be $A_{N,K}$ algebras. Then:*

(a) *Homotopy is an equivalence relation on $A_{N,K}$ morphisms $\mathfrak{f}:(A,\mathcal{G},\mathfrak{m})\to(B,\mathcal{G},\mathfrak{n})$.*
(b) *Homotopy is an equivalence relation on $A_{N,K}$ algebras.*
(c) *An $A_{N,K}$ morphism $\mathfrak{f}:(A,\mathcal{G},\mathfrak{m})\to(B,\mathcal{G},\mathfrak{n})$ is a homotopy equivalence if and only if it is a weak homotopy equivalence.*

For simplicity, in the rest of the paper we will take $K=0$, and consider only $A_{N,0}$ algebras. These are sufficient for our purposes, and fixing $K=0$ reduces conditions (a) and (b) of Definition 3.21 to the single inequality $\|(\lambda,\mu)\|+k-1\leqslant N$.

If $\bar N\geqslant N\geqslant 0$ then any $A_{\bar N,0}$ algebra $(A,\mathcal{G},\bar{\mathfrak{m}})$ induces an $A_{N,0}$ algebra $(A,\mathcal{G},\mathfrak{m})$ by taking $\mathfrak{m}$ to be the subset of $\bar{\mathfrak{m}}_k^{\lambda,\mu}$ with $\|(\lambda,\mu)\|+k-1\leqslant N$. Similarly, an $A_{\bar N,0}$ morphism $\bar{\mathfrak{f}}:(A,\mathcal{G},\bar{\mathfrak{m}})\to(B,\mathcal{G},\bar{\mathfrak{n}})$ restricts to an $A_{N,0}$ morphism $\mathfrak{f}:(A,\mathcal{G},\mathfrak{m})\to(B,\mathcal{G},\mathfrak{n})$ on the corresponding $A_{N,0}$ algebras. Conversely, we can ask about extending $A_{N,0}$ algebras and $A_{N,0}$ morphisms to $A_{\bar N,0}$ algebras and $A_{\bar N,0}$ morphisms. Our next theorem follows from Fukaya et al. [**8**, Th. 7.2.72 & Lem. 7.2.128].

Theorem 3.23. *Let $\mathfrak{f}:(A,\mathcal{G},\mathfrak{m})\to(B,\mathcal{G},\mathfrak{n})$ be an $A_{N,0}$ morphism of $A_{N,0}$ algebras which is a weak homotopy equivalence. Suppose $\bar N\geqslant N$, and $(B,\mathcal{G},\bar{\mathfrak{n}})$ is an $A_{\bar N,0}$ algebra extending $(B,\mathcal{G},\mathfrak{n})$. Then:*

(a) *there exists an $A_{\bar N,0}$ algebra $(A,\mathcal{G},\bar{\mathfrak{m}})$ extending $(A,\mathcal{G},\mathfrak{m})$, and an $A_{\bar N,0}$ morphism $\bar{\mathfrak{f}}:(A,\mathcal{G},\bar{\mathfrak{m}})\to(B,\mathcal{G},\bar{\mathfrak{n}})$ extending $\mathfrak{f}$ which is a weak homotopy equivalence; and*
(b) *if $(A,\mathcal{G},\bar{\mathfrak{m}})$ is an $A_{\bar N,0}$ algebra extending $(A,\mathcal{G},\mathfrak{m})$, and $\bar{\mathfrak{g}}:(A,\mathcal{G},\bar{\mathfrak{m}})\to(B,\mathcal{G},\bar{\mathfrak{n}})$ is an $A_{\bar N,0}$ morphism which restricts to an $A_{N,0}$ morphism $\mathfrak{g}:(A,\mathcal{G},\mathfrak{m})\to(B,\mathcal{G},\mathfrak{n})$ which is $A_{N,0}$ homotopic to $\mathfrak{f}$, then $\mathfrak{f}$ extends to an $A_{\bar N,0}$ morphism $\bar{\mathfrak{f}}:(A,\mathcal{G},\bar{\mathfrak{m}})\to(B,\mathcal{G},\bar{\mathfrak{n}})$ which is $A_{\bar N,0}$ homotopic to $\bar{\mathfrak{g}}$.*

All of §3.5–§3.7 also works over Λ^0_{CY} rather than Λ^0_{nov}, with the obvious changes.

4. Moduli spaces

Next we discuss moduli spaces of isomorphism classes of stable maps from a genus 0 prestable bordered Riemann surface with immersed Lagrangian boundary conditions. Most of the arguments are the same as in the embedded case of Fukaya et al. [**8**, §7.1] and Liu [**17**], but we put some extra data on the boundary of our stable maps.

4.1. Definition of the moduli spaces $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$. We first define stable J-holomorphic maps from prestable holomorphic discs with marked points.

Definition 4.1. Let (M,ω) be a compact $2n$-dimensional symplectic manifold with a compatible almost complex structure J, and $\iota : L \to M$ a compact Lagrangian immersion. Suppose that all the self-intersection points of the immersion ι are transverse double self-intersections.

Let Σ be a genus 0 prestable bordered Riemann surface, that is, Σ is a possibly singular Riemann surface with boundary $\partial\Sigma$ such that the double $\Sigma \cup_{\partial\Sigma} \bar{\Sigma}$ is a connected and simply connected compact singular Riemann surface whose only singularities are nodes. Let k be a nonnegative integer, and choose mutually distinct smooth points $z_0,\ldots,z_k$ on $\partial\Sigma$ (that is, $z_0,\ldots,z_k$ are not nodes), and write $\vec{z} = (z_0,\ldots,z_k)$. Let $u : \Sigma \to M$ be a J-holomorphic map with $u(\partial\Sigma) \subset \iota(L)$. We call the triple $(\Sigma,\vec{z},u)$ *stable* if the automorphism group $\mathrm{Aut}(\Sigma,\vec{z},u)$ of biholomorphisms $f : \Sigma \to \Sigma$ with $u \circ f = u$ and $f(z_i) = z_i$ for $i = 0,\ldots,k$ is finite. Equivalently, $(\Sigma,\vec{z},u)$ is stable if for each irreducible component Σ' of Σ, $u|_{\Sigma'}$ is not constant, or

- the number of singular points on Σ' is at least 3 when Σ' is diffeomorphic to a sphere,
- the number of marked or singular points on $\partial\Sigma'$ plus twice the number of singular points on $\Sigma' \setminus \partial\Sigma'$ is at least 3 when Σ' is diffeomorphic to a disc.

For $(\Sigma,\vec{z},u)$ as above, we would like to think of the boundary $\partial\Sigma$ as a circle, but this is not true if Σ has boundary nodes. Let $\mathcal{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ be a circle with the counter-clockwise orientation. The boundary $\partial\Sigma$ has the orientation induced by the complex structure, and there is a continuous and orientation-preserving map $l : \mathcal{S}^1 \to \partial\Sigma$ unique up to reparameterization such that

- the inverse image of a singular point of $\partial\Sigma$ consists of two points,
- the inverse image of a smooth point of $\partial\Sigma$ consists of one point.

Write $\zeta_i = l^{-1}(z_i)$, for $i = 0,\ldots,k$.

In the embedded case [**8**, §2.1], one defines moduli spaces $\overline{\mathcal{M}}_{k+1}(\beta,J)$ of isomorphism classes $[\Sigma,\vec{z},u]$ of triples $(\Sigma,\vec{z},u)$. But in our immersed case, we need to keep track of some extra information. In Definition 4.1, $u \circ l$ is a continuous map $\mathcal{S}^1 \to \iota(L)$. We want to know whether this can be locally lifted to a continuous map $\bar{u} : \mathcal{S}^1 \to L$ with $\iota \circ \bar{u} \equiv u \circ l$. This is only a problem at the self-intersection points of $\iota(L)$. For such a point $p \in M$ we have $\iota^{-1}(p) = \{p_+, p_-\}$, that is, two points p_+, p_- in L map to one point p in M, and $\iota(L)$ near p in M has two sheets, the images under ι of disjoint open neighbourhoods of p_+ and p_-.

If $u \circ l(\zeta) = p$ for some $\zeta \in \mathcal{S}^1$, it can happen that $u \circ l$ jumps at ζ between the two sheets of $\iota(L)$ near p in M, and so $u \circ l$ cannot be lifted to a continuous $\bar{u}: \mathcal{S}^1 \to L$ near ζ, since $\bar{u}$ would have to jump discontinuously between p_+ and p_- at ζ. The meaning of the next definition is that we consider triples $(\Sigma, \vec{z}, u)$ in which $u \circ l$ jumps at ζ between two sheets of $\iota(L)$ in this way if and only if $\zeta = \zeta_i$ for i in a fixed subset $I \subseteq \{0, \ldots, k\}$, and that we also prescribe $p = u(\zeta_i)$ and the limits p_+, p_- of $\bar{u}(\zeta')$ as $\zeta' \to \zeta_i$ in $\mathcal{S}^1$ from either direction.

Definition 4.2. Let (M, ω) be a compact $2n$-dimensional symplectic manifold with a compatible almost complex structure J, and $\iota: L \to M$ a compact Lagrangian immersion with only transverse double self-intersections. Define R to be the set of ordered pairs $(p_-, p_+) \in L \times L$ such that $p_- \neq p_+$ and $\iota(p_-) = \iota(p_+)$, and define an involution $\sigma: R \to R$ by $\sigma(p_-, p_+) = (p_+, p_-)$.

Fix $k \geqslant 0$. Let $I \subset \{0, \ldots, k\}$ be a subset, $\alpha: I \to R$ a map, and $\beta \in H_2(M, \iota(L); \mathbb{Z})$ a relative homology class. Consider quintuples $(\Sigma, \vec{z}, u, l, \bar{u})$, where Σ is a genus 0 prestable bordered Riemann surface, and $\vec{z} = (z_0, \ldots, z_k)$ for distinct smooth points $z_0, \ldots, z_k$ on $\partial\Sigma$ (that is, $z_0, \ldots, z_k$ are not nodes), and $u: \Sigma \to M$ is a J-holomorphic map with $u(\partial\Sigma) \subset \iota(L)$ and $(\Sigma, \vec{z}, u)$ stable, and $l: \mathcal{S}^1 \to \partial\Sigma$ is as in Definition 4.1 with $\zeta_i = l^{-1}(z_i)$ for all i, and $\bar{u}: \mathcal{S}^1 \setminus \{\zeta_i : i \in I\} \to L$ is a continuous map, satisfying the following conditions:

- $u_*([\Sigma]) = \beta \in H_2(M, \iota(L); \mathbb{Z})$, with $[\Sigma] \in H_2(\Sigma, \partial\Sigma; \mathbb{Z})$ the fundamental class;
- $\zeta_0, \ldots, \zeta_k$ are ordered counter-clockwise on $\mathcal{S}^1$;
- $\iota \circ \bar{u} \equiv u \circ l$ on $\mathcal{S}^1 \setminus \{\zeta_i : i \in I\}$; and
- $(\lim_{\theta \uparrow 0} \bar{u}(e^{\sqrt{-1}\,\theta}\zeta_i), \lim_{\theta \downarrow 0} \bar{u}(e^{\sqrt{-1}\,\theta}\zeta_i)) = \alpha(i)$ in R, for all $i \in I$.

We call two quintuples $(\Sigma, \vec{z}, u, l, \bar{u})$ and $(\Sigma', \vec{z}', u', l', \bar{u}')$ *isomorphic* if there exist a biholomorphic map $\varphi: \Sigma \to \Sigma'$ and an orientation-preserving homeomorphism $\bar{\varphi}: \mathcal{S}^1 \to \mathcal{S}^1$ such that

- $u' \circ \varphi = u$, and $\varphi(z_i) = z_i'$ for $i = 0, \ldots, k$,
- $\varphi \circ l = l' \circ \bar{\varphi}$, and $\bar{u}' \circ \bar{\varphi} = \bar{u}$ on $\mathcal{S}^1 \setminus \{\zeta_i : i \in I\}$.

Denote by $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha, \beta, J)$ the set of the isomorphism classes $[\Sigma, \vec{z}, u, l, \bar{u}]$ of such quintuples $(\Sigma, \vec{z}, u, l, \bar{u})$. Then we may define a natural, compact, Hausdorff topology on $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha, \beta, J)$ called the C^∞ *topology*, following Fukaya et al. [**8**, §7.1] and Liu [**17**, §5.2].

Define the *evaluation maps* $\mathrm{ev}_i: \overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha, \beta, J) \to L \amalg R$ by

$$\mathrm{ev}_i\bigl([\Sigma, \vec{z}, u, l, \bar{u}]\bigr) = \begin{cases} \bar{u}(\zeta_i) \in L, & i \notin I, \\ \alpha(i) \in R, & i \in I, \end{cases} \tag{24}$$

for $i = 0, \ldots, k$, and $\mathrm{ev}: \overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha, \beta, J) \to L \amalg R$ by

$$\mathrm{ev}\bigl([\Sigma, \vec{z}, u, l, \bar{u}]\bigr) = \begin{cases} \bar{u}(\zeta_0) \in L, & 0 \notin I, \\ \sigma \circ \alpha(0) \in R, & 0 \in I, \end{cases} \tag{25}$$

where $\sigma : R \to R$ is the involution above. Following Fukaya et al. [**8**, §3.4 & §7.1] and Liu [**17**] we may define a *Kuranishi structure* on $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$, with corners and a tangent bundle, and the continuous maps $\mathrm{ev}_i, \mathrm{ev}$ extend to *strong submersions* $\mathbf{ev}_i, \mathbf{ev} : \overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J) \to L \amalg R$.

We shall also write

$$\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J) = \coprod\nolimits_{I\subseteq\{0,\ldots,k\},\ \alpha:I\to R} \overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J). \tag{26}$$

Since by (33) below the virtual dimension of $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$ depends on I,α, this is technically not a Kuranishi space, only a disjoint union of Kuranishi spaces of different dimensions. We define strong submersions $\mathbf{ev}_i, \mathbf{ev} : \overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J) \to L \amalg R$ to be $\mathbf{ev}_i, \mathbf{ev}$ on each component $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$.

4.2. The boundary of $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$. Following Fukaya et al. [**8**, §7.2] we can give an expression for the *boundaries* of our moduli spaces. We postpone discussing the *orientations* in (27) until §5.

Theorem 4.3. *In the situation of Definition* 4.2*, there is an isomorphism of unoriented Kuranishi spaces, using the fibre product of Definition* 2.6:

$$\partial\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J) \cong \coprod_{\substack{k_1+k_2=k+1,\ 1\leqslant i\leqslant k_1,\ I_1\cup_i I_2=I,\\ \alpha_1\cup_i\alpha_2=\alpha,\ \beta_1+\beta_2=\beta}} \overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J) \times_{\mathbf{ev},L\amalg R,\mathbf{ev}_i} \overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J), \tag{27}$$

where we define $I_1 \cup_i I_2 \subseteq \{0,\ldots,k\}$ and $\alpha_1\cup_i\alpha_2 : I_1\cup_i I_2 \to R$ by

$$\begin{aligned} I_1 \cup_i I_2 = &\{j : j\in I_1,\ j<i\} \cup \{j+i-1 : j\in I_2,\ 0<j\} \\ &\cup \{j+k_2-1 : j\in I_1,\ i<j\}, \end{aligned}$$

$$(\alpha_1\cup_i\alpha_2)(j) = \begin{cases} \alpha_1(j), & \textit{for } 0\leqslant j<i,\\ \alpha_2(j-i+1), & \textit{for } 1\leqslant j-i+1\leqslant k_2,\\ \alpha_1(j-k_2+1), & \textit{for } i<j-k_2+1\leqslant k_1, \end{cases} \tag{28}$$

and we use the same notation for the evaluation maps $\mathbf{ev}_i : \overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J) \to L\amalg R$ and $\mathbf{ev} : \overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J) \to L\amalg R$.

Here (27) is a fairly straightforward consequence of the construction of the Kuranishi structure on $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$, as near the boundary strata of $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$ the Kuranishi neighbourhoods $(V_p,\ldots,\psi_p)$ are built from Kuranishi neighbourhoods on terms in the right hand side of (27), using gluing theorems to desingularize boundary nodes in Σ. In (27) we choose to write the fibre product as $\overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J) \times_{\mathbf{ev},L\amalg R,\mathbf{ev}_i} \overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J)$, although it would be more obvious to write it as $\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J) \times_{\mathbf{ev}_i,L\amalg R,\mathbf{ev}} \overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J)$, following Fukaya et al.

[**8**, Prop. 8.3.3]. As we will explain in Remark 5.14(b), because of peculiarities of the immersed case, when we orient our moduli spaces in §5, the signs in our formulae will look simpler and more natural with the fibre product order in (27).

4.3. The virtual dimension of $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$. We shall compute the *virtual dimension* of $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$, modifying Fukaya [**7**, Th. 3.2], who calculates the virtual dimension of moduli spaces of holomorphic discs with boundary attached to a union $L_0\cup\cdots\cup L_k$ of transversely intersecting embedded Lagrangians, so that $L_0\cup\cdots\cup L_k$ is an immersed Lagrangian submanifold with transverse double self-intersections, and also Fukaya et al. [**8**, Prop. 3.7.59], who perform the same calculation for $L_0\cup L_1$.

Definition 4.4. Let

$$Y=\bigl\{(x,y)\in\mathbb{R}^2 : \text{either } x\leqslant 0,\ x^2+y^2\leqslant 1 \text{ or } x\geqslant 0,\ |y|\leqslant 1\bigr\}. \tag{29}$$

Choose a smooth family $\lambda_{(p_-,p_+)}=\{\lambda_{(p_-,p_+)}(x,y)\}_{(x,y)\in\partial Y}$ of Lagrangian subspaces of T_pM for each $(p_-,p_+)\in R$, where $p=\iota(p_-)=\iota(p_+)$, with

$$\lambda_{(p_-,p_+)}(x,y)=\begin{cases}\mathrm{d}\iota(T_{p_-}L), & \text{if } y=1,\\ \mathrm{d}\iota(T_{p_+}L), & \text{if } y=-1.\end{cases}$$

If $(p_-,p_+)\in R$ then $\sigma(p_-,p_+)=(p_+,p_-)\in R$, and we require $\lambda_{(p_-,p_+)}$ and $\lambda_{(p_+,p_-)}$ to be related by $\lambda_{(p_+,p_-)}(x,y)\equiv\lambda_{(p_-,p_+)}(x,-y)$. When L is *oriented*, as it will be from §5 onwards, we take $\lambda_{(p_-,p_+)}$ to be a smooth family of oriented Lagrangian subspaces, which agree with $\mathrm{d}\iota(T_{p_\mp}L)$ as oriented subspaces when $y=\pm1$.

Consider the differential operator

$$\begin{aligned}\bar\partial_{\lambda_{(p_-,p_+)}}=\frac{\partial}{\partial x}+J_p\frac{\partial}{\partial y} : W^{1,q}(Y,\partial Y;T_pM,\lambda_{(p_-,p_+)})&\\ \longrightarrow L^q(Y;T_pM\otimes\Lambda^{0,1}Y),&\end{aligned} \tag{30}$$

for $q>2$, where $W^{1,q}(Y,\partial Y;T_pM,\lambda_{(p_-,p_+)})$ is the Sobolev space of the $W^{1,q}$-maps $\xi:Y\to T_pM$ with $\xi(x,y)\in\lambda_{(p_-,p_+)}(x,y)$, for $(x,y)\in\partial Y$, and $L^q(Y;T_pM\otimes\Lambda^{0,1}Y)$ is the one of the L^q-maps $\xi:Y\to T_pM\otimes\Lambda^{0,1}Y$. Following [**8**, Def. 3.7.62], define

$$\eta_{(p_-,p_+)}=\operatorname{ind}\bar\partial_{\lambda_{(p_-,p_+)}}, \tag{31}$$

the Fredholm index of (30). Since $\lambda_{(p_+,p_-)}(x,y)\equiv\lambda_{(p_-,p_+)}(x,-y)$, it is easy to check that

$$\eta_{(p_-,p_+)}+\eta_{(p_+,p_-)}=n. \tag{32}$$

Note that $\eta_{(p_-,p_+)}$ depends on the choice of $\lambda_{(p_-,p_+)}$ up to isotopy. When $\lambda_{(p_-,p_+)}$ is a family of oriented Lagrangian subspaces, different choices of $\lambda_{(p_-,p_+)}$ add an even number to $\eta_{(p_-,p_+)}$. Thus the only invariant information is whether $\eta_{(p_-,p_+)}$ is even or odd, which depends on whether the transverse, oriented subspaces $\mathrm{d}\iota(T_{p_-}L)$ and $\mathrm{d}\iota(T_{p_+}L)$ intersect positively or negatively in T_pM.

In §4.6 we will use this freedom to require that $\eta_{(p_-,p_+)} \geqslant 0$ for all $(p_-,p_+) \in R$, and ask that $\lambda_{(p_-,p_+)}$ is chosen generically, which ensures that $\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}}$ has dimension $\eta_{(p_-,p_+)}$, and $\operatorname{Coker}\bar\partial_{\lambda_{(p_-,p_+)}} = 0$. This is not strictly necessary, but it simplifies the arguments.

There is an important case in which it is natural to fix the $\eta_{(p_-,p_+)}$, however, to be discussed in §12. Suppose that (M,ω) is the symplectic manifold underlying a *Calabi–Yau* manifold, and that L is a *graded* immersed Lagrangian submanifold, in the sense of Definition 12.1. Then we can choose $\lambda_{(p_-,p_+)}$ to be a family of graded Lagrangian subspaces of T_pM, which agree with $\mathrm{d}\iota(T_{p_\mp}L)$ as graded Lagrangian subspaces when $y = \pm 1$. This requirement determines $\eta_{(p_-,p_+)}$ uniquely in $\mathbb{Z}$, independently of the choice of $\lambda_{(p_-,p_+)}$. Also in this case the Maslov index $\mu_L(\beta)$ below is automatically zero, provided the $\lambda_{(p_-,p_+)}$ are taken to be graded.

We can now define the *Maslov index* $\mu_L(\beta)$, and compute the virtual dimension of $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$.

Definition 4.5. For $[\Sigma,\vec z,u,l,\bar u] \in \overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$, we take $\varepsilon > 0$ and a continuous map $\psi : \mathcal{S}^1 \to \mathcal{S}^1$ such that

- $\psi : \mathcal{S}^1 \setminus \bigcup_{i\in I}\{e^{\sqrt{-1}\theta}\zeta_i : \theta \in [-\varepsilon,\varepsilon]\} \to \mathcal{S}^1 \setminus \{\zeta_i : i \in I\}$ is an orientation preserving homeomorphism,
- $\psi(\{e^{\sqrt{-1}\theta}\zeta_i : \theta \in [-\varepsilon,\varepsilon]\}) = \zeta_i$, for $i \in I$,

and define

$$A_{\alpha,\beta}(z) = \begin{cases} \mathrm{d}\iota(T_{\bar u\circ\psi(z)}L), & \text{for } z \in \mathcal{S}^1 \setminus \bigcup_{i\in I}\{e^{\sqrt{-1}\theta}\zeta_i : \theta \in (-\varepsilon,\varepsilon)\}, \\ \lambda_{\alpha(i)} \circ h_i(z), & \text{for } z \in \{e^{\sqrt{-1}\theta}\zeta_i : \theta \in (-\varepsilon,\varepsilon)\} \text{ with } i \in I, \end{cases}$$

where $h_i : \{e^{\sqrt{-1}\theta}\zeta_i : \theta \in (-\varepsilon,\varepsilon)\} \to \partial Y$ is a diffeomorphism with

$$\lim_{\theta\to-\varepsilon} h_i(e^{\sqrt{-1}\theta}\zeta_i) = (\infty,1) \text{ and } \lim_{\theta\to\varepsilon} h_i(e^{\sqrt{-1}\theta}\zeta_i) = (\infty,-1).$$

The symplectic vector bundle $u^*(TM)$ with $u^*(\omega)$ is isomorphic to the trivial one $\Sigma\times\mathbb{C}^n \to \Sigma$. Denote this trivialization by $f : u^*(TM) \to \mathbb{C}^n$, and $f \circ A_{\alpha,\beta}$ is a loop in the Grassmannian of Lagrangian subspaces in $\mathbb{C}^n$.

Write $\mu_L(\beta)$ for the *Maslov index* of $f \circ A_{\alpha,\beta}$, in the sense of Fukaya et al. [**8**, §2.1.1]. That is, $\mu_L(\beta) \in \mathbb{Z}$ is the contraction of the homology class of $f \circ A_{\alpha,\beta}$ with a certain class in the 1-cohomology of the Grassmannian of Lagrangian subspaces in $\mathbb{C}^n$. If L is oriented, as it will be from §5 onwards, then $\mu_L(\beta)$ is even. As above and in §12, if (M,ω) is Calabi–Yau and L is *graded*, we can define the $\lambda_{(p_-,p_+)}$ using graded Lagrangian subspaces, and then $\mu_L(\beta)=0$ for all β.

Now $\mu_L(\beta)$ depends on the choices of families $\lambda_{(p_-,p_+)}$ for $(p_-,p_+)\in R$ above up to isotopy, and hence in effect on the $\eta_{(p_-,p_+)}$. We regard these as fixed once and for all, and suppress the dependence of the Maslov index on them in our notation. In fact $\mu_L(\beta)$ is independent of the other choices involved, except β, which justifies our writing it as $\mu_L(\beta)$. That is, $\mu_L(\beta)$ is independent of $k, I, \alpha, [\Sigma, \vec{z}, u, l, \bar{u}], \psi, h_i$, and the trivialization of $\bigl(u^*(TM), u^*(\omega)\bigr)$. To see this, note that morally $\mu_L(\beta)=\beta\cdot c_1\bigl(M,\iota(L)\bigr)$, where $\beta\in H_2\bigl(M,\iota(L);\mathbb{Z}\bigr)$ and $c_1\bigl(M,\iota(L)\bigr)\in H^2\bigl(M,\iota(L);\mathbb{Z}\bigr)$ is the *relative first Chern class* for ω on $\bigl(M,\iota(L)\bigr)$. The reason $\mu_L(\beta)$ can be independent of I,α is that β partially determines I,α, enough so that the dependence of $\mu_L(\beta)$ on I,α is determined by β.

The following proposition is a straightforward modification of Fukaya [**7**, Th. 3.2] and Fukaya et al. [**8**, Prop. 7.1.1] to the immersed case, following [**8**, Prop. 3.7.59]. In effect, in constructing $\psi, A_{\alpha,\beta}$ above we are defining a desingularized moduli problem, with embedded Lagrangian boundary conditions. The virtual dimension of this desingularized moduli problem is computed as in [**8**, Prop. 7.1.1], and is the right hand side of (33) omitting the term $-\sum_{i\in I}\eta_{\alpha(i)}$. But the effect of desingularizing by gluing in $\lambda_{\alpha(i)}$ at z_i is to increase the virtual dimension by $\eta_{\alpha(i)}$, so to recover the virtual dimension of the original moduli problem we subtract $\sum_{i\in I}\eta_{\alpha(i)}$.

Proposition 4.6. *The virtual dimension of* $\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$ *is*

$$\operatorname{vdim}\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)=\mu_L(\beta)+k-2+n-\textstyle\sum_{i\in I}\eta_{\alpha(i)}. \tag{33}$$

4.4. The moduli spaces $\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\dots,f_k)$.

We add *smooth simplicial chains* to our moduli spaces.

Definition 4.7. For $i=1,\dots,k$, let $a_i\geqslant 0$ and $f_i:\Delta_{a_i}\to L\amalg R$ be a smooth map, where Δ_{a_i} is the a_i-simplex of (6), so that $f_i\in C^{\mathrm{si}}_{a_i}(L\amalg R)$ is a smooth simplicial chain. Define the Kuranishi space $\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\dots,f_k)$ to be the fibre product

$$\begin{aligned}&\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\dots,f_k)=\\&\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)\times_{\mathbf{ev}_1\times\cdots\times\mathbf{ev}_k,(L\amalg R)^k,f_1\times\cdots\times f_k}(\Delta_{a_1}\times\cdots\times\Delta_{a_k}).\end{aligned}\tag{34}$$

Here $\mathbf{ev}_i$ maps to L if $i\notin I$ and to R if $i\in I$. Also, the fibre product is over $1,\dots,k$ although $I\subseteq\{0,\dots,k\}$, so we have to exclude 0. Thus, (34) is in effect a fibre product over the manifold

$\prod_{i\in\{1,\ldots,k\}\setminus I} L \times \prod_{i\in I\setminus\{0\}} R$, which has dimension $n(k-|I\setminus\{0\}|)$. So we see from (33) and Definition 2.6 that

$$\begin{aligned}&\operatorname{vdim}\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)=\\&\mu_L(\beta)+k-2+n-\textstyle\sum_{i\in I}\eta_{\alpha(i)}+\sum_{0\neq i\notin I}(a_i-n)+\sum_{0\neq i\in I}a_i.\end{aligned}\tag{35}$$

Let $f:\Delta_a\to L\amalg R$ be a smooth map. Since f is connected, it must map either to L, or to some unique (p_-,p_+) in R. Define the *shifted cohomological degree* of $f:\Delta_a\to L\amalg R$ to be

$$\deg f=\begin{cases}n-a-1, & f(\Delta_a)\subseteq L,\\ \eta_{(p_-,p_+)}-a-1, & f(\Delta_a)=\{(p_-,p_+)\}\subset R.\end{cases}\tag{36}$$

In effect, we are defining a *new grading* on the chains $C^{\mathrm{si}}_*(L\amalg R;\mathbb{Q})=C^{\mathrm{si}}_*(L;\mathbb{Q})\oplus\bigoplus_{(p_-,p_+)\in R}C^{\mathrm{si}}_*(\{(p_-,p_+)\};\mathbb{Q})$, such that $\deg C^{\mathrm{si}}_a(L;\mathbb{Q})=n-a-1$ and $\deg C^{\mathrm{si}}_a(\{(p_-,p_+)\};\mathbb{Q})=\eta_{(p_-,p_+)}-a-1$.

Note that our notation differs from that of Fukaya et al. [**8**] in the embedded case. Fukaya et al. define the *cohomological degree* of $f:\Delta_a\to L$ in $C^{\mathrm{si}}_a(L;\mathbb{Q})$ to be $\deg f=n-a$, that is, $\deg f$ is in effect the codimension of $f(\Delta_a)$ in L. But then they work throughout with the *shifted complex* $C^{\mathrm{si}}_*(L;\mathbb{Q})[1]$ in which f has grading $\deg' f=\deg f-1$, as in [**8**, §3.2.1]. So our $\deg f$ corresponds to Fukaya et al.'s shifted degree $\deg' f$, which is why we call it the *shifted* cohomological degree.

We prefer this convention as it simplifies many of the dimensions and signs expressed in terms of $\deg f_1,\ldots,\deg f_n$ below, and also the shifted complexes $C^{\mathrm{si}}_*(L;\mathbb{Q})[1],\mathbb{Q}\mathcal{X}[1]$ which are ubiquitous in [**8**] are replaced below by unshifted complexes $C^{\mathrm{si}}_*(L;\mathbb{Q}),\mathbb{Q}\mathcal{X}$, simplifying the notation. We undo the shift when we define Lagrangian Floer cohomology in (144). We will explain the reason for grading $f:\Delta_a\to\{(p_-,p_+)\}$ by $\deg f=\eta_{(p_-,p_+)}-a-1$ in Definition 4.14.

Observe that $\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)=\emptyset$ unless $f_i:\Delta_{a_i}\to L\amalg R$ maps to L if $i\notin I$, and to $\alpha(i)\in R$ if $i\in I$. Then combining (35) and (36) yields

$$\begin{aligned}&\operatorname{vdim}\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)=\\&\begin{cases}\mu_L(\beta)-2+n-\sum_{i=1}^k\deg f_i, & 0\notin I,\\ \mu_L(\beta)-2+n-\sum_{i=1}^k\deg f_i-\eta_{\alpha(0)}, & 0\in I.\end{cases}\end{aligned}\tag{37}$$

This also holds in the other cases, as then $\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)=\emptyset$.

From (5) and (27), $\partial\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)$ is given without orientations by

$$\begin{aligned}&\coprod_{i=1}^k\coprod_{j=0}^{a_i}\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_{i-1},f_i\circ F_j^{a_i},f_{i+1},\ldots,f_k)\\&\amalg\partial\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)\times_{\mathbf{ev}_1\times\cdots\times\mathbf{ev}_k,(L\amalg R)^k,f_1\times\cdots\times f_k}(\Delta_{a_1}\times\cdots\times\Delta_{a_k}),\end{aligned}\tag{38}$$

where $F_j^{a_i}:\Delta_{a_i-1}\to\Delta_{a_i}$ is as in §2.6.

Write $\bar{\mathcal M}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\dots,f_{i-1};f_{i+k_2},\dots,f_k)$ for the fibre product

$$\begin{aligned}&\bar{\mathcal M}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J)\times_{\mathbf{ev}_1\times\cdots\times\mathbf{ev}_{i-1}\times\mathbf{ev}_{i+1}\times\cdots\times\mathbf{ev}_{k_1},(L\amalg R)^{k_1-1},}\\&{}_{f_1\times\cdots\times f_{i-1}\times f_{i+k_2}\times\cdots\times f_k}(\Delta_{a_1}\times\cdots\times\Delta_{a_{i-1}}\times\Delta_{a_{i+k_2}}\times\cdots\times\Delta_{a_k}),\end{aligned}\tag{39}$$

where $k_1+k_2=k+1$. Then as for (37) we calculate that

$$\operatorname{vdim}\bar{\mathcal M}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\dots,f_{i-1};f_{i+k_2},\dots,f_k)=\tag{40}$$
$$\begin{cases}\mu_L(\beta_1)-1+n-\sum_{j=1}^{i-1}\deg f_j-\sum_{j=i+k_2}^{k}\deg f_j, & 0,i\notin I_1,\\ \mu_L(\beta_1)-1+n-\sum_{j=1}^{i-1}\deg f_j-\sum_{j=i+k_2}^{k}\deg f_j-\eta_{\alpha_1(0)}, & 0\in I_1,\ i\notin I_1,\\ \mu_L(\beta_1)-1+n-\sum_{j=1}^{i-1}\deg f_j-\sum_{j=i+k_2}^{k}\deg f_j-\eta_{\alpha_1(i)}, & 0\notin I_1,\ i\in I_1,\\ \mu_L(\beta_1)-1+n-\sum_{j=1}^{i-1}\deg f_j-\sum_{j=i+k_2}^{k}\deg f_j-\eta_{\alpha_1(0)}-\eta_{\alpha_1(i)}, & 0,i\in I_1,\end{cases}$$

Combining (27), (38) and (39) shows that

$$\begin{aligned}&\partial\bar{\mathcal M}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\dots,f_k)\cong\\&\coprod_{i=1}^{k}\coprod_{j=0}^{a_i}\bar{\mathcal M}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\dots,f_{i-1},f_i\circ F_j^{a_i},f_{i+1},\dots,f_k)\\&\amalg\coprod_{\substack{k_1+k_2=k+1,\ 1\leqslant i\leqslant k_1,\\ I_1\cup_i I_2=I,\ \alpha_1\cup_i\alpha_2=\alpha,\\ \beta_1+\beta_2=\beta}}\bar{\mathcal M}^{\mathrm{main}}_{k_2+1}(\alpha_2,\beta_2,J,f_i,\dots,f_{i+k_2-1})\times_{\mathbf{ev},L\amalg R,\mathbf{ev}_i}\\&\qquad\bar{\mathcal M}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\dots,f_{i-1};f_{i+k_2},\dots,f_k),\end{aligned}\tag{41}$$

in unoriented Kuranishi spaces.

As for (26), we shall also write

$$\bar{\mathcal M}^{\mathrm{main}}_{k+1}(\beta,J,f_1,\dots,f_k)=\coprod_{\substack{I\subseteq\{0,\dots,k\},\\ \alpha:I\to R}}\bar{\mathcal M}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\dots,f_k).\tag{42}$$

Again, this is a disjoint union of Kuranishi spaces of different dimensions. We define a strong submersion $\mathbf{ev}:\bar{\mathcal M}^{\mathrm{main}}_{k+1}(\beta,J,f_1,\dots,f_k)\to L\amalg R$ to be $\mathbf{ev}$ on each component $\bar{\mathcal M}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\dots,f_k)$.

4.5. Adding families of almost complex structures. We can generalize all the material above to *smooth families* of almost complex structures J_t for $t\in\mathcal T$, with $\mathcal T$ a smooth manifold. We will need this in §8–§9 with $\mathcal T=[0,1]$, and in §10 with $\mathcal T$ a semicircle S and a triangle T.

Definition 4.8. Suppose (M,ω) is a compact $2n$-dimensional symplectic manifold, $\mathcal T$ an oriented smooth manifold, which may be noncompact and may have boundary or corners, and J_t for $t\in\mathcal T$ a smooth family of almost complex structures on M compatible with ω. Let $\iota:L\to M$ be a compact Lagrangian immersion. Suppose that all the self-intersection points of the immersion ι are transverse double self-intersections.

Generalizing Definition 4.2, define $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T})$ to be the set of $\bigl(t,[\Sigma,\vec z,u,l,\bar u]\bigr)$ for $t\in\mathcal{T}$ and $[\Sigma,\vec z,u,l,\bar u]\in\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t)$. Define $\pi_{\mathcal{T}}:\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T})\to\mathcal{T}$ by $\pi_{\mathcal{T}}:\bigl(t,[\Sigma,\vec z,u,l,\bar u]\bigr)\mapsto t$ and $\mathrm{ev}_i,\mathrm{ev}:\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T})\to L\amalg R$ by $\mathrm{ev}_i,\mathrm{ev}:\bigl(t,[\Sigma,\vec z,u,l,\bar u]\bigr)\mapsto\mathrm{ev}_i,\mathrm{ev}\bigl([\Sigma,\vec z,u,l,\bar u]\bigr)$.

As for the case of $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$ in §4.1, we may define a natural, Hausdorff topology on $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T})$ called the C^∞ *topology*, such that $\pi_{\mathcal{T}},\mathrm{ev}_i,\mathrm{ev}$ are continuous. If $\mathcal{T}$ is compact then $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T})$ is compact.

We can then define a *Kuranishi structure* on $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T})$, with corners and a tangent bundle, and $\pi_{\mathcal{T}},\mathrm{ev}_i,\mathrm{ev}$ extend to *strong submersions* $\boldsymbol{\pi}_{\mathcal{T}},\mathbf{ev}_i,\mathbf{ev}$. For each $t'\in\mathcal{T}$ there is an isomorphism of Kuranishi spaces

$$\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_{t'})\cong\{t'\}\times_{\iota,\mathcal{T},\boldsymbol{\pi}_{\mathcal{T}}}\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T}),\tag{43}$$

where $\iota:\{t'\}\to\mathcal{T}$ is the inclusion, and the right hand side is a fibre product of Kuranishi spaces, which is well-defined as $\boldsymbol{\pi}_{\mathcal{T}}$ is a strong submersion.

There is one subtle point here: the Kuranishi structures on each side depend on choices made during the constructions, and (43) holds provided the choices for the Kuranishi structures on $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_{t'})$ and $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T})$ are compatible. If $\mathcal{T}=[0,1]$ then for any allowed choices for $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_0)$ and $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_1)$, we can choose the Kuranishi structure on $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T})$ so that (43) holds when $t'=0,1$. We will usually suppress this issue of needing to make compatible choices of Kuranishi structures.

Here are the generalizations of Theorem 4.3 and Proposition 4.6.

Theorem 4.9. *In the situation of Definition* 4.8*, there is an isomorphism of unoriented Kuranishi spaces:*

$$\begin{aligned}\partial\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T})\cong{}&\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\partial\mathcal{T})\amalg\\&\coprod_{\substack{k_1+k_2=k+1,\;1\leqslant i\leqslant k_1,\\ I_1\cup_i I_2=I,\;\alpha_1\cup_i\alpha_2=\alpha,\\ \beta_1+\beta_2=\beta}}\overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J_t:t\in\mathcal{T})\times_{\boldsymbol{\pi}_{\mathcal{T}}\times\mathbf{ev},\mathcal{T}\times(L\amalg R),\boldsymbol{\pi}_{\mathcal{T}}\times\mathbf{ev}_i}\\&\qquad\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J_t:t\in\mathcal{T}),\qquad\textit{and}\end{aligned}\tag{44}$$

$$\begin{aligned}&\mathrm{vdim}\,\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T})\\&\qquad=\mu_L(\beta)+k-2+n-\textstyle\sum_{i\in I}\eta_{\alpha(i)}+\dim\mathcal{T}.\end{aligned}\tag{45}$$

We can also add smooth simplicial chains, following Definition 4.7. The obvious way to do this is to start with $f_i:\Delta_{a_i}\to L\amalg R$ for $i=1,\dots,k$, and take $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T})\times_{\mathbf{ev}_1\times\cdots\times\mathbf{ev}_k,(L\amalg R)^k,f_1\times\cdots\times f_k}(\Delta_{a_1}\times\cdots\times\Delta_{a_k})$ as in (34). But for our later purposes we need to do something different: we use simplicial chains on $\mathcal{T} \times (L \amalg R)$, so that f_i maps $\Delta_{a_i} \to \mathcal{T} \times (L \amalg R)$, and then we define $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha, \beta, J_t : t \in \mathcal{T}, f_1, \ldots, f_k)$ by a fibre product over $(\mathcal{T} \times (L \amalg R))^k$. Thus, roughly speaking we want to write

$$\begin{aligned} \overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha, \beta, J_t : t \in \mathcal{T}, f_1, \ldots, f_k) &= \overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha, \beta, J_t : t \in \mathcal{T}) \\ &\times_{(\boldsymbol{\pi}_{\mathcal{T}} \times \mathbf{ev}_1) \times \cdots \times (\boldsymbol{\pi}_{\mathcal{T}} \times \mathbf{ev}_k), (\mathcal{T} \times (L \amalg R))^k, f_1 \times \cdots \times f_k} (\Delta_{a_1} \times \cdots \times \Delta_{a_k}). \end{aligned} \tag{46}$$

However, there is a problem with (46). Although $\boldsymbol{\pi}_{\mathcal{T}} \times \mathbf{ev}_1 \times \cdots \times \mathbf{ev}_k : \overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha, \beta, J_t : t \in \mathcal{T}) \to \mathcal{T} \times (L \amalg R)^k$ is a strong submersion, if $\dim \mathcal{T} > 0$ and $k > 1$ then $(\boldsymbol{\pi}_{\mathcal{T}} \times \mathbf{ev}_1) \times \cdots \times (\boldsymbol{\pi}_{\mathcal{T}} \times \mathbf{ev}_k) : \overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha, \beta, J_t : t \in \mathcal{T}) \to (\mathcal{T} \times (L \amalg R))^k$ is *not* a strong submersion, as it does not locally map onto $\mathcal{T}^k$, but only onto the diagonal $\{(t, \ldots, t) \in \mathcal{T}^k : t \in \mathcal{T}\}$. Since $f_1 \times \cdots \times f_k$ may also not be a strong submersion, the fibre product in (46) is not well-defined.

We fix this by including an extra factor in the fibre product, which modifies the Kuranishi structures and makes the strongly smooth maps into strong submersions. The same problem holds for the moduli spaces $\mathcal{M}^{\mathrm{main}}_{k+1}(M', L', \{J_{1,s}\}_s : \beta; \mathrm{twp}(x); \vec{\mathcal{P}})$ in Fukaya et al. [**8**, §4.6.2], but appears to the authors to have been overlooked.

Definition 4.10. First suppose for simplicity that $\mathcal{T}$ is of dimension m and embedded in $\mathbb{R}^m$. For $k \geqslant 0$, define a new Kuranishi structure κ^m_k on $\mathbb{R}^m$ by the global Kuranishi neighbourhood $(V^m_k, E^m_k, s^m_k, \psi^m_k)$, where $V^m_k = (\mathbb{R}^m)^{k+1}$, and $E^m_k = (\mathbb{R}^m)^{k+1} \times (\mathbb{R}^m)^k$, the trivial vector bundle over V^m_k with fibre $(\mathbb{R}^m)^k$. Define $s^m_k : V^m_k \to E^m_k$ by $s^m_k : (\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k) \mapsto \big((\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k), (\boldsymbol{v}_1 - \boldsymbol{v}_0, \ldots, \boldsymbol{v}_k - \boldsymbol{v}_0)\big)$, for $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k \in \mathbb{R}^m$. Then $(s^m_k)^{-1}(0) = \{(\boldsymbol{v}, \ldots, \boldsymbol{v}) \in (\mathbb{R}^m)^{k+1} : \boldsymbol{v} \in \mathbb{R}^m\}$. Define $\psi^m_k : (s^m_k)^{-1}(0) \to \mathbb{R}^m$ by $\psi^m_k : (\boldsymbol{v}, \ldots, \boldsymbol{v}) \mapsto \boldsymbol{v}$. Define $\pi_i : V^m_k \to \mathbb{R}^m$ for $i = 0, \ldots, k$ by $\pi_i : (\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k) \mapsto \boldsymbol{v}_i$. Then π_i represents a strongly smooth map $\boldsymbol{\pi}_i : (\mathbb{R}^m, \kappa^m_k) \to \mathbb{R}^m$, with $\boldsymbol{\pi}_0 \times \cdots \times \boldsymbol{\pi}_k : (\mathbb{R}^m, \kappa^m_k) \to (\mathbb{R}^m)^{k+1}$ a strong submersion.

Now for $i = 1, \ldots, k$, let $a_i \geqslant 0$ and $f_i : \Delta_{a_i} \to \mathcal{T} \times (L \amalg R)$ be a smooth map. Define the Kuranishi space

$$\begin{aligned} &\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha, \beta, J_t : t \in \mathcal{T}, f_1, \ldots, f_k) = \\ &\quad \big((\mathbb{R}^m, \kappa^m_k) \times_{\boldsymbol{\pi}_0, \mathbb{R}^m, \boldsymbol{\pi}_{\mathcal{T}}} \overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha, \beta, J_t : t \in \mathcal{T})\big) \\ &\quad\quad \times_{(\boldsymbol{\pi}_1 \times \mathbf{ev}_1) \times \cdots \times (\boldsymbol{\pi}_k \times \mathbf{ev}_k), (\mathcal{T} \times (L \amalg R))^k, f_1 \times \cdots \times f_k} (\Delta_{a_1} \times \cdots \times \Delta_{a_k}). \end{aligned} \tag{47}$$

Unlike (46), this is well-defined, as $\boldsymbol{\pi}_0$ and $(\boldsymbol{\pi}_1 \times \mathbf{ev}_1) \times \cdots \times (\boldsymbol{\pi}_k \times \mathbf{ev}_k)$ are strong submersions. Also, the Kuranishi structure of $(\mathbb{R}^m, \kappa^m_k)$ is unchanged by diffeomorphisms of $\mathbb{R}^m$. Thus, by composing the embedding $\mathcal{T} \hookrightarrow \mathbb{R}^m$ with a diffeomorphism of $\mathbb{R}^m$, we see that the Kuranishi structure of $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha, \beta, J_t : t \in \mathcal{T}, f_1, \ldots, f_k)$ is locally independent of the choice of embedding of $\mathcal{T}$ in $\mathbb{R}^m$. In fact, since the Kuranishi structure depends only locally on $\mathcal{T} \hookrightarrow \mathbb{R}^m$, and any $\mathcal{T}$ can be locally embedded in $\mathbb{R}^m$, the Kuranishi structure of $\overline{\mathcal{M}}_{k+1}^{\rm main}(\alpha, \beta, J_t : t \in \mathcal{T}, f_1, \ldots, f_k)$ is well-defined even if $\mathcal{T}$ cannot be globally embedded in $\mathbb{R}^m$.

As for (35), but using (45), (47) and $\mathrm{vdim}(\mathbb{R}^m, \kappa_k^m) = m$, we see that

$$\begin{aligned} &\mathrm{vdim}\, \overline{\mathcal{M}}_{k+1}^{\rm main}(\alpha, \beta, J_t : t \in \mathcal{T}, f_1, \ldots, f_k) = (1-k) \dim \mathcal{T} + \\ &\mu_L(\beta) + k - 2 + n - \textstyle\sum_{i \in I} \eta_{\alpha(i)} + \sum_{0 \neq i \notin I} (a_i - n) + \sum_{0 \neq i \in I} a_i. \end{aligned} \tag{48}$$

As in §4.4, it is convenient to rewrite this using a notion of *shifted cohomological degree*. Let $f : \Delta_a \to \mathcal{T} \times (L \amalg R)$ be a smooth map. Generalizing (36), define

$$\deg f = \begin{cases} \dim \mathcal{T} + n - a - 1, & f(\Delta_a) \subseteq \mathcal{T} \times L, \\ \dim \mathcal{T} + \eta_{(p_-, p_+)} - a - 1, & f(\Delta_a) \subseteq \mathcal{T} \times \{(p_-, p_+)\} \subset \mathcal{T} \times R. \end{cases} \tag{49}$$

Then combining (48) and (49) yields a generalization of (37):

$$\begin{aligned} &\mathrm{vdim}\, \overline{\mathcal{M}}_{k+1}^{\rm main}(\alpha, \beta, J_t : \in \mathcal{T}, f_1, \ldots, f_k) = \\ &\begin{cases} \mu_L(\beta) - 2 + \dim \mathcal{T} + n - \sum_{i=1}^k \deg f_i, & 0 \notin I, \\ \mu_L(\beta) - 2 + \dim \mathcal{T} + n - \sum_{i=1}^k \deg f_i - \eta_{\alpha(0)}, & 0 \in I. \end{cases} \end{aligned} \tag{50}$$

This illustrates something we will see in §5.5, that to generalize from one complex structure J to a family $J_t : t \in \mathcal{T}$, in dimensions or signs we usually change n to $\dim \mathcal{T} + n$, and make no other changes.

Write $\overline{\mathcal{M}}_{k_1+1}^{\rm main}(\alpha_1, \beta_1, J_t : t \in \mathcal{T}, f_1, \ldots, f_{i-1}; f_{i+k_2}, \ldots, f_k)$ for the fibre product

$$\begin{aligned} &\bigl((\mathbb{R}^m, \kappa_{k_1}^m) \times_{\boldsymbol{\pi}_0, \mathbb{R}^m, \boldsymbol{\pi}_{\mathcal{T}}} \overline{\mathcal{M}}_{k_1+1}^{\rm main}(\alpha_1, \beta_1, J_t : t \in \mathcal{T})\bigr) \\ &\times_{(\boldsymbol{\pi}_1 \times \mathbf{ev}_1) \times \cdots (\boldsymbol{\pi}_{i-1} \times \mathbf{ev}_{i-1}) \times (\boldsymbol{\pi}_{i+1} \times \mathbf{ev}_{i+1}) \times \cdots \times (\boldsymbol{\pi}_{k_1} \times \mathbf{ev}_{k_1}), (\mathcal{T} \times (L \amalg R))^{k_1 - 1},} \\ &{}_{f_1 \times \cdots \times f_{i-1} \times f_{i+k_2} \times \cdots \times f_k} (\Delta_{a_1} \times \cdots \times \Delta_{a_{i-1}} \times \Delta_{a_{i+k_2}} \times \cdots \times \Delta_{a_k}). \end{aligned} \tag{51}$$

Its virtual dimension is given by the sum of (40) with $\dim \mathcal{T}$. As for (41) but using (44), and requiring $k > 0$, we find that

$$\begin{aligned} &\partial \overline{\mathcal{M}}_{k+1}^{\rm main}(\alpha, \beta, J_t : t \in \mathcal{T}, f_1, \ldots, f_k) \cong \\ &\coprod_{i=1}^{k} \coprod_{j=0}^{a_i} \overline{\mathcal{M}}_{k+1}^{\rm main}(\alpha, \beta, J_t : t \in \mathcal{T}, f_1, \ldots, f_{i-1}, f_i \circ F_j^{a_i}, f_{i+1}, \ldots, f_k) \\ &\amalg \coprod_{\substack{k_1 + k_2 = k+1,\ 1 \leqslant i \leqslant k_1, \\ I_1 \cup_i I_2 = I,\ \alpha_1 \cup_i \alpha_2 = \alpha, \\ \beta_1 + \beta_2 = \beta}} \begin{aligned}[t] &\overline{\mathcal{M}}_{k_2+1}^{\rm main}(\alpha_2, \beta_2, J_t : t \in \mathcal{T}, f_i, \ldots, f_{i+k_2-1}) \times_{\boldsymbol{\pi}_0 \times \mathbf{ev}, \mathcal{T} \times (L \amalg R), \boldsymbol{\pi}_i \times \mathbf{ev}_i} \\ &\overline{\mathcal{M}}_{k_1+1}^{\rm main}(\alpha_1, \beta_1, J_t : t \in \mathcal{T}, f_1, \ldots, f_{i-1}; f_{i+k_2}, \ldots, f_k), \end{aligned} \end{aligned} \tag{52}$$

in unoriented Kuranishi spaces. Here from (47), the first line of (52) involves a fibre product with $(\mathbb{R}^m, \kappa_k^m)$, but the third line involves fibre products with $(\mathbb{R}^m, \kappa_{k_1}^m)$ and $(\mathbb{R}^m, \kappa_{k_2}^m)$. To match these up, we construct an explicit isomorphism $(\mathbb{R}^m, \kappa_k^m) \cong (\mathbb{R}^m, \kappa_{k_2}^m) \times_{\boldsymbol{\pi}_0, \mathbb{R}^m, \boldsymbol{\pi}_i} (\mathbb{R}^m, \kappa_{k_1}^m)$.

Note that unlike (44), as $k>0$, there are no special contributions to (52) from the boundary $\partial\mathcal{T}$. As for (42), we shall also write

$$\begin{aligned}\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta, J_t: t\in\mathcal{T}, f_1,\ldots,f_k) = \\ \textstyle\coprod_{I\subseteq\{0,\ldots,k\},\ \alpha:I\to R} \bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T},f_1,\ldots,f_k).\end{aligned} \tag{53}$$

This is a disjoint union of Kuranishi spaces of different dimensions.

Remark 4.11. In §8 and §10 the following question will be important. Suppose $\mathcal{T}$ has boundary $\partial\mathcal{T}$, and for each $i=1,\ldots,k$ we have smooth $f_i:\Delta_{a_i}\to\mathcal{T}\times(L\amalg R)$ such that for some $b_i=0,\ldots,a_i$, the boundary map $g_i=f_i\circ F_{b_i}^{a_i}:\Delta_{a_i-1}\to\mathcal{T}\times(L\amalg R)$ maps to $\partial\mathcal{T}\times(L\amalg R)$, and that f_i maps $\Delta_{a_i}\setminus F_{b_i}^{a_i}(\Delta_{a_i-1})$ to $\mathcal{T}^\circ\times(L\amalg R)$, where $\mathcal{T}^\circ$ is the interior of $\mathcal{T}$. Then, what is the relation between $\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T},f_1,\ldots,f_k)$ and $\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\partial\mathcal{T},g_1,\ldots,g_k)$?

The answer is complicated, as if we locally embed $\mathcal{T}\hookrightarrow\mathbb{R}^m$ such that $\partial\mathcal{T}\hookrightarrow\mathbb{R}^{m-1}$, then the definition (47) of $\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T},f_1,\ldots,f_k)$ involves $(\mathbb{R}^m,\kappa_k^m)$, but for $\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\partial\mathcal{T},g_1,\ldots,g_k)$ it involves $(\mathbb{R}^{m-1},\kappa_k^{m-1})$. To give a satisfactory relation we need to impose an extra *transversality condition* for $f_1,\ldots,f_k$ over $\partial\mathcal{T}$:

Condition 4.12. *Assume that $\pi_{\mathcal{T}}\circ f_i:\Delta_{a_i}\to\mathcal{T}$ is transverse to $\partial\mathcal{T}$ along $F_{b_i}^{a_i}(\Delta_{a_i-1})$ for each $i=1,\ldots,k$. That is, for each $p\in F_{b_i}^{a_i}(\Delta_{a_i-1})$ we require $\mathrm{d}(\pi_{\mathcal{T}}\circ f_i)(T_p\Delta_{a_i})+T_{\pi_{\mathcal{T}}\circ f_i(p)}(\partial\mathcal{T})=T_{\pi_{\mathcal{T}}\circ f_i(p)}\mathcal{T}$.*

Supposing that $\mathcal{T}$ is embedded in $\mathbb{R}^m$ such that $\partial\mathcal{T}$ is embedded in $\mathbb{R}^{m-1}\subset\mathbb{R}^m$ locally, and using Condition 4.12, we have isomorphisms

$$\begin{aligned}&\partial\mathcal{T}\times_{i,\mathbb{R}^m,\boldsymbol{\pi}_0}\big((\mathbb{R}^m,\kappa_k^m)\times_{\boldsymbol{\pi}_1\times\cdots\times\boldsymbol{\pi}_k,(\mathbb{R}^m)^k,(\pi_{\mathcal{T}}\circ f_1)\times\cdots\times(\pi_{\mathcal{T}}\circ f_k)}\big(\Delta_{a_1}\times\cdots\times\Delta_{a_k}\big)\cong\\ &\big((\mathbb{R}^{m-1},\kappa_k^{m-1})\times_{\boldsymbol{\pi}_1\times\cdots\times\boldsymbol{\pi}_k,(\mathbb{R}^{m-1})^k,(\pi_{\partial\mathcal{T}}\circ g_1)\times\cdots\times(\pi_{\partial\mathcal{T}}\circ g_k)}\big(\Delta_{a_1-1}\times\cdots\times\Delta_{a_k-1}\big)\big)\\ &\qquad\times\big[\{0\}\times_{i,\mathbb{R},\boldsymbol{\pi}_0}\big((\mathbb{R},\kappa_k^1)\times_{\boldsymbol{\pi}_1\times\cdots\times\boldsymbol{\pi}_k,\mathbb{R}^k,i}[0,\infty)^k\big)\big],\end{aligned} \tag{54}$$

$$\begin{aligned}&(\mathbb{R}^m,\kappa_k^m)\times_{\boldsymbol{\pi}_1\times\cdots\times\boldsymbol{\pi}_k,(\mathbb{R}^m)^k,(\pi_{\mathcal{T}}\circ f_1)\times\cdots\times(\pi_{\mathcal{T}}\circ f_{j-1})\times(\pi_{\mathcal{T}}\circ g_j)\times(\pi_{\mathcal{T}}\circ f_{j+1})\times\cdots\times(\pi_{\mathcal{T}}\circ f_k)}\\ &\qquad\big(\Delta_{a_1}\times\cdots\times\Delta_{a_{j-1}}\times\Delta_{a_j-1}\times\Delta_{a_{j+1}}\times\cdots\times\Delta_{a_k}\big)\cong\\ &\big((\mathbb{R}^{m-1},\kappa_k^{m-1})\times_{\boldsymbol{\pi}_1\times\cdots\times\boldsymbol{\pi}_k,(\mathbb{R}^{m-1})^k,(\pi_{\partial\mathcal{T}}\circ g_1)\times\cdots\times(\pi_{\partial\mathcal{T}}\circ g_k)}\big(\Delta_{a_1-1}\times\cdots\times\Delta_{a_k-1}\big)\big)\\ &\qquad\times\big[(\mathbb{R},\kappa_k^1)\times_{\boldsymbol{\pi}_1\times\cdots\times\boldsymbol{\pi}_k,\mathbb{R}^k,i}[0,\infty)^{j-1}\times\{0\}\times[0,\infty)^{k-j}\big],\end{aligned} \tag{55}$$

for $j=1,\ldots,k$, where i denotes inclusion maps. To prove (54) and (55), we use the isomorphism $(\mathbb{R}^m,\kappa_k^m)\cong(\mathbb{R}^{m-1},\kappa_k^{m-1})\times(\mathbb{R},\kappa_k^1)$ and the isomorphism $\Delta_{a_j}\cong\Delta_{a_j-1}\times[0,\infty)$ near $F_{b_j}^{a_j}(\Delta_{a_j-1})$. Condition 4.12 ensures that the factor $[0,\infty)$ in $\Delta_{a_j}\cong\Delta_{a_j-1}\times[0,\infty)$ locally submerses to the factor $\mathbb{R}$ in $\mathbb{R}^m\cong\mathbb{R}^{m-1}\times\mathbb{R}$.

Equations (47), (54), (55) and properties of fibre products yield isomorphisms

$$\begin{aligned}&\partial\mathcal{T}\times_{i,\mathcal{T},\boldsymbol{\pi}_0}\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J_t:t\in\mathcal{T},f_1,\ldots,f_k)\\ &\quad\cong\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J_t:t\in\partial\mathcal{T},g_1,\ldots,g_k)\\ &\qquad\times\bigl[\{0\}\times_{i,\mathbb{R},\boldsymbol{\pi}_0}\bigl((\mathbb{R},\kappa^1_k)\times_{\boldsymbol{\pi}_1\times\cdots\times\boldsymbol{\pi}_k,\mathbb{R}^k,i}[0,\infty)^k\bigr)\bigr],\end{aligned}\tag{56}$$

$$\begin{aligned}&\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J_t:t\in\mathcal{T},f_1,\ldots,f_{j-1},g_j,f_{j+1},\ldots,f_k)\\ &\quad\cong\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J_t:t\in\partial\mathcal{T},g_1,\ldots,g_k)\\ &\qquad\times\bigl[(\mathbb{R},\kappa^1_k)\times_{\boldsymbol{\pi}_1\times\cdots\times\boldsymbol{\pi}_k,\mathbb{R}^k,i}[0,\infty)^{j-1}\times\{0\}\times[0,\infty)^{k-j}\bigr],\end{aligned}\tag{57}$$

for all $j=1,\ldots,k$. These are the relations we seek between $\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J_t:t\in\mathcal{T},f_1,\ldots,f_k)$ and $\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J_t:t\in\partial\mathcal{T},g_1,\ldots,g_k)$.

Note that the third lines of (56) and (57) are each a point $\{0\}$ with an unusual Kuranishi structure, of virtual dimension 0. Since the Kuranishi maps of these Kuranishi structures are already transverse, when we choose *perturbation data* as in §2.7, they do not need to be perturbed. Hence, from (57), a choice of perturbation data for $\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J_t:t\in\partial\mathcal{T},g_1,\ldots,g_k)$ determines perturbation data for $\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J_t:t\in\mathcal{T},f_1,\ldots,f_{j-1},g_j,f_{j+1},\ldots,f_k)$, which have the same virtual chains. This will enable us to relate $A_{N,K}$ algebras of singular chains on $\mathcal{T}\times(L\amalg R)$ to $A_{N,K}$ algebras of singular chains on $\partial\mathcal{T}\times(L\amalg R)$ in §8–§10.

4.6. Modified moduli spaces $\tilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$. We will see in §5 that defining and computing with orientations on $\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$ is rather complicated. This is mostly to do with the rôle of the operators $\bar\partial_{\lambda_{(p_-,p_+)}}$. We will now define modified, noncompact spaces $\tilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$ whose dimensions and orientations behave in a simpler, more natural way. To compute the sign in some orientation problem for the $\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\cdots)$, it is usually simpler to first work out the answer for the $\tilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\cdots)$. Also, the $\tilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\cdots)$ provide geometric explanations for the notions of *grading* and *shifted cohomological degree* introduced in §4.4.

Definition 4.13. In Definition 4.4, suppose that the families $\lambda_{(p_-,p_+)}$ for all (p_-,p_+) in R have been chosen such that $\eta_{(p_-,p_+)}\geqslant 0$, and $\lambda_{(p_-,p_+)}$ is generic. This genericity implies that $\dim\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}}$ and $\dim\operatorname{Coker}\bar\partial_{\lambda_{(p_-,p_+)}}$ are both as small as possible, so $\dim\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}}=\eta_{(p_-,p_+)}$ and $\dim\operatorname{Coker}\bar\partial_{\lambda_{(p_-,p_+)}}=0$, since $\eta_{(p_-,p_+)}=\operatorname{ind}\bar\partial_{\lambda_{(p_-,p_+)}}\geqslant 0$.

Consider the linear map $\mathrm{ev}_{(-1,0)}:\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}}\to\lambda_{(p_-,p_+)}(-1,0)$ mapping $\mathrm{ev}_{(-1,0)}:\xi\mapsto\xi(-1,0)$. We have $\dim\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}}\leqslant n=\dim\lambda_{(p_-,p_+)}(-1,0)$, since $0\leqslant\eta_{(p_-,p_+)}\leqslant n$ by (32). Thus, genericness implies that $\mathrm{ev}_{(-1,0)}$ is injective, so $\mathrm{ev}_{(-1,0)}\bigl(\mathrm{Ker}\,\bar\partial_{\lambda_{(p_-,p_+)}}\bigr)$ is a vector subspace of $\lambda_{(p_-,p_+)}(-1,0)$ of dimension $\eta_{(p_-,p_+)}$.

As $\lambda_{(p_+,p_-)}(x,y)\equiv\lambda_{(p_-,p_+)}(x,-y)$ we have $\lambda_{(p_+,p_-)}(-1,0)=\lambda_{(p_-,p_+)}(-1,0)$. Hence $\mathrm{ev}_{(-1,0)}\bigl(\mathrm{Ker}\,\bar\partial_{\lambda_{(p_-,p_+)}}\bigr)$ and $\mathrm{ev}_{(-1,0)}\bigl(\mathrm{Ker}\,\bar\partial_{\lambda_{(p_+,p_-)}}\bigr)$ are subspaces of $\lambda_{(p_-,p_+)}(-1,0)\cong\mathbb{R}^n$, of dimensions $\eta_{(p_-,p_+)}$ and $\eta_{(p_+,p_-)}=n-\eta_{(p_-,p_+)}$. By genericness they intersect transversely, so that

$$\lambda_{(p_-,p_+)}(-1,0)=\mathrm{ev}_{(-1,0)}\bigl(\mathrm{Ker}\,\bar\partial_{\lambda_{(p_-,p_+)}}\bigr)\oplus\mathrm{ev}_{(-1,0)}\bigl(\mathrm{Ker}\,\bar\partial_{\lambda_{(p_+,p_-)}}\bigr). \tag{58}$$

In §5 we will choose orientations for the $\mathrm{Ker}\,\bar\partial_{\lambda_{(p_-,p_+)}}$, and so we can ask whether or not (58) holds in oriented vector spaces.

In the situation of Definition 4.2, define

$$\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)=\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)\times\textstyle\prod_{i\in I}\mathrm{Ker}\,\bar\partial_{\lambda_{\alpha(i)}}. \tag{59}$$

We write elements of $\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$ as $\bigl([\Sigma,\vec z,u,l,\bar u],\xi_i:i\in I\bigr)$, for $[\Sigma,\vec z,u,l,\bar u]\in\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$ and $\xi_i\in\mathrm{Ker}\,\bar\partial_{\lambda_{\alpha(i)}}$. When for computing orientations we need to regard (59) as an *ordered* product, since $I\subseteq\{0,\ldots,k\}$ we regard the product $\prod_{i\in I}$ as occurring in the natural order $\leqslant$ on I. We interpret $\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$ as a *Kuranishi space*, since the $\mathrm{Ker}\,\bar\partial_{\lambda_{\alpha(i)}}$ are manifolds of dimension $\eta_{\alpha(i)}$ and $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$ is a Kuranishi space from §4.1. Equation (33) implies the simpler equation

$$\mathrm{vdim}\,\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)=\mu_L(\beta)+k-2+n. \tag{60}$$

This is *independent of* α, and agrees with Fukaya et al. [**8**, Prop. 7.1.1] in the embedded case.

Define $\tilde R=\coprod_{(p_-,p_+)\in R}\bigl(\{(p_-,p_+)\}\times\lambda_{(p_-,p_+)}(-1,0)\bigr)$. Then $\tilde R$ is an n-manifold, as each $\lambda_{(p_-,p_+)}(-1,0)\cong\mathbb{R}^n$. Thus $L\amalg\tilde R$ is an n-manifold. It is nicer to work with than $L\amalg R$, the disjoint union of an n-manifold and a 0-manifold. Define *modified evaluation maps* $\widetilde{\mathrm{ev}}_i:\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)\to L\amalg\tilde R$ by

$$\widetilde{\mathrm{ev}}_i\bigl([\Sigma,\vec z,u,l,\bar u],\xi_i:i\in I\bigr)=\begin{cases}\bar u(\zeta_i)\in L, & i\notin I,\\ \bigl(\alpha(i),\mathrm{ev}_{(-1,0)}(\xi_i)\bigr)\in\tilde R, & i\in I,\end{cases} \tag{61}$$

for $i=0,\ldots,k$, and $\widetilde{\mathrm{ev}}:\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)\to L\amalg\tilde R$ by

$$\widetilde{\mathrm{ev}}\bigl([\Sigma,\vec z,u,l,\bar u],\xi_i:i\in I\bigr)=\begin{cases}\bar u(\zeta_0)\in L, & 0\notin I,\\ \bigl(\sigma\circ\alpha(0),\mathrm{ev}_{(-1,0)}(\xi_0)\bigr)\in\tilde R, & 0\in I.\end{cases} \tag{62}$$

As for $\mathrm{ev}_i,\mathrm{ev}$, these extend to *strongly smooth maps* $\widetilde{\mathbf{ev}}_i,\widetilde{\mathbf{ev}}:\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)\to L\amalg\tilde R$ at the Kuranishi space level. They are not strong submersions, since the maps $\mathrm{ev}_{(-1,0)}:\mathrm{Ker}\,\bar\partial_{\lambda_{(p_-,p_+)}}\to\lambda_{(p_-,p_+)}(-1,0)$ are not submersions, but this will not matter in the fibre products in (63) and elsewhere, because of the transverseness of the subspaces in (58).

We can now generalize (27) to an isomorphism of unoriented Kuranishi spaces:

$$\partial\widetilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)\cong\coprod_{\substack{k_1+k_2=k+1,\;1\leqslant i\leqslant k_1,\;I_1\cup_i I_2=I,\\ \alpha_1\cup_i\alpha_2=\alpha,\;\beta_1+\beta_2=\beta}}\begin{aligned}&\widetilde{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J)\times_{\widetilde{\mathbf{ev}},L\amalg\tilde R,\widetilde{\mathbf{ev}}_i}\\ &\widetilde{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J).\end{aligned}\tag{63}$$

Note that if $i\in I_1$ and $0\notin I_2$, or if $i\notin I_1$ and $0\in I_2$, then the fibre products in (27) and (63) are empty, since one side maps to L, and the other to R or $\tilde R$. Thus, to deduce (63) from (27), for fixed $i,\ldots,\beta_2$ we may divide into the two cases (a) $i\notin I_1$ and $0\notin I_2$, and (b) $i\in I_1$ and $0\in I_2$.

In case (a), the right hand sides of (27) and (63) are both fibre products over L, and to see they are isomorphic we have to give an isomorphism between the extra factors $\prod_{j\in I}\operatorname{Ker}\bar\partial_{\lambda_{\alpha(j)}}$ from $\widetilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$ on the left, and $\prod_{j\in I_1}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(j)}}\times\prod_{j\in I_2}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(j)}}$ from $\widetilde{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J)$ and $\widetilde{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J)$ on the right. In this case, (28) defines an isomorphism between I and $I_1\amalg I_2$ which identifies α and $\alpha_1\amalg\alpha_2$, which induces an isomorphism between $\prod_{j\in I}\operatorname{Ker}\bar\partial_{\lambda_{\alpha(j)}}$ and $\prod_{j\in I_1}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(j)}}\times\prod_{j\in I_2}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(j)}}$ from $\widetilde{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J)$.

In case (b), equation (27) is a fibre product over R, and equation (63) a fibre product over $\tilde R$. By (24)–(25) and (61)–(62), both can only be nonempty if $\alpha_1(i)=\sigma\circ\alpha_2(0)$, so we suppose this. Set $\alpha_1(i)=(p_-,p_+)$ in R, so that $\alpha_2(0)=(p_+,p_-)$, and let $p=\iota(p_-)=\iota(p_+)$. Then the term in (27) is a fibre product over the point $\{(p_-,p_+)\}$, that is, it is just a product. The term in (63) is a fibre product over the Lagrangian subspace $\lambda_{(p_-,p_+)}(-1,0)$ in T_pM, and $\widetilde{\mathbf{ev}}_i$ maps the factor $\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(i)}}$ from $\widetilde{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J)$ to $\lambda_{(p_-,p_+)}(-1,0)$ by $\mathrm{ev}_{(-1,0)}$, and $\widetilde{\mathbf{ev}}$ maps the factor $\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(0)}}$ from $\widetilde{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J)$ to $\lambda_{(p_-,p_+)}(-1,0)$ by $\mathrm{ev}_{(-1,0)}$.

Since (58) is a direct sum, and $\mathrm{ev}_{(-1,0)}$ are embeddings, the fibre product of these two factors over $\lambda_{(p_-,p_+)}(-1,0)$ is just a point. The remaining extra factors $\prod_{i\neq j\in I_1}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(j)}}\times\prod_{0\neq j\in I_2}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(j)}}$ from $\widetilde{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J),\widetilde{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J)$ are identified with $\prod_{j\in I}\operatorname{Ker}\bar\partial_{\lambda_{\alpha(j)}}$ from $\widetilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$ using (28) as in case (a). This proves (63). Note that (63) is a fibre product over the n-manifold $L\amalg\tilde R$. This makes it easier to work with than (27), which is a fibre product over the disjoint union of an n-manifold L, and a 0-manifold R.

As for (26) and (42), we shall write

$$\widetilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J)=\coprod\nolimits_{\substack{I\subseteq\{0,\ldots,k\},\\ \alpha:I\to R}}\widetilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J).\tag{64}$$

Since by (60) the virtual dimension of $\widetilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$ is independent of I,α, this is a Kuranishi space, possibly noncompact because of the vector space $\prod_{i\in I}\operatorname{Ker}\bar\partial_{\lambda_{\alpha(i)}}$, of virtual dimension (60), another illustration of how the $\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$ are better behaved that the $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$. We define strong smooth maps $\widetilde{\mathbf{ev}}_i,\widetilde{\mathbf{ev}}:\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J)\to L\amalg\tilde R$ to be $\widetilde{\mathbf{ev}}_i,\widetilde{\mathbf{ev}}$ on each $\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$.

4.7. The moduli spaces $\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)$. We define modified versions $\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)$, $\widetilde{\mathcal{M}}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)$ of the moduli spaces of §4.4, in a similar way to §4.6.

Definition 4.14. In the situation of Definition 4.7, define

$$\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)=\begin{cases}\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k), & 0\notin I,\\ \overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha(0)}}, & 0\in I,\end{cases}\tag{65}$$

$$\widetilde{\mathcal{M}}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)=\tag{66}$$
$$\begin{cases}\overline{\mathcal{M}}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k), & 0,i\notin I_1,\\ \overline{\mathcal{M}}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(0)}}, & 0\in I_1,\ i\notin I_1,\\ \overline{\mathcal{M}}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(i)}}, & 0\notin I_1,\ i\in I_1,\\ \overline{\mathcal{M}}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(0)}}\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(i)}}, & 0,i\in I_1.\end{cases}$$

Then equations (37) and (40) imply the simpler formulae

$$\begin{aligned}&\operatorname{vdim}\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)=\mu_L(\beta)-2+n-\textstyle\sum_{i=1}^k\deg f_i,\\ &\operatorname{vdim}\widetilde{\mathcal{M}}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)=\\ &\qquad\mu_L(\beta_1)-1+n-\textstyle\sum_{j=1}^{i-1}\deg f_j-\sum_{j=i+k_2}^k\deg f_j.\end{aligned}\tag{67}$$

Suppose now that $f_i:\Delta_{a_i}\to L\amalg R$ maps to L if $i\notin I$, and to $\alpha(i)\in R$ if $i\in I$. As above, if this does not hold then $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)=\emptyset=\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)$. Then (65) is equivalent to

$$\begin{aligned}\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)&=\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)\\ &\times_{\prod_{i=1}^k\widetilde{\mathbf{ev}}_i,(L\amalg\tilde R)^k,\prod_{i=1}^k\left\{\begin{smallmatrix}f_i, & i\notin I\\ f_i\times\mathrm{ev}_{(-1,0)}, & i\in I\end{smallmatrix}\right\}}\prod_{i=1}^k\begin{Bmatrix}\Delta_{a_i}, & i\notin I\\ \Delta_{a_i}\times\operatorname{Ker}\bar\partial_{\lambda_{\sigma\circ\alpha(i)}}, & i\in I\end{Bmatrix}.\end{aligned}\tag{68}$$

The difference between (65) and (68) is that in (68) we have extra factors $\prod_{0\neq i\in I}\operatorname{Ker}\bar\partial_{\lambda_{\alpha(i)}}$ in $\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$ (we exclude 0 because of the factor $\operatorname{Ker}\bar\partial_{\lambda_{\alpha(0)}}$ in (65)) and $\prod_{0\neq i\in I}\operatorname{Ker}\bar\partial_{\lambda_{\sigma\circ\alpha(i)}}$ in $\prod_{0\neq i\in I}\Delta_{a_i}\times\operatorname{Ker}\bar\partial_{\lambda_{\sigma\circ\alpha(i)}}$. However, we are taking a fibre product over $(L\amalg\tilde R)^k$ rather than $(L\amalg R)^k$. The effect of this is that for each $0\neq i\in I$, in (68) we take the fibre product $\operatorname{Ker}\bar\partial_{\lambda_{\alpha(i)}}\times_{\mathrm{ev}_{(-1,0)},\lambda_{\alpha(i)}(-1,0),\mathrm{ev}_{(-1,0)}}\operatorname{Ker}\bar\partial_{\lambda_{\sigma\circ\alpha(i)}}$, which is just a point by (58) and injectivity of the $\mathrm{ev}_{(-1,0)}$. Thus (65) and (68) differ only by the product with $|I\setminus\{0\}|$ points, so they are equivalent.

Similarly, using (39) we find (66) is equivalent to the fibre product

$$\begin{aligned}(69)\quad &\widetilde{\mathcal{M}}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)=\widetilde{\mathcal{M}}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J)\\ &\times_{\widetilde{\mathbf{ev}}_1\times\cdots\times\widetilde{\mathbf{ev}}_{i-1}\times\widetilde{\mathbf{ev}}_{i+1}\times\cdots\times\widetilde{\mathbf{ev}}_{k_1},(L\amalg\tilde R)^{k_1-1},\prod_{\substack{j=1,\ldots,k:\\ j<i\text{ or }j\geqslant i+k_2}}\left\{\begin{smallmatrix}f_j, & j\notin I\\ f_j\times\mathrm{ev}_{(-1,0)}, & j\in I\end{smallmatrix}\right\}}\\ &\prod_{\substack{j=1,\ldots,k:\\ j<i\text{ or }j\geqslant i+k_2}}\begin{Bmatrix}\Delta_{a_j}, & j\notin I\\ \Delta_{a_j}\times\operatorname{Ker}\bar\partial_{\lambda_{\sigma\circ\alpha(j)}}, & j\in I\end{Bmatrix}.\end{aligned}$$

Combining (63), (68), (69) we find that by analogy with (41) we have

$$\begin{aligned}(70)\qquad &\partial\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)\cong\\ &\coprod_{i=1}^{k}\coprod_{j=0}^{a_i}\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_{i-1},f_i\circ F_j^{a_i},f_{i+1},\ldots,f_k)\\ &\amalg\coprod_{\substack{k_1+k_2=k+1,\ 1\leqslant i\leqslant k_1,\\ I_1\cup_i I_2=I,\ \alpha_1\cup_i\alpha_2=\alpha,\\ \beta_1+\beta_2=\beta}}\begin{aligned}[t]&\widetilde{\mathcal{M}}^{\mathrm{main}}_{k_2+1}(\alpha_2,\beta_2,J,f_i,\ldots,f_{i+k_2-1})\times_{\widetilde{\mathbf{ev}},L\amalg\tilde R,\widetilde{\mathbf{ev}}_i}\\ &\widetilde{\mathcal{M}}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k),\end{aligned}\end{aligned}$$

in unoriented Kuranishi spaces.

As for (26), (42) and (64) we shall also write

$$\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J,f_1,\ldots,f_k)=\coprod_{\substack{I\subseteq\{0,\ldots,k\},\\ \alpha:I\to R}}\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k).$$

This is a Kuranishi space, of dimension (67), which may be noncompact because of the factor $\operatorname{Ker}\bar\partial_{\lambda_{\alpha(0)}}$ in (65). We define $\widetilde{\mathbf{ev}}:\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J,f_1,\ldots,f_k)\to L\amalg\tilde R$ to be $\widetilde{\mathbf{ev}}$ on each $\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)$.

We can now explain the notion of *shifted cohomological degree* in Definition 4.7, and the grading it induces on $C^{\mathrm{si}}_*(L\amalg R;\mathbb{Q})$. Suppose $f:\Delta_a\to L\amalg R$ is smooth. By (36), if f maps to L then $\deg f=n-a-1$, which is the (virtual) codimension of $f(\Delta_a)$ in L minus one. But if f maps to (p_-,p_+) in R then $\deg f=\eta_{(p_-,p_+)}-a-1$. Here is a good way to understand this. Morally, we want to lift f to a map $\tilde f$ to the n-manifold $L\amalg\tilde R$. Since f maps to $\{(p_-,p_+)\}\subset R$, the lift $\tilde f$ should map to $\{(p_-,p_+)\}\times\lambda_{(p_-,p_+)}(-1,0)\subset\tilde R$. But the domain of f should not be Δ_a. Motivated by (68), we see that $f:\Delta_a\to\{(p_-,p_+)\}\subset R$ should lift to

$$\tilde f=f\times\mathrm{ev}_{(-1,0)}:\Delta_a\times\operatorname{Ker}\bar\partial_{\lambda_{(p_+,p_-)}}\longrightarrow\{(p_-,p_+)\}\times\lambda_{(p_-,p_+)}(-1,0)\subset\tilde R.$$

This is not a chain in $C^{\mathrm{si}}_*(L\amalg\tilde R;\mathbb{Q})$, as $\Delta_a\times\operatorname{Ker}\bar\partial_{\lambda_{\sigma(p_-,p_+)}}$ is not a simplex. But it does justify the change in degree in (36). We have $\dim\bigl(\Delta_a\times\operatorname{Ker}\bar\partial_{\lambda_{(p_+,p_-)}}\bigr)=a+\eta_{(p_+,p_-)}=a+n-\eta_{(p_-,p_+)}$ by (32). Thus, the (virtual) codimension of $\tilde f\bigl(\Delta_a\times\operatorname{Ker}\bar\partial_{\lambda_{(p_+,p_-)}}\bigr)$ in $\tilde R$ minus one is $n-(a+n-\eta_{(p_-,p_+)})-1=\eta_{(p_-,p_+)}-a-1=\deg f$. Hence, when we lift to modified moduli spaces in this way, the shifted cohomological degree $\deg f$ is the genuine shifted cohomological degree of the 'chain' $\tilde{f}$ in $L \amalg \tilde{R}$.

We could also easily define modified versions of the moduli spaces of §4.5 for families of complex structures, but we will not, as we only need the modified spaces for motivation anyway.

5. Orientations

We now define orientations on the Kuranishi spaces defined in §4, and prove formulae for their boundaries in oriented Kuranishi spaces, so computing the appropriate signs in (27), (41), (63), and (70).

5.1. Orientations on $\widetilde{\mathcal{M}}^{\text{main}}_{k+1}(\alpha,\beta,J)$.

Fukaya et al. [**8**, Def. 8.1.2] define *relative spin structures* on L. We adapt their definition to the immersed case.

Definition 5.1. Let $\iota : L \to M$ be an immersed submanifold with transverse self-intersections in M. Fix triangulations of L and M compatible under ι. This can be done by triangulating the self-intersection of $\iota(L)$ in M, then extending this to a triangulation of $\iota(L)$ which pulls back to one of L, and then extending the triangulation of $\iota(L)$ to one of M. A *relative spin structure* for $\iota : L \to M$ consists of an orientation on L; a class $\mathrm{st} \in H^2(M;\mathbb{Z}_2)$ such that $\iota^*(\mathrm{st}) = w_2(L) \in H^2(L;\mathbb{Z}_2)$, the second Stiefel–Whitney class of L; an oriented vector bundle V on the 3-skeleton $M_{[3]}$ of M with $w_2(V) = \mathrm{st}$; and a spin structure on $(TL \oplus \iota^*(V))|_{L_{[2]}}$.

Here $L_{[2]}$ is the 2-skeleton of L, and as $w_2(V|_{L_{[2]}}) = \iota^*(\mathrm{st})|_{L_{[2]}} = w_2(L)|_{L_{[2]}}$ we have $w_2\big((TL\oplus\iota^*(V))|_{L_{[2]}}\big) = 0$, so $(TL\oplus\iota^*(V))|_{L_{[2]}}$ admits a spin structure. If L is spin then $w_2(L) = 0$, so we can take $\mathrm{st} = 0$ and $V = 0$ and the spin structure on $TL|_{L_{[2]}}$ to be the restriction of that on TL. Hence, an orientation and spin structure on L induce a relative spin structure for $\iota : L \to M$.

We first construct orientations on the modified spaces $\widetilde{\mathcal{M}}^{\text{main}}_{k+1}(\alpha,\beta,J)$ of §4.6.

Theorem 5.2. *Let (M,ω) be a compact symplectic manifold with compatible almost complex structure J, and $\iota : L \to M$ a compact Lagrangian immersion with only transverse double self-intersections. Then choices of a relative spin structure for $\iota : L \to M$, and of $\lambda_{(p_-,p_+)}$ for $(p_-,p_+) \in R$ as in* §4.3, *determine orientations on the modified Kuranishi spaces $\widetilde{\mathcal{M}}^{\text{main}}_{k+1}(\alpha,\beta,J)$ of* §4.6 *for all k,α,β.*

Proof. Let $[\Sigma,\vec{z},u,l,\bar{u}] \in \overline{\mathcal{M}}^{\text{main}}_{k+1}(\alpha,\beta,J)$, so that $\vec{z} = (z_0,\dots,z_k)$ with $z_0,\dots,z_k$ distinct smooth points of $\partial\Sigma$. For each $i \in I$ we choose a small open neighbourhood U_i of z_i in Σ, such that $U_i \cap U_j = \emptyset$ if $i \neq j \in I$, and $z_j \notin U_i$ if $i \neq j \notin I$, and $U_i \setminus \{z_i\}$ is biholomorphic to $(-\infty, 0) \times [-1, 1]$, where z_i corresponds to $-\infty$. We identify U_i with $\{-\infty\} \cup (-\infty, 0) \times [-1, 1]$, and define

$$U_i^r = \{-\infty\} \cup (-\infty, -r) \times [-1, 1] \subset U_i,$$

for $r > 0$. For $i \in I$ we also define

$$Y_i^r = \bigl\{(x,y) \in \mathbb{R}^2 : \text{either } x \leqslant 0,\ x^2+y^2 \leqslant 1 \text{ or } 0 \leqslant x \leqslant r,\ |y| \leqslant 1\bigr\} \subset Y,$$

where Y is as in (29), and we set $y_i = (-1, 0) \in Y_i^r$. For $j \notin I$ we define $y_j = z_j \in \Sigma \setminus \bigcup_{i \in I} U_i$. Glue $\Sigma \setminus \bigcup_{i \in I} U_i^r$ and $\bigcup_{i \in I} Y_i^r$ by identifying $\{-r\} \times [-1, 1] \subset U_i$ with $\{r\} \times [-1, 1] \subset Y_i^r$ to make $(\Sigma^r, y_0, \ldots, y_k)$, which is diffeomorphic to $(\Sigma, z_0, \ldots, z_k)$. Consider the linearized Cauchy–Riemann operator

$$\begin{aligned} D_u\bar\partial : W^{1,q}\bigl(\Sigma \setminus \{z_i : i \in I\}, \partial\Sigma \setminus \{z_i : i \in I\}; u^*(TM), u^*(\mathrm{d}\iota(TL)\bigr) \\ \longrightarrow L^q\bigl(\Sigma \setminus \{z_i : i \in I\}; u^*(TM) \otimes \Lambda^{0,1}(\Sigma \setminus \{z_i : i \in I\})\bigr), \end{aligned}$$

for $q > 2$, and define the virtual vector space

$$\operatorname{Ind} D_u\bar\partial = \operatorname{Ker} D_u\bar\partial \ominus \operatorname{Coker} D_u\bar\partial.$$

Here for a Fredholm operator P, we will write $\operatorname{ind} P = \dim \operatorname{Ker} P - \dim \operatorname{Coker} P$ in $\mathbb{Z}$, and $\operatorname{Ind} P = \operatorname{Ker} P \ominus \operatorname{Coker} P$ as a virtual vector space.

By a suitable partition of unity, we define differential operators

$$D_{u,\lambda_\alpha} : W^{1,q}(\Sigma^r, \partial\Sigma^r; E_u, F_{u,\lambda_\alpha}) \longrightarrow L^q(\Sigma^r; E_u \otimes \Lambda^{0,1}\Sigma^r),$$

for $q > 2$ and large r, whose restrictions to $\Sigma \setminus \bigcup_{i \in I} U_i^{r-1}$ and Y_i^{r-1} coincide with $D_u\bar\partial$ and $\bar\partial_{\lambda_{\alpha(i)}}$, respectively, and we define the virtual vector space

$$\operatorname{Ind} D_{u,\lambda_\alpha} = \operatorname{Ker} D_{u,\lambda_\alpha} \ominus \operatorname{Coker} D_{u,\lambda_\alpha}.$$

Here $E_u \to \Sigma^r$ is a complex vector bundle agreeing with $u^*(TM)$ on $\Sigma \setminus \bigcup_{i \in I} U_i^r$, and is trivial with fibre $T_{p_i}M$ on Y_i^r for $i \in I$, where $\alpha(i) = (p_-, p_+) \in R$ with $\iota(p_-) = \iota(p_+) = p_i$. Also F_{u,λ_α} is a real vector subbundle of $E_u|_{\partial\Sigma^r}$ which agrees with $\mathrm{d}\iota(TL)$ on $\partial\Sigma \setminus \bigcup_{i \in I} U_i^r$, and with $\lambda_{\alpha(i)}$ on ∂Y_i^r for $i \in I$, except near $\{-r\} \times [-1, 1]$ where we interpolate between these two values. The notation λ_α in D_{u,λ_α} and F_{u,λ_α} denotes that these depend on the choice of $\lambda_{\alpha(i)}$ for all $i \in I$, where $\alpha(i) = (p_-, p_+) \in R$ and $\lambda_{(p_-,p_+)}$ is as in §4.3. Then, by a gluing theorem for large r, we have an isomorphism of virtual vector spaces

$$\operatorname{Ind} D_u\bar\partial \oplus \textstyle\bigoplus_{i \in I} \operatorname{Ker} \bar\partial_{\lambda_{\alpha(i)}} \cong \operatorname{Ind} D_{u,\lambda_\alpha}, \tag{71}$$

since $\operatorname{Coker} \bar\partial_{\lambda_{\alpha(i)}} = 0$ as in §4.6. Really this holds in the limit $r \to \infty$.

The *virtual tangent bundle* of $\overline{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J)$ is

$$\bigcup_{[\Sigma,\vec z,u,l,\bar u]\in\overline{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J)}(\operatorname{Ind} D_u\bar\partial\oplus T_{[\Sigma,\vec z]}\overline{\mathcal{M}}^{\rm main}_{k+1})\longrightarrow\overline{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J),$$

where $\overline{\mathcal{M}}^{\rm main}_{k+1}$ is the moduli space of isomorphism classes of genus 0 prestable bordered Riemann surfaces with $k+1$ distinct smooth boundary marked points ordered counter-clockwise. Combining this with (59) and using (71) shows that in the limit $r\to\infty$, the *virtual tangent bundle* of $\widetilde{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J)$ is

$$\bigcup_{(\xi_i:i\in I,[\Sigma,\vec z,u,l,\bar u])\in\widetilde{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J)}(\operatorname{Ind} D_{u,\lambda_\alpha}\oplus T_{[\Sigma,\vec z]}\overline{\mathcal{M}}^{\rm main}_{k+1})\longrightarrow\widetilde{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J). \tag{72}$$

Since $\overline{\mathcal{M}}^{\rm main}_{k+1}$ is oriented [**8**, §2.2.2], [**17**, §4.5], the factor $T_{[\Sigma,\vec z]}\overline{\mathcal{M}}^{\rm main}_{k+1}$ in (72) is oriented. As in the embedded case [**8**, §8.1], a relative spin structure for $\iota:L\to M$ canonically determines a homotopy type of trivializations of F_{u,λ_α}, which gives an orientation of $\operatorname{Ind} D_{u,\lambda_\alpha}$. This is obtained by gluing in $\lambda_{\alpha(i)}$ at z_i for $i\in I$, and so also depends on the choice of $\lambda_{(p_-,p_+)}$ for (p_-,p_+) in R. Combining these two gives an orientation for the virtual tangent bundle (72), and hence an orientation on the Kuranishi space $\widetilde{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J)$. q.e.d.

With these orientations, we compute the signs in (63).

Theorem 5.3. *In the situation of* §4.6, *with the orientations for* $\widetilde{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J)$ *in Theorem* 5.2 *and the conventions of* §2.4, *the orientations of* $\partial\widetilde{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J)$ *and* $\widetilde{\mathcal{M}}^{\rm main}_{k_1+1}(\alpha_1,\beta_1,J)\times_{\widetilde{\mathbf{ev}}_i,L\amalg\tilde R,\widetilde{\mathbf{ev}}}\widetilde{\mathcal{M}}^{\rm main}_{k_2+1}(\alpha_2,\beta_2,J)$ *in* (63) *differ by a factor* $(-1)^{(k_1-i)(k_2-1)+(n+k_1)}$, *so that in oriented Kuranishi spaces we have*

$$\begin{aligned}\partial\widetilde{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J)\cong\coprod_{\substack{k_1+k_2=k+1,\ 1\leqslant i\leqslant k_1,\\ I_1\cup_i I_2=I,\\ \alpha_1\cup_i\alpha_2=\alpha,\ \beta_1+\beta_2=\beta}}&(-1)^{n+i+ik_2}\widetilde{\mathcal{M}}^{\rm main}_{k_2+1}(\alpha_2,\beta_2,J)\\ &\times_{\widetilde{\mathbf{ev}},L\amalg\tilde R,\widetilde{\mathbf{ev}}_i}\widetilde{\mathcal{M}}^{\rm main}_{k_1+1}(\alpha_1,\beta_1,J).\end{aligned} \tag{73}$$

Proof. Suppose $[\Sigma,\vec z,u,l,\bar u]$ in $\partial\overline{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J)$ is identified in (27) with a point in $\overline{\mathcal{M}}^{\rm main}_{k_2+1}(\alpha_2,\beta_2,J)\times_{\mathbf{ev},L\amalg R,\mathbf{ev}_i}\overline{\mathcal{M}}^{\rm main}_{k_1+1}(\alpha_1,\beta_1,J)$ represented by $[\Sigma_1,\vec z_1,u_1,l_1,\bar u_1]\in\overline{\mathcal{M}}^{\rm main}_{k_1+1}(\alpha_1,\beta_1,J)$ and $[\Sigma_2,\vec z_2,u_2,l_2,\bar u_2]\in\overline{\mathcal{M}}^{\rm main}_{k_2+1}(\alpha_2,\beta_2,J)$. Then $\vec z_1=(z^1_0,\dots,z^1_{k_1})$ and $\vec z_2=(z^2_0,\dots,z^2_{k_2})$, and as the point lies in the fibre product we have $u_1(z^1_i)=u_2(z^2_0)$ in $\iota(L)$, and either $i\notin I_1$ and $0\notin I_2$, or $i\in I_1$ and $0\in I_2$ and $\alpha_1(i)=\sigma\circ\alpha_2(0)$ in R, noting the differing definitions of $\mathrm{ev}_i,\mathrm{ev}$ in (24), (25). From these we make $(\Sigma^r,y_0,\dots,y_k),(\Sigma^r_1,y^1_0,\dots,y^1_{k_1}),(\Sigma^r_2,y^2_0,\dots,y^2_{k_2})$, and smoothed operators $D_{u,\lambda_\alpha},D_{u_1,\lambda_{\alpha_1}},D_{u_2,\lambda_{\alpha_2}}$ upon them, as in the proof of Theorem 5.2. The following lemma is then proved as in [**8**, Lem. 8.3.5]:

Lemma 5.4. *We have an isomorphism of oriented virtual vector spaces*

$$\operatorname{Ind} D_{u,\lambda_\alpha} \cong \operatorname{Ind} D_{u_2,\lambda_{\alpha_2}} \times_{\bar{\mathrm{ev}},T(L\amalg\tilde R),\bar{\mathrm{ev}}_i} \operatorname{Ind} D_{u_1,\lambda_{\alpha_1}}, \tag{74}$$

where, for $\xi_1 \in W^{1,q}(\Sigma_1^r, \partial\Sigma_1^r; E_{u_1}, F_{u_1,\lambda_{\alpha_1}})$,

$$\bar{\mathrm{ev}}_i(\xi_1) = \begin{cases} \xi_1(y_i^1) \in T_{u_1(z_i^1)}L, & i \notin I_1, \\ \xi_1(y_i^1) \in \lambda_{\alpha_1(i)}(-1,0), & i \in I_1, \end{cases} \tag{75}$$

and, for $\xi_2 \in W^{1,q}(\Sigma_2^r, \partial\Sigma_2^r; E_{u_2}, F_{u_2,\lambda_{\alpha_2}})$, *we define*

$$\bar{\mathrm{ev}}(\xi_2) = \begin{cases} \xi_2(y_0^2) \in T_{u_2(z_0^2)}L, & 0 \notin I_2, \\ \xi_2(y_0^2) \in \lambda_{\sigma\circ\alpha_2(0)}(-1,0), & 0 \in I_2. \end{cases} \tag{76}$$

Since L is oriented and $\lambda_{(p_-,p_+)}$ is compatible with orientations, $\mu_L(\beta)$ is even. Thus we obtain the following corollary, proved as in Fukaya et al. [**8**, Prop. 8.3.3]. For reasons to be explained in Remark 5.14(b), we have reversed the order of their fibre product, as for (27) in §4.3, so the sign in (73) is not the same as that in [**8**, Prop. 8.3.3]; the difference can be computed using the second line of (5).

Corollary 5.5. *We have isomorphisms of oriented virtual vector spaces*

$$\begin{aligned} &\operatorname{Ind} D_{u,\lambda_\alpha} \oplus T_{[\Sigma,z_0,\ldots,z_k]}\partial\overline{\mathcal{M}}{}^{\mathrm{main}}_{k+1} \cong \\ &(-1)^{n+i+ik_2}(\operatorname{Ind} D_{u_2,\lambda_{\alpha_2}} \oplus T_{[\Sigma_2,z_0^2,\ldots,z_{k_2}^2]}\overline{\mathcal{M}}{}^{\mathrm{main}}_{k_2+1}) \\ &\times_{\bar{\mathrm{ev}},T(L\amalg\tilde R),\bar{\mathrm{ev}}_i} (\operatorname{Ind} D_{u_1,\lambda_{\alpha_1}} \oplus T_{[\Sigma_1,z_0^1,\ldots,z_{k_1}^1]}\overline{\mathcal{M}}{}^{\mathrm{main}}_{k_1+1}). \end{aligned} \tag{77}$$

By (72), in the limit $r \to \infty$ the three terms in (77) are the virtual tangent bundles of $\partial\widetilde{\mathcal{M}}{}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$, $\widetilde{\mathcal{M}}{}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J)$, $\widetilde{\mathcal{M}}{}^{\mathrm{main}}_{k_2+1}(\alpha_2,\beta_2,J)$. By comparing (61)–(62) and (75)–(76) we see that in the limit $r \to \infty$, the fibre product '$\cdots \times_{\bar{\mathrm{ev}},T(L\amalg\tilde R),\bar{\mathrm{ev}}_i} \cdots$' in (77) becomes that induced on virtual tangent bundles by the fibre product '$\cdots \times_{\widetilde{\mathbf{ev}},L\amalg\tilde R,\widetilde{\mathbf{ev}}_i} \cdots$' in (63) and (73). Taking the limit $r \to \infty$, equation (77) now implies the oriented virtual tangent bundle version of (73), so Theorem 5.3 follows from this and (63). q.e.d.

5.2. Orientations on $\widetilde{\mathcal{M}}{}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)$. Next we orient the spaces of §4.7.

Definition 5.6. In the situation of §4.3, choose orientations $o_{(p_-,p_+)}$ on the vector spaces $\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}}$ for all (p_-,p_+) in R. In equation (58), $\lambda_{(p_-,p_+)}(-1,0)$ is an oriented Lagrangian subspace of T_pM, and the maps $\mathrm{ev}_{(-1,0)}$ are injective, so our orientations $o_{(p_\mp,p_\pm)}$ on $\operatorname{Ker}\bar\partial_{\lambda_{(p_\mp,p_\pm)}}$ induce orientations on $\mathrm{ev}_{(-1,0)}\bigl(\operatorname{Ker}\bar\partial_{\lambda_{(p_\mp,p_\pm)}}\bigr)$. Thus, all three vector spaces in (58) are oriented. Define $\epsilon_{(p_-,p_+)}=1$ if (58) is true in oriented vector spaces, and $\epsilon_{(p_-,p_+)}=-1$ otherwise, for all $(p_-,p_+)\in R$.

The subspaces on the r.h.s. of (58) have dimensions $\eta_{(p_-,p_+)}, n-\eta_{(p_-,p_+)}$, so swapping them changes signs by $(-1)^{\eta_{(p_-,p_+)}(n-\eta_{(p_-,p_+)})}$. Thus

$$\epsilon_{(p_-,p_+)}\epsilon_{(p_+,p_-)}=(-1)^{\eta_{(p_-,p_+)}(n-\eta_{(p_-,p_+)})}. \tag{78}$$

If n is *odd* then one of $\eta_{(p_-,p_+)}, n-\eta_{(p_-,p_+)}$ is even, so (78) gives $\epsilon_{(p_-,p_+)}\epsilon_{(p_+,p_-)}=1$. In this case, we can choose the orientations on the $\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}}$ so that $\epsilon_{(p_-,p_+)}=1$ for all (p_-,p_+) in R, which simplifies some formulae below. But if n is *even* and some $\eta_{(p_-,p_+)}$ is odd then (78) gives $\epsilon_{(p_-,p_+)}\epsilon_{(p_+,p_-)}=-1$, so we cannot choose the orientations on the $\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}}$ to make all $\epsilon_{(p_-,p_+)}=1$.

We work in the situation of Definition 4.14 with orientations on $\widetilde{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J)$ from Theorem 5.2, and $o_{(p_-,p_+)}$ on $\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}}$. Define an orientation on $\widetilde{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J,f_1,\dots,f_k)$ by the fibre product of oriented Kuranishi spaces:

$$\begin{aligned}&\widetilde{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J,f_1,\dots,f_k)=(-1)^{(n+1)\sum_{l=1}^k(k-l)(\deg f_l+1)}\\&\widetilde{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J)\times_{\prod_{i=1}^k\widetilde{\mathbf{ev}}_i,(L\amalg\tilde R)^k,\prod_{i=1}^k\left\{\begin{smallmatrix}f_i, & i\notin I,\\ f_i\times\mathrm{ev}_{(-1,0)}, & i\in I\end{smallmatrix}\right\}}\\&\prod_{i=1}^k\begin{Bmatrix}\Delta_{a_i}, & i\notin I,\\ \Delta_{a_i}\times\operatorname{Ker}\bar\partial_{\lambda_{\sigma\circ\alpha(i)}}, & i\in I\end{Bmatrix},\end{aligned} \tag{79}$$

which is (68) with a choice of sign taken from Fukaya et al. [**8**, Def. 8.4.1]. Roughly speaking, the sign $(-1)^{(n+1)\sum_{l=1}^k(k-l)(\deg f_l+1)}$ is chosen so that in the A_∞ algebra we will construct later, $\mathfrak{m}_k(f_1,\dots,f_k)$ is a virtual chain for the oriented Kuranishi space $\widetilde{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J,f_1,\dots,f_k)$. But we actually define $\mathfrak{m}_k(f_1,\dots,f_k)$ using the $\overline{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J,f_1,\dots,f_k)$, and the calculations in this section are just motivation for the complicated choice of orientation on $\overline{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J,f_1,\dots,f_k)$ in §5.4.

Similarly, define an orientation on $\widetilde{\mathcal{M}}^{\rm main}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\dots,f_{i-1};f_{i+k_2},\dots,f_k)$ by

$$\begin{aligned}&\widetilde{\mathcal{M}}^{\rm main}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\dots,f_{i-1};f_{i+k_2},\dots,f_k)=(-1)^{n\sum_{l=1}^{i-1}(\deg f_l+1)}\\&\cdot(-1)^{(n+1)\sum_{l=1}^{i-1}(k-k_2+1-l)(\deg f_l+1)}(-1)^{(n+1)\sum_{l=i+k_2}^{k}(k-l)(\deg f_l+1)}\cdot\\&\widetilde{\mathcal{M}}^{\rm main}_{k_1+1}(\alpha_1,\beta_1,J)\times_{\prod_{i\neq j=1}^{k_1}\widetilde{\mathbf{ev}}_j,(L\amalg\tilde R)^{k_1-1},\prod_{\substack{j=1,\dots,k:\\ j<i\text{ or }j\geqslant i+k_2}}\left\{\begin{smallmatrix}f_j, & j\notin I\\ f_j\times\mathrm{ev}_{(-1,0)}, & j\in I\end{smallmatrix}\right\}}\\&\prod_{\substack{j=1,\dots,k:\\ j<i\text{ or }j\geqslant i+k_2}}\begin{Bmatrix}\Delta_{a_j}, & j\notin I\\ \Delta_{a_j}\times\operatorname{Ker}\bar\partial_{\lambda_{\sigma\circ\alpha(j)}}, & j\in I\end{Bmatrix},\end{aligned} \tag{80}$$

which is (69) with a sign inserted, chosen to achieve a simple form for the signs in (81) and (82) below.

We can now add orientations to equation (70).

Theorem 5.7. *In the situation of Definition* 4.14, *with the orientations of Definition* 5.6, *in oriented Kuranishi spaces we have*

$$
\begin{aligned}
(81)\qquad & \partial\widetilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\dots,f_k)\cong \\
& \coprod_{i=1}^{k}\coprod_{j=0}^{a_i}(-1)^{j+1+\sum_{l=1}^{i-1}\deg f_l} \\
&\qquad \widetilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\dots,f_{i-1},f_i\circ F_j^{a_i},f_{i+1},\dots,f_k) \\
& \amalg \coprod_{\substack{k_1+k_2=k+1,\ 1\leqslant i\leqslant k_1,\\ I_1\cup_i I_2=I,\ \alpha_1\cup_i\alpha_2=\alpha,\\ \beta_1+\beta_2=\beta}} (-1)^{n+\left(1+\sum_{l=1}^{i-1}\deg f_l\right)\left(1+\sum_{l=i}^{i+k_2-1}\deg f_l\right)} \\
&\qquad \widetilde{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J,f_i,\dots,f_{i+k_2-1})\times_{\widetilde{\mathbf{ev}},L\amalg\tilde R,\widetilde{\mathbf{ev}}_i} \\
&\qquad \widetilde{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\dots,f_{i-1};f_{i+k_2},\dots,f_k).
\end{aligned}
$$

Also, if $f:\Delta_a\to L\amalg R$ *is smooth then in oriented Kuranishi spaces we have*

$$
\begin{aligned}
(82)\qquad & \begin{Bmatrix}\Delta_a, & f(\Delta_a)\subset L \\ \Delta_a\times\operatorname{Ker}\bar\partial_{\lambda_{(p_+,p_-)}}, & f(\Delta_a)=\{(p_-,p_+)\}\subset R\end{Bmatrix} \\
& \times_{\left\{\begin{smallmatrix} f, & f(\Delta_a)\subset L \\ f\times\mathrm{ev}_{(-1,0)}, & f(\Delta_a)\subset R\end{smallmatrix}\right\},L\amalg\tilde R,\widetilde{\mathbf{ev}}_i}\widetilde{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\dots,f_{i-1};f_{i+k_2},\dots,f_k) \\
& =(-1)^{(1+\deg f)\left(1+\sum_{j=1}^{i-1}\deg f_j\right)}\widetilde{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\dots,f_{i-1},f,f_{i+k_2},\dots,f_k).
\end{aligned}
$$

Here (81) is proved by a sign calculation using equations (70) and (79)–(80), Proposition 2.10, Theorem 5.3, and the formula $\partial\Delta_{a_i}=\sum_{j=0}^{a_i}(-1)^jF_j^{a_i}(\Delta_{a_i-1})$ in oriented manifolds with corners, in the notation of §2.6, and (82) follows in a similar way from equations (79)–(80) and Proposition 2.10.

5.3. Orientations on $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$.

Definition 5.8. Choose a relative spin structure for $\iota:L\to M$, so that Theorem 5.2 gives orientations on the $\widetilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$. Inserting signs in (59), define the orientation on $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$ to be such that

$$
\begin{aligned}
(83)\qquad \widetilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J) &= \textstyle\prod_{0\neq j\in I}\epsilon_{\alpha(j)}\,(-1)^{\sum_{0\neq j\in I}\eta_{\alpha(j)}\left[k-j+\sum_{l\in I:l>j}\eta_{\alpha(l)}\right]} \\
&\qquad \overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)\times\textstyle\prod_{i\in I}\operatorname{Ker}\bar\partial_{\lambda_{\alpha(i)}}
\end{aligned}
$$

holds as a product of oriented Kuranishi spaces. This orientation on $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$ depends on the choices of a relative spin structure for $\iota:L\to M$, and the $\lambda_{(p_-,p_+)}$ in §4.2, and the orientations $o_{(p_-,p_+)}$ for the $\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}}$ in §5.2. The complicated choice of sign in (83) will be explained in Remark 5.14(c). One thing it does is achieve a fairly simple form for the sign in (84) below.

We compute the orientations in Theorem 4.3.

Theorem 5.9. *Using the orientations of Definition* 5.8, *the isomorphism* (27) *in oriented Kuranishi spaces becomes:*

$$\partial\overline{\mathcal{M}}{}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)\cong\coprod_{\substack{k_1+k_2=k+1,\ 1\leqslant i\leqslant k_1,\\ I_1\cup_i I_2=I,\ \alpha_1\cup_i\alpha_2=\alpha,\\ \beta_1+\beta_2=\beta}}\zeta_1\,\overline{\mathcal{M}}{}^{\mathrm{main}}_{k_2+1}(\alpha_2,\beta_2,J)\times_{\mathbf{ev},L\amalg R,\mathbf{ev}_i}\overline{\mathcal{M}}{}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J),\tag{84}$$

$$\textit{where}\quad \zeta_1=(-1)^{n+\left(i+\sum_{j\in I:0<j<i}\eta_{\alpha(j)}\right)\left(1+k_2+\sum_{l\in I:i\leqslant l<i+k_2}\eta_{\alpha(l)}\right)}\tag{85}$$

if $i\notin I_1$ *and* $0\notin I_2$, *and*

$$\zeta_1=(-1)^{n+\left(i+\sum_{j\in I:0<j<i}\eta_{\alpha(j)}\right)\left(\eta_{\alpha_1(i)}+1+k_2+\sum_{l\in I:i\leqslant l<i+k_2}\eta_{\alpha(l)}\right)}\tag{86}$$

if $i\in I_1$, $0\in I_2$, *and* $\alpha_2(0)=\sigma\circ\alpha_1(i)$. *Note that in the cases not covered by* (85) *and* (86) *we have* $\overline{\mathcal{M}}{}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J)\times_{\mathbf{ev}_i,L\amalg R,\mathbf{ev}}\overline{\mathcal{M}}{}^{\mathrm{main}}_{k_2+1}(\alpha_2,\beta_2,J)=\emptyset$, *so we do not need to define* ζ_1.

Proof. Substitute (83) into (73) three times for k,α,β and k_1,α_1,β_1 and k_2,α_2,β_2. This yields

$$\begin{aligned}
&\partial\overline{\mathcal{M}}{}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)\times\prod_{j\in I}\operatorname{Ker}\bar\partial_{\lambda_{\alpha(j)}}=\textstyle\coprod_{\substack{k_1+k_2=k+1,\ 1\leqslant i\leqslant k_1,\\ I_1\cup_i I_2=I,\\ \alpha_1\cup_i\alpha_2=\alpha,\ \beta_1+\beta_2=\beta}}(-1)^{n+i+ik_2}\\
&\prod_{0\neq j\in I}\epsilon_{\alpha(j)}\prod_{0\neq j\in I_1}\epsilon_{\alpha_1(j)}\prod_{0\neq j\in I_2}\epsilon_{\alpha_2(j)}(-1)^{\sum_{0\neq j\in I}\eta_{\alpha(j)}[k-j+\sum_{l\in I:l>j}\eta_{\alpha(l)}]}\\
&(-1)^{\sum_{0\neq j\in I_1}\eta_{\alpha_1(j)}[k_1-j+\sum_{l\in I_1:l>j}\eta_{\alpha_1(l)}]}(-1)^{\sum_{0\neq j\in I_2}\eta_{\alpha_2(j)}[k_2-j+\sum_{l\in I_2:l>j}\eta_{\alpha_2(l)}]}\\
&\quad\big(\overline{\mathcal{M}}{}^{\mathrm{main}}_{k_2+1}(\alpha_2,\beta_2,J)\times\textstyle\prod_{j\in I_2}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(j)}}\big)\times_{\widetilde{\mathbf{ev}},L\amalg\tilde R,\widetilde{\mathbf{ev}}_i}\\
&\qquad\big(\overline{\mathcal{M}}{}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J)\times\textstyle\prod_{j\in I_1}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(j)}}\big).
\end{aligned}\tag{87}$$

The left hand side is $\partial\widetilde{\mathcal{M}}{}^{\mathrm{main}}_{k+1}(\alpha,\beta,J)$. Fix $i,\ldots,\beta_2$ in (87), and first consider the case $i\notin I_1$ and $0\notin I_2$. Then we have

$$\begin{aligned}
&\big(\overline{\mathcal{M}}{}^{\mathrm{main}}_{k_2+1}(\alpha_2,\beta_2,J)\times\textstyle\prod_{j\in I_2}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(j)}}\big)\\
&\qquad\times_{\widetilde{\mathbf{ev}},L\amalg\tilde R,\widetilde{\mathbf{ev}}_i}\big(\overline{\mathcal{M}}{}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J)\times\textstyle\prod_{j\in I_1}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(j)}}\big)
\end{aligned}\tag{88}$$

$$\begin{aligned}
&=(-1)^{k_1\sum_{l\in I_2}\eta_{\alpha_2(l)}}\overline{\mathcal{M}}{}^{\mathrm{main}}_{k_2+1}(\alpha_2,\beta_2,J)\times_{\mathbf{ev},L\amalg R,\mathbf{ev}_i}\\
&\quad\overline{\mathcal{M}}{}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J)\times\big(\textstyle\prod_{j\in I_1}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(j)}}\big)\times\big(\textstyle\prod_{j\in I_2}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(j)}}\big)\\
&=(-1)^{k_1\sum_{l\in I_2}\eta_{\alpha_2(l)}}(-1)^{(\sum_{j\in I_1:j>i}\eta_{\alpha_1(j)})(\sum_{l\in I_2}\eta_{\alpha_2(l)})}\\
&\quad\overline{\mathcal{M}}{}^{\mathrm{main}}_{k_2+1}(\alpha_2,\beta_2,J)\times_{\mathbf{ev},L\amalg R,\mathbf{ev}_i}\overline{\mathcal{M}}{}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J)\times\big(\textstyle\prod_{j\in I}\operatorname{Ker}\bar\partial_{\lambda_{\alpha(j)}}\big).
\end{aligned}$$

Here in the first step we pull $\prod_{j\in I_1}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(j)}}$ and $\prod_{j\in I_2}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(j)}}$, which are not involved in the fibre product, out to the right. Since $\prod_{j\in I_1}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(j)}}$ is already on the right, it causes no sign changes. Moving $\prod_{j\in I_2}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(j)}}$ past $L\amalg\tilde R$ and then past $\widetilde{\mathcal M}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J)$ changes orientations by $(-1)^{\dim(\prod_{j\in I_2}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(j)}})(\dim L\amalg\tilde R+\dim\widetilde{\mathcal M}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J))}$. Using (60) to compute $\dim\widetilde{\mathcal M}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J)$ and omitting even terms 2, $2n$ and $\mu_L(\beta_1)$ in $\dim L\amalg\tilde R+\dim\widetilde{\mathcal M}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J)$ gives the sign on the third line of (88). The fifth and sixth lines reorder $\bigl(\prod_{j\in I_1}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(j)}}\bigr)\times\bigl(\prod_{j\in I_2}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(j)}}\bigr)$ to obtain $\bigl(\prod_{j\in I}\operatorname{Ker}\bar\partial_{\lambda_{\alpha(j)}}\bigr)$. By (28), this means swapping over factors $\prod_{j\in I_1:j>i}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(j)}}$ and $\prod_{l\in I_2}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(l)}}$, and so contributes the sign $(-1)^{(\sum_{j\in I_1:j>i}\eta_{\alpha_1(j)})(\sum_{l\in I_2}\eta_{\alpha_2(l)})}$ in the fifth line. Combining signs in (87) and (88) we obtain (85), proving the theorem in the case $i\notin I_1$ and $0\notin I_2$. The second case is similar. q.e.d.

Remark 5.10. If we reverse the order of the fibre product in (84) using Proposition 2.10(a) and (33), noting that the fibre product is over L with $\dim L=n$ in the case $i\notin I_1$, $0\notin I_2$, and over R with $\dim R=0$ in the case $i\in I_1$, $0\in I_2$, we obtain

$$\partial\overline{\mathcal M}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)\cong\coprod_{\substack{k_1+k_2=k+1,\ 1\leqslant i\leqslant k_1,\ I_1\cup_i I_2=I,\\ \alpha_1\cup_i\alpha_2=\alpha,\ \beta_1+\beta_2=\beta}}\begin{array}{r}\zeta_2\,\overline{\mathcal M}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J)\times_{\mathbf{ev}_i,L\amalg R,\mathbf{ev}}\\ \overline{\mathcal M}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J)\end{array}\tag{89}$$

in oriented Kuranishi spaces, where

$$\zeta_2=(-1)^{n+i+\sum_{j\in I:0<j<i}\eta_{\alpha(j)}}\cdot\begin{cases}(-1)^{\left(k_2+\sum_{j\in I:i\leqslant j<i+k_2}\eta_{\alpha(j)}\right)\left(k_1+i+\sum_{l\in I:i+k_2\leqslant l\leqslant k}\eta_{\alpha(l)}\right)}, & 0\notin I,\\ (-1)^{\left(k_2+\sum_{j\in I:i\leqslant j<i+k_2}\eta_{\alpha(j)}\right)\left(k_1+i+\eta_{\alpha(0)}+\sum_{l\in I:i+k_2\leqslant l\leqslant k}\eta_{\alpha(l)}\right)}, & 0\in I,\end{cases}$$

if $i\notin I_1$ and $0\notin I_2$, and

$$\zeta_2=(-1)^{n+i+\sum_{j\in I:0<j<i}\eta_{\alpha(j)}}\cdot\begin{cases}(-1)^{\left(\eta_{\alpha_1(i)}+k_2+\sum_{j\in I:i\leqslant j<i+k_2}\eta_{\alpha(j)}\right)\left(\eta_{\alpha_2(0)}+k_1+i+\sum_{l\in I:i+k_2\leqslant l\leqslant k}\eta_{\alpha(l)}\right)}, & 0\notin I,\\ (-1)^{\left(\eta_{\alpha_1(i)}+k_2+\sum_{j\in I:i\leqslant j<i+k_2}\eta_{\alpha(j)}\right)\left(\eta_{\alpha_2(0)}+k_1+i+\sum_{l\in I:i+k_2\leqslant l\leqslant k}\eta_{\alpha(l)}\right)}, & 0\in I,\end{cases}$$

if $i\in I_1$, $0\in I_2$, and $\alpha_2(0)=\sigma\circ\alpha_1(i)$. In the embedded case, when $I=\emptyset$, the sign ζ_2 reduces to $(-1)^{n+i+k_2(k_1+i)}$, which agrees with that calculated by Fukaya et al. in [**8**, Prop. 8.3.3 & Rem. 8.3.4] when $i=1$.

5.4. Orientations on $\overline{\mathcal M}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\dots,f_k)$.

Definition 5.11. We work in the situation of Definitions 4.7 and 4.14 with the orientations on the $\widetilde{\mathcal M}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\dots,f_k)$ from Definition 5.6, and $o_{(p_-,p_+)}$ on $\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}}$ from §5.2. Define $\overline{\mathcal M}_{k+1}^{\mathrm{main}}(\alpha,\beta,$

$J, f_1, \ldots, f_k)$ to have the unique orientation such that

$$\widetilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k) = \begin{cases} \overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k), & 0\notin I, \\ \overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)\times \operatorname{Ker}\bar\partial_{\lambda_{\alpha(0)}}, & 0\in I, \end{cases} \tag{90}$$

holds, in oriented Kuranishi spaces. This is just (65), with no extra sign added. Similarly, adding signs to (66), let $\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J, f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)$ have the unique orientation for which in oriented Kuranishi spaces we have

$$\widetilde{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k) = \tag{91}$$
$$\begin{cases} \overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k), & 0,i\notin I_1, \\ \overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(0)}}, & 0\in I_1,\ i\notin I_1, \\ (-1)^{\eta_{\alpha_1(i)}\left(1+\sum_{j=1}^{i-1}\deg f_j+\sum_{j=i+k_2}^{k}\deg f_j\right)}\epsilon_{\alpha_1(i)} & \\ \quad \overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(i)}}, & 0\notin I_1,\ i\in I_1, \\ (-1)^{\eta_{\alpha_1(i)}\left(1+\sum_{j=1}^{i-1}\deg f_j+\sum_{j=i+k_2}^{k}\deg f_j\right)}\epsilon_{\alpha_1(i)} & \\ \quad \overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(0)}}\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(i)}}, & 0,i\in I_1. \end{cases}$$

Reordering the factors using (5), (40) and (78) gives

$$\widetilde{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k) = \tag{92}$$
$$\begin{cases} \overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k), & 0,i\notin I_1, \\ \overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(0)}}, & 0\in I_1,\ i\notin I_1, \\ \epsilon_{\sigma\circ\alpha_1(i)}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(i)}}\times\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k), & 0\notin I_1,\ i\in I_1, \\ \epsilon_{\sigma\circ\alpha_1(i)}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(i)}}\times\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k) & \\ \qquad\qquad\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(0)}}, & 0,i\in I_1. \end{cases}$$

Combining equations (79), (83) and (90) and calculating using Proposition 2.10, the definition of $\epsilon_{(p_-,p_+)}$ and (78) to determine the signs, we prove that:

Theorem 5.12. *An alternative way to define the orientations in Definition* 5.11, *in terms of the orientation on* $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$ *given in Definition* 5.8, *is that*

$$\begin{aligned} &\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k) = \\ &\zeta_3\,\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)\times_{\mathbf{ev}_1\times\cdots\times\mathbf{ev}_k,(L\amalg R)^k,f_1\times\cdots\times f_k}(\Delta_{a_1}\times\cdots\times\Delta_{a_k}) \end{aligned} \tag{93}$$

in oriented Kuranishi spaces, which is (34) *with signs inserted, where*

$$\begin{aligned} \zeta_3 = {} & (-1)^{\sum_{0\neq i\in I}(n-\eta_{\alpha(i)})\left[\sum_{j=1}^{i}(\deg f_j+1)-\sum_{j\in I:0<j\leqslant i}\eta_{\alpha(j)}\right]} \\ & (-1)^{(n+1)\left[\sum_{i=1}^{k}(k-i)(\deg f_i+1)-\sum_{0\neq i\in I}(k-i)\eta_{\alpha(i)}\right]} \\ & \cdot\begin{cases} 1, & 0\notin I, \\ (-1)^{\eta_{\alpha(0)}\left[\sum_{i=1}^{k}(\deg f_i+1)-\sum_{0\neq i\in I}\eta_{\alpha(i)}\right]}, & 0\in I. \end{cases} \end{aligned} \tag{94}$$

We can now prove an analogue of Theorem 5.7. Note that the signs in equations (95)–(96) are exactly the same as those in (81)–(82).

Theorem 5.13. *In the situation of Definition* 4.14, *with the orientations of Definition* 5.6, *in oriented Kuranishi spaces we have*

$$\partial\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)\cong \tag{95}$$
$$\coprod_{i=1}^{k}\coprod_{j=0}^{a_i}(-1)^{j+1+\sum_{l=1}^{i-1}\deg f_l}\quad \overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_{i-1},f_i\circ F_j^{a_i},f_{i+1},\ldots,f_k)$$
$$\amalg\coprod_{\substack{k_1+k_2=k+1,\ 1\leqslant i\leqslant k_1,\\ I_1\cup_i I_2=I,\ \alpha_1\cup_i\alpha_2=\alpha,\\ \beta_1+\beta_2=\beta}}(-1)^{n+\left(1+\sum_{l=1}^{i-1}\deg f_l\right)\left(1+\sum_{l=i}^{i+k_2-1}\deg f_l\right)}\quad \overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J,f_i,\ldots,f_{i+k_2-1})\times_{\mathbf{ev},L\amalg R,\mathbf{ev}_i}\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k).$$

Also, if $f:\Delta_a\to L\amalg R$ *is smooth then in oriented Kuranishi spaces we have*

$$\Delta_a\times_{f,L\amalg R,\mathbf{ev}_i}\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k) \tag{96}$$
$$=(-1)^{(1+\deg f)\left(1+\sum_{j=1}^{i-1}\deg f_j\right)}\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1},f,f_{i+k_2},\ldots,f_k).$$

Proof. To prove (95), we substitute (90) and (92) into (81). We must consider separately the cases $0\notin I$ and $0\in I$ in (90). As $0\in I$ if and only if $0\in I_1$ by (27) and (28), these determine whether or not $0\in I_1$, but for each $i,\ldots,\beta_2$ in (95) we must still consider separately the cases $i\notin I_1$ and $i\in I_1$ in (92), so there are four cases to consider. We explain the most complicated case $0\in I$ and $0,i\in I_1$. Then substituting (90) and (92) into (81) yields in oriented Kuranishi spaces

$$\partial\big(\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha(0)}}\big)=\big(\partial(\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)\big)\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha(0)}}$$

on the left hand side, using Proposition 2.10(a) and $\partial\big(\operatorname{Ker}\bar\partial_{\lambda_{\alpha(0)}}\big)=\emptyset$, and

$$(-1)^{j+1+\sum_{l=1}^{i-1}\deg f_l}\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_{i-1},f_i\circ F_j^{a_i},f_{i+1},\ldots,f_k)\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha(0)}}$$

for the first term on the right hand side, for each j, and

$$
\begin{aligned}
&(-1)^{n+\left(1+\sum_{l=1}^{i-1}\deg f_l\right)\left(1+\sum_{l=i}^{i+k_2-1}\deg f_l\right)}\\
&\big(\overline{\mathcal{M}}_{k_2+1}^{\rm main}(\alpha_2,\beta_2,J,f_i,\ldots,f_{i+k_2-1})\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(0)}}\big)\times_{\widetilde{\mathbf{ev}},L\amalg\tilde R,\widetilde{\mathbf{ev}}_i}\\
&\big(\epsilon_{\sigma\circ\alpha_1(i)}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(i)}}\times\overline{\mathcal{M}}_{k_1+1}^{\rm main}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(0)}}\big)\\
&=(-1)^{n+\left(1+\sum_{l=1}^{i-1}\deg f_l\right)\left(1+\sum_{l=i}^{i+k_2-1}\deg f_l\right)}\,\overline{\mathcal{M}}_{k_2+1}^{\rm main}(\alpha_2,\beta_2,J,f_i,\ldots,f_{i+k_2-1})\times\\
&\big(\epsilon_{\sigma\circ\alpha_1(i)}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(0)}}\times_{\lambda_{\alpha_2(0)}(-1,0)}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(i)}}\big)\times\\
&\overline{\mathcal{M}}_{k_1+1}^{\rm main}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(0)}}\\
&=(-1)^{n+\left(1+\sum_{l=1}^{i-1}\deg f_l\right)\left(1+\sum_{l=i}^{i+k_2-1}\deg f_l\right)}\\
&\big(\overline{\mathcal{M}}_{k_2+1}^{\rm main}(\alpha_2,\beta_2,J,f_i,\ldots,f_{i+k_2-1})\times_{\mathbf{ev},L\amalg R,\mathbf{ev}_i}\\
&\overline{\mathcal{M}}_{k_1+1}^{\rm main}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)\big)\times\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(0)}}
\end{aligned}
$$

for the final term on the right hand side, for fixed $i,\ldots,\beta_2$ with $i\in I_1$. Here we use the fact that $\operatorname{Ker}\bar\partial_{\lambda_{\alpha_2(0)}}\times_{\lambda_{\alpha_2(0)}(-1,0)}\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(i)}}$ is a point with sign $\epsilon_{\alpha_2(0)}$, and as $\alpha_2(0)=\sigma\circ\alpha_1(i)$ this cancels with $\epsilon_{\sigma\circ\alpha_1(i)}$, so that the fifth line is just a point with sign 1. In the last two lines, the fibre product over $L\amalg R$ is actually a fibre product over the point $\alpha_1(i)$ in R, so it is a product, as in the fifth to seventh lines.

The last three equations are the oriented products of the corresponding terms in (95) with $\operatorname{Ker}\bar\partial_{\lambda_{\alpha_1(0)}}$. This proves (95) in the case $0\in I$ and $0,i\in I_1$. The other cases follow by similar but simpler arguments. To prove (96) we substitute (90) and (92) into (82), and use the same method. q.e.d.

Remark 5.14. (a) Theorem 5.13 is the main result of this section. It is important that the signs in (95) and (96) depend only on n,i,j,k_2 and the *shifted cohomological degrees* $\deg f_j,\deg f$. In particular, they do not involve the $\epsilon_{\alpha(j)},\eta_{\alpha(j)}$ or a_j. Because of this, in the rest of the paper we will be able to write all our signs in terms of $\deg f_j,\deg f$, without any correction factors involving $\epsilon_{\alpha(j)},\eta_{\alpha(j)},a_j$. This was one aim of the careful definition of orientations above.

Theorem 5.13 is an analogue in the immersed case of Fukaya et al. [**8**, Prop. 8.5.1]; roughly speaking, if we substitute (96) into (95), then we get [**8**, Prop. 8.5.1], with the same signs, noting that our definition of $\deg f_i$ differs by 1 from that of [**8**]. Since our signs are compatible with those of Fukaya et al. [**8**], we can follow their proof to construct an A_∞ algebra, and there will be no new orientation issues, provided we grade our complexes using shifted cohomological degrees in (36).

(b) In equations (27),(41),(63),(70),(73),(81),(84),(87) and (95) above, we chose to order the fibre products as $\overline{\mathcal{M}}_{k_2+1}^{\rm main}(\alpha_2,\ldots)\times_{\ldots}\overline{\mathcal{M}}_{k_1+1}^{\rm main}(\alpha_1,\ldots)$ rather than the other way round; this order was reversed in (89). Fukaya et al. adopt the opposite order to us, in [**8**, Prop. 8.3.3] for instance.

We can now explain why we chose this order for our fibre products. Using (5),(37) and (40) we may rewrite (95) and (96) with the other fibre product order, which yields:

$$
\begin{aligned}
&\partial\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)= \qquad (97)\\
&\coprod_{i=1}^{k}\coprod_{j=0}^{a_i}(-1)^{j+1+\sum_{l=1}^{i-1}\deg f_l}\,\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_{i-1},f_i\circ F_j^{a_i},f_{i+1},\ldots,f_k)\\
&\amalg\coprod_{\substack{k_1+k_2=k+1,\ 1\leqslant i\leqslant k_1,\\ I_1\cup_i I_2=I,\ \alpha_1\cup_i\alpha_2=\alpha,\\ \beta_1+\beta_2=\beta}}(-1)^{n+1+i+ik_2+\sum_{l=1}^{i-1}\deg f_l+\sum_{l=i}^{i+k_2-1}\deg f_l\sum_{l=i+k_2}^{k}\deg f_l}\cdot\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)\\
&\qquad\times_{\mathbf{ev}_i,L\amalg R,\mathbf{ev}}\overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J,f_i,\ldots,f_{i+k_2-1})\cdot\\
&\left\{\begin{matrix}
1, & 0,i\notin I_1,\\
(-1)^{\eta_{\alpha_1(0)}\sum_{l=i}^{i+k_2-1}\deg f_l}, & 0\in I_1,\ i\notin I_1,\\
(-1)^{\eta_{\alpha_1(i)}\left[1+\sum\limits_{l=1}^{i-1}\deg f_l+\sum\limits_{l=i+k_2}^{k}\deg f_l\right]}(-1)^{\eta_{\sigma\circ\alpha_1(i)}\left[\eta_{\alpha_1(i)}+\sum\limits_{l=i}^{i+k_2-1}\deg f_l\right]}, & 0\notin I_1,\ i\in I_1,\\
(-1)^{\eta_{\alpha_1(i)}\left[1+\sum\limits_{l=1}^{i-1}\deg f_l+\sum\limits_{l=i+k_2}^{k}\deg f_l\right]}(-1)^{(\eta_{\alpha_1(0)}+\eta_{\sigma\circ\alpha_1(i)})\left[\eta_{\alpha_1(i)}+\sum\limits_{l=i}^{i+k_2-1}\deg f_l\right]}, & 0,i\in I_1,
\end{matrix}\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)\times_{\mathbf{ev}_i,L\amalg R,f}\Delta_a \qquad (98)\\
&=(-1)^{(\deg f+1)\sum\limits_{l=i+k_2}^{k}\deg f_l}\,\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1},f,f_{i+k_2},\ldots,f_k)\cdot\\
&\left\{\begin{matrix}
1, & 0,i\notin I_1,\\
(-1)^{\eta_{\alpha_1(0)}(\deg f+1)}, & 0\in I_1,\ i\notin I_1,\\
(-1)^{\eta_{\alpha_1(i)}\left[1+\sum\limits_{l=1}^{i-1}\deg f_l+\sum\limits_{l=i+k_2}^{k}\deg f_l\right]}(-1)^{\eta_{\sigma\circ\alpha_1(i)}(\deg f+1+\eta_{\alpha_1(i)})}, & 0\notin I_1,\ i\in I_1,\\
(-1)^{\eta_{\alpha_1(i)}\left[1+\sum\limits_{l=1}^{i-1}\deg f_l+\sum\limits_{l=i+k_2}^{k}\deg f_l\right]}(-1)^{(\eta_{\alpha_1(0)}+\eta_{\sigma\circ\alpha_1(i)})(\deg f+1+\eta_{\alpha_1(i)})}, & 0,i\in I_1,
\end{matrix}\right\}.
\end{aligned}
$$

Observe that equations (97)–(98) have complicated extra sign terms involving $\eta_{\alpha_1(0)},\eta_{\alpha_1(i)},\eta_{\sigma\circ\alpha_1(i)}$, so they are not simply written in terms of n,i,j,k_2 and $\deg f_j,\deg f$, as (95)–(96) were. Thus we prefer the fibre product order in (95)–(96). One might guess that by changing the signs in (90) and (91), altering the orientations of $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)$, $\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)$, one could eliminate the troublesome terms in (97) and (98), to get signs depending only on $n,i,j,k_1,k_2,\deg f_j,\deg f$. However, calculations by the authors indicate that this is impossible, at least with the orientation conventions of §2.4.

(c) We defined the orientation on $\overline{\mathcal{M}}_{k+1}(\alpha,\beta,J)$ in §5.3 by (83), which includes a complicated choice of sign. We chose this particular sign by requiring that if $a_i=n$ for $i\notin I$ and $a_i=0$ for $i\in I$, so that $\deg f_i=-1$ for $i\notin I$ and $\deg f_i=\eta_{\alpha(i)}-1$ for $i\in I$, then the sign ζ_3 in (93) and (94) should be 1. The sign in (83) was then determined as in the proof of

Theorem 5.13. The motivation for this choice is that we have found natural orientations for the moduli spaces $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)$, with good properties under boundaries as in Theorem 5.13. Now we have

$$\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)= \\ \overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)\times_{\prod_{i=1}^k \mathbf{ev}_i,(L\amalg R)^k,\prod_{i=1}^k\left\{\begin{smallmatrix}\mathrm{id}_L, & i\notin I\\ \mathrm{id}_{\{\alpha(i)\}}, & i\in I\end{smallmatrix}\right\}}\prod_{i=1}^k\begin{Bmatrix}L, & i\notin I\\ \{\alpha(i)\}, & i\in I\end{Bmatrix}.$$

This is like the fibre product (34) defining $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)$, but replacing $f_i:\Delta_{a_i}\to L\amalg R$ by $\mathrm{id}_L:L\to L\amalg R$ for $i\notin I$ and $\mathrm{id}_{\{\alpha(i)\}}:\{\alpha(i)\}\to L\amalg R$ for $i\in I$. Thus we can think of $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$ as a generalization of $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)$ in which $a_i=n=\dim L$ for $i\notin I$ and $a_i=0=\dim\{\alpha(i)\}$ for $i\in I$, and so we should arrange to get $\zeta_3=1$ in (94) in this case.

The orientations on $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)$ depend on the choice of paths $\lambda_{(p_-,p_+)}$ in §4.3 and orientations $o_{(p_-,p_+)}$ on $\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}}$ in §5.2, for $(p_-,p_+)\in R$. Suppose $\tilde\lambda_{(p_-,p_+)},\tilde o_{(p_-,p_+)}$ are alternative choices. Changing $\lambda_{(p_-,p_+)}$ to $\tilde\lambda_{(p_-,p_+)}$ alters the index $\eta_{(p_-,p_+)}$ in (31) to $\tilde\eta_{(p_-,p_+)}$, and this changes the shifted cohomological degree $\deg f$ in (36).

As $\lambda_{(p_-,p_+)},\tilde\lambda_{(p_-,p_+)}$ are paths in *oriented* Lagrangian spaces, $\eta_{(p_-,p_+)}$, $\tilde\eta_{(p_-,p_+)}$ differ by an even number, so we may write $\tilde\eta_{(p_-,p_+)}=\eta_{(p_-,p_+)}+2d_{(p_-,p_+)}$ for $d_{(p_-,p_+)}\in\mathbb{Z}$. So degrees in (36) change by $\deg f\mapsto\deg f+2d_{(p_-,p_+)}$ if $f:\Delta_a\to\{(p_-,p_+)\}$. Since the changes in $\eta_{(p_-,p_+)},\deg f$ are even, all the signs above, such as those in (95) and (96), are unchanged. Here is how changing to alternative choices $\tilde\lambda_{(p_-,p_+)},\tilde o_{(p_-,p_+)}$ affects the orientations on $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)$.

Proposition 5.15. *In the situation above, suppose we replace the paths $\lambda_{(p_-,p_+)}$ in §4.3 and orientations $o_{(p_-,p_+)}$ on $\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}}$ in §5.2 by alternative choices $\tilde\lambda_{(p_-,p_+)},\tilde o_{(p_-,p_+)}$, for all (p_-,p_+) in R, so that $\eta_{(p_-,p_+)}$ is replaced by $\tilde\eta_{(p_-,p_+)}$, but we make no other changes. Then for all $(p_-,p_+)\in R$ there exist $\xi_{(p_-,p_+)}=\pm1$ depending only on $\lambda_{(p_-,p_+)}$, $o_{(p_-,p_+)},\tilde\lambda_{(p_-,p_+)},\tilde o_{(p_-,p_+)}$, such that for all $k,\alpha,\beta,f_1,\ldots,f_k$ the orientation on $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)$ changes by a factor*

$$\prod_{0\neq i\in I}\xi_{\sigma\circ\alpha(i)}\cdot\begin{cases}\xi_{\alpha(0)}, & 0\in I,\\ 1, & 0\notin I.\end{cases}\tag{99}$$

Proof. When we change only the $o_{(p_-,p_+)}$, so that $\tilde\lambda_{(p_-,p_+)}=\lambda_{(p_-,p_+)}$, using (79), (90) and the fact that the orientation of $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J)$ is independent of the $o_{(p_-,p_+)}$, we see that changing from $o_{(p_-,p_+)}$ to $\tilde o_{(p_-,p_+)}$ for all $(p_-,p_+)\in R$ changes the orientation of $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)$ by a factor (99), with the $\xi_{(p_-,p_+)}$ determined by $\tilde o_{(p_-,p_+)}=$

$\xi_{(p_-,p_+)} o_{(p_-,p_+)}$. For the general case, we must also consider how the *virtual tangent bundle* of $\widetilde{\mathcal{M}}_{k+1}(\alpha,\beta,J)$ in §5.1 changes when we replace $\lambda_{(p_-,p_+)}$ by $\tilde{\lambda}_{(p_-,p_+)}$.

The virtual tangent bundle changes by direct sum with $\bigoplus_{i\in I} V_{\alpha(i)}$, where $V_{(p_-,p_+)} = \operatorname{Ind}\bar{\partial}_{(p_-,p_+)}$ for $(p_-,p_+) \in R$ are oriented virtual vector spaces, and $\bar{\partial}_{(p_-,p_+)}$ is an elliptic operator on the disc $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leqslant 1\}$ with boundary conditions $\lambda_{(p_-,p_+)}(x,y)$ on the semicircle $x \leqslant 0$ and $\tilde{\lambda}_{(p_-,p_+)}(-x,y)$ on the semicircle $x \geqslant 0$. There is an isomorphism $\xi_{(p_-,p_+)} V_{(p_-,p_+)} \cong \operatorname{Ker}\bar{\partial}_{\tilde{\lambda}_{(p_-,p_+)}} \ominus \operatorname{Ker}\bar{\partial}_{\lambda_{(p_-,p_+)}}$, where $\operatorname{Ker}\bar{\partial}_{\lambda_{(p_-,p_+)}}, \operatorname{Ker}\bar{\partial}_{\tilde{\lambda}_{(p_-,p_+)}}$ have orientations $o_{(p_-,p_+)}, \tilde{o}_{(p_-,p_+)}$ and $\xi_{(p_-,p_+)} = \pm 1$, and the proposition holds with these $\xi_{(p_-,p_+)}$. q.e.d.

5.5. Adding families of almost complex structures. We can generalize the material above to the moduli spaces with smooth families of almost complex structures in §4.5. First we explain how to orient the moduli spaces $\overline{\mathcal{M}}^{\text{main}}_{k+1}(\alpha,\beta,J_t : t\in\mathcal{T})$ of Definition 4.8, generalizing Definition 5.8.

Definition 5.16. We work in the situation of §4.5, with M, L, and $J_t : t \in \mathcal{T}$, with the additional assumptions of §5.1–§5.3, that is, that we have chosen a relative spin structure for $\iota : L \to M$, and orientations for $\operatorname{Ker}\bar{\partial}_{\lambda_{(p_+,p_-)}}$ for all $(p_+,p_-) \in R$. We also suppose that $\mathcal{T}$ is oriented. At a point $\big(t,[\Sigma,\vec{z},u,l,\bar{u}]\big)$ of $\overline{\mathcal{M}}^{\text{main}}_{k+1}(\alpha,\beta,J_t : t\in\mathcal{T})$, we have an isomorphism of virtual vector spaces

$$\begin{aligned} T_{(t,[\Sigma,\vec{z},u,l,\bar{u}])}&\overline{\mathcal{M}}^{\text{main}}_{k+1}(\alpha,\beta,J_t : t\in\mathcal{T}) \\ &\cong T_t\mathcal{T} \oplus T_{[\Sigma,\vec{z},u,l,\bar{u}]}\overline{\mathcal{M}}^{\text{main}}_{k+1}(\alpha,\beta,J_t). \end{aligned} \tag{100}$$

In Definition 5.8 we constructed an orientation on $\overline{\mathcal{M}}^{\text{main}}_{k+1}(\alpha,\beta,J_t)$, and hence on $T_{[\Sigma,\vec{z},u,l,\bar{u}]}\overline{\mathcal{M}}^{\text{main}}_{k+1}(\alpha,\beta,J_t)$. As $\mathcal{T}$ is oriented, $T_t\mathcal{T}$ is oriented. Define $\overline{\mathcal{M}}^{\text{main}}_{k+1}(\alpha,\beta,J_t : t\in\mathcal{T})$ to have the orientation such that (100) holds in oriented virtual vector spaces.

A special case of this which is useful for computing signs in formulae is to take $J_t = J$ for some almost complex structure J and all $t \in \mathcal{T}$. Then

$$\overline{\mathcal{M}}^{\text{main}}_{k+1}(\alpha,\beta,J_t : t\in\mathcal{T}) \cong \mathcal{T} \times \overline{\mathcal{M}}^{\text{main}}_{k+1}(\alpha,\beta,J) \tag{101}$$

holds in oriented Kuranishi spaces.

Here is the analogue of Theorem 5.9. We can prove it by the same method; alternatively, we can take $J_t = J$ for $t \in \mathcal{T}$, so that (101) holds, and then deduce the signs in (102) from Proposition 2.10 and (84).

Theorem 5.17. *Using the orientations of Definition* 5.16*, the isomorphism* (44) *in oriented Kuranishi spaces becomes:*

$$\partial\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T})\cong\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\partial\mathcal{T}) \tag{102}$$

$$\coprod_{\substack{k_1+k_2=k+1,\ 1\leqslant i\leqslant k_1,\\ I_1\cup_i I_2=I,\ \alpha_1\cup_i\alpha_2=\alpha,\\ \beta_1+\beta_2=\beta}}\zeta_4\,\bar{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J_t:t\in\mathcal{T})\times_{\boldsymbol{\pi}_{\mathcal{T}}\times\mathbf{ev},\mathcal{T}\times(L\amalg R),\boldsymbol{\pi}_{\mathcal{T}}\times\mathbf{ev}_i}\bar{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J_t:t\in\mathcal{T}),$$

where $\zeta_4=(-1)^{\dim\mathcal{T}+n+\left(i+\sum_{j\in I:0<j<i}\eta_{\alpha(j)}\right)\left(1+k_2+\sum_{l\in I:i\leqslant l<i+k_2}\eta_{\alpha(l)}\right)}$

if $i\notin I_1$ *and* $0\notin I_2$, *and*

$$\zeta_4=(-1)^{\dim\mathcal{T}+n+\left(i+\sum_{j\in I:0<j<i}\eta_{\alpha(j)}\right)\left(\eta_{\alpha_1(i)}+1+k_2+\sum_{l\in I:i\leqslant l<i+k_2}\eta_{\alpha(l)}\right)}$$

if $i\in I_1$, $0\in I_2$, *and* $\alpha_2(0)=\sigma\circ\alpha_1(i)$.

Next we add simplicial chains, and orient the moduli spaces $\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T},f_1,\dots,f_k)$ of Definition 4.10.

Definition 5.18. In the situation of Definition 5.16, for $i=1,\dots,k$, let $a_i\geqslant 0$ and $f_i:\Delta_{a_i}\to\mathcal{T}\times(L\amalg R)$ be a smooth map, as in Definition 4.10. Since we have not defined modified moduli spaces $\tilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T},f_1,\dots,f_k)$, we cannot define an orientation on $\tilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T},f_1,\dots,f_k)$ following (90). Instead, we will take the analogue of Theorem 5.12 to be our definition. Inserting signs in (47) motivated by (93)–(94), define $\tilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T},f_1,\dots,f_k)$ to have the orientation given in oriented Kuranishi spaces by

$$\begin{aligned}&\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T},f_1,\dots,f_k)=\\&\zeta_5\big((\mathbb{R}^m,\kappa_k^m)\times_{\boldsymbol{\pi}_0,\mathbb{R}^m,\boldsymbol{\pi}_{\mathcal{T}}}\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T})\big)\\&\times_{(\boldsymbol{\pi}_1\times\mathbf{ev}_1)\times\cdots\times(\boldsymbol{\pi}_k\times\mathbf{ev}_k),(\mathcal{T}\times(L\amalg R))^k,f_1\times\cdots\times f_k}(\Delta_{a_1}\times\cdots\times\Delta_{a_k}),\end{aligned} \tag{103}$$

where $(\mathbb{R}^m,\kappa_k^m)$ and $\mathbb{R}^m$ have their natural orientations, the orientation of $\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t:t\in\mathcal{T})$ is as in Definition 5.16, and

$$\begin{aligned}\zeta_5=&(-1)^{\sum_{0\neq i\in I}(n-\eta_{\alpha(i)})\left[\sum_{j=1}^{i}(\deg f_j+1)-\sum_{j\in I:0<j\leqslant i}\eta_{\alpha(j)}\right]}\\&(-1)^{(\dim\mathcal{T}+n+1)\left[\sum_{i=1}^{k}(k-i)(\deg f_i+1)-\sum_{0\neq i\in I}(k-i)\eta_{\alpha(i)}\right]}\\&\cdot\begin{cases}1, & 0\notin I,\\ (-1)^{\eta_{\alpha(0)}\left[\sum_{i=1}^{k}(\deg f_i+1)-\sum_{0\neq i\in I}\eta_{\alpha(i)}\right]}, & 0\in I,\end{cases}\end{aligned} \tag{104}$$

where the degrees $\deg f_i$ are as in (49). Similarly, we orient $\bar{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J_t:t\in\mathcal{T},f_1,\dots,f_{i-1};f_{i+k_2},\dots,f_k)$ by inserting signs in (51). We will not write this sign down explicitly, but we choose it to satisfy (106) below.

Calculation using equations (102)–(104) and Proposition 2.10 then yields an analogue of Theorem 5.13:

Theorem 5.19. *In the situation of Definition* 4.10, *with the orientations of Definition* 5.18 *and degrees in* (49), *for* $k > 0$ *in oriented Kuranishi spaces we have*

$$\partial\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t : t\in\mathcal{T},f_1,\ldots,f_k) \cong \tag{105}$$

$$\coprod_{i=1}^{k}\coprod_{j=0}^{a_i} (-1)^{j+1+\sum_{l=1}^{i-1}\deg f_l} \overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J_t : t\in\mathcal{T},f_1,\ldots,f_{i-1},f_i\circ F_j^{a_i},f_{i+1},\ldots,f_k)$$

$$\amalg \coprod_{\substack{k_1+k_2=k+1,\ 1\leqslant i\leqslant k_1,\\ I_1\cup_i I_2=I,\ \alpha_1\cup_i\alpha_2=\alpha,\\ \beta_1+\beta_2=\beta}} (-1)^{\dim\mathcal{T}+n+\left(1+\sum_{l=1}^{i-1}\deg f_l\right)\left(1+\sum_{l=i}^{i+k_2-1}\deg f_l\right)} \overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J_t : t\in\mathcal{T},f_i,\ldots,f_{i+k_2-1}) \times_{\boldsymbol{\pi}_{\mathcal{T}}\times\mathbf{ev},\mathcal{T}\times(L\amalg R),\boldsymbol{\pi}_{\mathcal{T}}\times\mathbf{ev}_i} \overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J_t : t\in\mathcal{T},f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k).$$

When $k=0$ *this holds with an extra term* $\overline{\mathcal{M}}_1^{\mathrm{main}}(\alpha,\beta,J_t : t\in\partial\mathcal{T})$ *on the right hand side, as in* (102). *Also, if* $f:\Delta_a\to\mathcal{T}\times(L\amalg R)$ *is smooth, in oriented Kuranishi spaces we have*

$$\Delta_a\times_{f,\mathcal{T}\times(L\amalg R),\boldsymbol{\pi}_{\mathcal{T}}\times\mathbf{ev}_i}\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J_t : t\in\mathcal{T},f_1,\ldots,f_{i-1};f_{i+k_2},\ldots,f_k)$$
$$= (-1)^{(\deg f+1)\left(1+\sum_{j=1}^{i-1}\deg f_j\right)} \overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J_t : t\in\mathcal{T},f_1,\ldots,f_{i-1},f,f_{i+k_2},\ldots,f_k). \tag{106}$$

6. Perturbation data and virtual chains

We shall now choose *perturbation data* $\mathfrak{s}_{\beta,J,f_1,\ldots,f_k}$ for families of moduli spaces $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J,f_1,\ldots,f_k)$ in (42), as in §2.7, which are compatible at the boundaries with choices made for the boundary strata appearing in (41). Technically $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J,f_1,\ldots,f_k)$ may not be a Kuranishi space, as the components $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)$ in (42) may have different virtual dimensions. Perturbation data for $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J,f_1,\ldots,f_k)$ means perturbation data for $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\ldots,f_k)$ for all I,α in (42), in the obvious way.

Our goal is to define $A_{N,0}$ algebras and gapped filtered A_∞ algebras, which are filtered using $\mathcal{G}\subset[0,\infty)\times\mathbb{Z}$. It is convenient to introduce $\mathcal{G}$ at this point. Choose $\mathcal{G}\subset[0,\infty)\times\mathbb{Z}$ to satisfy the conditions:

(i) $\mathcal{G}$ is closed under addition with $\mathcal{G}\cap(\{0\}\times\mathbb{Z})=\{(0,0)\}$, and $\mathcal{G}\cap([0,C]\times\mathbb{Z})$ is finite for any $C\geqslant 0$; and

(ii) If $\beta\in H_2(M,\iota(L);\mathbb{Z})$, $\overline{\mathcal{M}}_1^{\mathrm{main}}(\beta,J)\neq\emptyset$ then $\bigl([\omega]\cdot\beta,\frac{1}{2}\mu_L(\beta)\bigr)\in\mathcal{G}$.

Here (i) is as in §3.5 and §3.7. If we define $\mathcal{G}_J$ to be the smallest subset of $[0,\infty)\times\mathbb{Z}$ containing $\bigl([\omega]\cdot\beta,\frac{1}{2}\mu_L(\beta)\bigr)$ for all $\beta\in H_2(M,\iota(L);\mathbb{Z})$ with

$\bar{\mathcal{M}}_1^{\mathrm{main}}(\beta,J) \neq \emptyset$ and closed under addition, then $\mathcal{G}_J \cap (\{0\}\times\mathbb{Z}) = \{(0,0)\}$ is immediate as $[\omega]\cdot\beta = 0$, $\bar{\mathcal{M}}_1^{\mathrm{main}}(\beta,J)\neq\emptyset$ imply $\beta=0$, and $\mathcal{G}_J\cap([0,C]\times\mathbb{Z})$ finite for any $C\geqslant 0$ follows from compactness for the family of stable J-holomorphic curves with area $\leqslant C$.

Thus there exists at least one subset $\mathcal{G}$ satisfying (i),(ii). However, we do not want to fix $\mathcal{G}=\mathcal{G}_J$, since in §8–§10 we will vary the complex structure J, and we will want $\mathcal{G}$ to be independent of J. So for the moment we take $\mathcal{G}$ satisfying (i),(ii) to be given. If $\beta\in H_2(M,\iota(L);\mathbb{Z})$ and $\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J)\neq\emptyset$ for any $k\geqslant 0$ then $\bigl([\omega]\cdot\beta,\frac{1}{2}\mu_L(\beta)\bigr)\in\mathcal{G}$. Write $\|\beta\| = \bigl\|([\omega]\cdot\beta,\frac{1}{2}\mu_L(\beta))\bigr\|$, using the notation of (23). Then $\|\beta\|\geqslant 0$, and if $\beta=\beta_1+\beta_2$ for $\beta_1,\beta_2\in H_2(M,\iota(L);\mathbb{Z})$ with $\bar{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\beta_1,J)$, $\bar{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\beta_2,J)\neq\emptyset$ then $\|\beta\|\geqslant\|\beta_1\|+\|\beta_2\|$. With this notation we prove:

Theorem 6.1. *For a given $N\in\mathbb{N}$, there are $\mathcal{X}_0\subset\cdots\subset\mathcal{X}_N$ and $\{\mathfrak{s}_{\beta,J,f_1,\dots,f_k}\}$ which satisfy the following conditions:*

$(N1)$ *$\mathcal{X}_0,\dots,\mathcal{X}_N$ are countable sets of smooth simplicial chains $f:\Delta_a\to L\amalg R$ such that*
- (a) *if $f:\Delta_a\to L\amalg R$ lies in $\mathcal{X}_i$ and $a>0$ then $f\circ F_j^a:\Delta_{a-1}\to L\amalg R$ lies in $\mathcal{X}_i$ for all $j=0,\dots,a$, using the notation of §2.6; and*
- (b) *part* (a) *implies that $\mathbb{Q}\mathcal{X}_i$ is closed under ∂, and a subcomplex of the singular chains $C_*^{\mathrm{si}}(L\amalg R;\mathbb{Q})$. We require that the natural projection $H_*(\mathbb{Q}\mathcal{X}_i,\partial)\to H_*^{\mathrm{si}}(L\amalg R;\mathbb{Q})$ should be an isomorphism.*

$(N2)$ *For all $k\geqslant 0$, $f_1\in\mathcal{X}_{i_1},\dots,f_k\in\mathcal{X}_{i_k}$ and $\beta\in H_2(M,\iota(L);\mathbb{Z})$ with $i_1+\cdots+i_k+\|\beta\|+k-1\leqslant N$ and $\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J,f_1,\dots,f_k)\neq\emptyset$, $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ is perturbation data for $\bigl(\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J,f_1,\dots,f_k),\mathbf{ev}\bigr)$ in the sense of §2.7, and the simplices of $VC\bigl(\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J,f_1,\dots,f_k),\mathbf{ev},\mathfrak{s}_{\beta,J,f_1,\dots,f_k}\bigr)$ lie in $\mathcal{X}_{i_1+\cdots+i_k+\|\beta\|+k-1}$. At the boundary $\partial\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J,f_1,\dots,f_k)$, given by the union of* (41) *over all I,α, this $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ must be compatible with:*
- (i) *the choices of $\mathfrak{s}_{\beta,J,f_1,\dots,f_{i-1},f_i\circ F_j^{a_i},f_{i+1},\dots,f_k}$ for the term $\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha,\beta,J,f_1,\dots,f_{i-1},f_i\circ F_j^{a_i},f_{i+1},\dots,f_k)$ in* (41);
- (ii) *the choices of $\mathfrak{s}_{\beta_2,J,f_i,\dots,f_{i+k_2-1}}$ for the term $\bar{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2,\beta_2,J,f_i,\dots,f_{i+k_2-1})\subset\bar{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\beta_2,J,f_i,\dots,f_{i+k_2-1})$ in* (41); *and*
- (iii) *for each $g:\Delta_a\to L\amalg R$ in $VC\bigl(\bar{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\beta_2,J,f_i,\dots,f_{i+k_2-1}),\mathbf{ev},\mathfrak{s}_{\beta_2,J,f_i,\dots,f_{i+k_2-1}}\bigr)$, the choices of $\mathfrak{s}_{\beta_1,J,f_1,\dots,f_{i-1},g,f_{i+k_2},\dots,f_k}$ for $\bar{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1,\beta_1,J,f_1,\dots,f_{i-1};f_{i+k_2},\dots,f_k)$ in* (41) *combined with $VC\bigl(\bar{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\beta_2,J,f_i,\dots,f_{i+k_2-1}),\mathbf{ev},\mathfrak{s}_{\beta_2,J,f_i,\dots,f_{i+k_2-1}}\bigr)$.*

This boundary compatibility implies that, for $f_1 : \Delta_{a_1} \to L \amalg R$ in $\mathcal{X}_{i_1}, \ldots, f_k : \Delta_{a_k} \to L \amalg R$ in $\mathcal{X}_{i_k}$ as above, we have

$$\begin{aligned}(107)\qquad &\partial VC\big(\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta, J, f_1, \ldots, f_k), \mathbf{ev}, \mathfrak{s}_{\beta,J,f_1,\ldots,f_k}\big) = \\ &\sum_{i=1}^{k}\sum_{j=0}^{a_i} (-1)^{j+1+\sum_{l=1}^{i-1}\deg f_l} VC\big(\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta, J, f_1, \ldots, f_{i-1}, f_i \circ F_j^{a_i}, f_{i+1}, \\ &\qquad \ldots, f_k), \mathbf{ev}, \mathfrak{s}_{\beta,J,f_1,\ldots,f_{i-1},f_i\circ F_j^{a_i},f_{i+1},\ldots,f_k}\big) \\ &+ \sum_{\substack{k_1+k_2=k+1,\\ 1\leqslant i\leqslant k_1,\\ \beta_1+\beta_2=\beta}} (-1)^{n+1+\sum_{l=1}^{i-1}\deg f_l} VC\big(\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\beta_1, J, f_1, \ldots, f_{i-1}, \\ &\qquad VC(\overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\beta_2, J, f_i, \ldots, f_{i+k_2-1}), \mathbf{ev}, \mathfrak{s}_{\beta_2,J,f_i,\ldots,f_{i+k_2-1}}), f_{i+k_2}, \\ &\qquad \ldots, f_k), \mathbf{ev}, \mathfrak{s}_{\beta_1,J,f_1,\ldots,f_{i-1},VC(\overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\beta_2,J,f_i,\ldots,f_{i+k_2-1}),\,\mathbf{ev},\mathfrak{s}_{\beta_2,J,f_i,\ldots,f_{i+k_2-1}}),f_{i+k_2},\ldots,f_k}\big).\end{aligned}$$

Here if $VC(\overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\beta_2,J,f_i,\ldots,f_{i+k_2-1}),\mathbf{ev},\mathfrak{s}_{\beta_2,J,f_i,\ldots,f_{i+k_2-1}}) = \sum_{a\in A}\sigma_a g_a$ for $\sigma_a \in \mathbb{Q}$ and g_a in $\mathcal{X}_{i_i+\cdots+i_{i+k_2-1}+\|\beta_2\|+k_2-1}$, the final term $VC(\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\ldots, VC(\ldots), \ldots), \mathbf{ev}, \mathfrak{s}_{\ldots,VC(\ldots),\ldots})$ in (107) is short for

$$(108)\qquad \sum_{a\in A}\sigma_a\, VC\big(\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\beta_1, J, f_1, \ldots, f_{i-1}, g_a, f_{i+k_2}, \ldots, f_k), \mathbf{ev}, \mathfrak{s}_{\beta_1,J,f_1,\ldots,f_{i-1},g_a,f_{i+k_2},\ldots,f_k}\big).$$

Proof. Our proof is based on Fukaya et al. [**8**, §7.2.5]. It involves a quadruple induction, an outer induction over $g = 0, \ldots, N$ in which we choose $\mathcal{X}_0, \ldots, \mathcal{X}_N$, and an inner triple induction over (j, k, l) during the construction of $\mathcal{X}_{g+1}$.

For the first step $g = 0$ of the outer induction, let $(\|\beta\|, k) = (1, 0)$. Since $\overline{\mathcal{M}}_1^{\mathrm{main}}(\beta, J)$ has no boundary, (i)–(iii) are trivial. Choose arbitrary (but 'small', in a sense discussed below) perturbation data $\mathfrak{s}_{\beta,J}$ for $\big(\overline{\mathcal{M}}_1^{\mathrm{main}}(\beta, J), \mathbf{ev}\big)$ for all $\beta \in H_2(M, \iota(L); \mathbb{Z})$ with $\|\beta\| = 1$ and $\overline{\mathcal{M}}_1^{\mathrm{main}}(\beta, J) \neq \emptyset$. There are only finitely many such β, and we can choose such $\mathfrak{s}_{\beta,J}$ as in §2.7. The virtual cycles $VC\big(\overline{\mathcal{M}}_1^{\mathrm{main}}(\beta, J), \mathbf{ev}, \mathfrak{s}_{\beta,J}\big)$ for all such β involve only finitely many simplices $f : \Delta_a \to L \amalg R$. We must choose $\mathcal{X}_0$ to contain all these simplices, and to satisfy (a),(b) in $(N1)$ above. This is possible by Proposition 2.13.

For the inductive step, suppose that we have chosen $\mathcal{X}_0 \subset \cdots \subset \mathcal{X}_g$ and $\{\mathfrak{s}_{\beta,J,f_1,\ldots,f_k}\}$, which satisfy $(N1)$ and $(N2)$ with $N = g$. We shall construct $\mathcal{X}_{g+1}$ and further choices of $\mathfrak{s}_{\beta,J,f_1,\ldots,f_k}$ satisfying $(N1)$ and $(N2)$ with $N = g + 1$. These choices are not independent of each other, but have to be made in a certain order. Consider triples of integers (j, k, l) such that $j \geqslant 0$, $k \geqslant 1$, $(j, k) \neq (0, 1)$, $j + k \leqslant g + 2$ and $l \geqslant 0$.

Define a total order $\leqslant$ on such triples (j, k, l) by $(j_1, k_1, l_1) \leqslant (j_2, k_2, l_2)$ if either

$(*_1)$ $j_1 + k_1 < j_2 + k_2$; or

$(*_2)$ $j_1 + k_1 = j_2 + k_2$ and $j_1 < j_2$; or
$(*_3)$ $j_1 + k_1 = j_2 + k_2$ and $j_1 = j_2$ and $l_1 \leqslant l_2$.

In a triple induction on (j, k, l), at step (j, k, l) we consider all possible choices of $\beta \in H_2(M, \iota(L); \mathbb{Z})$ with $\|\beta\| = j$ and $i_1, \dots, i_k \geqslant 0$ with $i_1 + \cdots + i_k + j + k - 1 = g + 1$ and $f_1 \in \mathcal{X}_{i_1}, f_2 \in \mathcal{X}_{i_2}, \dots, f_k \in \mathcal{X}_{i_k}$ with $f_i : \Delta_{a_i} \to L \amalg R$, where $a_1 + \cdots + a_k = l$, and such that $\mathcal{M}^{\mathrm{main}}_{k+1}(\beta, J, f_1, \dots, f_k) \neq \emptyset$. There are only finitely many possibilities for such $\beta, i_1, \dots, i_k$, and countably many possibilities for $f_1, \dots, f_k$. We will choose perturbation data $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ on such choices in the order $\leqslant$ on triples (j, k, l).

The important thing about this way of organizing our choices is that for given $\beta, i_1, \dots, i_k, f_1, \dots, f_k$ at step (j, k, l), the compatibilities (i)–(iii) on $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ in $(N2)$ depend only on $\mathfrak{s}_{\beta',J,f'_1,\dots,f'_{k'}}$ which were either chosen with $\mathcal{X}_{g'}$ for $g' \leqslant g$, or were chosen during this step $g + 1$, but for some (j', k', l') with $(j', k', l') < (j, k, l)$. So the boundary conditions on $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ always depend on choices we have already made, not on choices we have yet to make.

To see this, note that at step (g, j, k, l), part (i) involves choices made at step $(g, j, k, l - 1)$, part (ii) choices at step (g', j', k', l') for $g' \leqslant g$, $j' \leqslant j$, $k' \leqslant k$ and l' arbitrary, but with either $j' < j$ (if $\beta_1 \neq 0$) or $k' < k$ (if $\beta_1 = 0$), and part (iii) choices at step (g', j', k', l') for $g' \leqslant g$, $j' \leqslant j$, $k' \leqslant k + 1$, and l' arbitrary, but with either $j' < j$ (if $\beta_2 \neq 0$) or $k' < k$ (if $\beta_2 = 0$); this allows $(j', k') = (j - 1, k + 1)$. Here we use the fact that $\mathcal{M}^{\mathrm{main}}_{k+1}(0, J, f_1, \dots, f_k) = \emptyset$ unless $k \geqslant 2$. In each case $(j', k', l') < (j, k, l)$ by $(*_1)$–$(*_3)$ above.

So, at step (j, k, l) we must choose perturbation data $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ for $\big(\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta, J, f_1, \dots, f_k), \mathbf{ev}\big)$ for the finitely many possibilities for such $\beta, i_1, \dots, i_k$ and countably many possibilities for $f_1, \dots, f_k$ above, satisfying compatibilities (i)–(iii) above with previous choices, which should be 'small' in the sense below. Essentially, (i)–(iii) prescribe $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ over $\partial\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta, J, f_1, \dots, f_k)$, and we have to extend these values over the interior of $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta, J, f_1, \dots, f_k)$.

Because of the definition of boundaries of Kuranishi spaces in §2.2, regarded as subspaces of $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta, J, f_1, \dots, f_k)$, the disjoint components of (39) do actually intersect in $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta, J, f_1, \dots, f_k)$, in the codimension 2 corners of $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta, J, f_1, \dots, f_k)$ which lift to $\partial^2\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta, J, f_1, \dots, f_k)$. But by induction (i)–(iii) prescribe consistent values for $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ on these codimension 2 corners, since the boundary values $\mathfrak{s}_{\beta,J,f_1,\dots,f_{i-1},f_i\circ F_j^{a_i},f_{i+1},\dots,f_k}$, $\mathfrak{s}_{\beta_2,J,f_i,\dots,f_{i+k_2-1}}$, $\mathfrak{s}_{\beta_1,J,f_1,\dots,f_{i-1},f,f_{i+k_2},\dots,f_k}$ appearing in (i)–(iii) themselves satisfy (i)–(iii).

Therefore, the discussion of §2.7 shows that we can choose perturbation data $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ satisfying boundary compatibilities (i)–(iii), but with one caveat. In Definition 2.15 and Remark 2.16(a) we said that a set of perturbation data $\mathfrak{s}_X$ for a Kuranishi space involves a finite cover of X by Kuranishi neighbourhoods $(V^i, E^i, \Gamma^i, s^i, \psi^i)$ and smooth, transverse multisections $\mathfrak{s}^i$ on $(V^i, \dots, \psi^i)$ such that *each* $\mathfrak{s}^i$ *is sufficiently close to* s^i *in* C^0. Here the definition of 'sufficiently close' is rather vague; it has to do with ensuring that the perturbed Kuranishi spaces remain compact.

Now it is conceivable that conditions (i)–(iii) on $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ might be incompatible with this requirement that the multisections $\mathfrak{s}^i$ be 'sufficiently close' to s^i in C^0. That is, in effect (i)–(iii) prescribe $\mathfrak{s}^i$ over ∂V^i, and if these prescribed values are not 'sufficiently close' to $s^i|_{\partial V^i}$ in C^0, then we cannot choose $\mathfrak{s}^i$ on V^i 'sufficiently close' to s^i in C^0 with these values on s^i. In this case, we could not choose $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ satisfying all the necessary conditions.

A version of this problem is described in [**8**, §7.2.3]. The solution adopted by Fukaya et al. [**8**, §7.2.3–§7.2.5], which we follow, is that at every step we choose the perturbations $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ to be 'small', by which we mean that the $\mathfrak{s}^i$ should be close enough to s^i in C^0 that not only does the construction of $VC\big(\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta, J, f_1, \dots, f_k), \mathbf{ev}, \mathfrak{s}_{\beta,J,f_1,\dots,f_k}\big)$ work, as in Definition 2.15, but also, for all the conditions (i)–(iii) involving $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ at later inductive steps in the proof, the prescribed values for $\mathfrak{s}^i$ on ∂V^i should be sufficiently close to $s^i|_{V^i}$ that the later constructions of $VC(\cdots)$ also work. We will discuss this in Remark 6.2.

Thus following this method, at each step (j, k, l) in our triple induction, we can choose perturbation data $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ satisfying (i)–(iii) for all the finitely many choices of β and countably many choices of $f_1, \dots, f_k$ required. This completes the inner induction on (j, k, l). To finish the outer inductive step on g, it remains to choose $\mathcal{X}_{g+1}$. The conditions on $\mathcal{X}_{g+1}$ are that it should contain $\mathcal{X}_g$, and that it should contain the simplices of $VC\big(\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta, J, f_1, \dots, f_k), \mathbf{ev}, \mathfrak{s}_{\beta,J,f_1,\dots,f_k}\big)$ for all the finitely many β and countably many $f_1, \dots, f_k$ we have just considered over all (j, k, l), which is a countable set of simplices, and that it should satisfy (a),(b). This is possible by Proposition 2.13. So we can choose $\mathcal{X}_{g+1}$ satisfying all the conditions. This completes the inductive step for $g = 0, \dots, N$.

We have now constructed $\mathcal{X}_0 \subset \cdots \subset \mathcal{X}_N$ and $\{\mathfrak{s}_{\beta,J,f_1,\dots,f_k}\}$ satisfying $(N1)$ and $(N2)$. It remains only to prove (107). Essentially, this is equation (95), summed over all α, perturbed using the $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$, and regarded as an equation in virtual cycles in $C_*^{\mathrm{si}}(L \amalg R; \mathbb{Q})$ rather than in oriented Kuranishi spaces. However, since we have not chosen perturbation data for $\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1, \beta_1, J, f_1, \dots, f_{i-1}; f_{i+k_2}, \dots, f_k)$, we have to treat the final term of (95) differently. We perturb $\overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2, \beta_2, J, f_i, \dots, f_{i+k_2-1})$ in the fifth line of (95), summed over all α_2, using

$\mathfrak{s}_{\beta_2,J,f_i,\dots,f_{i+k_2-1}}$, so that it becomes a virtual cycle

$$VC(\overline{\mathcal{M}}^{\mathrm{main}}_{k_2+1}(\beta_2,J,f_i,\dots,f_{i+k_2-1}),\mathbf{ev},\mathfrak{s}_{\beta_2,J,f_i,\dots,f_{i+k_2-1}}) = \textstyle\sum_{a\in A}\sigma_a\, g_a. \tag{109}$$

Then in the fibre product in the fifth and sixth lines of (95), we substitute each g_a which is part of the perturbed $\overline{\mathcal{M}}^{\mathrm{main}}_{k_2+1}(\alpha_2,\beta_2,J,f_i,\dots,f_{i+k_2-1})$ into $\overline{\mathcal{M}}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\dots,f_{i-1};f_{i+k_2},\dots,f_k)$. This gives $(-1)^{\cdots}\overline{\mathcal{M}}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\dots,f_{i-1},g_a,f_{i+k_2},\dots,f_k)$ by (96), and we perturb this using $\mathfrak{s}_{\beta_1,J,f_1,\dots,f_{i-1},g_a,f_{i+k_2},\dots,f_k}$, and take its virtual cycle. Considering (i)–(iii) above, we see that modifying (95) in this way to give an equation in virtual cycles is valid, because it corresponds exactly to the conditions (i)–(iii) on $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$, which equate to boundary conditions on $VC\big(\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J,f_1,\dots,f_k),\mathbf{ev},\mathfrak{s}_{\beta,J,f_1,\dots,f_k}\big)$.

Thus (107) follows from (95) summed over all α and (96), except for the signs in (107), which we have not yet computed. The sign on the second line of (107) is the sign on the second line of (95). The sign on the second line of (107) is the combination of the sign on the fourth line on (95), and the sign in (96) when we substitute g_a into $\overline{\mathcal{M}}^{\mathrm{main}}_{k_1+1}(\alpha_1,\beta_1,J,f_1,\dots,f_{i-1};f_{i+k_2},\dots,f_k)$. To calculate this, we need to know $\deg g_a$ for the g_a in (109). We have $g_a:\Delta_b\to L\amalg R$, where $b=\mathrm{vdim}\,\overline{\mathcal{M}}^{\mathrm{main}}_{k_2+1}(\beta_2,J,f_i,\dots,f_{i+k_2-1})$, which is given by (37). Then $\deg g_a$ is given in terms of b by (36). Both equations are divided into cases $0\notin I_2$ and $0\in I_2$, and involve $\eta_{\alpha_2(0)}$ if $0\in I_2$. But combining them, these contributions cancel out, so in every case we have

$$\deg g_a = 1-\mu_L(\beta_2)+\textstyle\sum_{l=i}^{i+k_2-1}\deg f_l. \tag{110}$$

Therefore the overall sign in the fourth line of (107) should be

$$\begin{aligned}&(-1)^{n+\left(1+\sum_{l=1}^{i-1}\deg f_l\right)\left(1+\sum_{l=i}^{i+k_2-1}\deg f_l\right)}\cdot\\ &(-1)^{\left(2-\mu_L(\beta_2)+\sum_{l=i}^{i+k_2-1}\deg f_l\right)\left(1+\sum_{l=1}^{i-1}\deg f_l\right)},\end{aligned} \tag{111}$$

where the first line comes from the fourth line of (95), and the second line from (96), with g_a in place of f and (110) in place of $\deg f$. Noting that $2-\mu_L(\beta_2)$ is even, (111) simplifies to $(-1)^{n+1+\sum_{l=1}^{i-1}\deg f_l}$. This completes the proof. q.e.d.

Remark 6.2. In Theorem 6.1, we had to fix a *finite* $N\geqslant 0$, and then choose $\mathcal{X}_0,\dots,\mathcal{X}_N$ and $\{\mathfrak{s}_{\beta,J,f_1,\dots,f_k}\}$. The conditions on $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ with $f_1\in\mathcal{X}_{i_1},\dots,f_k\in\mathcal{X}_{i_k}$ and $i_1+\cdots+i_k+\|\beta\|+k-1\leqslant g$ for $g\leqslant N$ really do depend not just on the $\mathcal{X}_1,\dots,\mathcal{X}_g$, *but also on the choice of* N, because we had to choose $\mathfrak{s}_{\beta,J,f_1,\dots,f_k}$ to be 'small', that is, the multisections $\mathfrak{s}^i$ must be sufficiently close to s^i in C^0, and this notion of 'sufficiently close' depends not just on $\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J,f_1,\dots,f_k)$, but on all the other fibre products involving $\mathfrak{s}_{\beta,J,f_1,\ldots,f_k}$ in their boundary conditions in the later inductive steps $g+1, g+2, \ldots, N$.

Because of this, we cannot prove the theorem for $N=\infty$, that is, we cannot get an infinite sequence $\mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots$ and an infinite set of choices of perturbation data $\{\mathfrak{s}_{\beta,J,f_1,\ldots,f_k}\}$ satisfying $(N1)$, $(N2)$. Taking the limit $N\to\infty$ does not work, since the $\mathcal{X}_g$ for $g<N$ and $\mathfrak{s}_{\beta,J,f_1,\ldots,f_k}$ in the theorem depend on N. This is discussed by Fukaya et al. [**8**, §7.2.3–§7.2.5], see [**8**, (7.2.56.1–5) & Rem. 7.2.57], and we follow them.

The issue is that it may not be possible to impose both *infinitely many smallness conditions*, and a *transversality condition*, on a single choice of perturbation. As a simple analogy, consider finding $\epsilon\in\mathbb{R}$ satisfying $|\epsilon|\leqslant 1/n$ for $n=1,2,\ldots$ (infinitely many smallness conditions) and $\epsilon\neq 0$ (a transversality condition). This is clearly not possible. Since the proof of Theorem 6.1 does involve making infinitely many linked choices of transverse perturbations with smallness conditions, we have to be careful that these choices are possible. In the method of [**8**], by imposing a fixed upper limit N for $i_1+\cdots+i_k+\|\beta\|+k-1$, each choice of perturbation data $\mathfrak{s}_{\beta,J,f_1,\ldots,f_k}$ is involved in only finitely many inductive steps, and so must satisfy only finitely many smallness conditions.

7. $A_{N,0}$ algebras from immersed Lagrangian submanifolds

Definition 7.1. Let $\mathcal{G}$ be as in §6, and $\|\,.\,\|:\mathcal{G}\to\mathbb{N}$ be as in (23). For a given $N\in\mathbb{N}$, let $\mathcal{X}_0\subset\cdots\subset\mathcal{X}_N$ and $\{\mathfrak{s}_{\beta,J,f_1,\ldots,f_k}\}$ be as in Theorem 6.1. Suppose $k\geqslant 0$, $(\lambda,\mu)\in\mathcal{G}$, and $i_1,\ldots,i_k=0,\ldots,N$ with $i_1+\cdots+i_k+\|(\lambda,\mu)\|+k-1\leqslant N$. Define a $\mathbb{Q}$-multilinear map $\mathfrak{m}_{k,\mathrm{geo}}^{\lambda,\mu}:\mathbb{Q}\mathcal{X}_{i_1}\times\cdots\times\mathbb{Q}\mathcal{X}_{i_k}\to\mathbb{Q}\mathcal{X}_{i_1+\cdots+i_k+\|(\lambda,\mu)\|+k-1}$ by

$$\begin{aligned}&\mathfrak{m}_{1,\mathrm{geo}}^{0,0}(f_1)=(-1)^n\partial f_1,\\ &\mathfrak{m}_{k,\mathrm{geo}}^{\lambda,\mu}(f_1,\ldots,f_k)=\sum_{\substack{\beta\in H_2(M,\iota(L);\mathbb{Z}):\\ [\omega]\cdot\beta=\lambda,\ \mu_L(\beta)=2\mu,\\ \overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J,f_1,\ldots,f_k)\neq\emptyset}} VC\bigl(\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J,f_1,\ldots,f_k),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\mathbf{ev},\mathfrak{s}_{\beta,J,f_1,\ldots,f_k}\bigr),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad(k,\lambda,\mu)\neq(1,0,0).\end{aligned}\tag{112}$$

Combining (36), (37) and $\mu_L(\beta)=2\mu$ shows that the shifted cohomological degree in (112) is

$$\begin{aligned}\deg VC\bigl(\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J,f_1,\ldots,f_k),\mathbf{ev},\mathfrak{s}_{\beta,J,f_1,\ldots,f_k}\bigr)&\\ =1-2\mu+\textstyle\sum_{i=1}^k\deg f_i.&\end{aligned}\tag{113}$$

Thus $\mathfrak{m}_{k,\mathrm{geo}}^{\lambda,\mu}:\mathbb{Q}\mathcal{X}_{i_1}\times\cdots\times\mathbb{Q}\mathcal{X}_{i_k}\to\mathbb{Q}\mathcal{X}_{i_1+\cdots+i_k+\|(\lambda,\mu)\|+k-1}$ has degree $1-2\mu$.

Then (107) immediately implies the following:

Proposition 7.2. *For $k \in \mathbb{N}$, $(\lambda,\mu) \in \mathcal{G}$ and $f_1 \in \mathcal{X}_{i_1}, \dots, f_k \in \mathcal{X}_{i_k}$ with $i_1 + \cdots + i_k + \|(\lambda,\mu)\| + k - 1 \leqslant N$, we have*

$$\sum_{\substack{k_1+k_2=k+1,\ 1\leqslant i\leqslant k_1,\ k_2\geqslant 0,\\ (\lambda_1,\mu_1),(\lambda_2,\mu_2)\in\mathcal{G},\ (\lambda_1,\mu_1)+(\lambda_2,\mu_2)=(\lambda,\mu)}} (-1)^{\sum_{l=1}^{i-1}\deg f_l}\mathfrak{m}_{k_1,\mathrm{geo}}^{\lambda_1,\mu_1}\big(f_1,\dots,f_{i-1},\mathfrak{m}_{k_2,\mathrm{geo}}^{\lambda_2,\mu_2}(f_i,\dots,f_{i+k_2-1}), f_{i+k_2}\dots,f_k\big) = 0. \tag{114}$$

Equation (114) is just (17) for the $\mathfrak{m}_{k,\mathrm{geo}}^{\lambda,\mu}$. Thus, the data $\mathbb{Q}\mathcal{X}_0 \subset \cdots \subset \mathbb{Q}\mathcal{X}_N$ and $\mathfrak{m}_{k,\mathrm{geo}}^{\lambda,\mu}$ are a finite approximation to a gapped filtered A_∞ algebras, as for $A_{N,K}$ algebras in §3.7. But it is not an $A_{N,K}$ algebra, as the conditions for when $\mathfrak{m}_{k,\mathrm{geo}}^{\lambda,\mu}(f_1,\dots,f_k)$ is defined are different. We can apply purely algebraic methods from Fukaya et al. [**8**, §5.4.4, §7.2.7] to define $A_{N,0}$ algebras. We use the method of sums over planar trees from §3.3, based on the construction of $\mathfrak{n}$ in Definition 3.8.

Definition 7.3. For a given $N \in \mathbb{N}$, we take $N' = N(N+2)$. Let $\mathcal{X}_0 \subset \cdots \subset \mathcal{X}_{N'}$ and $\{s_{\beta,J,f_1,\dots,f_k}\}$ be as in Theorem 6.1 with N' in place of N. Since the homologies of $(\mathbb{Q}\mathcal{X}_N, \partial), (\mathbb{Q}\mathcal{X}_{N'}, \partial)$ are isomorphic, we can find some linear subspace $A \subset \mathbb{Q}\mathcal{X}_{N'}$ such that $\mathbb{Q}\mathcal{X}_{N'} = \mathbb{Q}\mathcal{X}_N \oplus A \oplus \partial A$ and $\partial : A \to \partial A$ is an isomorphism. Let $\Pi : \mathbb{Q}\mathcal{X}_{N'} \to \mathbb{Q}\mathcal{X}_N$ for the projection, and define linear $H : \mathbb{Q}\mathcal{X}_{N'} \to \mathbb{Q}\mathcal{X}_{N'}$ by

$$H(x) = \begin{cases} 0, & \text{for } x \in \mathbb{Q}\mathcal{X}_N \oplus A, \\ \partial^{-1}x, & \text{for } x \in \partial A. \end{cases} \tag{115}$$

Then $\mathrm{id} - \Pi = \partial H + H\partial$, as in §3.4 with $\mathfrak{m}_1 = \partial$.

Suppose $k \geqslant 0$ and $(\lambda,\mu) \in \mathcal{G}$ with $\|(\lambda,\mu)\| + k - 1 \leqslant N$. Let T be a rooted planar tree with k leaves, and $(\boldsymbol{\lambda},\boldsymbol{\mu})$ be a family of $(\lambda_v,\mu_v) \in \mathcal{G}$ for each internal vertex v of T, such that $\sum_v(\lambda_v,\mu_v) = (\lambda,\mu)$, and $(\lambda_v,\mu_v) = (0,0)$ implies that v has at least 2 incoming edges. We shall define a graded multilinear operator $\mathfrak{m}_{k,T}^{(\boldsymbol{\lambda},\boldsymbol{\mu})} : (\mathbb{Q}\mathcal{X}_N)^{\times^k} \to \mathbb{Q}\mathcal{X}_N$ of degree $-2\mu + 1$. Let $f_1,\dots,f_k \in \mathcal{X}_N$. Assign objects and operators to the vertices and edges of T:

- assign $f_1,\dots,f_k$ to the leaf vertices $1,\dots,k$ respectively.
- for each internal vertex v with 1 outgoing edge and n incoming edges, assign $\mathfrak{m}_{n,\mathrm{geo}}^{\lambda_v,\mu_v}$. (Here by assumption $(\lambda_v,\nu_v) = (0,0)$ implies $n \geqslant 2$, so we never assign the special case $\mathfrak{m}_{1,\mathrm{geo}}^{0,0}$ in (112).)
- assign id to each leaf edge.
- assign Π to the root edge.
- assign $(-1)^{n+1}H$ to each internal edge.

Then define $\mathfrak{m}_{k,T}^{(\boldsymbol{\lambda},\boldsymbol{\mu})}(f_1,\dots,f_k)$ to be the composition of all these objects and morphisms, as in §3.3. Define a $\mathbb{Q}$-multilinear map $\mathfrak{m}_k^{\lambda,\mu} : (\mathbb{Q}\mathcal{X}_N)^{\times^k} \to \mathbb{Q}\mathcal{X}_N$ graded of degree $1 - 2\mu$ by $\mathfrak{m}_1^{0,0} = \mathfrak{m}_{1,\mathrm{geo}}^{0,0} = (-1)^n\partial$

and $\mathfrak{m}_k^{\lambda,\mu} = \sum_{T,(\boldsymbol{\lambda},\boldsymbol{\mu})} \mathfrak{m}_{k,T}^{(\boldsymbol{\lambda},\boldsymbol{\mu})}$ for $(k,\lambda,\mu) \neq (1,0,0)$, where the sum is over all $T,(\boldsymbol{\lambda},\boldsymbol{\mu})$ as above.

We can now associate an $A_{N,0}$ algebra to L. It depends on the choices of almost complex structure J, perturbation data $\mathfrak{s}_{\beta,J,f_1,\ldots,f_k}$, and $N, N', \mathcal{X}_N, \mathcal{X}_{N'}, A$ above.

Theorem 7.4. **(a)** *In Definition* 7.3, *the* $\mathfrak{m}_k^{\lambda,\mu}$ *satisfy equation* (17) *for all* $k \geqslant 0$ *and* $(\lambda,\mu) \in \mathcal{G}$ *with* $\|(\lambda,\mu)\| + k - 1 \leqslant N$. *Thus* $(\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m})$ *is an* $A_{N,0}$ *algebra in the sense of Definition* 3.21, *where* $\mathfrak{m} = \big(\mathfrak{m}_k^{\lambda,\mu} : k \geqslant 0,\, (\lambda,\mu) \in \mathcal{G},\, \|(\lambda,\mu)\|+k-1 \leqslant N\big)$, *and* $\mathbb{Q}\mathcal{X}_N$ *is graded by shifted cohomological degree in* (36).

(b) *If* $f_1 \in \mathcal{X}_{i_1}, \ldots, f_k \in \mathcal{X}_{i_k}$ *with* $i_1+\cdots+i_k+\|(\lambda,\mu)\|+k-1 \leqslant N$ *then*

$$\mathfrak{m}_k^{\lambda,\mu}(f_1,\ldots,f_k) = \mathfrak{m}_{k,\mathrm{geo}}^{\lambda,\mu}(f_1,\ldots,f_k). \tag{116}$$

Proof. The proof of (a) follows the first parts of those of Theorems 3.9 and 3.17, as in [**8**, §7.2.7]. For (b), suppose $f_1 \in \mathcal{X}_{i_1}, \ldots, f_k \in \mathcal{X}_{i_k}$ with $i_1+\cdots+i_k+\|(\lambda,\mu)\|+k-1 \leqslant N$ and $T,(\boldsymbol{\lambda},\boldsymbol{\mu})$ are as in Definition 7.3, where T has at least one internal edge. Then $\mathfrak{m}_{k,T}^{(\boldsymbol{\lambda},\boldsymbol{\mu})}(f_1,\ldots,f_k)$ includes an expression $-H \circ \mathfrak{m}_{n,\mathrm{geo}}^{\lambda_v,\mu_v}(f_{a+1},\ldots,f_{a+n})$, with $-H$ from an internal edge, and $\mathfrak{m}_{n,\mathrm{geo}}^{\lambda_v,\mu_v}(f_{a+1},\ldots,f_{a+n})$ lies in $\mathbb{Q}\mathcal{X}_{i_{a+1}+\cdots+i_{a+n}+\|(\lambda_v,\mu_v)\|+n-1}$, and so in $\mathbb{Q}\mathcal{X}_N$, as $n \leqslant k$ and $\|(\lambda_v,\mu_v)\| \leqslant \|(\lambda,\mu)\|$. But $H = 0$ on $\mathbb{Q}\mathcal{X}_N$, so $-H \circ \mathfrak{m}_{n,\mathrm{geo}}^{\lambda_v,\mu_v}(f_{a+1},\ldots,f_{a+n}) = 0$, and $\mathfrak{m}_{k,T}^{(\boldsymbol{\lambda},\boldsymbol{\mu})}(f_1,\ldots,f_k) = 0$.

Therefore $\mathfrak{m}_{k,T}^{(\boldsymbol{\lambda},\boldsymbol{\mu})}(f_1,\ldots,f_k) = 0$ if T has an internal edge, so for $(k,\lambda,\mu) \neq (1,0,0)$ the only nonzero contribution to $\mathfrak{m}_k^{\lambda,\mu}(f_1,\ldots,f_k)$ comes from the unique planar tree T with one internal vertex and k leaves, which gives $\Pi \circ \mathfrak{m}_{k,\mathrm{geo}}^{\lambda,\mu}(f_1,\ldots,f_k)$. But $\mathfrak{m}_{k,\mathrm{geo}}^{\lambda,\mu}(f_1,\ldots,f_k) \in \mathbb{Q}\mathcal{X}_N$, so Π acts as the identity on it, proving (116). When $(k,\lambda,\mu) = (1,0,0)$, equation (116) holds by definition. q.e.d.

8. Choosing perturbation data for $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta, J_t : t \in [0,1], f_1, \ldots, f_k)$

The $A_{N,0}$ algebras of §7 depended on a choice of almost complex structure J. In §9 we will show that for two choices J_0, J_1 for J, the resulting $A_{N,0}$ algebras are homotopy equivalent. We do this by choosing a smooth 1-parameter family $J_t : t \in [0,1]$ of almost complex structures interpolating between J_0 and J_1, and using the moduli spaces $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta, J_t : t \in [0,1], f_1, \ldots, f_k)$.

In this section we generalize Theorem 6.1 to choose perturbation data for the $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta, J_t : t \in [0,1], f_1, \ldots, f_k)$. Choose $\mathcal{G} \subset [0,\infty) \times \mathbb{Z}$ to satisfy the conditions:

(i) $\mathcal{G}$ is closed under addition with $\mathcal{G} \cap (\{0\} \times \mathbb{Z}) = \{(0,0)\}$, and $\mathcal{G} \cap ([0,C] \times \mathbb{Z})$ is finite for any $C \geqslant 0$; and

(ii) if $\beta \in H_2(M, \iota(L); \mathbb{Z})$, and $\overline{\mathcal{M}}_1^{\mathrm{main}}(\beta, J_t : t \in [0,1]) \neq \emptyset$ then $\bigl([\omega] \cdot \beta, \frac{1}{2}\mu_L(\beta)\bigr) \in \mathcal{G}$.

As for $\mathcal{G}_J$ in §6, there exists a unique smallest subset $\mathcal{G}_{J_t:t\in[0,1]}$ satisfying (i),(ii), but we do not necessarily take $\mathcal{G} = \mathcal{G}_{J_t:t\in[0,1]}$. Write $\|\beta\| = \bigl\|([\omega] \cdot \beta, \frac{1}{2}\mu_L(\beta))\bigr\|$, using (23) for this $\mathcal{G}$. With this notation we prove:

Theorem 8.1. *Let (M,ω) be a compact $2n$-dimensional symplectic manifold and $\iota : L \to M$ be a compact Lagrangian immersion with only transverse double self-intersections. Suppose J_t for $t \in [0,1]$ is a smooth family of almost complex structures on M compatible with ω. Define compact Kuranishi spaces $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha, \beta, J_t : t \in [0,1], f_1, \dots, f_k)$ as in §4.5 with $\mathcal{T} = [0,1]$, and orient them as in §5.5. Then for a given $N \in \mathbb{N}$, there are $\mathcal{X}_0^{[0,1]} \subset \cdots \subset \mathcal{X}_N^{[0,1]}$ and $\{\mathfrak{s}_{\beta, J_t:t\in[0,1], f_1,\dots,f_k}\}$ which satisfy the following conditions:*

(N1) *$\mathcal{X}_0^{[0,1]}, \dots, \mathcal{X}_N^{[0,1]}$ are countable sets of smooth simplicial chains $f : \Delta_a \to [0,1] \times (L \amalg R)$ such that*

(a) *there is a decomposition $\mathcal{X}_i^{[0,1]} = \mathcal{X}_i^0 \amalg \mathcal{X}_i^{(0,1)} \amalg \mathcal{X}_i^1$ for $i = 0, \dots, N$, where $f \in \mathcal{X}_i^0$ implies $f(\Delta_a) \subseteq \{0\} \times (L \amalg R)$, and $f \in \mathcal{X}_i^1$ implies $f(\Delta_a) \subseteq \{1\} \times (L \amalg R)$, and $f \in \mathcal{X}_i^{(0,1)}$ implies $f(\Delta_a^\circ) \subseteq (0,1) \times (L \amalg R)$, where Δ_a° is the interior of Δ_a, and $\pi_{[0,1]} \circ f : \Delta_a \to [0,1]$ is a submersion near $(\pi_{[0,1]} \circ f)^{-1}(\{0,1\})$. (This is equivalent to Condition 4.12.) We shall sometimes regard $\mathcal{X}_i^{(0,1)}$ as singular chains on $[0,1] \times (L \amalg R)$ **relative to** $\{0,1\} \times (L \amalg R)$, that is, we project to $C_*^{\mathrm{si}}\bigl([0,1] \times (L \amalg R), \{0,1\} \times (L \amalg R); \mathbb{Q}\bigr)$.*

(b) *if $f : \Delta_a \to [0,1] \times (L \amalg R)$ lies in $\mathcal{X}_i^{[0,1]}$ and $a > 0$ then $f \circ F_b^a : \Delta_{a-1} \to [0,1] \times (L \amalg R)$ lies in $\mathcal{X}_i^{[0,1]}$ for all $b = 0, \dots, a$, using the notation of §2.6. If $g : \Delta_{a-1} \to [0,1] \times (L \amalg R)$ lies in $\mathcal{X}_i^0$ or $\mathcal{X}_i^1$ then $g = f \circ F_b^a$ for some $f : \Delta_a \to [0,1] \times (L \amalg R)$ in $\mathcal{X}_i^{(0,1)}$ and $b \in \{0, \dots, a\}$. If $f : \Delta_a \to [0,1] \times (L \amalg R)$ in $\mathcal{X}_i^{(0,1)}$ then $f \circ F_b^a$ lies in $\mathcal{X}_i^0$ or $\mathcal{X}_i^1$ for at most one $b = 0, \dots, a$.*

(c) *(a),(b) imply that $\mathbb{Q}\mathcal{X}_i^0, \mathbb{Q}\mathcal{X}_i^1$ and $\mathbb{Q}\mathcal{X}_i^{(0,1)}$ are subcomplexes of the (relative) singular chains $C_*^{\mathrm{si}}\bigl(\{0\} \times (L \amalg R); \mathbb{Q}\bigr)$, $C_*^{\mathrm{si}}\bigl(\{1\} \times (L \amalg R); \mathbb{Q}\bigr)$ and $C_*^{\mathrm{si}}\bigl([0,1] \times (L \amalg R), \{0,1\} \times (L \amalg R); \mathbb{Q}\bigr)$ respectively. We require that the corresponding three natural projections should be isomorphisms:*

$$
\begin{gathered}
H_*(\mathbb{Q}\mathcal{X}_i^0, \partial^0) \xrightarrow{\cong} H_*^{\mathrm{si}}(L \amalg R; \mathbb{Q}), \quad H_*(\mathbb{Q}\mathcal{X}_i^1, \partial^1) \xrightarrow{\cong} H_*^{\mathrm{si}}(L \amalg R; \mathbb{Q}), \\
H_*\bigl(\mathbb{Q}\mathcal{X}_i^{(0,1)}, \partial^{(0,1)}\bigr) \xrightarrow{\cong} H_*^{\mathrm{si}}\bigl([0,1] \times (L \amalg R), \{0,1\} \times (L \amalg R); \mathbb{Q}\bigr),
\end{gathered}
\tag{117}
$$

identifying $\{0\} \times (L \amalg R)$ and $\{1\} \times (L \amalg R)$ with $L \amalg R$.

(N2) *For all* $k \geqslant 0$, $f_1 \in \mathcal{X}_{i_1}^{[0,1]}, \dots, f_k \in \mathcal{X}_{i_k}^{[0,1]}$ *and* $\beta \in H_2(M, \iota(L); \mathbb{Z})$ *with* $i_1 + \cdots + i_k + \|\beta\| + k - 1 \leqslant N$ *and* $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta, J_t : t \in [0,1], f_1, \dots, f_k) \neq \emptyset$, $\mathfrak{s}_{\beta, J_t : t\in[0,1], f_1,\dots,f_k}$ *is perturbation data for* $\bigl(\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta, J_t : t\in[0,1], f_1, \dots, f_k), \boldsymbol{\pi}_{[0,1]} \times \mathbf{ev}\bigr)$, *and the simplices of* $VC\bigl(\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta, J_t : t\in[0,1], f_1, \dots, f_k), \boldsymbol{\pi}_{[0,1]} \times \mathbf{ev}, \mathfrak{s}_{\beta, J_t : t\in[0,1], f_1,\dots,f_k}\bigr)$ *lie in* $\mathcal{X}^{[0,1]}_{i_1+\cdots+i_k+\|\beta\|+k-1}$. *At the boundary* $\partial\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta, J_t : t \in [0,1], f_1, \dots, f_k)$, *given by the union of* (52) *over all* I, α, *this* $\mathfrak{s}_{\beta, J_t : t\in[0,1], f_1,\dots,f_k}$ *must be compatible with:*

(i) *the choices of* $\mathfrak{s}_{\beta, J_t : t\in[0,1], f_1,\dots,f_{i-1}, f_i\circ F_j^{a_i}, f_{i+1},\dots,f_k}$ *for* $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha, \beta, J_t : t \in [0,1], f_1, \dots, f_{i-1}, f_i \circ F_j^{a_i}, f_{i+1}, \dots, f_k)$ *in* (52);

(ii) *the choices of* $\mathfrak{s}_{\beta_2, J_t : t\in[0,1], f_i,\dots,f_{i+k_2-1}}$ *for the term* $\overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\alpha_2, \beta_2, J_t : t \in [0,1], f_i, \dots, f_{i+k_2-1})$ *in* (52); *and*

(iii) *for each* $g : \Delta_a \to [0,1] \times (L \amalg R)$ *appearing in* $VC\bigl(\overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\beta_2, J_t : t \in [0,1], f_i, \dots, f_{i+k_2-1}), \boldsymbol{\pi}_{[0,1]} \times \mathbf{ev}, \mathfrak{s}_{\beta_2, J_t : t\in[0,1], f_i,\dots,f_{i+k_2-1}}\bigr)$, *the choices of* $\mathfrak{s}_{\beta_1, J_t : t\in[0,1], f_1,\dots,f_{i-1}, g, f_{i+k_2},\dots,f_k}$ *for* $\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\alpha_1, \beta_1, J_t : t \in [0,1], f_1, \dots, f_{i-1}; f_{i+k_2}, \dots, f_k)$ *in* (52) *combined with* $VC\bigl(\overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\beta_2, J_t : t \in [0,1], f_i, \dots, f_{i+k_2-1}), \boldsymbol{\pi}_{[0,1]} \times \mathbf{ev}, \mathfrak{s}_{\beta_2, J_t : t\in[0,1], f_i,\dots,f_{i+k_2-1}}\bigr)$.

This boundary compatibility implies that, for $f_1 : \Delta_{a_1} \to [0,1] \times (L \amalg R)$ *in* $\mathcal{X}_{i_1}^{[0,1]}, \dots, f_k : \Delta_{a_k} \to L \amalg R$ *in* $\mathcal{X}_{i_k}^{[0,1]}$ *as above, when* $k > 0$ *we have*

$$
\begin{aligned}
&\partial VC\bigl(\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta, J_t : t\in[0,1], f_1, \dots, f_k), \boldsymbol{\pi}_{[0,1]} \times \mathbf{ev}, \mathfrak{s}_{\beta, J_t : t\in[0,1], f_1,\dots,f_k}\bigr) = \\
&\sum_{i=1}^{k}\sum_{j=0}^{a_i} (-1)^{j+1+\sum_{l=1}^{i-1}\deg f_l} VC\bigl(\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta, J_t : t\in[0,1], f_1, \dots, f_{i-1}, f_i\circ F_j^{a_i}, \\
&\qquad f_{i+1}, \dots, f_k), \boldsymbol{\pi}_{[0,1]} \times \mathbf{ev}, \mathfrak{s}_{\beta, J_t : t\in[0,1], f_1,\dots,f_{i-1}, f_i\circ F_j^{a_i}, f_{i+1},\dots,f_k}\bigr) \\
&+ \sum_{\substack{k_1+k_2=k+1,\\ 1\leqslant i\leqslant k_1,\\ \beta_1+\beta_2=\beta}} (-1)^{n+\sum_{l=1}^{i-1}\deg f_l} VC\bigl(\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\beta_1, J_t : t \in [0,1], f_1, \dots, f_{i-1}, \\
&\qquad VC(\overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\beta_2, J_t : t\in[0,1], f_i, \dots, f_{i+k_2-1}), \boldsymbol{\pi}_{[0,1]} \times \mathbf{ev}, \\
&\qquad \mathfrak{s}_{\beta_2, J_t : t\in[0,1], f_i,\dots,f_{i+k_2-1}}), f_{i+k_2}, \dots, f_k), \boldsymbol{\pi}_{[0,1]} \times \mathbf{ev}, \\
&\qquad \mathfrak{s}_{\substack{\beta_1, J_t : t\in[0,1], f_1,\dots,f_{i-1}, VC(\overline{\mathcal{M}}_{k_2+1}^{\mathrm{main}}(\beta_2, J_t : t\in[0,1], f_i,\dots,f_{i+k_2-1}),\\ \boldsymbol{\pi}_{[0,1]}\times\mathbf{ev}, \mathfrak{s}_{\beta_2, J_t : t\in[0,1], f_i,\dots,f_{i+k_2-1}}), f_{i+k_2},\dots,f_k}}\bigr),
\end{aligned}
\tag{118}
$$

using notation $VC(\overline{\mathcal{M}}_{k_1+1}^{\mathrm{main}}(\dots, VC(\dots), \dots), \mathbf{ev}, \mathfrak{s}_{\dots, VC(\dots), \dots})$ *as in* (108).

When $k = 0$ *equation* (118) *holds with the addition of an extra term supported on* $\{0,1\} \times (L \amalg R)$ *corresponding to* $\overline{\mathcal{M}}_1^{\mathrm{main}}(\beta, J_t : t \in \{0,1\})$, *as in Theorem* 5.19. *Thus, if we project to relative chains* $C_*^{\mathrm{si}}\bigl([0,1] \times (L \amalg R), \{0,1\} \times (L \amalg R); \mathbb{Q}\bigr)$, *then* (118) *holds for all* $k \geqslant 0$.

($N3$) *As well as the boundary compatibilities* ($N2$)(i)–(iii), *we can impose compatibilities at the boundary* $\{0,1\}\times(L\amalg R)$ *of* $[0,1]\times(L\amalg R)$, *as follows. Suppose* $g_1\in\mathcal{X}^0_{i_1},\dots,g_k\in\mathcal{X}^0_{i_k}$, *where* $g_j:\Delta_{a_j-1}\to\{0\}\times(L\amalg R)$. *We shall also abuse notation by regarding* g_j *as mapping* $\Delta_{a_j-1}\to L\amalg R$. *Then* ($N1$)(b) *implies that there exist* $f_1\in\mathcal{X}^{(0,1)}_{i_1},\dots,f_k\in\mathcal{X}^{(0,1)}_{i_k}$ *and* $b_1,\dots,b_k$ *such that* $b_j=0,\dots,a_j$ *and* $g_j=f_j\circ F^{a_j}_{b_j}$ *for* $j=1,\dots,k$. *Then using the notation of Remark* 4.11 *and inserting signs in* (56)–(57), *there are natural isomorphisms of oriented Kuranishi spaces:*

$$\begin{aligned}&\{0\}\times_{i,[0,1],\boldsymbol{\pi}_{[0,1]}}\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J_t:t\in[0,1],f_1,\dots,f_k)\\ &\quad\cong(-1)^{k+\sum_{j=1}^k b_j}\,\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J_0,g_1,\dots,g_k)\\ &\qquad\times\bigl[\{0\}\times_{i,\mathbb{R},\boldsymbol{\pi}_0}\bigl((\mathbb{R},\kappa^1_k)\times_{\boldsymbol{\pi}_1\times\cdots\times\boldsymbol{\pi}_k,\mathbb{R}^k,i}[0,\infty)^k\bigr)\bigr],\end{aligned}\tag{119}$$

$$\begin{aligned}&\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J_t:t\in[0,1],f_1,\dots,f_{j-1},g_j,f_{j+1},\dots,f_k)\\ &\quad\cong\pm\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J_0,g_1,\dots,g_k)\\ &\qquad\quad\times\bigl[(\mathbb{R},\kappa^1_k)\times_{\boldsymbol{\pi}_1\times\cdots\times\boldsymbol{\pi}_k,\mathbb{R}^k,i}[0,\infty)^{j-1}\times\{0\}\times[0,\infty)^{k-j}\bigr],\end{aligned}\tag{120}$$

for all $j=1,\dots,k$, *where* $i:\{0\}\to[0,1]$ *is the inclusion. The analogues for* $g_1\in\mathcal{X}^1_{i_1},\dots,g_k\in\mathcal{X}^1_{i_k}$ *and* $i:\{1\}\to[0,1]$ *are*

$$\begin{aligned}&\{1\}\times_{i,[0,1],\boldsymbol{\pi}_{[0,1]}}\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J_t:t\in[0,1],f_1,\dots,f_k)\\ &\quad\cong(-1)^{\sum_{j=1}^k b_j}\,\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J_1,g_1,\dots,g_k)\\ &\qquad\times\bigl[\{0\}\times_{i,\mathbb{R},\boldsymbol{\pi}_0}\bigl((\mathbb{R},\kappa^1_k)\times_{\boldsymbol{\pi}_1\times\cdots\times\boldsymbol{\pi}_k,\mathbb{R}^k,i}[0,\infty)^k\bigr)\bigr],\end{aligned}\tag{121}$$

$$\begin{aligned}&\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J_t:t\in[0,1],f_1,\dots,f_{j-1},g_j,f_{j+1},\dots,f_k)\\ &\quad\cong\pm\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J_1,g_1,\dots,g_k)\\ &\qquad\quad\times\bigl[(\mathbb{R},\kappa^1_k)\times_{\boldsymbol{\pi}_1\times\cdots\times\boldsymbol{\pi}_k,\mathbb{R}^k,i}[0,\infty)^{j-1}\times\{0\}\times[0,\infty)^{k-j}\bigr].\end{aligned}\tag{122}$$

Suppose $\tilde{\mathcal{X}}^0_0\subset\cdots\subset\tilde{\mathcal{X}}^0_N$ *and* $\{\mathfrak{s}^0_{\beta,J_0,f_1,\dots,f_k}\}$ *are possible choices in Theorem* 6.1 *with* J_0 *in place of* J, *and that* $\tilde{\mathcal{X}}^1_0\subset\cdots\subset\tilde{\mathcal{X}}^1_N$ *and* $\{\mathfrak{s}^1_{\beta,J_1,f_1,\dots,f_k}\}$ *are possible choices in Theorem* 6.1 *with* J_1 *in place of* J. *Then we can choose* $\mathcal{X}^{[0,1]}_0\subset\cdots\subset\mathcal{X}^{[0,1]}_N$ *and* $\{\mathfrak{s}_{\beta,J_t:t\in[0,1],f_1,\dots,f_k}\}$ *above such that*

(a) $\mathcal{X}^0_i=\tilde{\mathcal{X}}^0_i$ *and* $\mathcal{X}^1_i=\tilde{\mathcal{X}}^1_i$ *for* $i=1,\dots,N$.

(b) *For all* $g_1\in\tilde{\mathcal{X}}^0_{i_1},\dots,g_k\in\tilde{\mathcal{X}}^0_{i_k}$, *and choices of* $f_1,\dots,f_k,b_1,\dots,b_k,j$ *above, the perturbation data* $\mathfrak{s}_{\beta,J_t:t\in[0,1],f_1,\dots,f_{j-1},g_j,f_{j+1},\dots,f_k}$ *for* $\bigl(\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J_t:t\in[0,1],f_1,\dots,f_{j-1},g_j,f_{j+1},\dots,f_k),\boldsymbol{\pi}_{[0,1]}\times\mathbf{ev}\bigr)$ *over* $\{0\}\times(L\amalg R)$ *is identified with the perturbation data* $\mathfrak{s}^0_{\beta,J_0,g_1,\dots,g_k}$ *for* $\bigl(\overline{\mathcal{M}}^{\mathrm{main}}_{k+1}(\beta,J_0,g_1,\dots,g_k),\mathbf{ev}\bigr)$ *over* $L\amalg R$ *under the isomorphism* (120) *and the identification* $\{0\}\times(L\amalg R)\cong$

$L \amalg R$, noting that $\big[\{0\} \times_{i,\mathbb{R},\boldsymbol{\pi}_0} \big((\mathbb{R},\kappa^1_k) \times_{\boldsymbol{\pi}_1\times\cdots\times\boldsymbol{\pi}_k,\mathbb{R}^k,i} [0,\infty)^k\big)\big]$ is a single point whose Kuranishi structure has transverse Kuranishi map, so it needs no perturbation, and $\mathfrak{s}^0_{\beta,J_0,g_1,\ldots,g_k}$ induces perturbation data $\mathfrak{s}_{\beta,J_t:t\in[0,1],f_1,\ldots,f_{j-1},g_j,f_{j+1},\ldots,f_k}$ with the same virtual chain, up to sign.

(c) *The analogue of* (b) *holds for $g_1 \in \tilde{\mathcal{X}}^1_{i_1},\ldots,g_k \in \tilde{\mathcal{X}}^1_{i_k}$ and J_1.*

Proof. Most of the proof is a straightforward generalization of that of Theorem 6.1, so we will just comment on the differences. As in $(N3)$, we suppose some choices $\tilde{\mathcal{X}}^0_0 \subset \cdots \subset \tilde{\mathcal{X}}^0_N$, $\{\mathfrak{s}^0_{\beta,J_0,f_1,\ldots,f_k}\}$ and $\tilde{\mathcal{X}}^1_0 \subset \cdots \subset \tilde{\mathcal{X}}^1_N$, $\{\mathfrak{s}^1_{\beta,J_1,f_1,\ldots,f_k}\}$ are given for the outcomes of Theorem 6.1 with J_0, J_1 in place of J. Then $(N3)$(a) determines $\mathcal{X}^0_0,\ldots,\mathcal{X}^0_N$ and $\mathcal{X}^1_0,\ldots,\mathcal{X}^1_N$, and in the inductive proof we are only free to choose $\mathcal{X}^{(0,1)}_0,\ldots,\mathcal{X}^{(0,1)}_N$. Also, $(N3)$(b),(c) determine $\mathfrak{s}_{\beta,J_t:t\in[0,1],f_1,\ldots,f_k}$ if any f_j lies in $\mathcal{X}^0_{i_j}$ or $\mathcal{X}^1_{i_j}$, so in the inductive proof we are only free to choose $\mathfrak{s}_{\beta,J_t:t\in[0,1],f_1,\ldots,f_k}$ when $f_j \in \mathcal{X}^{(0,1)}_{i_j}$ for all $j = 1,\ldots,k$.

As in Theorem 6.1, we perform a quadruple induction in which we choose $\mathcal{X}^{(0,1)}_0 \subset \cdots \subset \mathcal{X}^{(0,1)}_N$ and $\mathfrak{s}_{\beta,J_t:t\in[0,1],f_1,\ldots,f_k}$ for all $k \geqslant 0$, $f_1 \in \mathcal{X}^{(0,1)}_{i_1},\ldots,f_k \in \mathcal{X}^{(0,1)}_{i_k}$ and $\beta \in H_2(M,\iota(L);\mathbb{Z})$ with $i_1+\cdots+i_k+\|\beta\|+k-1 \leqslant N$ and $\overline{\mathcal{M}}^{\rm main}_{k+1}(\beta,J_t : t\in[0,1],f_1,\ldots,f_k) \neq \emptyset$. At the point when we choose $\mathfrak{s}_{\beta,J_t:t\in[0,1],f_1,\ldots,f_k}$, we have already chosen perturbation data for every components in $\partial\overline{\mathcal{M}}^{\rm main}_{k+1}(\beta,J_t : t\in[0,1],f_1,\ldots,f_k)$, which are consistent on corners of codimension 2 and higher, and we must extend these choices over the interior of $\overline{\mathcal{M}}^{\rm main}_{k+1}(\beta,J_t : t\in[0,1],f_1,\ldots,f_k)$. In this proof, for the components of $\partial\overline{\mathcal{M}}^{\rm main}_{k+1}(\beta,J_t : t\in[0,1],f_1,\ldots,f_k)$ lying over $t=0$ or $t=1$ in $[0,1]$ the choice of perturbation data is given by some $\{\mathfrak{s}^0_{\beta,J_0,g_1,\ldots,g_l}\}$ or $\{\mathfrak{s}^1_{\beta,J_1,g_1,\ldots,g_l}\}$, but in Theorem 6.1, all the boundary choices were made at previous steps in the quadruple induction.

Since each f_j maps $\Delta^\circ_{a_j} \to (0,1)\times(L\amalg R)$, it is immediate that the interior of each simplex in $VC\big(\overline{\mathcal{M}}^{\rm main}_{k+1}(\beta,J_t : t\in[0,1],f_1,\ldots,f_k),\boldsymbol{\pi}_{[0,1]}\times\mathbf{ev},\mathfrak{s}_{\beta,J_t:t\in[0,1],f_1,\ldots,f_k}\big)$ maps to $(0,1)\times(L\amalg R)$, and so satisfies the conditions in $(N1)$(a) to lie in $\mathcal{X}^{(0,1)}_i$. Thus, in the final step in the outer induction when we have to choose $\mathcal{X}^{(0,1)}_{g+1}$, there will be a countable set $\mathcal{W}$ of smooth simplices $f : \Delta_a \to [0,1]\times(L\amalg R)$ with $f(\Delta^\circ_a) \subseteq (0,1)\times(L\amalg R)$ that are the new simplices introduced in virtual cycles for $\overline{\mathcal{M}}^{\rm main}_{k+1}(\beta,J_t : t\in[0,1],f_1,\ldots,f_k)$ in this step, and we must choose $\mathcal{X}^{(0,1)}_{g+1}$ with $\mathcal{W} \subseteq \mathcal{X}^{(0,1)}_{g+1}$ and $\mathcal{X}^{(0,1)}_g \subseteq \mathcal{X}^{(0,1)}_{g+1}$ to satisfy $(N1)$(a)–(c). This is possible by a relative version of Proposition 2.13, given the properties of $\mathcal{X}^0_0,\ldots,\mathcal{X}^0_N$ and $\mathcal{X}^1_0,\ldots,\mathcal{X}^1_N$ in Theorem 6.1, and the fact that any face of $f : \Delta_a \to [0,1]\times(L\amalg R)$ either lies in $\mathcal{X}^0_{g+1}$ or $\mathcal{X}^1_{g+1}$, or its interior maps to $(0,1)\times(L\amalg R)$.

In equation (118), the sign on the fourth line is $(-1)^{n+\sum_{l=1}^{i-1}\deg f_l}$, rather than $(-1)^{n+1+\sum_{l=1}^{i-1}\deg f_l}$ in the fourth line of (107), because of the factor $(-1)^{\dim\mathcal{T}} = -1$ in the fourth line of (105), which does not occur in the corresponding equation (95) used to deduce (107). The extra term in (118) when $k = 0$ supported on $\{0,1\} \times (L \amalg R)$ comes from the extra $\partial\mathcal{T}$ term in Theorem 5.19 when $k = 0$.

It remains only to justify the isomorphisms (119)–(122). These are given in unoriented Kuranishi spaces in (56)–(57), and we do not specify signs in (120) and (122), so we only have to compute the signs in (119) and (121). This is done by going through the proof of (56) inserting orientations. The signs $(-1)^{k+\sum_{j=1}^k b_j}$ and $(-1)^{\sum_{j=1}^k b_j}$ come from the isomorphisms of oriented manifolds

$$\begin{aligned}(123)\qquad &\{0\}^k \times_{i,[0,1]^k,(\pi_{[0,1]}\circ f_1)\times\cdots\times(\pi_{[0,1]}\circ f_k)} \big(\Delta_{a_1}\times\cdots\times\Delta_{a_k}\big)\\ &\quad\cong (-1)^{k+\sum_{j=1}^k b_j}\Delta_{a_1-1}\times\cdots\times\Delta_{a_k-1},\\ (124)\qquad &\{1\}^k \times_{i,[0,1]^k,(\pi_{[0,1]}\circ f_1)\times\cdots\times(\pi_{[0,1]}\circ f_k)} \big(\Delta_{a_1}\times\cdots\times\Delta_{a_k}\big)\\ &\quad\cong (-1)^{\sum_{j=1}^k b_j}\Delta_{a_1-1}\times\cdots\times\Delta_{a_k-1},\end{aligned}$$

for (119) and (121) respectively. Here the factors $(-1)^{\sum_{j=1}^k b_j}$ arise since $F_{b_j}^{a_j} : \Delta_{a_j-1} \to \partial\Delta_{a_j}$ multiplies orientations by $(-1)^{b_j}$, and the extra $(-1)^k$ in (123) is the coefficient -1 of $\{0\}$ in $\partial[0,1] = -\{0\}\amalg\{1\}$, raised to the power k. q.e.d.

In fact it is not difficult to extend Theorem 8.1 from a family $J_t : t \in [0,1]$ to a general family $J_t : t \in \mathcal{T}$ for $\mathcal{T}$ a compact manifold with corners, and we will use this extension in §10 when $\mathcal{T}$ is a closed semicircle or triangle in $\mathbb{R}^2$. But the statement of this generalization is even more complex, with special treatment for the codimension k corners of $\mathcal{T}$ for $k = 0, 1, \ldots, \dim\mathcal{T}$, and the analogue of $(N3)$ referring recursively to the outcome of Theorem 8.1 with $\partial\mathcal{T}$ in place of $\mathcal{T}$, rather than just to the outcome of Theorem 6.1. For simplicity, it seemed better just to state the result for $\mathcal{T} = [0,1]$.

9. $A_{N,0}$ morphisms from J_0 to J_1 $A_{N,0}$ algebras

We work in the situation of §8, with J_t for $t \in [0,1]$ a smooth family of almost complex structures on M compatible with ω. We begin by constructing an $A_{N,0}$ algebra of relative chains $C_*^{\mathrm{si}}\big([0,1]\times(L\amalg R), \{0,1\}\times(L\amalg R);\mathbb{Q}\big)$ depending on the whole family $J_t : t \in [0,1]$. Here are the analogues of Definitions 7.1 and 7.3, Proposition 7.2 and Theorem 7.4.

Definition 9.1. Let $\mathcal{G}$ be as in §8, and $\|\,.\,\| : \mathcal{G} \to \mathbb{N}$ be as in (23). For a given $N \in \mathbb{N}$, let $\mathcal{X}_i^{[0,1]} = \mathcal{X}_i^0 \amalg \mathcal{X}_i^{(0,1)} \amalg \mathcal{X}_i^1$ for $i = 0, \ldots, N$, and $\{\mathfrak{s}_{\beta,J_t:t\in[0,1],f_1,\ldots,f_k}\}$ be as in Theorem 8.1. Write $\Pi^{(0,1)} : \mathbb{Q}\mathcal{X}_i^{[0,1]} \to$

$\mathbb{Q}\mathcal{X}_i^{(0,1)}$ for the projection, with kernel $\mathbb{Q}\mathcal{X}_i^0 \oplus \mathbb{Q}\mathcal{X}_i^1$. Suppose $k \geqslant 0$, $(\lambda,\mu) \in \mathcal{G}$, and $i_1,\ldots,i_k = 0,\ldots,N$ with $i_1+\cdots+i_k+\|(\lambda,\mu)\|+k-1 \leqslant N$. Generalizing (112), define a $\mathbb{Q}$-multilinear map $\mathfrak{m}_{k,\mathrm{geo}}^{(0,1)\lambda,\mu}$: $\mathbb{Q}\mathcal{X}_{i_1}^{(0,1)} \times \cdots \times \mathbb{Q}\mathcal{X}_{i_k}^{(0,1)} \to \mathbb{Q}\mathcal{X}_{i_1+\cdots+i_k+\|(\lambda,\mu)\|+k-1}^{(0,1)}$ of degree $1-2\mu$ by

$$\begin{aligned} &\mathfrak{m}_{1,\mathrm{geo}}^{(0,1)0,0}(f_1) = \Pi^{(0,1)}\bigl[(-1)^{n+1}\partial f_1\bigr] = (-1)^{n+1}\partial^{(0,1)} f_1,\\ &\mathfrak{m}_{k,\mathrm{geo}}^{(0,1)\lambda,\mu}(f_1,\ldots,f_k) = \sum_{\substack{\beta\in H_2(M,\iota(L);\mathbb{Z}):\\ [\omega]\cdot\beta=\lambda,\ \mu_L(\beta)=2\mu,\\ \overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J_t:t\in[0,1],f_1,\ldots,f_k)\neq\emptyset}} \Pi^{(0,1)}\bigl[VC\bigl(\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J_t:t\in[0,1],f_1,\ldots,f_k),\\ &\qquad\qquad \mathbf{ev},\mathfrak{s}_{\beta,J_t:t\in[0,1],f_1,\ldots,f_k}\bigr)\bigr],\\ &\qquad\qquad (k,\lambda,\mu)\neq(1,0,0). \end{aligned} \tag{125}$$

Now applying $\Pi^{(0,1)} : \mathbb{Q}\mathcal{X}_i^{[0,1]} \to \mathbb{Q}\mathcal{X}_i^{(0,1)}$ is equivalent to projecting to *relative* singular chains to $C_*^{\mathrm{si}}\bigl([0,1]\times(L\amalg R),\{0,1\}\times(L\amalg R);\mathbb{Q}\bigr)$, so we can regard $\mathbb{Q}\mathcal{X}_i^{(0,1)}$ as a space of relative chains. As in Theorem 8.1($N2$), equation (118) holds in relative chains for all $k \geqslant 0$. Note the two sign differences compared to §6–§7: the signs in the fourth lines of (107) and (118) differ by -1, and the signs on the first lines of (112) and (125) differ by -1. Both of these are really $(-1)^{\dim\mathcal{T}}$, where $\mathcal{T}=[0,1]$. In proving (126) below, these two sign differences cancel out, so that the signs in (114) and (126) are the same. Thus as for Proposition 7.2 we deduce:

Proposition 9.2. *For $k\in\mathbb{N}$, $(\lambda,\mu)\in\mathcal{G}$ and $f_1\in\mathcal{X}_{i_1}^{(0,1)},\ldots,f_k\in\mathcal{X}_{i_k}^{(0,1)}$ with $i_1+\cdots+i_k+\|(\lambda,\mu)\|+k-1\leqslant N$, we have*

$$\begin{aligned} \sum_{\substack{k_1+k_2=k+1,\ 1\leqslant i\leqslant k_1,\ k_2\geqslant 0,\\ (\lambda_1,\mu_1),(\lambda_2,\mu_2)\in\mathcal{G},\\ (\lambda_1,\mu_1)+(\lambda_2,\mu_2)=(\lambda,\mu)}} (-1)^{\sum_{l=1}^{i-1}\deg f_l}\mathfrak{m}_{k_1,\mathrm{geo}}^{(0,1)\lambda_1,\mu_1}\bigl(f_1,\ldots,f_{i-1},\mathfrak{m}_{k_2,\mathrm{geo}}^{(0,1)\lambda_2,\mu_2}(f_i,\ldots,\\ f_{i+k_2-1}),f_{i+k_2}\ldots,f_k\bigr)=0. \end{aligned} \tag{126}$$

Definition 9.3. For a given $N\in\mathbb{N}$, we take $N'=N(N+2)$. Let $\mathcal{X}_i^{[0,1]}=\mathcal{X}_i^0\amalg\mathcal{X}_i^{(0,1)}\amalg\mathcal{X}_i^1$ for $i=0,\ldots,N'$ and $\{s_{\beta,J_t:t\in[0,1],f_1,\ldots,f_k}\}$ be as in Theorem 8.1 with N' in place of N. Since by (117) the homologies of $(\mathbb{Q}\mathcal{X}_N^{(0,1)},\partial^{(0,1)}),(\mathbb{Q}\mathcal{X}_{N'}^{(0,1)},\partial^{(0,1)})$ are isomorphic, we can find some linear subspace $A^{(0,1)}\subset\mathbb{Q}\mathcal{X}_{N'}^{(0,1)}$ such that

- $\mathbb{Q}\mathcal{X}_{N'}^{(0,1)}=\mathbb{Q}\mathcal{X}_N^{(0,1)}\oplus A^{(0,1)}\oplus\partial^{(0,1)}A^{(0,1)}$; and
- $\partial^{(0,1)}:A^{(0,1)}\to\partial^{(0,1)}A^{(0,1)}$ is an isomorphism.

Later we will take $A^{(0,1)}$ compatible with choices of A in Definition 7.3 for J_0,J_1. Define a linear map $H^{(0,1)}:\mathbb{Q}\mathcal{X}_{N'}^{(0,1)}\to\mathbb{Q}\mathcal{X}_{N'}^{(0,1)}$ by

$$H^{(0,1)}(x)=\begin{cases}0, & \text{for } x\in\mathbb{Q}\mathcal{X}_N\oplus A^{(0,1)},\\ (\partial^{(0,1)})^{-1}x, & \text{for } x\in\partial^{(0,1)}A^{(0,1)}.\end{cases} \tag{127}$$

Write $\Pi:\mathbb{Q}\mathcal{X}_{N'}^{(0,1)}\to\mathbb{Q}\mathcal{X}_N^{(0,1)}$ for the projection. Then $\mathrm{id}-\Pi=\partial^{(0,1)}H^{(0,1)}+H^{(0,1)}\partial^{(0,1)}$.

Suppose $k \geqslant 0$ and $(\lambda,\mu) \in \mathcal{G}$ with $\|(\lambda,\mu)\| + k - 1 \leqslant N$. Let T be a rooted planar tree with k leaves, and $(\boldsymbol{\lambda},\boldsymbol{\mu})$ be a family of $(\lambda_v,\mu_v) \in \mathcal{G}$ for each internal vertex v of T, such that $\sum_v(\lambda_v,\mu_v) = (\lambda,\mu)$, and $(\lambda_v,\mu_v) = (0,0)$ implies that v has at least 2 incoming edges. We shall define a graded multilinear operator $\mathfrak{m}_{k,T}^{(0,1)(\boldsymbol{\lambda},\boldsymbol{\mu})} : (\mathbb{Q}\mathcal{X}_N^{(0,1)})^{\times k} \to \mathbb{Q}\mathcal{X}_N^{(0,1)}$ of degree $1-2\mu$. Let $f_1,\ldots,f_k \in \mathcal{X}_N^{(0,1)}$. Assign objects and operators to the vertices and edges of T:

- assign $f_1,\ldots,f_k$ to the leaf vertices $1,\ldots,k$ respectively.
- for each internal vertex v with 1 outgoing edge and n incoming edges, assign $\mathfrak{m}_{n,\mathrm{geo}}^{(0,1)\lambda_v,\mu_v}$.
- assign id to each leaf edge.
- assign Π to the root edge.
- assign $(-1)^n H^{(0,1)}$ to each internal edge.

Then define $\mathfrak{m}_{k,T}^{(0,1)(\boldsymbol{\lambda},\boldsymbol{\mu})}(f_1,\ldots,f_k)$ to be the composition of all these. Define a $\mathbb{Q}$-multilinear map $\mathfrak{m}_k^{(0,1)\lambda,\mu} : (\mathbb{Q}\mathcal{X}_N^{(0,1)})^{\times k} \to \mathbb{Q}\mathcal{X}_N^{(0,1)}$ graded of degree $1-2\mu$ by $\mathfrak{m}_1^{(0,1)0,0} = \mathfrak{m}_{1,\mathrm{geo}}^{(0,1)0,0} = (-1)^{n+1}\partial^{(0,1)}$ and $\mathfrak{m}_k^{(0,1)\lambda,\mu} = \sum_{T,(\boldsymbol{\lambda},\boldsymbol{\mu})} \mathfrak{m}_{k,T}^{(0,1)(\boldsymbol{\lambda},\boldsymbol{\mu})}$ for $(k,\lambda,\mu) \neq (1,0,0)$.

Theorem 9.4. **(a)** *In Definition* 9.3, *the* $\mathfrak{m}_k^{(0,1)\lambda,\mu}$ *satisfy equation* (17) *for all* $k \geqslant 0$ *and* $(\lambda,\mu) \in \mathcal{G}$ *with* $\|(\lambda,\mu)\| + k - 1 \leqslant N$. *Thus* $(\mathbb{Q}\mathcal{X}_N^{(0,1)},\mathcal{G},\mathfrak{m}^{(0,1)})$ *is an* $A_{N,0}$ *algebra in the sense of Definition* 3.21, *where* $\mathfrak{m}^{(0,1)} = \big(\mathfrak{m}_k^{(0,1)\lambda,\mu} : k \geqslant 0,\ (\lambda,\mu) \in \mathcal{G},\ \|(\lambda,\mu)\|+k-1 \leqslant N\big)$, *and* $\mathbb{Q}\mathcal{X}_N^{(0,1)}$ *is graded by shifted cohomological degree in* (49).

(b) *If* $f_1 \in \mathcal{X}_{i_1}^{(0,1)},\ldots,f_k \in \mathcal{X}_{i_k}^{(0,1)}$ *with* $i_1+\cdots+i_k+\|(\lambda,\mu)\|+k-1 \leqslant N$ *then*

$$\mathfrak{m}_k^{(0,1)\lambda,\mu}(f_1,\ldots,f_k) = \mathfrak{m}_{k,\mathrm{geo}}^{(0,1)\lambda,\mu}(f_1,\ldots,f_k). \tag{128}$$

Now let $(\mathbb{Q}\mathcal{X}_N^0,\mathcal{G},\mathfrak{m}^0)$ and $(\mathbb{Q}\mathcal{X}_N^1,\mathcal{G},\mathfrak{m}^1)$ be $A_{N,0}$ algebras constructed in Theorem 7.4 with $J = J_0$ and $J = J_1$. We shall construct strict, surjective $A_{N,0}$ morphisms $\mathfrak{p}^0 : (\mathbb{Q}\mathcal{X}_N^{(0,1)},\mathcal{G},\mathfrak{m}^{(0,1)}) \to (\mathbb{Q}\mathcal{X}_N^0,\mathcal{G},\mathfrak{m}^0)$ and $\mathfrak{p}^1 : (\mathbb{Q}\mathcal{X}_N^{(0,1)},\mathcal{G},\mathfrak{m}^{(0,1)}) \to (\mathbb{Q}\mathcal{X}_N^1,\mathcal{G},\mathfrak{m}^1)$, and show they are homotopy equivalences.

Definition 9.5. Let J_0,J_1 be complex structures on M compatible with ω, and $J_t : t \in [0,1]$ a smooth family of complex structures on M compatible with ω interpolating between them. Fix once and for all $N \in \mathbb{N}$, $N' = N(N+2)$ and $\mathcal{G} \subset [0,\infty)\times\mathbb{Z}$ satisfying conditions (i),(ii) of §8. This implies that $\mathcal{G}$ satisfies conditions (i),(ii) of §6 for $J = J_0$ and $J = J_1$.

With these $N,N',\mathcal{G}$, suppose $\mathcal{X}_0^0 \subset \cdots \subset \mathcal{X}_{N'}^0$, $\{\mathfrak{s}_{\beta,J_0,f_1,\ldots,f_k}^0\}$ are possible choices in Theorem 6.1 with J_0,N' in place of J,N, and $\mathcal{X}_0^1 \subset \cdots \subset \mathcal{X}_{N'}^1$, $\{\mathfrak{s}_{\beta,J_1,f_1,\ldots,f_k}^1\}$ possible choices in Theorem 6.1 with J_1,N'. Let

$(\mathbb{Q}\mathcal{X}^0_N, \mathcal{G}, \mathfrak{m}^0)$ and $(\mathbb{Q}\mathcal{X}^1_N, \mathcal{G}, \mathfrak{m}^1)$ be possible $A_{N,0}$ algebras constructed in Theorem 7.4 from this data for each of J_0, J_1. As in Definition 7.3, this involves additional choices of subspace A and corresponding operator H, which we write as A^0, H^0 and A^1, H^1 respectively.

Let $\mathcal{X}^{[0,1]}_i = \mathcal{X}^0_i \amalg \mathcal{X}^{(0,1)}_i \amalg \mathcal{X}^1_i$ for $i = 0, \ldots, N'$ and $\{s_{\beta,J_t:t\in[0,1],f_1,\ldots,f_k}\}$ be possible choices in Theorem 8.1 with N' in place of N, and compatible in $(N3)$ with the above choices of $\mathcal{X}^0_0 \subset \cdots \subset \mathcal{X}^0_{N'}$, $\{\mathfrak{s}^0_{\beta,J_0,f_1,\ldots,f_k}\}$ and $\mathcal{X}^1_0 \subset \cdots \subset \mathcal{X}^1_{N'}$, $\{\mathfrak{s}^1_{\beta,J_1,f_1,\ldots,f_k}\}$, dropping the distinction between $\tilde{\mathcal{X}}^0_i, \tilde{\mathcal{X}}^1_i$ and $\mathcal{X}^0_i, \mathcal{X}^1_i$. Suppose $(\mathbb{Q}\mathcal{X}^{(0,1)}_N, \mathcal{G}, \mathfrak{m}^{(0,1)})$ is a possible $A_{N,0}$ algebra constructed in Theorem 9.4 from this data. This involves an additional choice of $A^{(0,1)}$, yielding $H^{(0,1)}$. We will shortly require $A^{(0,1)}, H^{(0,1)}$ to be compatible with A^0, H^0 and A^1, H^1.

Write $\partial^0, \partial^1, \partial^{[0,1]}, \partial^{(0,1)}$ for the boundary operators on $\mathbb{Q}\mathcal{X}^0_i, \mathbb{Q}\mathcal{X}^1_i$, $\mathbb{Q}\mathcal{X}^{[0,1]}_i, \mathbb{Q}\mathcal{X}^{(0,1)}_i$ respectively, where we regard $\partial^{(0,1)} : \mathbb{Q}\mathcal{X}^{(0,1)}_i \to \mathbb{Q}\mathcal{X}^{(0,1)}_i$ as acting on $\mathbb{Q}\mathcal{X}^{(0,1)}_i$ as a subspace of the relative chains $C^{\mathrm{si}}_*\big([0,1]\times(L\amalg R), \{0,1\}\times(L\amalg R);\mathbb{Q}\big)$. But we will also regard $\mathbb{Q}\mathcal{X}^{(0,1)}_i$ as a subspace of $\mathbb{Q}\mathcal{X}^{[0,1]}_i$, so that $\partial^{[0,1]}$ maps $\mathbb{Q}\mathcal{X}^{(0,1)}_i \to \mathbb{Q}\mathcal{X}^{[0,1]}_i$.

Define linear maps $P^0 : \mathbb{Q}\mathcal{X}^{(0,1)}_i \to \mathbb{Q}\mathcal{X}^0_i$ and $P^1 : \mathbb{Q}\mathcal{X}^{(0,1)}_i \to \mathbb{Q}\mathcal{X}^1_i$ for $i = 0, \ldots, N'$ by $P^0 = -\Pi^0 \circ \partial^{[0,1]}$ and $P^1 = \Pi^1 \circ \partial^{[0,1]}$, where $\Pi^0, \Pi^1 : \mathbb{Q}\mathcal{X}^{[0,1]}_i \to \mathbb{Q}\mathcal{X}^0_i, \mathbb{Q}\mathcal{X}^1_i$ are the projections coming from the decomposition $\mathcal{X}^{[0,1]}_i = \mathcal{X}^0_i \amalg \mathcal{X}^{(0,1)}_i \amalg \mathcal{X}^1_i$. Observe that although $\partial^{[0,1]}$ reduces dimension of singular chains by one, $\mathbb{Q}\mathcal{X}^0_i, \mathbb{Q}\mathcal{X}^1_i$ are graded by $\deg f$ in (36), but $\mathbb{Q}\mathcal{X}^{(0,1)}_i$ is graded by $\deg f$ in (49) with $\dim \mathcal{T} = 1$. Therefore P^0, P^1 are actually *graded of degree zero.*

Considering the components of $\partial^{[0,1]} : \mathbb{Q}\mathcal{X}^{[0,1]}_i \to \mathbb{Q}\mathcal{X}^{[0,1]}_i$ in the splitting $\mathbb{Q}\mathcal{X}^{[0,1]}_i = \mathbb{Q}\mathcal{X}^0_i \oplus \mathbb{Q}\mathcal{X}^{(0,1)}_i \oplus \mathbb{Q}\mathcal{X}^1_i$, we see that $\partial^{[0,1]} = \partial^0 + \partial^{(0,1)} + \partial^1 - P^0 + P^1$. Since $(\partial^{[0,1]})^2 = 0$, taking components of $(\partial^{[0,1]})^2$ mapping from $\mathbb{Q}\mathcal{X}^{(0,1)}_i$ to $\mathbb{Q}\mathcal{X}^0_i, \mathbb{Q}\mathcal{X}^1_i$ shows that

$$P^0 \circ \partial^{(0,1)} + \partial^0 \circ P^0 = 0 \quad \text{and} \quad P^1 \circ \partial^{(0,1)} + \partial^1 \circ P^1 = 0. \tag{129}$$

Thus P^0, P^1 are morphisms of complexes $(\mathbb{Q}\mathcal{X}^{(0,1)}_i, \partial^{(0,1)})$, $(\mathbb{Q}\mathcal{X}^0_i, \partial^0)$, $(\mathbb{Q}\mathcal{X}^1_i, \partial^1)$, and induce maps P^0_*, P^1_* on cohomology. But by assumption (117) are isomorphisms. Under these, P^0_* corresponds (though not with gradings) to the natural map

$$H^{\mathrm{si}}_*\big([0,1]\times(L\amalg R), \{0,1\}\times(L\amalg R);\mathbb{Q}\big) \longrightarrow H^{\mathrm{si}}_{*-1}\big(\{0\}\times(L\amalg R);\mathbb{Q}\big).$$

Since this is an isomorphism, P^0_* and similarly P^1_* are isomorphisms.

Theorem 8.1$(N1)$(b) implies that if $g \in \mathcal{X}^0_i$ then there exists $f \in \mathcal{X}^{(0,1)}_i$ with $\Pi^0(f) = \pm g$, and also $\Pi^1(f) = 0$. Similarly, if $g \in \mathcal{X}^1_i$ there exists $f \in \mathcal{X}^{(0,1)}_i$ with $\Pi^1(f) = \pm g$, and also $\Pi^0(f) = 0$. Therefore $\Pi^0 \oplus \Pi^1 : \mathbb{Q}\mathcal{X}^{(0,1)}_i \to \mathbb{Q}\mathcal{X}^0_i \oplus \mathbb{Q}\mathcal{X}^1_i$ is surjective. Combining this with (129), one can show that in Definition 9.3, one can choose $A^{(0,1)}$ so that $\Pi^0(A^{(0,1)}) = A^0$ and $\Pi^1(A^{(0,1)}) = A^1$. Combining this with (115), (127) and (129), we see that

$$P^0 \circ H^{(0,1)} + H^0 \circ P^0 = 0 \quad \text{and} \quad P^1 \circ H^{(0,1)} + H^1 \circ P^1 = 0. \tag{130}$$

Now define $\mathfrak{p}_1^{0\,0,0} : \mathbb{Q}\mathcal{X}_N^{(0,1)} \to \mathbb{Q}\mathcal{X}_N^0$ by $\mathfrak{p}_1^{0\,0,0} = P^0$, and for all $k \geqslant 0$ and $(\lambda,\mu) \in \mathcal{G}$ with $\|(\lambda,\mu)\| + k - 1 \leqslant N$ and $(k,\lambda,\mu) \neq (1,0,0)$, define $\mathfrak{p}_k^{0\,\lambda,\mu} : (\mathbb{Q}\mathcal{X}_N^{(0,1)})^{\times k} \to \mathbb{Q}\mathcal{X}_N^0$ by $\mathfrak{p}_k^{0\,\lambda,\mu} = 0$. Write $\mathfrak{p}^0 = \big(\mathfrak{p}_k^{0\,\lambda,\mu} : k \geqslant 0,\ (\lambda,\mu) \in \mathcal{G},\ \|(\lambda,\mu)\| + k - 1 \leqslant N\big)$. Similarly, define $\mathfrak{p}^1 = \big(\mathfrak{p}_k^{1\,\lambda,\mu} : k \geqslant 0,\ (\lambda,\mu) \in \mathcal{G},\ \|(\lambda,\mu)\| + k - 1 \leqslant N\big)$ by $\mathfrak{p}_1^{1\,0,0} = P^1$ and $\mathfrak{p}_k^{1\,\lambda,\mu} = 0$ for $(k,\lambda,\mu) \neq (1,0,0)$.

Theorem 9.6. *In Definition* 9.5, $\mathfrak{p}^0 : (\mathbb{Q}\mathcal{X}_N^{(0,1)}, \mathcal{G}, \mathfrak{m}^{(0,1)}) \to (\mathbb{Q}\mathcal{X}_N^0, \mathcal{G}, \mathfrak{m}^0)$ *and* $\mathfrak{p}^1 : (\mathbb{Q}\mathcal{X}_N^{(0,1)}, \mathcal{G}, \mathfrak{m}^{(0,1)}) \to (\mathbb{Q}\mathcal{X}_N^1, \mathcal{G}, \mathfrak{m}^1)$ *are strict, surjective* $A_{N,0}$ *morphisms, and weak homotopy equivalences.*

Proof. Combining (112), (125) and (129), and noting the difference in signs $(-1)^n$, $(-1)^{n+1}$ in the first lines of (112) and (125) gives

$$\mathfrak{m}_{1,\mathrm{geo}}^{0\,0,0} \circ P^0 = P^0 \circ \mathfrak{m}_{1,\mathrm{geo}}^{(0,1)\,0,0} \quad \text{and} \quad \mathfrak{m}_{1,\mathrm{geo}}^{1\,0,0} \circ P^1 = P^1 \circ \mathfrak{m}_{1,\mathrm{geo}}^{(0,1)\,0,0}. \tag{131}$$

We will prove the analogue of (131) for $\mathfrak{m}_{k,\mathrm{geo}}^{0\,\lambda,\mu}, \mathfrak{m}_{k,\mathrm{geo}}^{1\,\lambda,\mu}$, $(k,\lambda,\mu) \neq (1,0,0)$, using equations (119) and (121). To do this we relate P^0, P^1 to the fibre products $\{0\} \times_{i,[0,1],\ldots} \cdots$, $\{1\} \times_{i,[0,1],\ldots} \cdots$ used in (119) and (121).

Suppose $f : \Delta_a \to [0,1] \times (L \amalg R)$ lies in $\mathcal{X}_N^{(0,1)}$ for $a > 0$. Then $\partial^{[0,1]} f = \sum_{b=0}^{a} (-1)^b f \circ F_b^a$. By Theorem 8.1($N1$)(b), $f \circ F_b^a \in \mathcal{X}_N^0$ for at most one $b = 0, \ldots, a$. Suppose $f \circ F_b^a \in \mathcal{X}_N^0$. Then $P^0(f) = -\Pi^0(\partial^{[0,1]} f) = (-1)^{1+b} f \circ F_b^a$. But as in the proof of Theorem 8.1, we have $\{0\} \times_{i,[0,1],\pi_1 \circ f} \Delta_a \cong (-1)^{1+b} \Delta_{a-1}$, and the restriction of f to this Δ_{a-1} is $f \circ F_b^a$. Thus it is natural to identify $P^0(f)$ with $(\{0\} \times_{i,[0,1],\pi_1 \circ f} \Delta_a, f \circ \pi_{\Delta_a})$, as signed singular simplices. This is also valid if $f \circ F_b^a \notin \mathcal{X}_N^0$ for any $b = 0, \ldots, a$, since then $P^0(f) = 0$ and $\{0\} \times_{i,[0,1],\pi_1 \circ f} \Delta_a = \emptyset$.

Therefore $P^0 : \mathbb{Q}\mathcal{X}_N^{(0,1)} \to \mathbb{Q}\mathcal{X}_N^0$ is essentially equivalent, with signs, to the fibre product $\{0\} \times_{i,[0,1],\ldots} \cdots$, that is, P^0 takes $f : \Delta_a \to [0,1] \times (L \amalg R)$ to $f \circ \pi_{\Delta_a} : \{0\} \times_{i,[0,1],\pi_1 \circ f} \Delta_a \to [0,1] \times (L \amalg R)$. In the same way, $P^1 : \mathbb{Q}\mathcal{X}_N^{(0,1)} \to \mathbb{Q}\mathcal{X}_N^1$ is essentially equivalent, with signs, to the fibre product $\{1\} \times_{i,[0,1],\ldots} \cdots$.

Now suppose as in ($N3$) that $g_1 \in \mathcal{X}_N^0, \ldots, g_k \in \mathcal{X}_N^0$ and $f_1 \in \mathcal{X}_N^{(0,1)}, \ldots, f_k \in \mathcal{X}_N^{(0,1)}$ with $f_j : \Delta_{a_j} \to [0,1] \times (L \amalg R)$, $g_j : \Delta_{a_j - 1} \to \{0\} \times (L \amalg R)$ and $g_j = f_j \circ F_{b_j}^{a_j}$ for $b_j = 0, \ldots, a_j$ and $j = 1, \ldots, k$. Then $P^0(f_j) = (-1)^{1+b_j} g_j$, as above. Let $k \geqslant 0$ and $(\lambda,\mu) \in \mathcal{G}$ with

$\|(\lambda,\mu)\|+k-1\leqslant N$ and $(k,\lambda,\mu)\neq(1,0,0)$. Then

$$\begin{aligned}
(132)\qquad & P^0\circ\mathfrak{m}_{k,\mathrm{geo}}^{(0,1)\lambda,\mu}(f_1,\ldots,f_k)\\
&= \sum_{\substack{\beta\in H_2(M,\iota(L);\mathbb{Z}):\ [\omega]\cdot\beta=\lambda,\ \mu_L(\beta)=2\mu,\\ \bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J_t:t\in[0,1],f_1,\ldots,f_k)\neq\emptyset}} P^0\circ\Pi^{(0,1)}\bigl[VC\bigl(\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J_t:t\in[0,1],f_1,\ldots,f_k),\\
&\qquad\qquad \mathbf{ev},\mathfrak{s}_{\beta,J_t:t\in[0,1],f_1,\ldots,f_k}\bigr)\bigr]\\
&= (-1)^{k+\sum_{j=1}^k b_j}\sum_{\substack{\beta\in H_2(M,\iota(L);\mathbb{Z}):[\omega]\cdot\beta=\lambda,\\ \mu_L(\beta)=2\mu,\ \bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J_0,g_1,\ldots,g_k)\neq\emptyset}} VC\bigl(\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J_0,g_1,\ldots,g_k),\mathbf{ev},\\
&\qquad\qquad \mathfrak{s}^0_{\beta,J_0,g_1,\ldots,g_k}\bigr)\\
&= (-1)^{k+\sum_{j=1}^k b_j}\mathfrak{m}_{k,\mathrm{geo}}^{0\lambda,\mu}(g_1,\ldots,g_k)=\mathfrak{m}_{k,\mathrm{geo}}^{0\lambda,\mu}\bigl(P^0(f_1),\ldots,P^0(f_k)\bigr),
\end{aligned}$$

using (125), (112) in the first and third steps, and $P^0(f_j)=(-1)^{1+b_j}g_j$ in the fourth.

In the second, most difficult step of (132) we use the essential equivalence of P^0 with the fibre product $\{0\}\times_{i,[0,1],\ldots}\cdots$, equations (119) and (120), and the identification of $\mathfrak{s}_{\beta,J_t:t\in[0,1],f_1,\ldots,f_{j-1},g_j,f_{j+1},\ldots,f_k},\mathfrak{s}^0_{\beta,J_0,g_1,\ldots,g_k}$ under (120) in Theorem 8.1($N3$). The idea here is that because of this compatibility of perturbation data, the two operations of taking fibre product $\{0\}\times_{i,[0,1],\ldots}\cdots$ (basically P^0), and taking virtual chains using perturbation data, commute when applied to $\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J_t:t\in[0,1],f_1,\ldots,f_k)$. That is, we can take virtual chains first and then apply P^0, giving the r.h.s. of the first line of (132). Or we can apply $\{0\}\times_{i,[0,1],\ldots}\cdots$ first, giving $(-1)^{k+\sum_{j=1}^k b_j}\bar{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J_0,g_1,\ldots,g_k)$ by (119), and then take virtual chains, giving the second line of (132). Since the two operations commute, the two expressions are equal.

Now let T be a rooted planar tree with k leaves, and $(\boldsymbol{\lambda},\boldsymbol{\mu})$ be as in Definition 9.3. Then Definitions 7.3 and 9.3 define $\mathfrak{m}_{k,T}^{0(\boldsymbol{\lambda},\boldsymbol{\mu})}:(\mathbb{Q}\mathcal{X}_N^0)^{\times k}\to\mathbb{Q}\mathcal{X}_N^{(0,1)}$ and $\mathfrak{m}_{k,T}^{(0,1)(\boldsymbol{\lambda},\boldsymbol{\mu})}:(\mathbb{Q}\mathcal{X}_N^{(0,1)})^{\times k}\to\mathbb{Q}\mathcal{X}_N^{(0,1)}$, where $\mathfrak{m}_{k,T}^{0(\boldsymbol{\lambda},\boldsymbol{\mu})}$ assigns $\mathfrak{m}_{n,\mathrm{geo}}^{0\lambda_v,\mu_v}$ to an internal vertex and $(-1)^{n+1}H^0$ to an internal edge, and $\mathfrak{m}_{k,T}^{(0,1)(\boldsymbol{\lambda},\boldsymbol{\mu})}$ assigns $\mathfrak{m}_{n,\mathrm{geo}}^{(0,1)\lambda_v,\mu_v}$ to an internal vertex and $(-1)^nH^{(0,1)}$ to an internal edge.

Equation (132) gives $P^0\circ\mathfrak{m}_{n,\mathrm{geo}}^{(0,1)\lambda_v,\mu_v}=\mathfrak{m}_{n,\mathrm{geo}}^{0\lambda_v,\mu_v}\circ(P^0\times\cdots\times P^0)$, and (130) implies that $P^0\circ(-1)^nH^{(0,1)}=(-1)^{n+1}H^0\circ P^0$. Combining these we see that $P^0\circ\mathfrak{m}_{k,T}^{(0,1)(\boldsymbol{\lambda},\boldsymbol{\mu})}=\mathfrak{m}_{k,T}^{0(\boldsymbol{\lambda},\boldsymbol{\mu})}\circ(P^0\times\cdots\times P^0)$. Summing this over $T,(\boldsymbol{\lambda},\boldsymbol{\mu})$ and using (131) now shows that $P^0\circ\mathfrak{m}_k^{(0,1)\lambda,\mu}=\mathfrak{m}_k^{0\lambda,\mu}\circ(P^0\times\cdots\times P^0)$ for all $k\geqslant 0$ and $(\lambda,\mu)\in\mathcal{G}$ with $\|(\lambda,\mu)\|+k-1\leqslant N$. This and the definition of $\mathfrak{p}^0$ imply that $\mathfrak{p}^0$ is a *strict* $A_{N,0}$ *morphism*, as we have to prove.

From Definition 9.3, $P^0:\mathbb{Q}\mathcal{X}_N^{(0,1)}\to\mathbb{Q}\mathcal{X}_N^0$ is surjective, and $P^0_*:H^*\bigl(\mathbb{Q}\mathcal{X}_N^{(0,1)},\partial^{(0,1)}\bigr)\to H^*\bigl(\mathbb{Q}\mathcal{X}_N^0,\partial^0\bigr)$ is an isomorphism. As $\mathfrak{p}_1^{00,0}=P^0$,

$\mathfrak{m}_1^{00,0} = (-1)^n \partial^0$ and $\mathfrak{m}_1^{(0,1)0,0} = (-1)^{n+1}\partial^{(0,1)}$, it follows that $\mathfrak{p}^0$ is *surjective*, and a *weak homotopy equivalence*, as we have to prove. The proof for $\mathfrak{p}^1$ is the same, apart from sign differences $P^1(f_j) = (-1)^{b_j} g_j$ and between (119) and (121). q.e.d.

By the $A_{N,0}$ version of Corollary 3.18, we deduce:

Corollary 9.7. *We can construct explicit $A_{N,0}$ morphisms $\mathfrak{i}^0 : (\mathbb{Q}\mathcal{X}_N^0, \mathcal{G}, \mathfrak{m}^0) \to (\mathbb{Q}\mathcal{X}_N^{(0,1)}, \mathcal{G}, \mathfrak{m}^{(0,1)})$ and $\mathfrak{i}^1 : (\mathbb{Q}\mathcal{X}_N^1, \mathcal{G}, \mathfrak{m}^1) \to (\mathbb{Q}\mathcal{X}_N^{(0,1)}, \mathcal{G}, \mathfrak{m}^{(0,1)})$ which are homotopy inverses for $\mathfrak{p}^0, \mathfrak{p}^1$ respectively, using sums over planar trees. Hence $\mathfrak{f}^{01} = \mathfrak{p}^1 \circ \mathfrak{i}^0 : (\mathbb{Q}\mathcal{X}_N^0, \mathcal{G}, \mathfrak{m}^0) \to (\mathbb{Q}\mathcal{X}_N^1, \mathcal{G}, \mathfrak{m}^1)$ is an $A_{N,0}$ morphism and a homotopy equivalence, with homotopy inverse $\mathfrak{f}^{10} = \mathfrak{p}^0 \circ \mathfrak{i}^1$.*

This is important, as it shows that the $A_{N,0}$ algebras we associated to L in §7 are independent of the almost complex structure J and other choices, up to homotopy equivalence. We can now compare our proof of this with analogous results in Fukaya et al. [**8**, §4.6.1 & §7.2.9]. In effect, in [**8**, Th. 4.6.1] Fukaya et al. construct a version $\mathfrak{f}$ of our homotopy equivalence $\mathfrak{f}^{01}$ directly, without introducing an intermediate $A_{N,0}$ algebra $(\mathbb{Q}\mathcal{X}_N^{(0,1)}, \mathcal{G}, \mathfrak{m}^{(0,1)})$ as we do.

Since our $\mathfrak{i}^0$ involves a sum over planar trees, one would expect their $\mathfrak{f}$ also to involve sums over planar trees, and it does, though this is not made very explicit. In [**8**, Def. 4.6.8], Fukaya et al. define complicated moduli spaces $\mathcal{M}_{k+1}^{\mathrm{main}}(M', L', \{J_\rho\}_\rho : \beta; \mathrm{top}(\rho))$ which are in effect disjoint unions over planar trees T with k leaves of multiple fibre products over T of Kuranishi spaces, where to each internal vertex of T we associate $\overline{\mathcal{M}}_{n+1}^{\mathrm{main}}(\beta_v, J_t : t \in [0,1])$ in our notation, and to each internal edge of T we associate $\{(s,t) \in [0,1]^2 : s \leqslant t\}$. Here the fibre product '$\{(s,t) \in [0,1]^2 : s \leqslant t\} \times_{\pi_1,[0,1],\ldots} \cdots$' is an analogue of our H, an explicit partial inverse for ∂.

All these sums and products over trees happen at the level of Kuranishi spaces, not complexes $\mathbb{Q}\mathcal{X}_i$. To extend them to complexes, Fukaya et al. [**8**, Prop. 4.6.14, §7.2.9] choose perturbation data $\mathfrak{s}^{\mathrm{top}(\rho)}$ for the moduli spaces $\mathcal{M}_{k+1}^{\mathrm{main}}(M', L', \{J_\rho\}_\rho : \beta; \mathrm{top}(\rho))$, and further chain complexes $\mathbb{Q}\mathcal{X}_i'$, satisfying many compatibility conditions. This adds an extra layer of complexity to the proof. We believe our method in §8–§9 is preferable to that of [**8**], as it is shorter and more transparent.

10. Homotopies between $A_{N,0}$ morphisms

In §7 we constructed $A_{N,0}$ algebras $(\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m})$ from L using a choice of almost complex structure J, and in §9, given two such $A_{N,0}$ algebras $(\mathbb{Q}\mathcal{X}_N^0, \mathcal{G}, \mathfrak{m}^0), (\mathbb{Q}\mathcal{X}_N^1, \mathcal{G}, \mathfrak{m}^1)$ from J_0, J_1, we constructed a homotopy equivalence $\mathfrak{f}^{01} : (\mathbb{Q}\mathcal{X}_N^0, \mathcal{G}, \mathfrak{m}^0) \to (\mathbb{Q}\mathcal{X}_N^1, \mathcal{G}, \mathfrak{m}^1)$. We will now show that such $\mathfrak{f}^{01}$ are *unique up to homotopy*, and also that they *form commutative triangles up to homotopy.*

10.1. Uniqueness of $\mathfrak{f}^{01}$ in Corollary 9.7 up to homotopy. Let J_0, J_1 be complex structures on M compatible with ω. Fix $N \geqslant 0$, $N' = N(N+2)$ and $\mathcal{G}$, which must satisfy some conditions below, once and for all. Suppose $(\mathbb{Q}\mathcal{X}_N^0, \mathcal{G}, \mathfrak{m}^0)$, $(\mathbb{Q}\mathcal{X}_N^1, \mathcal{G}, \mathfrak{m}^1)$ are possible outcomes for the $A_{N,0}$ algebra of Theorem 7.4 with $J = J_0$ and $J = J_1$ respectively, and $N, N', \mathcal{G}$ as above.

Suppose $J_t : t \in [0,1]$ and $\hat{J}_t : t \in [0,1]$ are smooth 1-parameter families of almost complex structures on M compatible with ω interpolating between J_0 and J_1, so that $\hat{J}_0 = J_0$ and $\hat{J}_1 = J_1$. Let $(\mathbb{Q}\mathcal{X}_N^{(0,1)}, \mathcal{G}, \mathfrak{m}^{(0,1)})$, $(\mathbb{Q}\hat{\mathcal{X}}_N^{(0,1)}, \mathcal{G}, \hat{\mathfrak{m}}^{(0,1)})$ be possible outcomes for the $A_{N,0}$ algebra of Theorem 9.4 using $J_t : t \in [0,1]$ and $\hat{J}_t : t \in [0,1]$, and $\mathfrak{p}^0, \mathfrak{p}^1, \mathfrak{i}^0, \mathfrak{f}^{01}, \hat{\mathfrak{p}}^0, \hat{\mathfrak{p}}^1, \hat{\mathfrak{i}}^0, \hat{\mathfrak{f}}^{01}$ corresponding outcomes for the $A_{N,0}$ morphisms $\mathfrak{p}^0, \mathfrak{p}^1, \mathfrak{i}^0, \mathfrak{f}^{01}$ of Theorem 9.6 and Corollary 9.7.

Then $\mathfrak{f}^{01} = \mathfrak{p}^1 \circ \mathfrak{i}^0$ and $\hat{\mathfrak{f}}^{01} = \hat{\mathfrak{p}}^1 \circ \hat{\mathfrak{i}}^0$ are $A_{N,0}$ morphisms $(\mathbb{Q}\mathcal{X}_N^0, \mathcal{G}, \mathfrak{m}^0) \to (\mathbb{Q}\mathcal{X}_N^1, \mathcal{G}, \mathfrak{m}^1)$. We shall construct a *homotopy* $\mathfrak{H} : \mathfrak{f}^{01} \Rightarrow \hat{\mathfrak{f}}^{01}$. This implies that the $A_{N,0}$ morphism $\mathfrak{f}^{01} : (\mathbb{Q}\mathcal{X}_N^0, \mathcal{G}, \mathfrak{m}^0) \to (\mathbb{Q}\mathcal{X}_N^1, \mathcal{G}, \mathfrak{m}^1)$ in Corollary 9.7 is *independent of choices up to homotopy*, and thus that the $A_{N,0}$ algebra $(\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m})$ in Theorem 7.4 is independent of J and other choices *up to canonical homotopy equivalence*, rather than just up to homotopy equivalence.

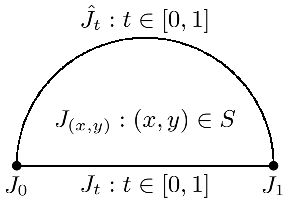
To construct $\mathfrak{H}$ we need to choose a 2-parameter family of almost complex structures $J_s : s \in S$ interpolating between $J_t : t \in [0,1]$ and $\hat{J}_t : t \in [0,1]$. The most obvious way to do this is, as in Fukaya et al. [**8**, §4.6.2], is to take $S = [0,1]^2$, with boundary conditions $J_{(0,t)} = J_t$, $J_{(1,t)} = \hat{J}_t$, $J_{(s,0)} = J_0$ and $J_{(s,1)} = J_1$ for $s, t \in [0,1]$. But for us there is a better choice: we take S to be the *semicircle*

$$S = \big\{(x,y) \in \mathbb{R}^2 : 0 \leqslant x^2 + y^2 \leqslant 1, \ \ y \geqslant 0\big\},$$

and $J_{(x,y)} : (x,y) \in S$ a smooth family of almost complex structures on M compatible with ω, with the boundary conditions

$$J_{(-1,0)} = J_0, \ \ J_{(1,0)} = J_1, \ \ J_{(2t-1,0)} = J_t, \ \ J_{(-\cos \pi t, \sin \pi t)} = \hat{J}_t, \ t \in [0,1].$$

Here we regard S as a 2-manifold with corners. It has two corners $(\mp 1, 0)$ to which we assign J_0, J_1, and two edges, a straight edge E to which we assign $J_t : t \in [0,1]$, and a semicircle $\hat{E}$ to which we assign $\hat{J}_t : t \in [0,1]$. This is illustrated in Figure 10.1(a). The semicircle is preferable because our method will associate an $A_{N,0}$ algebra to each face, edge and vertex of S. Using the square $[0,1]^2$ we would have to deal with $1+4+4 = 9$ $A_{N,0}$ algebras, but the semicircle gives only $1+2+2 = 5$ $A_{N,0}$ algebras, leading to a simpler proof.



$$\begin{tikzcd}
 & (\mathbb{Q}\hat{\mathcal{X}}_N^{(0,1)}, \mathcal{G}, \hat{\mathfrak{m}}^{(0,1)}) \arrow[ddl, bend right, "\hat{\mathfrak{p}}^1"'] \arrow[ddr, bend left, "\hat{\mathfrak{p}}^1"] & \\
 & (\mathbb{Q}\mathcal{X}_N^S, \mathcal{G}, \mathfrak{m}^S) \arrow[u, "\hat{\mathfrak{p}}^{(0,1)}"] \arrow[d, "\mathfrak{p}^{(0,1)}"'] & \\
(\mathbb{Q}\mathcal{X}_N^0, \mathcal{G}, \mathfrak{m}^1) & (\mathbb{Q}\mathcal{X}_N^{(0,1)}, \mathcal{G}, \mathfrak{m}^{(0,1)}) \arrow[l, "\mathfrak{p}^0"'] \arrow[r, "\mathfrak{p}^1"] & (\mathbb{Q}\mathcal{X}_N^1, \mathcal{G}, \mathfrak{m}^1)
\end{tikzcd}$$

Figure 10.1. (a) $J_s : s \in S$ (b) $A_{N,0}$ algebras and morphisms

We need the family $J_s : s \in S$ to be compatible with $\mathcal{G}$ in the sense that if $\beta \in H_2(M, \iota(L); \mathbb{Z})$ and $\overline{\mathcal{M}}_1^{\mathrm{main}}(\beta, J_s : s \in S) \neq \emptyset$ then $\big([\omega] \cdot \beta, \frac{1}{2}\mu_L(\beta)\big) \in \mathcal{G}$, generalizing condition (ii) of §6 and §8. One way to ensure this is always possible is to choose $\mathcal{G}$ as follows: let $J_t : t \in \mathcal{T}$ be a smooth family of complex structures on M compatible with ω, where $\mathcal{T}$ is a compact, connected, simply-connected manifold with boundary. We think of $J_t : t \in \mathcal{T}$ as a *large* family, with $\dim \mathcal{T} \gg 0$, the set of all almost complex structures we are interested in. Define $\mathcal{G} \subset [0, \infty) \times \mathbb{Z}$ to be the unique smallest subset satisfying the conditions:

(i) $\mathcal{G}$ is closed under addition with $\mathcal{G} \cap (\{0\} \times \mathbb{Z}) = \{(0,0)\}$, and $\mathcal{G} \cap ([0, C] \times \mathbb{Z})$ is finite for any $C \geqslant 0$; and

(ii) if $\beta \in H_2(M, \iota(L); \mathbb{Z})$, $\overline{\mathcal{M}}_1^{\mathrm{main}}(\beta, J_t : t \in \mathcal{T}) \neq \emptyset$ then $\big([\omega] \cdot \beta, \frac{1}{2}\mu_L(\beta)\big) \in \mathcal{G}$.

Then $\mathcal{G}$ satisfies conditions (i),(ii) in §6 and §8 and upon $J_s : s \in S$ above provided all the (families of) complex structures J, $J_t : t \in [0,1]$, $J_s : s \in S$ that we choose lie in $\mathcal{T}$. This problem of dependence of $\mathcal{G}$ on J will disappear in §11, since although we need to specify a particular $\mathcal{G}$ to define an $A_{N,K}$ algebra, we do not need to specify $\mathcal{G}$ to define a gapped filtered A_∞ algebra, there just has to exist some suitable $\mathcal{G}$.

Our next result generalizes the material of §8–§9 to our 2-parameter family $J_s : s \in S$. To write the details out in full would take pages, but the proof involves few new ideas, so we will just briefly indicate how to modify sections 8 and 9.

Theorem 10.1. *In the situation above, generalizing Theorem* 9.4 *we can define an $A_{N,0}$ algebra $(\mathbb{Q}\mathcal{X}_N^S, \mathcal{G}, \mathfrak{m}^S)$, and generalizing Theorem* 9.6 *we can define strict, surjective $A_{N,0}$ morphisms $\mathfrak{p}^{(0,1)} : (\mathbb{Q}\mathcal{X}_N^S, \mathcal{G}, \mathfrak{m}^S) \to (\mathbb{Q}\mathcal{X}_N^{(0,1)}, \mathcal{G}, \mathfrak{m}^{(0,1)})$ and $\hat{\mathfrak{p}}^{(0,1)} : (\mathbb{Q}\mathcal{X}_N^S, \mathcal{G}, \mathfrak{m}^S) \to (\mathbb{Q}\hat{\mathcal{X}}_N^{(0,1)}, \mathcal{G}, \hat{\mathfrak{m}}^{(0,1)})$ which are weak homotopy equivalences, such that Figure* 10.1(b) *is a commutative diagram of $A_{N,0}$ morphisms, that is,*

$$\mathfrak{p}^0 \circ \mathfrak{p}^{(0,1)} = \hat{\mathfrak{p}}^0 \circ \hat{\mathfrak{p}}^{(0,1)} \quad \textit{and} \quad \mathfrak{p}^1 \circ \mathfrak{p}^{(0,1)} = \hat{\mathfrak{p}}^1 \circ \hat{\mathfrak{p}}^{(0,1)}. \tag{133}$$

Proof. Here is how to modify Theorem 8.1 to the new $J_s : s \in S$. The conclusion is that for a given $N \in \mathbb{N}$, there are $\bar{\mathcal{X}}_0^S \subset \cdots \subset$

$\bar{\mathcal{X}}_N^S$ and $\{\mathfrak{s}_{\beta,J_s:s\in S,f_1,\dots,f_k}\}$ satisfying analogues of $(N1)$–$(N3)$. In $(N1)$, $\bar{\mathcal{X}}_0^S,\dots,\bar{\mathcal{X}}_N^S$ are countable sets of smooth simplicial chains $f:\Delta_a\to S\times(L\amalg R)$ with decompositions

$$\bar{\mathcal{X}}_i^S=\mathcal{X}_i^0\amalg\mathcal{X}_i^1\amalg\mathcal{X}_i^{(0,1)}\amalg\hat{\mathcal{X}}_i^{(0,1)}\amalg\mathcal{X}_i^S\quad\text{for } i=0,\dots,N,$$

where if $f\in\bar{\mathcal{X}}_i^S$ and $a>0$ then $f\circ F_b^a\in\bar{\mathcal{X}}_i^S$ for $b=0,\dots,a$, and

- $\mathcal{X}_i^0$ consists of $f:\Delta_a\to\{(-1,0)\}\times(L\amalg R)$, and are identified with choices of $\mathcal{X}_i$ in Theorem 6.1 with $J=J_1$ under $L\amalg R\cong\{(-1,0)\}\times(L\amalg R)$.
- $\mathcal{X}_i^1$ consists of $f:\Delta_a\to\{(1,0)\}\times(L\amalg R)$, and are identified with choices of $\mathcal{X}_i$ in Theorem 6.1 with $J=J_1$ under $L\amalg R\cong\{(1,0)\}\times(L\amalg R)$.
- $\mathcal{X}_i^{(0,1)}$ consists of $f:\Delta_a\to E\times(L\amalg R)$, and are identified with choices of $\mathcal{X}_i^{(0,1)}$ in Theorem 8.1 with for $J_t:t\in[0,1]$ under $[0,1]\times(L\amalg R)\cong E\times(L\amalg R)$ given by $t\mapsto(2t-1,0)$. Also f maps $\Delta_a^\circ\to E^\circ\times(L\amalg R)$ and $\pi_E\circ f$ is a submersion near $(\pi_E\circ f)^{-1}\big(\{(\pm1,0)\}\big)$, as in $(N1)$(a).
- $\hat{\mathcal{X}}_i^{(0,1)}$ consists of $f:\Delta_a\to\hat{E}\times(L\amalg R)$, and are identified with choices of $\mathcal{X}_i^{(0,1)}$ in Theorem 8.1 with for $\hat{J}_t:t\in[0,1]$ under $[0,1]\times(L\amalg R)\cong\hat{E}\times(L\amalg R)$ given by $t\mapsto(-\cos\pi t,\sin\pi t)$. Also f maps $\Delta_a^\circ\to\hat{E}^\circ\times(L\amalg R)$ and $\pi_{\hat{E}}\circ f$ is a submersion near $(\pi_{\hat{E}}\circ f)^{-1}\big(\{(\pm1,0)\}\big)$.
- $\mathcal{X}_i^S$ consists of $f:\Delta_a\to S\times(L\amalg R)$ such that f maps $\Delta_a^\circ\to S^\circ\times(L\amalg R)$ and $\pi_S\circ f$ is transverse to ∂S. That is, for each $p\in\partial\Delta_a$ with $\pi_S\circ f(p)\in\partial S$, we require that $\mathrm{d}(\pi_S\circ f)(T_p\Delta_a)+T_{\pi_S\circ f(p)}(\partial S)=T_{\pi_S\circ f(p)}S$. Furthermore, if $a>0$ then for each $b=0,\dots,a$ we have $f\circ F_b^a\in\mathcal{X}_i^{(0,1)},\hat{\mathcal{X}}_i^{(0,1)}$ or $\mathcal{X}_i^S$, that is, we do not allow $f\circ F_b^a\in\mathcal{X}_i^0$ or $\mathcal{X}_i^1$. Also, $f\circ F_b^a\in\mathcal{X}_i^{(0,1)}$ for at most one $b=0,\dots,a$, and $f\circ F_b^a\in\hat{\mathcal{X}}_i^{(0,1)}$ for at most one $b=0,\dots,a$.

Here the submersion and transversality conditions are equivalent to Condition 4.12, so that we can apply Remark 4.11.

We regard $\mathbb{Q}\mathcal{X}_i^S$ as a space of relative chains in $C_*^{\mathrm{si}}\big(S\times(L\amalg R),\partial S\times(L\amalg R);\mathbb{Q}\big)$. As for (117) we require the following maps to be isomorphisms:

$$\begin{aligned}
&H_*(\mathbb{Q}\mathcal{X}_i^0,\partial^0)\xrightarrow{\cong}H_*^{\mathrm{si}}(L\amalg R;\mathbb{Q}),\ H_*(\mathbb{Q}\mathcal{X}_i^1,\partial^1)\xrightarrow{\cong}H_*^{\mathrm{si}}(L\amalg R;\mathbb{Q}),\\
&H_*\big(\mathbb{Q}\mathcal{X}_i^{(0,1)},\partial^{(0,1)}\big)\xrightarrow{\cong}H_*^{\mathrm{si}}\big(E\times(L\amalg R),\{(\pm1,0)\}\times(L\amalg R);\mathbb{Q}\big),\\
&H_*\big(\mathbb{Q}\hat{\mathcal{X}}_i^{(0,1)},\hat{\partial}^{(0,1)}\big)\xrightarrow{\cong}H_*^{\mathrm{si}}\big(\hat{E}\times(L\amalg R),\{(\pm1,0)\}\times(L\amalg R);\mathbb{Q}\big),\\
&H_*\big(\mathbb{Q}\mathcal{X}_i^S,\partial^S\big)\xrightarrow{\cong}H_*^{\mathrm{si}}\big(S\times(L\amalg R),\partial S\times(L\amalg R);\mathbb{Q}\big).
\end{aligned}\tag{134}$$

In $(N2),(N3)$, for $k\geqslant0$, $f_1\in\bar{\mathcal{X}}_{i_1}^S,\dots,f_k\in\bar{\mathcal{X}}_{i_k}^S$ and $\beta\in H_2(M,\iota(L);\mathbb{Z})$ with $i_1+\cdots+i_k+\|\beta\|+k-1\leqslant N$ and $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\beta,J_s:s\in S,f_1,\dots,f_k)\neq\emptyset$,

$\mathfrak{s}_{\beta,J_s:s\in S,f_1,\ldots,f_k}$ is perturbation data for $\overline{\mathcal{M}}{}^{\rm main}_{k+1}(\beta,J_s:s\in S,f_1,\ldots,f_k)$, which should satisfy compatibilities both over the boundary of $\overline{\mathcal{M}}{}^{\rm main}_{k+1}(\beta,J_s:s\in S,f_1,\ldots,f_k)$, and with previous choices made in Theorem 6.1 for J_0,J_1 and Theorem 8.1 for $J_t:t\in[0,1]$ and $\hat J_t:t\in[0,1]$.

We modify Definition 9.1 to define $\mathbb{Q}$-multilinear maps $\mathfrak{m}^{S\lambda,\mu}_{k,\rm geo}:\mathbb{Q}\mathcal{X}^S_{i_1}\times\cdots\times\mathbb{Q}\mathcal{X}^S_{i_k}\to\mathbb{Q}\mathcal{X}^S_{i_1+\cdots+i_k+\|(\lambda,\mu)\|+k-1}$ of degree $1-2\mu$ by

$$\begin{aligned}&\mathfrak{m}^{S0,0}_{1,\rm geo}(f_1)=\Pi^S\bigl[(-1)^{n+2}\partial f_1\bigr]=(-1)^{n+2}\partial^S f_1,\\&\mathfrak{m}^{S\lambda,\mu}_{k,\rm geo}(f_1,\ldots,f_k)=\sum_{\substack{\beta\in H_2(M,\iota(L);\mathbb{Z}):\\ [\omega]\cdot\beta=\lambda,\ \mu_L(\beta)=2\mu,\\ \overline{\mathcal{M}}{}^{\rm main}_{k+1}(\beta,J_s:s\in S,f_1,\ldots,f_k)\neq\emptyset}}\Pi^S\bigl[VC\bigl(\overline{\mathcal{M}}{}^{\rm main}_{k+1}(\beta,J_s:s\in S,f_1,\ldots,f_k),\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\mathbf{ev},\mathfrak{s}_{\beta,J_s:s\in S,f_1,\ldots,f_k}\bigr)\bigr],\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(k,\lambda,\mu)\neq(1,0,0).\end{aligned}$$

The analogue of Proposition 9.2 holds. In our modification of Definition 9.3 we assign $(-1)^{n+1}H^S$ to each internal edge, and then the analogue of Theorem 9.4 holds, giving the $A_{N,0}$ algebra $(\mathbb{Q}\mathcal{X}^S_N,\mathcal{G},\mathfrak{m}^S)$.

The strict $A_{N,0}$-morphisms $\mathfrak{p}^{(0,1)}:(\mathbb{Q}\mathcal{X}^S_N,\mathcal{G},\mathfrak{m}^S)\to(\mathbb{Q}\mathcal{X}^{(0,1)}_N,\mathcal{G},\mathfrak{m}^{(0,1)})$ and $\hat{\mathfrak{p}}^{(0,1)}:(\mathbb{Q}\mathcal{X}^S_N,\mathcal{G},\mathfrak{m}^S)\to(\mathbb{Q}\hat{\mathcal{X}}^{(0,1)}_N,\mathcal{G},\hat{\mathfrak{m}}^{(0,1)})$ are defined as in Definition 9.5, but using the projections $P^{(0,1)}:\mathbb{Q}\mathcal{X}^S_i\to\mathbb{Q}\mathcal{X}^{(0,1)}_i$ and $\hat P^{(0,1)}:\mathbb{Q}\mathcal{X}^S_i\to\mathbb{Q}\hat{\mathcal{X}}^{(0,1)}_i$ defined by $P^{(0,1)}=\Pi^{(0,1)}\circ\bar\partial^S$ and $\hat P^{(0,1)}=-\hat\Pi^{(0,1)}\circ\bar\partial^S$, where $\bar\partial^S$ is the boundary operator on $\mathbb{Q}\bar{\mathcal{X}}^S_i$ and $\Pi^{(0,1)},\hat\Pi^{(0,1)}$ are the projections to $\mathbb{Q}\mathcal{X}^{(0,1)}_i,\mathbb{Q}\hat{\mathcal{X}}^{(0,1)}_i$. The difference in signs here is because in oriented manifolds we have $\partial S=E\amalg-\hat E$, where the orientations on $E,\hat E$ are determined by their identifications with $[0,1]$.

Then the analogue of Theorem 9.6 holds, so that $\mathfrak{p}^{(0,1)},\hat{\mathfrak{p}}^{(0,1)}$ are strict, surjective $A_{N,0}$ morphisms. Using (134) and the natural isomorphism

$$\begin{aligned}&H^{\rm si}_*\bigl(S\times(L\amalg R),\partial S\times(L\amalg R);\mathbb{Q}\bigr)\\&\xrightarrow{\cong}H^{\rm si}_{*-1}\bigl(E\times(L\amalg R),\{(\pm1,0)\}\times(L\amalg R);\mathbb{Q}\bigr),\end{aligned}$$

we find that $\mathfrak{p}^{(0,1)}$ is a weak homotopy equivalence, and similarly so is $\hat{\mathfrak{p}}^{(0,1)}$.

Equation (133) now follows immediately from the identities

$$P^0\circ P^{(0,1)}=\hat P^0\circ\hat P^{(0,1)}\qquad\text{and}\qquad P^1\circ P^{(0,1)}=\hat P^1\circ\hat P^{(0,1)}.\tag{135}$$

To prove these, suppose that $f:\Delta_a\to S\times(L\amalg R)$ lies in $\mathcal{X}^S_N$ with $P^0\circ P^{(0,1)}(f)\neq0$. Then there exist $b=0,\ldots,a$ with $f\circ F^a_b\in\mathcal{X}^{(0,1)}_N$ and $c=0,\ldots,a-1$ with $f\circ F^a_b\circ F^{a-1}_c\in\mathcal{X}^0_N$, where b,c are unique by the conditions on $\mathcal{X}^S_N$ above and the conditions on $\mathcal{X}^{(0,1)}_N$ in Theorem 8.1(N1)(b). Therefore $P^{(0,1)}(f)=(-1)^bf\circ F^a_b$, and $P^0\circ P^{(0,1)}(f)=(-1)^{1+b+c}f\circ F^a_b\circ F^{a-1}_c$, as $P^{(0,1)}=\Pi^{(0,1)}\circ\bar\partial^S$ and $P^0=-\Pi^0\circ\partial^{[0,1]}$.

If $c<b$ define $b'=c$ and $c'=b-1$, and if $c\geqslant b$ define $b'=c+1$ and $c'=b$. Then $f\circ F^a_b\circ F^{a-1}_c=f\circ F^a_{b'}\circ F^{a-1}_{c'}$, so $(f\circ F^a_{b'})\circ F^{a-1}_{c'}\in\mathcal{X}^0_N$. The conditions on $\mathcal{X}^S_N$ above give $f\circ F^a_{b'}\notin\mathcal{X}^S_N$, and also $f\circ F^a_{b'}\notin\mathcal{X}^0_N,\mathcal{X}^1_N$.

Thus $f\circ F^a_{b'}$ lies in $\mathcal{X}_N^{(0,1)}$ or $\hat{\mathcal{X}}_N^{(0,1)}$. But $f\circ F^a_b\in\mathcal{X}_N^{(0,1)}$, $b\neq b'$ and uniqueness of b in the conditions on $\mathcal{X}_N^S$ above imply that $f\circ F^a_{b'}\notin\mathcal{X}_N^{(0,1)}$. Hence $f\circ F^a_{b'}\in\hat{\mathcal{X}}_N^{(0,1)}$. The argument above now gives $\hat{P}^{(0,1)}(f)=(-1)^{1+b'}f\circ F^a_{b'}$, as $\hat{P}^{(0,1)}=-\hat{\Pi}^{(0,1)}\circ\bar{\partial}^S$, and $\hat{P}^0\circ\hat{P}^{(0,1)}(f)=(-1)^{b'+c'}f\circ F^a_{b'}\circ F^{a-1}_{c'}=(-1)^{1+b+c}f\circ F^a_b\circ F^{a-1}_c=P^0\circ P^{(0,1)}(f)$, as $\hat{P}^0=-\hat{\Pi}^0\circ\hat{\partial}^{[0,1]}$. Therefore if $P^0\circ P^{(0,1)}(f)\neq 0$ then $P^0\circ P^{(0,1)}(f)=\hat{P}^0\circ\hat{P}^{(0,1)}(f)$. By the same reasoning, if $\hat{P}^0\circ\hat{P}^{(0,1)}(f)\neq 0$ then $P^0\circ P^{(0,1)}(f)=\hat{P}^0\circ\hat{P}^{(0,1)}(f)$. This proves the first equation of (135). The second is similar. q.e.d.

Here is the main result of this section.

Theorem 10.2. *In the situation above, there is a homotopy $\mathfrak{H}:\mathfrak{f}^{01}\Rightarrow\hat{\mathfrak{f}}^{01}$ between the $A_{N,0}$ morphisms $\mathfrak{f}^{01},\hat{\mathfrak{f}}^{01}:(\mathbb{Q}\mathcal{X}_N^0,\mathcal{G},\mathfrak{m}^0)\to(\mathbb{Q}\mathcal{X}_N^1,\mathcal{G},\mathfrak{m}^1)$.*

Proof. As $\mathfrak{p}^{(0,1)},\hat{\mathfrak{p}}^{(0,1)}$ are weak homotopy equivalences by Theorem 10.2, they are homotopy equivalences by Theorem 3.22(c), so they have homotopy inverses $\mathfrak{i}^{(0,1)},\hat{\mathfrak{i}}^{(0,1)}$. Write $\mathfrak{f}\sim\mathfrak{g}$ when two $A_{N,0}$ morphisms are homotopic. Then we have

$$\mathfrak{f}^{01}=\mathfrak{p}^1\circ\mathfrak{i}^0=\mathfrak{p}^1\circ\mathrm{id}_{\mathbb{Q}\mathcal{X}_N^{(0,1)}}\circ\mathfrak{i}^0\sim\mathfrak{p}^1\circ\mathfrak{p}^{(0,1)}\circ\mathfrak{i}^{(0,1)}\circ\mathfrak{i}^0=$$
$$\hat{\mathfrak{p}}^1\circ\hat{\mathfrak{p}}^{(0,1)}\circ\mathfrak{i}^{(0,1)}\circ\mathfrak{i}^0\sim\hat{\mathfrak{p}}^1\circ\hat{\mathfrak{p}}^{(0,1)}\circ\hat{\mathfrak{i}}^{(0,1)}\circ\hat{\mathfrak{i}}^0\sim\hat{\mathfrak{p}}^1\circ\mathrm{id}_{\mathbb{Q}\hat{\mathcal{X}}_N^{(0,1)}}\circ\hat{\mathfrak{i}}^0=\hat{\mathfrak{p}}^1\circ\hat{\mathfrak{i}}^0=\hat{\mathfrak{f}}^{01}.$$

Here in the third step $\mathfrak{p}^{(0,1)}\circ\mathfrak{i}^{(0,1)}\sim\mathrm{id}_{\mathbb{Q}\mathcal{X}_N^{(0,1)}}$ as $\mathfrak{p}^{(0,1)},\mathfrak{i}^{(0,1)}$ are homotopy inverses, the sixth step is similar, and in the fourth step we use (133). For the fifth step, $\mathfrak{i}^{(0,1)}\circ\mathfrak{i}^0\sim\hat{\mathfrak{i}}^{(0,1)}\circ\hat{\mathfrak{i}}^0$ since these are homotopy inverses for $\mathfrak{p}^0\circ\mathfrak{p}^{(0,1)},\hat{\mathfrak{p}}^0\circ\hat{\mathfrak{p}}^{(0,1)}$, which are equal by (133). Thus $\mathfrak{H}$ exists, as homotopy is an equivalence relation. q.e.d.

If $\mathfrak{i}^0,\hat{\mathfrak{i}}^0$ are constructed by sums over planar trees as in the $A_{N,0}$ version of Corollary 3.18, then we can construct $\mathfrak{H}$ explicitly as a (complicated) sum over trees using the techniques of Markl [**18**]. Fukaya et al. [**8**, §4.6.2 & §7.2.10] prove results analogous to Theorem 10.2 by a rather more elaborate method. Their proof involves a family of almost complex structures $J_{\rho,s}$ for $(\rho,s)\in[0,1]^2$, four $A_{N,K}$ algebras of chains on L, and one $A_{N,K}$ algebra of chains on $(-\epsilon,1+\epsilon)\times L$.

To construct one of the $A_{N,0}$ morphisms between these, they define [**8**, eq. (4.6.27)] complicated moduli spaces $\mathcal{M}^{\mathrm{main}}_{k+1}(M',L',\{J_{\rho,s}\}_{\rho,s}:\beta;\mathrm{top}(\rho),\mathrm{twp}(s))$, which are in effect disjoint unions over planar trees T with k leaves of multiple fibre products over T of Kuranishi spaces, with $\overline{\mathcal{M}}^{\mathrm{main}}_{n+1}(\beta_v,J_{\rho,s}:\rho,s\in[0,1])$ at each internal vertex, and $\{(\rho_1,\rho_2)\in[0,1]^2:\rho_1\leqslant\rho_2\}$ at each internal edge. This sum over trees roughly speaking constructs an explicit homotopy inverse for the strict surjective $A_{N,K}$ morphism $\mathfrak{p}^0\circ\mathfrak{p}^{(0,1)}=\hat{\mathfrak{p}}^0\circ\hat{\mathfrak{p}}^{(0,1)}$ in our notation, using the method of §3.3.

10.2. Compositions of $\mathfrak{f}^{01}$ in Corollary 9.7 up to homotopy. Let J^a, J^b, J^c be complex structures on M compatible with ω. Fix $N \geqslant 0$, $N' = N(N+2)$ and $\mathcal{G}$, which must satisfy some conditions below, once and for all. Suppose $(\mathbb{Q}\mathcal{X}^a_N, \mathcal{G}, \mathfrak{m}^a)$, $(\mathbb{Q}\mathcal{X}^b_N, \mathcal{G}, \mathfrak{m}^b)$, $(\mathbb{Q}\mathcal{X}^c_N, \mathcal{G}, \mathfrak{m}^c)$ are possible outcomes for the $A_{N,0}$ algebra of Theorem 7.4 with $J = J^a, J^b, J^c$ respectively, and $N, N', \mathcal{G}$ as above.

Suppose $J^{ab}_t, J^{bc}_t, J^{ac}_t$ for $t \in [0,1]$ are smooth 1-parameter families of almost complex structures on M compatible with ω with $J^{ab}_0 = J^{ac}_0 = J^a$, $J^{ab}_1 = J^{bc}_0 = J^b$, $J^{bc}_1 = J^{ac}_1 = J^c$. Let $(\mathbb{Q}\mathcal{X}^{ab}_N, \mathcal{G}, \mathfrak{m}^{ab})$ be the $A_{N,0}$ algebra of Theorem 9.4 using $J^{ab}_t : t \in [0,1]$. Write $\mathfrak{p}^{ab,a}, \mathfrak{p}^{ab,b}, \mathfrak{i}^{a,ab}, \mathfrak{f}^{ab}$ respectively for the $A_{N,0}$ morphisms $\mathfrak{p}^0, \mathfrak{p}^1, \mathfrak{i}^0, \mathfrak{f}^{01}$ of Theorem 9.6 and Corollary 9.7 for $J^{ab}_t : t \in [0,1]$, so that $\mathfrak{p}^{ab,a} : (\mathbb{Q}\mathcal{X}^{ab}_N, \mathcal{G}, \mathfrak{m}^{ab}) \to (\mathbb{Q}\mathcal{X}^a_N, \mathcal{G}, \mathfrak{m}^a)$, and so on. Use the analogous notation for $J^{bc}_t, J^{ac}_t : t \in [0,1]$. Then $\mathfrak{f}^{ac}$ and $\mathfrak{f}^{bc} \circ \mathfrak{f}^{ab}$ are both $A_{N,0}$ morphisms $(\mathbb{Q}\mathcal{X}^a_N, \mathcal{G}, \mathfrak{m}^a) \to (\mathbb{Q}\mathcal{X}^c_N, \mathcal{G}, \mathfrak{m}^c)$. We shall construct a *homotopy* $\mathfrak{H} : \mathfrak{f}^{ac} \Rightarrow \mathfrak{f}^{bc} \circ \mathfrak{f}^{ab}$, using a very similar method to §10.1.

To construct $\mathfrak{H}$ we choose a 2-parameter family of almost complex structures $J_t : t \in T$ interpolating between $J^{ab}_t, J^{bc}_t, J^{ac}_t$ for $t \in [0,1]$. Let T be the triangle

$$T = \big\{(x,y) \in \mathbb{R}^2 : x \leqslant 1, \ \ y \geqslant 0, \ \ x \geqslant y\big\},$$

and $J_{(x,y)} : (x,y) \in T$ a smooth family of almost complex structures on M compatible with ω, with the boundary conditions

$$J_{(0,0)} = J^a, \quad J_{(1,0)} = J^b, \quad J_{(1,1)} = J^c,$$
$$J_{(t,0)} = J^{ab}_t, \quad J_{(1,t)} = J^{bc}_t, \quad J_{(t,t)} = J^{ac}_t, \quad t \in [0,1].$$

This is illustrated in Figure 10.2(a). We need the family $J_t : t \in T$ to be compatible with $\mathcal{G}$ in the sense that if $\beta \in H_2(M, \iota(L); \mathbb{Z})$ and $\bar{\mathcal{M}}^{\mathrm{main}}_1(\beta, J_t : t \in T) \neq \emptyset$ then $\big([\omega]\cdot\beta, \frac{1}{2}\mu_L(\beta)\big) \in \mathcal{G}$. We can ensure this as in §10.1.

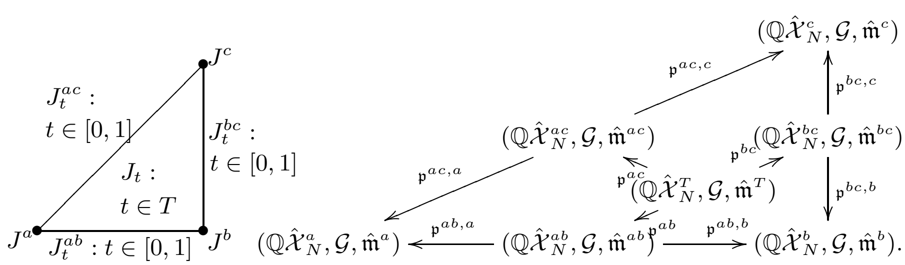


Figure 10.2. (a) $J_t : t \in T$ (b) $A_{N,0}$ algebras and morphisms

Then we prove analogues of Theorems 10.1 and 10.2 by the same methods:

Theorem 10.3. *In the situation above, we can define an $A_{N,0}$ algebra $(\mathbb{Q}\mathcal{X}_N^T, \mathcal{G}, \mathfrak{m}^T)$ and strict, surjective $A_{N,0}$ morphisms $\mathfrak{p}^{ab} : (\mathbb{Q}\mathcal{X}_N^T, \mathcal{G}, \mathfrak{m}^T) \to (\mathbb{Q}\mathcal{X}_N^{ab}, \mathcal{G}, \mathfrak{m}^{ab})$, $\mathfrak{p}^{bc} : (\mathbb{Q}\mathcal{X}_N^T, \mathcal{G}, \mathfrak{m}^T) \to (\mathbb{Q}\mathcal{X}_N^{bc}, \mathcal{G}, \mathfrak{m}^{bc})$, $\mathfrak{p}^{ac} : (\mathbb{Q}\mathcal{X}_N^T, \mathcal{G}, \mathfrak{m}^T) \to (\mathbb{Q}\mathcal{X}_N^{ac}, \mathcal{G}, \mathfrak{m}^{ac})$ which are weak homotopy equivalences, such that Figure* 10.2(b) *is a commutative diagram.*

Theorem 10.4. *In the situation above, there exists a homotopy $\mathfrak{H} : \mathfrak{f}^{ac} \Rightarrow \mathfrak{f}^{bc} \circ \mathfrak{f}^{ab}$ between the $A_{N,0}$ morphisms $\mathfrak{f}^{ac}, \mathfrak{f}^{bc} \circ \mathfrak{f}^{ab} : (\mathbb{Q}\mathcal{X}_N^a, \mathcal{G}, \mathfrak{m}^a) \to (\mathbb{Q}\mathcal{X}_N^c, \mathcal{G}, \mathfrak{m}^c)$.*

Fukaya et al. [**8**, §4.6.3] prove related results by a different method. In our notation, they suppose that the families $J_t^{ab}, J_t^{bc}, J_t^{ac}$ satisfy $J_t^{ac} = J_{2t}^{ab}$ for $t \leqslant \frac{1}{2}$ and $J_t^{ac} = J_{2t-1}^{bc}$ for $t \geqslant \frac{1}{2}$, and show that one can make choices in the constructions of $\mathfrak{f}^{ab}, \mathfrak{f}^{bc}, \mathfrak{f}^{ac}$ so that $\mathfrak{f}^{ac} = \mathfrak{f}^{bc} \circ \mathfrak{f}^{ab}$. Then for more general choices of $J_t^{ab}, J_t^{bc}, J_t^{ac}$ and $\mathfrak{f}^{ab}, \mathfrak{f}^{bc}, \mathfrak{f}^{ac}$, Theorem 10.4 follows from Theorem 10.2.

11. Gapped filtered A_∞ algebras from immersed Lagrangians

We can now, at last, associate a gapped filtered A_∞ algebra to L.

Definition 11.1. Suppose (M, ω) is a compact symplectic manifold, and $\iota : L \to M$ a compact immersed Lagrangian in M with only transverse double self-intersections. Let J be an almost complex structure on M compatible with ω. Choose a relative spin structure for $\iota : L \to M$ and orientations $o_{(p_-,p_+)}$ of the $\operatorname{Ker} \bar\partial_{\lambda_{(p_-,p_+)}}$ as in §5. Let $\mathcal{G} \subset [0,\infty) \times \mathbb{Z}$ satisfy conditions (i),(ii) of §6.

For each $N = 0, 1, 2, \ldots$, let $(\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m}_N)$ be an $A_{N,0}$ algebra constructed in Theorem 7.4 for these $J, \mathcal{G}$; we write $\mathfrak{m}_N$ rather than $\mathfrak{m}$ to make clear the dependence on N. We assume no relation between the choices made in constructing $(\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m}_N)$ and $(\mathbb{Q}\mathcal{X}_{N'}, \mathcal{G}, \mathfrak{m}_{N'})$ for $N \neq N'$, so the sets of simplices, perturbation data, and so on, can all be different.

As in §7, any $A_{N+1,0}$ algebra $(A, \mathcal{G}, \bar{\mathfrak{m}})$ can be truncated to an $A_{N,0}$ algebra $(A, \mathcal{G}, \mathfrak{m})$ by taking $\mathfrak{m}$ to be the subset of $\bar{\mathfrak{m}}_k^{\lambda,\mu}$ with $\|(\lambda,\mu)\| + k - 1 \leqslant N$. Write $(\mathbb{Q}\mathcal{X}_{N+1}, \mathcal{G}, \mathfrak{m}_{N+1})_N$ for the $A_{N,0}$ algebra truncation of $(\mathbb{Q}\mathcal{X}_{N+1}, \mathcal{G}, \mathfrak{m}_{N+1})$. Then $(\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m}_N)$ and $(\mathbb{Q}\mathcal{X}_{N+1}, \mathcal{G}, \mathfrak{m}_{N+1})_N$ are both possible outcomes for $A_{N,0}$ algebras constructed in Theorem 7.4 using $J, \mathcal{G}$. Applying the results of §8–§9 with $J_t = J$ for $t \in [0,1]$, Corollary 9.7 constructs an $A_{N,0}$ morphism $\mathfrak{f}^{01}$ that we will write as $\mathfrak{f}^N : (\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m}_N) \to (\mathbb{Q}\mathcal{X}_{N+1}, \mathcal{G}, \mathfrak{m}_{N+1})_N$, which is a homotopy equivalence. Putting $J_s \equiv J$ for $s \in S$ in §10, Theorem 10.2 implies that $\mathfrak{f}^N$ is independent of choices up to homotopy.

Set $\mathcal{X} = \mathcal{X}_0$. By induction on $N = 0, 1, 2, \ldots$ we shall construct $\mathfrak{m}^N$ such that $(\mathbb{Q}\mathcal{X}, \mathcal{G}, \mathfrak{m}^N)$ is an $A_{N,0}$ algebra, and an $A_{N,0}$ morphism $\mathfrak{g}^N$:

$(\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^N)\to(\mathbb{Q}\mathcal{X}_N,\mathcal{G},\mathfrak{m}_N)$ which is a weak homotopy equivalence, satisfying the conditions:

(i) $\mathfrak{m}^0=\mathfrak{m}_0$ and $\mathfrak{g}^0=\mathrm{id}_{\mathbb{Q}\mathcal{X}}$;

(ii) $\mathfrak{m}^{N+1}$ extends $\mathfrak{m}^N$ for all $N\geqslant 0$, that is, the $A_{N,0}$ algebra truncation $(\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^{N+1})_N$ of $(\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^{N+1})$ is $(\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^N)$; and

(iii) the truncation $(\mathfrak{g}^{N+1})_N:(\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^{N+1})_N\to(\mathbb{Q}\mathcal{X}_{N+1},\mathcal{G},\mathfrak{m}_{N+1})_N$ of g^{N+1} to an $A_{N,0}$ morphism satisfies $(\mathfrak{g}^{N+1})_N=\mathfrak{f}^N\circ\mathfrak{g}^N$ for all $N\geqslant 0$, using $(\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^{N+1})_N=(\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^N)$ from (ii).

For the first step, $\mathfrak{m}^0,\mathfrak{g}^0$ are given in (i). For the inductive step, suppose we have constructed $\mathfrak{m}^N,\mathfrak{g}^N$ satisfying (i)–(iii) for $N=0,1,\ldots,P$. Then $\mathfrak{f}^P\circ\mathfrak{g}^P:(\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^P)\to(\mathbb{Q}\mathcal{X}_{P+1},\mathcal{G},\mathfrak{m}_{P+1})_P$ is an $A_{P,0}$ morphism which is a weak homotopy equivalence, since $\mathfrak{f}^P,\mathfrak{g}^P$ are. Theorem 3.23(a) with $N=P$, $\bar N=P+1$ now shows that there exists an $A_{P+1,0}$ algebra $(\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^{P+1})$ extending $(\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^P)$ and an $A_{P+1,0}$ morphism $\mathfrak{g}^{P+1}:(\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^{P+1})\to(\mathbb{Q}\mathcal{X}_{P+1},\mathcal{G},\mathfrak{m}_{P+1})$ extending $\mathfrak{f}^P\circ\mathfrak{g}^P$ which is a weak homotopy equivalence. This proves the inductive step.

For all $k\geqslant 0$ and $(\lambda,\mu)\in\mathcal{G}$, define $\mathfrak{m}_k^{\lambda,\mu}:(\mathbb{Q}\mathcal{X})^{\times^k}\to\mathbb{Q}\mathcal{X}$ by $\mathfrak{m}_k^{\lambda,\mu}=\mathfrak{m}_k^{N,\lambda,\mu}$, where $N=\max(\|(\lambda,\mu)\|+k-1,0)$ and $\mathfrak{m}_k^{N,\lambda,\mu}$ is the (k,λ,μ) term in $\mathfrak{m}^N$. Then (ii) implies that $\mathfrak{m}_k^{\lambda,\mu}=\mathfrak{m}_k^{N',\lambda,\mu}$ for any $N'\geqslant N$. Since $(\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^N)$ is an $A_{N,0}$ algebra for all $N\geqslant 0$, equation (17) holds for the $\mathfrak{m}_k^{N,\lambda,\mu}$, so by independence of N, the $\mathfrak{m}_k^{\lambda,\mu}$ satisfy (17) for all $k\geqslant 0$, $(\lambda,\mu)\in\mathcal{G}$ and pure $a_1,\ldots,a_k\in\mathbb{Q}\mathcal{X}$.

Define $\mathbb{Q}$-multilinear maps $\mathfrak{m}_k:(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\mathrm{nov}})^{\times^k}\to\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\mathrm{nov}}$ for $k=0,1,\ldots$ by $\mathfrak{m}_k=\sum_{(\lambda,\mu)\in\mathcal{G}}T^\lambda e^\mu\mathfrak{m}_k^{\lambda,\mu}$. Write $\mathfrak{m}=(\mathfrak{m}_k)_{k\geqslant 0}$. Then Definition 3.13 implies that $(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\mathrm{nov}},\mathfrak{m})$ is a *gapped filtered A_∞ algebra.*

Definition 11.1 is similar to Fukaya et al. [**8**, §7.2.8]. Here is one of our main results, analogous to [**8**, Th.s 3.5.11, 4.1.1 & 4.1.2].

Theorem 11.2. **(a)** *In Definition* 11.1, $(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\mathrm{nov}},\mathfrak{m})$ *depends up to canonical homotopy equivalence only on* (M,ω), $\iota:L\to M$ *and its relative spin structure, and the indices* $\eta_{(p_-,p_+)}$ *in* §4.3, *and is independent of* $J,\mathcal{G}$, *changes of paths* $\lambda_{(p_-,p_+)}$ *in* §4.3 *which fix* $\eta_{(p_-,p_+)}$, *the orientations* $o_{(p_-,p_+)}$ *on* $\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}}$ *in* §5.2, *and all other choices.*

That is, if $(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\mathrm{nov}},\mathfrak{m}),(\mathbb{Q}\tilde{\mathcal{X}}\hat\otimes\Lambda^0_{\mathrm{nov}},\tilde{\mathfrak{m}})$ *are outcomes in Definition* 11.1 *depending on* $J,\mathcal{G},\lambda_{(p_-,p_+)},o_{(p_-,p_+)},\ldots$ *and* $\tilde J,\tilde{\mathcal{G}},\tilde\lambda_{(p_-,p_+)},\tilde o_{(p_-,p_+)},\ldots,$ *we can construct a gapped filtered A_∞ morphism* $\mathfrak{j}:(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\mathrm{nov}},\mathfrak{m})\to(\mathbb{Q}\tilde{\mathcal{X}}\hat\otimes\Lambda^0_{\mathrm{nov}},\tilde{\mathfrak{m}})$ *which is a homotopy equivalence. If* $\mathfrak{j},\mathfrak{j}'$ *are possibilities for* $\mathfrak{j}$ *there is a homotopy* $\mathfrak{H}:\mathfrak{j}\Rightarrow\mathfrak{j}'$.

(b) *If* $(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\mathrm{nov}},\mathfrak{m}),(\mathbb{Q}\tilde{\mathcal{X}}\hat\otimes\Lambda^0_{\mathrm{nov}},\tilde{\mathfrak{m}}),(\mathbb{Q}\check{\mathcal{X}}\hat\otimes\Lambda^0_{\mathrm{nov}},\check{\mathfrak{m}})$ *are possible outcomes in Definition* 11.1 *and* $\mathfrak{j}:(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\mathrm{nov}},\mathfrak{m})\to(\mathbb{Q}\tilde{\mathcal{X}}\hat\otimes\Lambda^0_{\mathrm{nov}},\tilde{\mathfrak{m}})$, $\mathfrak{j}':(\mathbb{Q}\tilde{\mathcal{X}}\hat\otimes\Lambda^0_{\mathrm{nov}},\tilde{\mathfrak{m}})\to(\mathbb{Q}\check{\mathcal{X}}\hat\otimes\Lambda^0_{\mathrm{nov}},\check{\mathfrak{m}})$, $\mathfrak{j}'':(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\mathrm{nov}},\mathfrak{m})\to(\mathbb{Q}\check{\mathcal{X}}\hat\otimes\Lambda^0_{\mathrm{nov}},$

$\check{\mathfrak{m}}$) *are corresponding gapped filtered* A_∞ *morphisms in part* (a), *then there is a homotopy* $\mathfrak{H} : \mathfrak{j}'' \Rightarrow \mathfrak{j}' \circ \mathfrak{j}$.

Proof. First we explain how to construct $\mathfrak{j}$ in (a) when $\mathcal{G}$, $\lambda_{(p_-,p_+)}$, $o_{(p_-,p_+)}$ are fixed, but other choices $J, \ldots$ vary. Suppose $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\rm nov}, \mathfrak{m})$, $(\mathbb{Q}\tilde{\mathcal{X}} \hat{\otimes} \Lambda^0_{\rm nov}, \tilde{\mathfrak{m}})$ are constructed using $\mathcal{G}, \lambda_{(p_-,p_+)}, o_{(p_-,p_+)}$ and other choices $J, \ldots$ and $\tilde{J}, \ldots$. Let $(\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m}_N), \mathfrak{f}^N, (\mathbb{Q}\mathcal{X}, \mathcal{G}, \mathfrak{m}^N), \mathfrak{g}^N$ and $(\mathbb{Q}\tilde{\mathcal{X}}_N, \mathcal{G}, \tilde{\mathfrak{m}}_N), \tilde{\mathfrak{f}}^N, (\mathbb{Q}\tilde{\mathcal{X}}, \mathcal{G}, \tilde{\mathfrak{m}}^N), \tilde{\mathfrak{g}}^N$, be the corresponding choices in Definition 11.1. Let $\tilde{\mathfrak{d}}^N, \tilde{\mathfrak{e}}^N$ be homotopy inverses for $\tilde{\mathfrak{f}}^N, \tilde{\mathfrak{g}}^N$.

Let $J_t : t \in [0,1]$ be a smooth family of almost complex structures on M compatible with ω, with $J_0 = J$ and $J_1 = \tilde{J}$. Suppose that $\mathcal{G}$ satisfies conditions (i),(ii) of §8 for $J_t : t \in [0,1]$; this implies that $\mathcal{G}$ also satisfies conditions (i),(ii) of §6 for $J, \tilde{J}$. If $\mathcal{G}$ does not satisfy (i),(ii), we can use the third part of the proof to change to a new $\mathcal{G}$ which does. Then Corollary 9.7 constructs an $A_{N,0}$ morphism $\mathfrak{f}^{01}$ that we will write as $\mathfrak{h}^N : (\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m}_N) \to (\mathbb{Q}\tilde{\mathcal{X}}_N, \mathcal{G}, \tilde{\mathfrak{m}}_N)$, which is a homotopy equivalence. Also, as $(\mathbb{Q}\tilde{\mathcal{X}}_{N+1}, \mathcal{G}, \tilde{\mathfrak{m}}_{N+1})_N$ is also a possible $A_{N,0}$ algebra from Theorem 7.4 with $\tilde{J}$, Corollary 9.7 constructs an $A_{N,0}$ morphism $\mathfrak{i}^N : (\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m}_N) \to (\mathbb{Q}\tilde{\mathcal{X}}_{N+1}, \mathcal{G}, \tilde{\mathfrak{m}}_{N+1})_N$, which is a homotopy equivalence. Thus we obtain the diagram of $A_{N,0}$ morphism homotopy equivalences:

$$
\begin{tikzcd}
(\mathbb{Q}\mathcal{X}, \mathcal{G}, \mathfrak{m}^N) \arrow[r, "\mathfrak{g}^N"'] \arrow[rr, bend left, "(\mathfrak{g}^{N+1})_N"] \arrow[d, dashed, "\mathfrak{j}^N"] & (\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m}_N) \arrow[r, "\mathfrak{f}^N"'] \arrow[d, "\mathfrak{h}^N"'] \arrow[dr, "\mathfrak{i}^N"] & (\mathbb{Q}\mathcal{X}_{N+1}, \mathcal{G}, \mathfrak{m}_{N+1})_N \arrow[d, "(\mathfrak{h}^{N+1})_N"] \\
(\mathbb{Q}\tilde{\mathcal{X}}, \mathcal{G}, \tilde{\mathfrak{m}}^N) \arrow[r, shift left, "\tilde{\mathfrak{g}}^N"] \arrow[rr, bend right, shift left, "(\tilde{\mathfrak{g}}^{N+1})_N"] & (\mathbb{Q}\tilde{\mathcal{X}}_N, \mathcal{G}, \tilde{\mathfrak{m}}_N) \arrow[l, shift left, "\tilde{\mathfrak{e}}^N"] \arrow[r, shift left, "\tilde{\mathfrak{f}}^N"] & (\mathbb{Q}\tilde{\mathcal{X}}_{N+1}, \mathcal{G}, \tilde{\mathfrak{m}}_{N+1})_N \arrow[l, shift left, "\tilde{\mathfrak{d}}^N"] \arrow[ll, bend left, shift left, "(\tilde{\mathfrak{e}}^{N+1})_N"]
\end{tikzcd}
\tag{136}
$$

Write $\mathfrak{f} \sim \mathfrak{g}$ when two $A_{N,0}$ morphisms are homotopic. Then we have

$$
\begin{aligned}
&\tilde{\mathfrak{e}}^N \circ \mathfrak{h}^N \circ \mathfrak{g}^N \sim \tilde{\mathfrak{e}}^N \circ \tilde{\mathfrak{d}}^N \circ \tilde{\mathfrak{f}}^N \circ \mathfrak{h}^N \circ \mathfrak{g}^N \sim \tilde{\mathfrak{e}}^N \circ \tilde{\mathfrak{d}}^N \circ \mathfrak{i}^N \circ \mathfrak{g}^N \sim \\
&\tilde{\mathfrak{e}}^N \circ \tilde{\mathfrak{d}}^N \circ (\mathfrak{h}^{N+1})_N \circ \mathfrak{f}^N \circ \mathfrak{g}^N \sim (\tilde{\mathfrak{e}}^{N+1})_N \circ (\mathfrak{h}^{N+1})_N \circ (\mathfrak{g}^{N+1})_N \\
&\qquad = (\tilde{\mathfrak{e}}^{N+1} \circ \mathfrak{h}^{N+1} \circ \mathfrak{g}^{N+1})_N,
\end{aligned}
\tag{137}
$$

using $\tilde{\mathfrak{d}}^N, \tilde{\mathfrak{f}}^N$ homotopy inverses in the first step, $\tilde{\mathfrak{f}}^N \circ \mathfrak{h}^N \sim \mathfrak{i}^N$ by Theorem 10.4 in the second, $(\mathfrak{h}^{N+1})_N \circ \mathfrak{f}^N \sim \mathfrak{i}^N$ by Theorem 10.4 in the third, and $(\mathfrak{g}^{N+1})_N = \mathfrak{f}^N \circ \mathfrak{g}^N$ and $\tilde{\mathfrak{e}}^N \circ \tilde{\mathfrak{d}}^N \sim (\tilde{\mathfrak{e}}^{N+1})_N$ which follows from $(\tilde{\mathfrak{g}}^{N+1})_N = \tilde{\mathfrak{f}}^N \circ \tilde{\mathfrak{g}}^N$ and $\tilde{\mathfrak{d}}^N, \tilde{\mathfrak{e}}^N, \tilde{\mathfrak{e}}^{N+1}$ homotopy inverses for $\mathfrak{f}^N, \mathfrak{g}^N, \mathfrak{g}^{N+1}$ in the fourth.

By induction on $N = 0, 1, 2, \ldots$ we now choose $A_{N,0}$ morphisms $\mathfrak{j}^N : (\mathbb{Q}\mathcal{X}, \mathcal{G}, \mathfrak{m}^N) \to (\mathbb{Q}\tilde{\mathcal{X}}, \mathcal{G}, \tilde{\mathfrak{m}}^N)$ which are homotopy equivalences, satisfying the conditions:

(i) $\mathfrak{j}^N$ is homotopic to $\tilde{\mathfrak{e}}^N \circ \mathfrak{h}^N \circ \mathfrak{g}^N$; and
(ii) the truncation $(\mathfrak{j}^{N+1})_N : (\mathbb{Q}\mathcal{X}, \mathcal{G}, \mathfrak{m}^{N+1})_N \to (\mathbb{Q}\tilde{\mathcal{X}}, \mathcal{G}, \tilde{\mathfrak{m}}^{N+1})_N$ of $\mathfrak{j}^{N+1}$ to an $A_{N,0}$ morphism satisfies $(\mathfrak{j}^{N+1})_N = \mathfrak{j}^N$ for all $N \geqslant 0$, using $(\mathbb{Q}\mathcal{X}, \mathcal{G}, \mathfrak{m}^{N+1})_N = (\mathbb{Q}\mathcal{X}, \mathcal{G}, \mathfrak{m}^N)$ and $(\mathbb{Q}\tilde{\mathcal{X}}, \mathcal{G}, \tilde{\mathfrak{m}}^{N+1})_N = (\mathbb{Q}\tilde{\mathcal{X}}, \mathcal{G}, \tilde{\mathfrak{m}}^N)$.

For the first step, we take $\mathfrak{j}^0 = \tilde{\mathfrak{e}}^N \circ \mathfrak{h}^N \circ \mathfrak{g}^N$, so that (i) for $N = 0$ is trivial. For the inductive step, suppose we have chosen $\mathfrak{j}^N$ satisfying (i),(ii) for $N = 0, 1, \ldots, P$. We shall construct $\mathfrak{j}^{P+1}$. Since $\mathfrak{j}^P$ is homotopic to $\tilde{\mathfrak{e}}^P \circ \mathfrak{h}^P \circ \mathfrak{g}^P$ by (i), and $\tilde{\mathfrak{e}}^P \circ \mathfrak{h}^P \circ \mathfrak{g}^P$ is homotopic to $(\tilde{\mathfrak{e}}^{P+1} \circ \mathfrak{h}^{P+1} \circ \mathfrak{g}^{P+1})_P$ by (137), $\mathfrak{j}^P$ is homotopic to $(\tilde{\mathfrak{e}}^{P+1} \circ \mathfrak{h}^{P+1} \circ \mathfrak{g}^{P+1})_P$. So Theorem 3.23(b) with $N = P$, $\bar{N} = P + 1$, $\mathfrak{f} = \mathfrak{j}^P$ and $\bar{\mathfrak{g}} = \tilde{\mathfrak{e}}^{P+1} \circ \mathfrak{h}^{P+1} \circ \mathfrak{g}^{P+1}$ gives $\mathfrak{j}^{P+1}$ satisfying (i),(ii). Therefore by induction $\mathfrak{j}^N$ exists for all N.

There is now a unique gapped filtered A_∞ morphism $\mathfrak{j} : (\mathbb{Q}\mathcal{X} \hat{\otimes} \Lambda^0_{\mathrm{nov}}, \mathfrak{m}) \to (\mathbb{Q}\tilde{\mathcal{X}} \hat{\otimes} \Lambda^0_{\mathrm{nov}}, \tilde{\mathfrak{m}})$ whose truncation to $A_{N,0}$ algebras is $\mathfrak{j}^N$ for $N = 0, 1, 2, \ldots$. It is a weak homotopy equivalence as the $\mathfrak{j}^N$ are, and so is a homotopy equivalence by Theorem 3.15(c). Regarding $\mathfrak{g}^N, \tilde{\mathfrak{g}}^N$ as fixed, $\tilde{\mathfrak{e}}^N$ above is independent of choices up to homotopy, and by Theorem 10.2, so is $\mathfrak{h}^N$. Thus, $\mathfrak{j}^N$ is independent of choices up to $A_{N,0}$ homotopy. As this holds for all N, $\mathfrak{j}$ is independent of choices up to homotopy. That is, if $\mathfrak{j}, \mathfrak{j}'$ are possible choices for $\mathfrak{j}$ then there is a homotopy $\mathfrak{H} : \mathfrak{j} \to \mathfrak{j}'$. We construct $\mathfrak{H}$ as the union of a family of $A_{N,0}$ homotopies $\mathfrak{H}^N : \mathfrak{j}^N \Rightarrow \mathfrak{j}'^N$ with $(\mathfrak{H}^{N+1})_N = \mathfrak{H}_N$, chosen using an analogue of Theorem 3.23(b) for homotopies. This proves (a) with $\mathcal{G}$ and $\lambda_{(p_-,p_+)}, o_{(p_-,p_+)}$ for $(p_-, p_+) \in R$ fixed.

Secondly, we prove (b) with $\mathcal{G}, \lambda_{(p_-,p_+)}, o_{(p_-,p_+)}$ fixed. Suppose $(\mathbb{Q}\mathcal{X} \hat{\otimes} \Lambda^0_{\mathrm{nov}}, \mathfrak{m})$, $(\mathbb{Q}\tilde{\mathcal{X}} \hat{\otimes} \Lambda^0_{\mathrm{nov}}, \tilde{\mathfrak{m}})$, $(\mathbb{Q}\check{\mathcal{X}} \hat{\otimes} \Lambda^0_{\mathrm{nov}}, \check{\mathfrak{m}})$ and $\mathfrak{j}, \mathfrak{j}', \mathfrak{j}''$ are as in (b), all constructed using the same $\mathcal{G}, \lambda_{(p_-,p_+)}, o_{(p_-,p_+)}$. Then with the obvious notation we have a diagram of $A_{N,0}$ morphism homotopy equivalences:

$$\begin{tikzcd}
(\mathbb{Q}\mathcal{X}, \mathcal{G}, \mathfrak{m}^N) \arrow[r, "\mathfrak{g}^N"'] \arrow[d, "\mathfrak{j}^N"] \arrow[dd, bend right=60, "\mathfrak{j}''^N"'] & (\mathbb{Q}\mathcal{X}_N, \mathcal{G}, \mathfrak{m}_N) \arrow[d, "\mathfrak{h}^N"'] \arrow[dd, bend left=60, "\mathfrak{h}''^N"] \\
(\mathbb{Q}\tilde{\mathcal{X}}, \mathcal{G}, \tilde{\mathfrak{m}}^N) \arrow[r, shift left, "\tilde{\mathfrak{g}}^N"] \arrow[d, "\mathfrak{j}'^N"] & (\mathbb{Q}\tilde{\mathcal{X}}_N, \mathcal{G}, \tilde{\mathfrak{m}}_N) \arrow[l, shift left, "\tilde{\mathfrak{e}}^N"] \arrow[d, "\mathfrak{h}'^N"'] \\
(\mathbb{Q}\check{\mathcal{X}}, \mathcal{G}, \check{\mathfrak{m}}^N) \arrow[r, shift left, "\check{\mathfrak{g}}^N"] & (\mathbb{Q}\check{\mathcal{X}}_N, \mathcal{G}, \check{\mathfrak{m}}_N) \arrow[l, shift left, "\check{\mathfrak{e}}^N"]
\end{tikzcd}$$

Theorem 10.4 shows that $\mathfrak{h}''^N \sim \mathfrak{h}'^N \circ \mathfrak{h}^N$. Since $\mathfrak{j}^N \sim \tilde{\mathfrak{e}}^N \circ \mathfrak{h}^N \circ \mathfrak{g}^N$, $\mathfrak{j}'^N \sim \check{\mathfrak{e}}^N \circ \mathfrak{h}'^N \circ \tilde{\mathfrak{g}}^N$ and $\tilde{\mathfrak{g}}^N, \tilde{\mathfrak{e}}^N$ are homotopy inverses, this implies that $\mathfrak{j}''^N \sim \mathfrak{j}'^N \circ \mathfrak{j}^N$. That is, the $A_{N,0}$ truncations of $\mathfrak{j}''$ and $\mathfrak{j}' \circ \mathfrak{j}$ are $A_{N,0}$ homotopic for all $N = 0, 1, \ldots$. We can now construct $\mathfrak{H} : \mathfrak{j}'' \Rightarrow \mathfrak{j}' \circ \mathfrak{j}$ as in the end of the first part of the proof.

Thirdly, we explain how to change $\mathcal{G}$ in (a) and (b). Suppose that $\mathcal{G} \subseteq \tilde{\mathcal{G}} \subset [0, \infty) \times \mathbb{Z}$, and $\mathcal{G}, \tilde{\mathcal{G}}$ are closed under addition, such that $\mathcal{G} \cap (\{0\} \times \mathbb{Z}) = \tilde{\mathcal{G}} \cap (\{0\} \times \mathbb{Z}) = \{(0, 0)\}$ and $\mathcal{G} \cap ([0, C] \times \mathbb{Z}), \tilde{\mathcal{G}} \cap ([0, C] \times \mathbb{Z})$

are finite for any $C \geqslant 0$. We shall define a functor from the 2-category of $A_{N,0}$ algebras with fixed $\mathcal{G}$ to the 2-category of $A_{N,0}$ algebras with fixed $\tilde{\mathcal{G}}$, which we call $\tilde{\mathcal{G}}$-*truncation*.

If $(\lambda,\mu) \in \mathcal{G}$ then as $\mathcal{G} \subseteq \tilde{\mathcal{G}}$, in (23) we can define $\|(\lambda,\mu)\|$ using either $\mathcal{G}$ or $\tilde{\mathcal{G}}$. Write these as $\|(\lambda,\mu)\|_{\mathcal{G}}, \|(\lambda,\mu)\|_{\tilde{\mathcal{G}}}$ to distinguish them. Then $\mathcal{G} \subseteq \tilde{\mathcal{G}}$ implies that $\|(\lambda,\mu)\|_{\mathcal{G}} \leqslant \|(\lambda,\mu)\|_{\tilde{\mathcal{G}}}$, as (λ,μ) can be split into more pieces in $\tilde{\mathcal{G}}$ than in $\mathcal{G}$. Thus for k, N given, $\|(\lambda,\mu)\|_{\tilde{\mathcal{G}}} + k - 1 \leqslant N$ implies that $\|(\lambda,\mu)\|_{\mathcal{G}} + k - 1 \leqslant N$.

Suppose $(A,\mathcal{G},\mathfrak{m})$ is an $A_{N,0}$ algebra, so that $\mathfrak{m} = \big(\mathfrak{m}_k^{\lambda,\mu} : k \geqslant 0,$ $(\lambda,\mu) \in \mathcal{G},\ \|(\lambda,\mu)\|_{\mathcal{G}} + k - 1 \leqslant N\big)$. Define an $A_{N,0}$ algebra $(A,\tilde{\mathcal{G}},\tilde{\mathfrak{m}})$, where $\tilde{\mathfrak{m}} = \big(\tilde{\mathfrak{m}}_k^{\lambda,\mu} : k \geqslant 0,\ (\lambda,\mu) \in \tilde{\mathcal{G}},\ \|(\lambda,\mu)\|_{\tilde{\mathcal{G}}} + k - 1 \leqslant N\big)$ by $\tilde{\mathfrak{m}}_k^{\lambda,\mu} = 0$ if $(\lambda,\mu) \in \tilde{\mathcal{G}} \setminus \mathcal{G}$, and $\tilde{\mathfrak{m}}_k^{\lambda,\mu} = \mathfrak{m}_k^{\lambda,\mu}$ if $(\lambda,\mu) \in \mathcal{G}$. Since $(\lambda,\mu) \in \mathcal{G}$ and $\|(\lambda,\mu)\|_{\tilde{\mathcal{G}}} + k - 1 \leqslant N$ implies that $\|(\lambda,\mu)\|_{\mathcal{G}} + k - 1 \leqslant N$, this is well-defined, and (17) holds for the $\tilde{\mathfrak{m}}_k^{\lambda,\mu}$ as it does for the $\mathfrak{m}_k^{\lambda,\mu}$. So $(A,\tilde{\mathcal{G}},\tilde{\mathfrak{m}})$ is an $A_{N,0}$ algebra.

Write $(A,\mathcal{G},\mathfrak{m})_{\tilde{\mathcal{G}}} = (A,\tilde{\mathcal{G}},\tilde{\mathfrak{m}})$, that is, $(A,\mathcal{G},\mathfrak{m})_{\tilde{\mathcal{G}}}$ is the $\tilde{\mathcal{G}}$-*truncation* of $(A,\mathcal{G},\mathfrak{m})$. In a similar way, if $\mathfrak{f} : (A,\mathcal{G},\mathfrak{m}) \to (B,\mathcal{G},\mathfrak{n})$ is an $A_{N,0}$ morphism of $A_{N,0}$ algebras with $\mathcal{G}$, then the $\tilde{\mathcal{G}}$-*truncation* $\mathfrak{f}_{\tilde{\mathcal{G}}} = \tilde{\mathfrak{f}} :$ $(A,\mathcal{G},\mathfrak{m})_{\tilde{\mathcal{G}}} \to (B,\mathcal{G},\mathfrak{n})_{\tilde{\mathcal{G}}}$ is an $A_{N,0}$ morphism of $A_{N,0}$ algebras with $\tilde{\mathcal{G}}$, where $\tilde{\mathfrak{f}}_k^{\lambda,\mu} = 0$ if $(\lambda,\mu) \in \tilde{\mathcal{G}} \setminus \mathcal{G}$, and $\tilde{\mathfrak{f}}_k^{\lambda,\mu} = \mathfrak{f}_k^{\lambda,\mu}$ if $(\lambda,\mu) \in \mathcal{G}$. If $\mathfrak{H} : \mathfrak{f} \to \mathfrak{g}$ is a homotopy of $A_{N,0}$ morphisms $\mathfrak{f},\mathfrak{g} : (A,\mathcal{G},\mathfrak{m}) \to (B,\mathcal{G},\mathfrak{n})$, then the $\tilde{\mathcal{G}}$-truncation $\mathfrak{H}_{\tilde{\mathcal{G}}} = \tilde{\mathfrak{H}} : \mathfrak{f}_{\tilde{\mathcal{G}}} \Rightarrow \mathfrak{g}_{\tilde{\mathcal{G}}}$ is a homotopy, where $\tilde{\mathfrak{H}}_k^{\lambda,\mu} = 0$ if $(\lambda,\mu) \in \tilde{\mathcal{G}} \setminus \mathcal{G}$, and $\tilde{\mathfrak{H}}_k^{\lambda,\mu} = \mathfrak{H}_k^{\lambda,\mu}$ if $(\lambda,\mu) \in \mathcal{G}$.

Now suppose that $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})$ is a gapped filtered A_∞ algebra constructed in Definition 11.1 using data $J, \mathcal{G}, \ldots$. We shall show how to construct *exactly the same* gapped filtered A_∞ algebra using $\tilde{\mathcal{G}}$ instead of $\mathcal{G}$. Use all the notation $(\mathbb{Q}\mathcal{X}_N,\mathcal{G},\mathfrak{m}_N), \mathfrak{f}^N, \mathfrak{g}^N, \mathfrak{m}^N, \ldots$ of Definition 11.1. Then it is easy to see that we may go through Definition 11.1 replacing $\mathcal{G}$ by $\tilde{\mathcal{G}}$, and all the $A_{N,0}$ algebras, morphisms and homotopies by their $\tilde{\mathcal{G}}$-truncations, and get a valid set of choices. That is, we replace $(\mathbb{Q}\mathcal{X}_N,\mathcal{G},\mathfrak{m}_N)$ by $(\mathbb{Q}\mathcal{X}_N,\tilde{\mathcal{G}},\tilde{\mathfrak{m}}_N) = (\mathbb{Q}\mathcal{X}_N,\mathcal{G},\mathfrak{m}_N)_{\tilde{\mathcal{G}}}$, $\mathfrak{f}^N, \mathfrak{g}^N$ by $\tilde{\mathfrak{f}}^N =$ $(\mathfrak{f}^N)_{\tilde{\mathcal{G}}}, \tilde{\mathfrak{g}}^N = (\mathfrak{g}^N)_{\tilde{\mathcal{G}}}$, and $(\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^N)$ by $(\mathbb{Q}\mathcal{X},\mathcal{G},\tilde{\mathfrak{m}}^N) = (\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^N)_{\tilde{\mathcal{G}}}$.

Since $\tilde{\mathcal{G}}$-truncation commutes with truncation of $A_{N+1,0}$ algebras to $A_{N,0}$ algebras, these satisfy $(\tilde{\mathfrak{g}}^{N+1})_N = \tilde{\mathfrak{f}}^N \circ \tilde{\mathfrak{g}}^N$, and so on. Thus, we obtain a gapped filtered A_∞ algebra $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\tilde{\mathfrak{m}})$ using $\tilde{\mathcal{G}}$, whose truncation to an $A_{N,0}$ algebra with $\tilde{\mathcal{G}}$ is $(\mathbb{Q}\mathcal{X},\mathcal{G},\tilde{\mathfrak{m}}^N) = (\mathbb{Q}\mathcal{X},\mathcal{G},\mathfrak{m}^N)_{\tilde{\mathcal{G}}}$ for all $N = 0, 1, \ldots$. Clearly this implies that $\tilde{\mathfrak{m}} = \mathfrak{m}$, and $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\tilde{\mathfrak{m}}) =$ $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})$. So we are always free to enlarge $\mathcal{G}$ to $\tilde{\mathcal{G}}$, and obtain not just two homotopic, but the same, gapped filtered A_∞ algebras.

To extend the proofs of the first two parts to allow $\mathcal{G}$ to vary, suppose in (a) that $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})$, $(\mathbb{Q}\tilde{\mathcal{X}}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\tilde{\mathfrak{m}})$ are constructed using

$J, \mathcal{G}, \lambda_{(p_-,p_+)}, o_{(p_-,p_+)}, \ldots$ and $\tilde{J}, \tilde{\mathcal{G}}, \lambda_{(p_-,p_+)}, o_{(p_-,p_+)}, \ldots$, with possibly different $\mathcal{G}, \tilde{\mathcal{G}}$. Choose a smooth 1-parameter family of almost complex structures $J_t : t \in [0,1]$ on M compatible with ω, with $J_0 = J$ and $J_1 = \tilde{J}$. Choose some $\check{\mathcal{G}} \subset [0,\infty) \times \mathbb{Z}$ such that $\mathcal{G} \subseteq \check{\mathcal{G}}$, and $\tilde{\mathcal{G}} \subseteq \check{\mathcal{G}}$, and conditions (i),(ii) of §8 hold for $\check{\mathcal{G}}$ and $J_t : t \in [0,1]$. This is possible, and there is a unique smallest such $\check{\mathcal{G}}$.

Now regard $(\mathbb{Q}\mathcal{X} \hat{\otimes} \Lambda^0_{\rm nov}, \mathfrak{m})$, $(\mathbb{Q}\tilde{\mathcal{X}} \hat{\otimes} \Lambda^0_{\rm nov}, \tilde{\mathfrak{m}})$ as having been constructed using $\check{\mathcal{G}}$ rather than $\mathcal{G}, \tilde{\mathcal{G}}$, as above. Then we can use the first part of the proof with $\check{\mathcal{G}}$ in place of $\mathcal{G}$ to construct $\mathfrak{j} : (\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\rm nov}, \mathfrak{m}) \to (\mathbb{Q}\tilde{\mathcal{X}}\hat{\otimes}\Lambda^0_{\rm nov}, \tilde{\mathfrak{m}})$ and prove (a). The extension of (b) to varying $\mathcal{G}$ is similar; we must choose $\check{\mathcal{G}}$ to contain $\mathcal{G}, \mathcal{G}', \mathcal{G}''$, and the choices of $\mathcal{G}$ used to define $\mathfrak{j}, \mathfrak{j}', \mathfrak{j}''$, and to be compatible with the family of almost complex structures $J_t : t \in T$ used in §10.2 to construct homotopies.

Finally we explain how to change the paths $\lambda_{(p_-,p_+)}$ and orientations $o_{(p_-,p_+)}$ on $\operatorname{Ker} \bar{\partial}_{\lambda_{(p_-,p_+)}}$ for $(p_-,p_+) \in R$. Let $\tilde{\lambda}_{(p_-,p_+)}, \tilde{o}_{(p_-,p_+)}$ be an alternative set of choices, which yield the same indices $\eta_{(p_-,p_+)}$. Then Proposition 5.15 shows how the orientation of $\overline{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)$ changes for these new choices, in terms of $\xi_{(p_-,p_+)} = \pm 1$ for $(p_-,p_+) \in R$. Let $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\rm nov}, \mathfrak{m})$ be constructed in Definition 11.1 using the $\lambda_{(p_-,p_+)}$, $o_{(p_-,p_+)}$, and $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\rm nov}, \tilde{\mathfrak{m}})$ be constructed using $\tilde{\lambda}_{(p_-,p_+)}$, $\tilde{o}_{(p_-,p_+)}$, but otherwise using *exactly the same choices*. That is, the chain complexes $\mathbb{Q}\mathcal{X}_N, \mathbb{Q}\mathcal{X}$ and choices of perturbation data are unchanged, but the other data of virtual chains, $\mathfrak{m}_N, \mathfrak{f}^N, \mathfrak{g}^N, \mathfrak{m}^N, \mathfrak{m}, \ldots$ change to $\tilde{\mathfrak{m}}_N, \tilde{\mathfrak{f}}^N$, $\tilde{\mathfrak{g}}^N, \tilde{\mathfrak{m}}^N, \tilde{\mathfrak{m}}, \ldots$ with various sign changes depending on the $\xi_{(p_-,p_+)}$.

But $\overline{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k) = \emptyset$ unless $f_i : \Delta_{a_i} \to L \amalg R$ maps to $\alpha(i) \in R$ if $i \in I$ and to L if $i \notin I$, and $\mathbf{ev} : \overline{\mathcal{M}}^{\rm main}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k) \to L \amalg R$ maps to $\sigma \circ \alpha(0)$ if $0 \in I$ and to L if $0 \notin I$. Because of this, if we define linear $\Xi : \mathbb{Q}\mathcal{X}_i \to \mathbb{Q}\mathcal{X}_i$ by

$$\Xi(f) = \begin{cases} \xi_{\sigma(p_-,p_+)} f, & f : \Delta_a \to \{(p_-,p_+)\} \subset R, \\ f, & f : \Delta_a \to L, \end{cases}$$

then in Definition 7.1 we have $\tilde{\mathfrak{m}}^{\lambda,\mu}_{k,\rm geo}\bigl(\Xi(f_1),\ldots,\Xi(f_k)\bigr) = \Xi \circ \mathfrak{m}^{\lambda,\mu}_{k,\rm geo}(f_1, \ldots, f_k)$, as $\mathfrak{m}^{\lambda,\mu}_{k,\rm geo}, \tilde{\mathfrak{m}}^{\lambda,\mu}_{k,\rm geo}$ are constructed from virtual chains for $\overline{\mathcal{M}}^{\rm main}_{k+1}$ $(\alpha,\beta,J,f_1,\ldots,f_k)$, which change signs as in Proposition 5.15.

Going through the constructions of §7–§10 and Definition 11.1, we find that everything commutes with Ξ in this way, so that eventually $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\rm nov}, \mathfrak{m})$ and $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\rm nov}, \tilde{\mathfrak{m}})$ satisfy $\tilde{\mathfrak{m}}_k\bigl(\hat{\Xi}(f_1),\ldots,\hat{\Xi}(f_k)\bigr) = \hat{\Xi} \circ \mathfrak{m}_k(f_1,\ldots,f_k)$, where $\hat{\Xi} : \mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\rm nov} \to \mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\rm nov}$ is the $\Lambda^0_{\rm nov}$-linear map induced by $\Xi : \mathbb{Q}\mathcal{X} \to \mathbb{Q}\mathcal{X}$. Thus $\hat{\Xi}$ induces a strict A_∞ isomorphism $\boldsymbol{\Xi} : (\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\rm nov}, \mathfrak{m}) \to (\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\rm nov}, \tilde{\mathfrak{m}})$. To include change of $\lambda_{(p_-,p_+)}, o_{(p_-,p_+)}$ in (a), we compose $\mathfrak{j}$ constructed above for fixed $o_{(p_-,p_+)}$ with this $\boldsymbol{\Xi}$ to get the new $\mathfrak{j}$. The same idea works for (b). q.e.d.

Remark 11.3. In Theorem 11.2(a), it is *nearly* true that $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}}, \mathfrak{m})$ is also independent of the indices $\eta_{(p_-,p_+)}$ in §4.3 up to canonical homotopy equivalence. This would be true if we relaxed the definition of gapped filtered A_∞ morphism in Definition 3.14 slightly. For $(p_-,p_+)\in R$, let $\lambda_{(p_-,p_+)},\tilde\lambda_{(p_-,p_+)}$ be possible choices in §4.3, let $\eta_{(p_-,p_+)},\tilde\eta_{(p_-,p_+)}$ be the corresponding indices (31), and let $o_{(p_-,p_+)},\tilde o_{(p_-,p_+)}$ be orientations on $\operatorname{Ker}\bar\partial_{\lambda_{(p_-,p_+)}},\operatorname{Ker}\bar\partial_{\tilde\lambda_{(p_-,p_+)}}$. As at the end of §5.4, we have $\tilde\eta_{(p_-,p_+)}=\eta_{(p_-,p_+)}+2d_{(p_-,p_+)}$ for $d_{(p_-,p_+)}\in\mathbb{Z}$.

We can now try to adapt the final part of the proof of Theorem 11.2, as follows. Suppose $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})$ is constructed in Definition 11.1 using $\lambda_{(p_-,p_+)},\eta_{(p_-,p_+)},o_{(p_-,p_+)}$, and $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\tilde{\mathfrak{m}})$ is constructed using $\tilde\lambda_{(p_-,p_+)},\tilde\eta_{(p_-,p_+)},\tilde o_{(p_-,p_+)}$, but otherwise using *exactly the same choices*. When we change from $\lambda_{(p_-,p_+)},\eta_{(p_-,p_+)},o_{(p_-,p_+)}$ to $\tilde\lambda_{(p_-,p_+)},\tilde\eta_{(p_-,p_+)},\tilde o_{(p_-,p_+)}$, the orientations of $\bar{\mathcal{M}}^{\mathrm{main}}_{k+1}(\alpha,\beta,J,f_1,\ldots,f_k)$ change as in Proposition 5.15, in terms of $\xi_{(p_-,p_+)}=\pm1$ for $(p_-,p_+)\in R$, and $\deg f$ in (36) changes by $\deg f\mapsto\deg f+2d_{(p_-,p_+)}$ if $f:\Delta_a\to\{(p_-,p_+)\}$. Define a Λ^0_{nov}-linear map $\hat\Xi:\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}}\to\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}}$ by

$$\hat\Xi(f)=\begin{cases} e^{-d_{(p_-,p_+)}}\xi_{\sigma(p_-,p_+)}f, & f:\Delta_a\to\{(p_-,p_+)\}\subset R,\\ f, & f:\Delta_a\to L,\end{cases}$$

where e is the formal variable in Λ^0_{nov} from §3.4.

Note that $\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}}$ is graded differently in $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})$ and $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\tilde{\mathfrak{m}})$, because of the change in $\deg f$. Since e has degree 2, the correction $e^{-d_{(p_-,p_+)}}$ ensures that $\hat\Xi$ is graded of degree 0 as a map $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})\to(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\tilde{\mathfrak{m}})$. As in the final part of the proof of Theorem 11.2, we find that $\tilde{\mathfrak{m}}_k\bigl(\hat\Xi(f_1),\ldots,\hat\Xi(f_k)\bigr)=\hat\Xi\circ\mathfrak{m}_k(f_1,\ldots,f_k)$ for all $f_1,\ldots,f_k\in\mathcal{X}$.

We would like to define a strict gapped filtered A_∞ isomorphism $\boldsymbol{\Xi}:(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})\to(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\tilde{\mathfrak{m}})$ by $\boldsymbol{\Xi}_1=\hat\Xi$ and $\boldsymbol{\Xi}_k=0$ for $k\neq1$, which would prove that $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})$ is independent of $\lambda_{(p_-,p_+)},\eta_{(p_-,p_+)}$ up to canonical homotopy equivalence. However, this $\boldsymbol{\Xi}$ has $\boldsymbol{\Xi}_1^{0,-d_{(p_-,p_+)}}\neq0$ for all $(p_-,p_+)\in R$, which contradicts the conditions on $\mathcal{G}'$ in Definition 3.14(i) if $d_{(p_-,p_+)}\neq0$. We could weaken Definition 3.14(i) to make $\boldsymbol{\Xi}$ a gapped filtered A_∞ morphism, but this would cause problems elsewhere, in particular, the definition of weak homotopy equivalence would no longer make sense.

By Theorem 3.17, the gapped filtered A_∞ algebra $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})$ of Definition 11.1 admits a *minimal model* $(\mathcal{H}\otimes\Lambda^0_{\mathrm{nov}},\mathfrak{n})$ with $\mathcal{H}\cong H^*(\mathbb{Q}\mathcal{X},\mathfrak{m}_1^{0,0})$. Here Theorem 6.1$(N1)$(b) implies that $H^*(\mathbb{Q}\mathcal{X},\mathfrak{m}_1^{0,0})\cong H_*^{\mathrm{si}}(L\amalg R;\mathbb{Q})$ as an ungraded vector space, and the grading is given by shifted cohomological degree in (36). As $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m})$ is unique up to canonical homotopy equivalence by Theorem 11.2, $(\mathcal{H}\otimes\Lambda^0_{\mathrm{nov}},\mathfrak{n})$

is unique up to canonical gapped filtered A_∞ isomorphism. Thus we deduce:

Corollary 11.4. *The gapped filtered A_∞ algebra $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda^0_{\mathrm{nov}}, \mathfrak{m})$ of Definition* 11.1 *has a minimal model $(\mathcal{H}\otimes\Lambda^0_{\mathrm{nov}}, \mathfrak{n})$, with graded $\mathbb{Q}$-vector space $\mathcal{H} = \bigoplus_{d\in\mathbb{Z}} \mathcal{H}^d$ given by*

$$\mathcal{H}^d = H_{n-d-1}(L;\mathbb{Q}) \oplus \bigoplus\nolimits_{\substack{(p_-,p_+)\in R:\\ d=\eta_{(p_-,p_+)}-1}} \mathbb{Q}(p_-,p_+), \tag{138}$$

where $\mathbb{Q}(p_-,p_+) \cong H_0\big(\{(p_-,p_+)\};\mathbb{Q}\big)$ is the $\mathbb{Q}$-vector space with basis $\{(p_-,p_+)\}$.

This $(\mathcal{H}\otimes\Lambda^0_{\mathrm{nov}}, \mathfrak{n})$ depends up to canonical gapped filtered A_∞ isomorphism only on (M,ω), $\iota : L \to M$ and its relative spin structure, and the indices $\eta_{(p_-,p_+)}$, and is otherwise independent of $J, \mathcal{G}, \lambda_{(p_-,p_+)}, o_{(p_-,p_+)}$ and other choices. That is, if $(\mathcal{H}\otimes\Lambda^0_{\mathrm{nov}}, \mathfrak{n})$, $(\mathcal{H}\otimes\Lambda^0_{\mathrm{nov}}, \tilde{\mathfrak{n}})$ are two possible outcomes, we can construct a gapped filtered A_∞ isomorphism $\mathfrak{j} : (\mathcal{H}\otimes\Lambda^0_{\mathrm{nov}}, \mathfrak{n}) \to (\mathcal{H}\otimes\Lambda^0_{\mathrm{nov}}, \tilde{\mathfrak{n}})$, and $\mathfrak{j}$ is unique up to homotopy. Furthermore, $\mathfrak{j}^{0,0}_1 : \mathcal{H} \to \mathcal{H}$ is the identity on $H_*(L;\mathbb{Q})$, and ± 1 on each (p_-,p_+) in R.

This is similar to Fukaya et al. [**8**, Th. A, §1.2].

12. Calabi–Yau manifolds and graded Lagrangian submanifolds

We now explain how the material of §4–§11 simplifies when (M,ω) is *Calabi–Yau*, and the Lagrangian L is *graded*. Floer cohomology of graded Lagrangian submanifolds in Calabi–Yau manifolds is important because of its rôle in the Homological Mirror Symmetry Conjecture of Kontsevich [**14**]. For background on Calabi–Yau manifolds, special Lagrangian submanifolds, and Mirror Symmetry see Joyce [**11**], and for graded Lagrangian submanifolds and Floer cohomology see Seidel [**19**] and Fukaya [**7**, Def. 2.9]. The framework we give can be generalized in various ways; see Joyce [**11**, §8.4] on *almost Calabi–Yau manifolds*, and Seidel [**19**] for a more general notion of grading, expressed in terms of covering spaces of bundles of Lagrangian Grassmannians.

Definition 12.1. A *Calabi–Yau n-fold* is a quadruple (M,J,ω,Ω) where (M,J) is a compact n-dimensional complex manifold, ω is the Kähler form of a Kähler metric g on M, and Ω is a non-vanishing holomorphic $(n,0)$-form on M satisfying

$$\omega^n/n! = (-1)^{n(n-1)/2}(i/2)^n\Omega\wedge\bar{\Omega}. \tag{139}$$

This implies that g is Ricci-flat with holonomy group contained in $\mathrm{SU}(n)$. Note that (M,ω) is a compact symplectic manifold, and J is an (almost) complex structure on M compatible with ω.

If (M,J) is a compact complex manifold with trivial canonical bundle K_M, then by Yau's proof of the Calabi Conjecture, every Kähler class on M contains a unique Ricci-flat Kähler metric g, with Kähler form ω. There exists Ω, unique up to phase change $\Omega \mapsto \mathrm{e}^{i\theta}\Omega$, such that (M,J,ω,Ω) is Calabi–Yau. One can construct many examples of such (M,J) using complex algebraic geometry.

Now let $\iota : L \to M$ be an oriented, immersed Lagrangian. Then $\iota^*(\Omega)$ is a complex n-form on L, and the normalization (139) implies that $|\iota^*(\Omega)| \equiv 1$, where $|\,.\,|$ is computed using $\iota^*(g)$. Hence $\iota^*(\Omega) = u\,\mathrm{vol}_L$ for some smooth $u : L \to \mathrm{U}(1)$, where vol_L is the volume form on L defined using $\iota^*(g)$ and the orientation. We call L *special Lagrangian with phase* $\mathrm{e}^{i\theta}$ for $\theta \in [0,2\pi)$ if $u \equiv \mathrm{e}^{i\theta}$.

A *grading* on L is a choice of smooth function $\phi : L \to \mathbb{R}$ such that $u \equiv \mathrm{e}^{i\phi}$. We call (L,ϕ) a *graded Lagrangian submanifold*. If a grading exists it is unique up to $\phi \mapsto \phi + 2\pi k$ for $k \in \mathbb{Z}$, provided L is connected. Special Lagrangian submanifolds with phase $\mathrm{e}^{i\theta}$ are automatically graded, with $\phi \equiv \theta$ constant. Let $\alpha \in H^1(\mathrm{U}(1);\mathbb{Z})$ be the generator with $\int_{\mathrm{U}(1)} \alpha = 1$. Then $u^*(\alpha) \in H^1(L;\mathbb{Z})$ is called the *Maslov class*, and L admits a grading if and only if $u^*(L) = 0$ in $H^1(L;\mathbb{Z})$, that is, if and only if L is *Maslov zero*.

Suppose that (M,J,ω,Ω) is Calabi–Yau and (L,ϕ) is an embedded graded Lagrangian in M. Then the *Maslov index* $\mu_L(\beta)$ of Definition 4.5 is zero for all $\beta \in H_2(M,L;\mathbb{Z})$. This is because $\mu_L(\beta) = \beta \cdot c_1\bigl(M,\iota(L)\bigr)$, where $c_1\bigl(M,\iota(L)\bigr) \in H^2\bigl(M,\iota(L);\mathbb{Z}\bigr)$ is the *relative first Chern class* for ω on (M,L), and the Calabi–Yau and graded conditions imply that $c_1\bigl(M,\iota(L)\bigr) = 0$.

To extend this to immersed graded Lagrangians, we require the paths $\lambda_{(p_-,p_+)}$ in Definition 4.4 to lift to paths $(\lambda_{(p_-,p_+)}, \psi_{(p_-,p_+)})$ in *graded* Lagrangian subspaces of T_pM. That is, $\lambda_{(p_-,p_+)} = \{\lambda_{(p_-,p_+)}(x,y)\}_{(x,y)\in\partial Y}$ is a smooth family of oriented Lagrangian subspaces of T_pM, where $p = \iota(p_-) = \iota(p_+)$, and $\psi_{(p_-,p_+)} : \partial Y \to \mathbb{R}$ is a smooth map, such that $\Omega_p|_{\lambda_{(p_-,p_+)}(x,y)} = \mathrm{e}^{i\psi_{(p_-,p_+)}(x,y)}\,\mathrm{vol}_{\lambda_{(p_-,p_+)}(x,y)}$ for all $(x,y) \in \partial Y$, and

$$\lambda_{(p_-,p_+)}(x,y) = \begin{cases} \mathrm{d}\iota(T_{p_-}L), & y = 1, \\ \mathrm{d}\iota(T_{p_+}L), & y = -1, \end{cases} \quad \psi_{(p_-,p_+)}(x,y) = \begin{cases} \phi(p_-), & y = 1, \\ \phi(p_+), & y = -1. \end{cases}$$

Then the same argument ensures $\mu_L(\beta) = 0$ for all $\beta \in H_2\bigl(M,\iota(L);\mathbb{Z}\bigr)$.

Requiring the $\lambda_{(p_-,p_+)}$ to lift to paths $(\lambda_{(p_-,p_+)}, \psi_{(p_-,p_+)})$ in graded Lagrangians determines the index $\eta_{(p_-,p_+)}$ in (31) uniquely, independently of choices in $(\lambda_{(p_-,p_+)}, \psi_{(p_-,p_+)})$. Calculation shows that we can give a simple local formula for $\eta_{(p_-,p_+)}$.

Proposition 12.2. *Let (M,J,ω,Ω) be a Calabi–Yau n-fold, and $(\iota : L \to M, \phi)$ be an immersed graded Lagrangian submanifold with only* *transverse double self-intersections. Suppose $p_-, p_+ \in L$ with $p_- \neq p_+$ and $\iota(p_-) = \iota(p_+) = p$. Then for any choice of path $(\lambda_{(p_-,p_+)}, \psi_{(p_-,p_+)})$ in graded Lagrangian subspaces of T_pM as above, the index $\eta_{(p_-,p_+)}$ in Definition 4.4 may be computed as follows. One can choose holomorphic coordinates $(z^1, \dots, z^n)$ near p in M in which*

$$\begin{aligned} &\omega|_p = \tfrac{i}{2}(\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 + \cdots + \mathrm{d}z^n \wedge \mathrm{d}\bar{z}^n), \;\; \Omega|_p = \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n, \\ &\mathrm{d}\iota(T_{p_-}L) = \big\{(\mathrm{e}^{i\phi_-^1}x^1, \dots, \mathrm{e}^{i\phi_-^n}x^n) : x^1, \dots, x^n \in \mathbb{R}\big\}, \quad \textit{and} \\ &\mathrm{d}\iota(T_{p_+}L) = \big\{(\mathrm{e}^{i\phi_+^1}x^1, \dots, \mathrm{e}^{i\phi_+^n}x^n) : x^1, \dots, x^n \in \mathbb{R}\big\}, \end{aligned} \tag{140}$$

where $\phi_\pm^1, \dots, \phi_\pm^n \in \mathbb{R}$ satisfy $\phi_\pm^1 + \cdots + \phi_\pm^n = \phi(p_\pm)$ and $\phi_+^j - \phi_-^j \notin \pi\mathbb{Z}$ for $j = 1, \dots, n$. For $x \in \mathbb{R}$, write $[x]$ for the greatest integer m with $m \leqslant x$. Then

$$\eta_{(p_-,p_+)} = n + \sum\nolimits_{j=1}^n \Big[\frac{\phi_+^j - \phi_-^j}{\pi}\Big]. \tag{141}$$

Since $\phi_+^j - \phi_-^j \notin \pi\mathbb{Z}$, we have $\big[\frac{\phi_+^j - \phi_-^j}{\pi}\big] + \big[\frac{\phi_-^j - \phi_+^j}{\pi}\big] = -1$ for $j = 1, \dots, n$. Thus exchanging p_-, p_+ and ϕ_-^j, ϕ_+^j we see from (141) that $\eta_{(p_-,p_+)} + \eta_{(p_+,p_-)} = n$, as in (32). Recall that in §4.6 we assumed that $\eta_{(p_-,p_+)} \geqslant 0$ for all $(p_-, p_+) \in R$. This is *not compatible* with requiring $\lambda_{(p_-,p_+)}$ to lift to graded Lagrangians, since then $\eta_{(p_-,p_+)}$ is determined by (141), and need not satisfy $\eta_{(p_-,p_+)} \geqslant 0$.

In fact we only used $\eta_{(p_-,p_+)} \geqslant 0$ to define the modified moduli spaces $\widetilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha, \beta, J), \widetilde{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha, \beta, J, f_1, \dots, f_k)$, which were only for motivation, and in the orientation calculations of §5. But as we explained in §5.4, changing the $\eta_{(p_-,p_+)}$ does not affect any of the signs in §5, as the $\eta_{(p_-,p_+)}$ change by even numbers, and Proposition 5.15 explains how changing $\lambda_{(p_-,p_+)}, \eta_{(p_-,p_+)}, o_{(p_-,p_+)}$ affects the orientations on $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha, \beta, J, f_1, \dots, f_k)$. Using this, we can define the orientations on $\overline{\mathcal{M}}_{k+1}^{\mathrm{main}}(\alpha, \beta, J, f_1, \dots, f_k)$ using choices $\tilde{\lambda}_{(p_-,p_+)}$ inducing indices $\tilde{\eta}_{(p_-,p_+)} \geqslant 0$, and then replace $\tilde{\eta}_{(p_-,p_+)}$ by $\eta_{(p_-,p_+)}$ in (141) defined using graded paths $(\lambda_{(p_-,p_+)}, \psi_{(p_-,p_+)})$, and the results of §5 such as Theorem 5.13 will still be valid.

To summarize our discussion so far: when (M, J, ω, Ω) is Calabi–Yau and $(\iota : L \to M, \phi)$ is an immersed graded Lagrangian with only transverse double self-intersections, by using graded paths $(\lambda_{(p_-,p_+)}, \psi_{(p_-,p_+)})$ in §4.3 the indices $\eta_{(p_-,p_+)}$ are uniquely determined by (141), for all $\beta \in H_2\big(M, \iota(L); \mathbb{Z}\big)$ the Maslov index $\mu_L(\beta)$ is zero, and the orientation results of §5 still hold.

We can now go through the whole of §6–§11 working over the Calabi–Yau Novikov ring Λ_{CY}^0 of §3.4, rather than over Λ_{nov}^0. The point is that terms $T^\lambda e^\mu$ in Λ_{nov}^0 are to keep track of holomorphic discs with area λ and Maslov index 2μ. But for graded Lagrangians all Maslov indices are zero, so we can work just with terms T^λ in $\Lambda^0_{\rm CY}$. Thus we prove analogues of Theorem 11.2 and Corollary 11.4:

Theorem 12.3. *Let (M,J,ω,Ω) be a Calabi–Yau n-fold and $(\iota:L\rightarrow M,\phi)$ a compact, immersed, graded Lagrangian with only transverse double self-intersections. Choose a relative spin structure for $\iota:L\rightarrow M$. Then:*

- **(a)** *By an analogue of Definition* 11.1*, we can construct a gapped filtered A_∞ algebra $(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\rm CY},\mathfrak{m})$, which depends up to canonical homotopy equivalence only on (M,ω), $\iota:L\rightarrow M$, and its relative spin structure.*

  That is, if $(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\rm CY},\mathfrak{m})$ and $(\mathbb{Q}\tilde{\mathcal{X}}\hat\otimes\Lambda^0_{\rm CY},\tilde{\mathfrak{m}})$ are outcomes depending on $J,\mathcal{G},\lambda_{(p_-,p_+)},\ldots$ and $\tilde J,\tilde{\mathcal{G}},\tilde\lambda_{(p_-,p_+)},\ldots$, we can construct a gapped filtered A_∞ morphism $\mathfrak{j}:(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\rm CY},\mathfrak{m})\rightarrow(\mathbb{Q}\tilde{\mathcal{X}}\hat\otimes\Lambda^0_{\rm CY},\tilde{\mathfrak{m}})$ which is a homotopy equivalence. If $\mathfrak{j},\mathfrak{j}'$ are possibilities for $\mathfrak{j}$ there is a homotopy $\mathfrak{H}:\mathfrak{j}\Rightarrow\mathfrak{j}'$.
- **(b)** *If $(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\rm CY},\mathfrak{m}),(\mathbb{Q}\tilde{\mathcal{X}}\hat\otimes\Lambda^0_{\rm CY},\tilde{\mathfrak{m}}),(\mathbb{Q}\check{\mathcal{X}}\hat\otimes\Lambda^0_{\rm CY},\check{\mathfrak{m}})$ and $\mathfrak{j}:(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\rm CY},\mathfrak{m})\rightarrow(\mathbb{Q}\tilde{\mathcal{X}}\hat\otimes\Lambda^0_{\rm CY},\tilde{\mathfrak{m}})$, $\mathfrak{j}':(\mathbb{Q}\tilde{\mathcal{X}}\hat\otimes\Lambda^0_{\rm CY},\tilde{\mathfrak{m}})\rightarrow(\mathbb{Q}\check{\mathcal{X}}\hat\otimes\Lambda^0_{\rm CY},\check{\mathfrak{m}})$, $\mathfrak{j}'':(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\rm CY},\mathfrak{m})\rightarrow(\mathbb{Q}\check{\mathcal{X}}\hat\otimes\Lambda^0_{\rm CY},\check{\mathfrak{m}})$ are as in* (a)*, there is a homotopy $\mathfrak{H}:\mathfrak{j}''\Rightarrow\mathfrak{j}'\circ\mathfrak{j}$.*
- **(c)** *The gapped filtered A_∞ algebra $(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\rm CY},\mathfrak{m})$ in* (a) *has a minimal model $(\mathcal{H}\otimes\Lambda^0_{\rm CY},\mathfrak{n})$, with $\mathcal{H}=\bigoplus_{d\in\mathbb{Z}}\mathcal{H}^d$ given by* (138)*.*

13. Bounding cochains and Lagrangian Floer cohomology

Finally we apply our results to define bounding cochains and Lagrangian Floer cohomology for immersed Lagrangians. We do this for one and two Lagrangians over $\Lambda^0_{\rm nov},\Lambda_{\rm nov}$ in §13.1–§13.2, and for graded Lagrangians in Calabi–Yau manifolds over $\Lambda^0_{\rm CY},\Lambda_{\rm CY}$ in §13.3. Sections 13.4–13.5 suggest some questions and conjectures for future research, concerning the invariance of Floer cohomology under *local Hamiltonian equivalence* of immersed Lagrangians, and on whether there exists a theory of *Legendrian Floer cohomology* for embedded Legendrians in contact manifolds which are U(1)-bundles over symplectic manifolds, that is invariant under embedded Legendrian isotopy.

13.1. Bounding cochains, and the Floer cohomology of one Lagrangian. As in §3.6, given a gapped filtered A_∞ algebra $(A\hat\otimes\Lambda^0_{\rm nov},\mathfrak{m})$, we can define *bounding cochains* b for $(A\hat\otimes\Lambda^0_{\rm nov},\mathfrak{m})$, and form *cohomology groups* $H^*(A\hat\otimes\Lambda^0_{\rm nov},\mathfrak{m}^b_1)$ and $H^*(A\hat\otimes\Lambda_{\rm nov},\mathfrak{m}^b_1)$ over $\Lambda^0_{\rm nov},\Lambda_{\rm nov}$. We can apply these ideas either to $(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\rm nov},\mathfrak{m})$ in Definition 11.1, or to its canonical model $(\mathcal{H}\otimes\Lambda^0_{\rm nov},\mathfrak{n})$ in Corollary 11.4. The results will be the same in both cases, since up to equivalence, bounding cochains and cohomology depend only on the homotopy type of the gapped filtered

A_∞ algebra. We choose to work with $(\mathcal{H}\otimes\Lambda^0_{\rm nov},\mathfrak{n})$, as the geometric interpretation is clearer, and the notion of equivalence of bounding cochains is better behaved.

Definition 13.1. Let $(\mathcal{H}\otimes\Lambda^0_{\rm nov},\mathfrak{n})$ be a gapped filtered A_∞ algebra in Corollary 11.4, constructed from (M,ω) and $\iota:L\to M$. As in Definition 3.19, a *bounding cochain* b for $(\mathcal{H}\otimes\Lambda^0_{\rm nov},\mathfrak{n})$ is $b\in F^\lambda(\mathcal{H}\otimes\Lambda^0_{\rm nov})^{(0)}$ for some $\lambda>0$, satisfying $\sum_{k\geqslant 0}\mathfrak{n}_k(b,\ldots,b)=0$. Fix some bounding cochain b for $(\mathcal{H}\otimes\Lambda^0_{\rm nov},\mathfrak{n})$.

We shall define Lagrangian Floer cohomology over both Novikov rings $\Lambda^0_{\rm nov}$ and $\Lambda_{\rm nov}$. For brevity we will use $\Lambda^*_{\rm nov}$ to mean either $\Lambda^0_{\rm nov}$ or $\Lambda_{\rm nov}$, the same for each occurrence. Define graded $\Lambda^*_{\rm nov}$-multilinear maps $\mathfrak{n}^b_k:(\mathcal{H}\otimes\Lambda^*_{\rm nov})^{\times^k}\to\mathcal{H}\otimes\Lambda^*_{\rm nov}$ for $k=0,1,2,\ldots$, of degree $+1$, by

$$
\begin{aligned}
\mathfrak{n}^b_k(a_1,\ldots,a_k)=\sum_{n_0,\ldots,n_k\geqslant 0}\mathfrak{n}_{k+n_0+\cdots+n_k}\bigl(&\overbrace{b,\ldots,b}^{n_0},a_1,\overbrace{b,\ldots,b}^{n_1},a_2,\\
&\overbrace{b,\ldots,b}^{n_2},\ldots,\overbrace{b,\ldots,b}^{n_{k-1}},a_k,\overbrace{b,\ldots,b}^{n_k}\bigr).
\end{aligned}
\tag{142}
$$

Then the $\mathfrak{n}^b_k$ satisfy the A_∞ relations (8), and $\mathfrak{n}^b_0=0$ as b is a bounding cochain, so for pure $a_1,a_2,a_3\in\mathcal{H}\otimes\Lambda^*_{\rm nov}$ we have

$$
\begin{aligned}
&(\mathfrak{n}^b_1)^2=0,\\
&\mathfrak{n}^b_2\bigl(\mathfrak{n}^b_1(a_1),a_2\bigr)+(-1)^{\deg a_1}\mathfrak{n}^b_2\bigl(a_1,\mathfrak{n}^b_1(a_2)\bigr)+\mathfrak{n}^b_1\circ\mathfrak{n}^b_2(a_1,a_2)=0,\\
&\mathfrak{n}^b_3\bigl(\mathfrak{n}^b_1(a_1),a_2,a_3\bigr)+(-1)^{\deg a_1}\mathfrak{n}^b_3\bigl(a_1,\mathfrak{n}^b_1(a_2),a_3\bigr)+\\
&(-1)^{\deg a_1+\deg a_2}\mathfrak{n}^b_3\bigl(a_1,a_2,\mathfrak{n}^b_1(a_3)\bigr)+\mathfrak{n}^b_2\bigl(\mathfrak{n}^b_2(a_1,a_2),a_3\bigr)+\\
&(-1)^{\deg a_1}\mathfrak{n}^b_2\bigl(a_1,\mathfrak{n}^b_2(a_2,a_3)\bigr)+\mathfrak{n}^b_1\circ\mathfrak{n}^b_3(a_1,a_2,a_3)=0.
\end{aligned}
\tag{143}
$$

The first equation of (143) implies that $(\mathcal{H}\otimes\Lambda^*_{\rm nov},\mathfrak{n}^b_1)$ is a complex. Define the *Lagrangian Floer cohomology groups* $HF^*\bigl((L,b);\Lambda^0_{\rm nov}\bigr)$ and $HF^*\bigl((L,b);\Lambda_{\rm nov}\bigr)$ by

$$
HF^k\bigl((L,b);\Lambda^*_{\rm nov}\bigr)=H^{k-1}\bigl(\mathcal{H}\otimes\Lambda^*_{\rm nov},\mathfrak{n}^b_1\bigr).
\tag{144}
$$

The grading is motivated by (138) and $H_k(L;\mathbb{Q})\cong H^{n-k}(L;\mathbb{Q})$ as L is oriented of dimension n, and implies $HF^k\bigl((L,b);\Lambda^*_{\rm nov}\bigr)$ is a modified version of ordinary cohomology $H^k(L;\Lambda^*_{\rm nov})$. Define a $\Lambda^*_{\rm nov}$-bilinear product $\bullet:HF^k\bigl((L,b);\Lambda^*_{\rm nov}\bigr)\times HF^l\bigl((L,b);\Lambda^*_{\rm nov}\bigr)\to HF^{k+l}\bigl((L,b);\Lambda^*_{\rm nov}\bigr)$ by

$$
\bigl(a_1+\operatorname{Im}\mathfrak{n}^b_1\bigr)\bullet\bigl(a_2+\operatorname{Im}\mathfrak{n}^b_1\bigr)=(-1)^{k(l+1)}\mathfrak{n}^b_2(a_1,a_2)+\operatorname{Im}\mathfrak{n}^b_1.
\tag{145}
$$

Here since $\mathfrak{n}^b_1(a_1)=\mathfrak{n}^b_1(a_2)=0$, the second equation of (143) implies that $\mathfrak{n}^b_1\bigl(\mathfrak{n}^b_2(a_1,a_2)\bigr)=0$, so the right hand side of (145) does lie in $HF^{k+l}\bigl((L,b);\Lambda^*_{\rm nov}\bigr)$. Using the second equation of (143) we see that replacing $a_1\mapsto a_1+\mathfrak{n}^b_1(c_1)$ changes $\mathfrak{n}^b_2(a_1,a_2)\mapsto\mathfrak{n}^b_2(a_1,a_2)-\mathfrak{n}^b_1\bigl(\mathfrak{n}^b_2(c_1,a_2)\bigr)$. So the right hand side of (145) is independent of the choice of representative a_1 for $a_1+\operatorname{Im}\mathfrak{n}^b_1$, and similarly for a_2. Thus $\bullet$ is well-defined.

Using the third equation of (143) we can show that $\bullet$ is *associative*. It is a modified version of the cup product on $H^*(L;\Lambda^*_{\rm nov})$.

One can also construct a *unit* for $\bigl(HF^*((L,b);\Lambda^*_{\rm nov}),\bullet\bigr)$, making it into a $\Lambda^*_{\rm nov}$-algebra. There is a complicated procedure for doing this in Fukaya et al. [**8**, §3.3], involving first finding a homotopy unit for $(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\rm nov},\mathfrak{m})$ in Definition 11.1. We will not explain it, as the immersed case introduces no new issues.

Remark 13.2. Although $HF^*\bigl((L,b);\Lambda^*_{\rm nov}\bigr)$ is graded by $k\in\mathbb{Z}$, multiplication by $e^d\in\Lambda^*_{\rm nov}$ induces an isomorphism $HF^k\bigl((L,b);\Lambda^*_{\rm nov}\bigr)\rightarrow HF^{k+2d}\bigl((L,b);\Lambda^*_{\rm nov}\bigr)$ for $d\in\mathbb{Z}$. So there are really only two groups $HF^0\bigl((L,b);\Lambda^*_{\rm nov}\bigr),HF^1\bigl((L,b);\Lambda^*_{\rm nov}\bigr)$, and it would be better to regard $HF^*\bigl((L,b);\Lambda^*_{\rm nov}\bigr)$ as *graded over* $\mathbb{Z}_2$, rather than over $\mathbb{Z}$.

We could rewrite most of the paper using $\mathbb{Z}_2$-graded spaces rather than $\mathbb{Z}$-graded spaces, and this would achieve some simplifications. In §3 we would work with $\mathbb{Z}_2$-graded vector spaces $A=A^0\oplus A^1$ rather than $A=\bigoplus_{d\in\mathbb{Z}}A^d$, and we would replace $\Lambda_{\rm nov},\Lambda^0_{\rm nov}$ by $\Lambda_{\rm CY},\Lambda^0_{\rm CY}$ throughout. For computing orientations and degrees, we would regard $\eta_{(p_-,p_+)},\deg f$ as lying in $\mathbb{Z}_2$ rather than $\mathbb{Z}$. Then $\eta_{(p_-,p_+)}\in\mathbb{Z}_2$ becomes independent of choice of $\lambda_{(p_-,p_+)}$, and the problem in Remark 11.3 disappears. We have not done this to keep our paper consistent with Fukaya et al. [**8**].

In §13.3 we will see that for graded Lagrangians (L,ϕ) in Calabi–Yau manifolds, Floer cohomology $HF^*\bigl((L,\phi,b);\Lambda^*_{\rm CY}\bigr)$ is truly $\mathbb{Z}$-graded rather than $\mathbb{Z}_2$-graded.

Next we explain in which sense Floer cohomology is independent of choices.

Definition 13.3. Let $(\mathcal{H}\otimes\Lambda^0_{\rm nov},\mathfrak{n})$ be as in Corollary 11.4. Write $\widehat{\mathcal{M}}_{\mathcal{H},\mathfrak{n}}$ for the set of bounding cochains b for $(\mathcal{H}\otimes\Lambda^0_{\rm nov},\mathfrak{n})$. Define $G_{\mathcal{H},\mathfrak{n}}$ to be the group of gapped filtered A_∞ isomorphisms $\mathfrak{j}:(\mathcal{H}\otimes\Lambda^0_{\rm nov},\mathfrak{n})\rightarrow(\mathcal{H}\otimes\Lambda^0_{\rm nov},\mathfrak{n})$ which are homotopic to the identity. We call $G_{\mathcal{H},\mathfrak{n}}$ the *gauge group*. For $\mathfrak{j}\in G_{\mathcal{H},\mathfrak{n}}$ and $b\in\widehat{\mathcal{M}}_{\mathcal{H},\mathfrak{n}}$, define $\mathfrak{j}\cdot b\in(\mathcal{H}\otimes\Lambda^0_{\rm nov})^{(0)}$ by $\mathfrak{j}\cdot b=\sum_{k\geqslant 0}\mathfrak{j}_k(b,\ldots,b)$. By summing (18) with $\mathfrak{j},\mathfrak{n}$ in place of $\mathfrak{f},\mathfrak{m}$ and $a_1=\cdots=a_k=b$ over all $k=0,1,\ldots$, we find that

$$\sum_{k=0}^{\infty}\mathfrak{n}_k(\mathfrak{j}\cdot b,\ldots,\mathfrak{j}\cdot b)=\sum_{l,m=0}^{\infty}\mathfrak{j}_{l+m+1}\Bigl(\overbrace{b,\ldots,b}^{l},\sum_{k=0}^{\infty}\mathfrak{n}_k(b,\ldots,b),\overbrace{b,\ldots,b}^{m}\Bigr)=0,$$

as b is a bounding cochain. Thus $\mathfrak{j}\cdot b$ is a bounding cochain, so $\mathfrak{j}\cdot b\in\widehat{\mathcal{M}}_{\mathcal{H},\mathfrak{n}}$, and this defines an action of $G_{\mathcal{H},\mathfrak{n}}$ on $\widehat{\mathcal{M}}_{\mathcal{H},\mathfrak{n}}$. Define the *moduli space of bounding cochains* to be $\mathcal{M}_{\mathcal{H},\mathfrak{n}}=\widehat{\mathcal{M}}_{\mathcal{H},\mathfrak{n}}/G_{\mathcal{H},\mathfrak{n}}$.

For $\mathfrak{j},b$ as above, define linear $\mathfrak{j}^b_1:\mathcal{H}\otimes\Lambda^*_{\rm nov}\rightarrow\mathcal{H}\otimes\Lambda^*_{\rm nov}$ by

$$\mathfrak{j}^b_1(a)=\textstyle\sum_{l,m=0}^{\infty}\mathfrak{j}_{l+m+1}\bigl(\overbrace{b,\ldots,b}^{l},a,\overbrace{b,\ldots,b}^{m}\bigr).\tag{146}$$

Now $\mathfrak{j}$ has an inverse $\mathfrak{j}^{-1}$ in $G_{\mathcal{H},\mathfrak{n}}$, and calculation shows that $(\mathfrak{j}^{-1})^b_1\circ\mathfrak{j}^b_1=\mathrm{id}$, so $\mathfrak{j}^b_1$ is an isomorphism. By summing (18) with $\mathfrak{j},\mathfrak{n}$ in place of $\mathfrak{f},\mathfrak{m}$,

$k = l + m + 1$ and $a_j = a$ for $j = l + 1$ and $a_j = b$ otherwise over all $l, m \geqslant 0$ we find that $\mathfrak{j}_1^b \circ \mathfrak{n}_1^b = \mathfrak{n}_1^{\mathfrak{j}\cdot b} \circ \mathfrak{j}_1^b$. Thus $\mathfrak{j}_1^b : (\mathcal{H} \otimes \Lambda_{\rm nov}^*, \mathfrak{n}_1^b) \to (\mathcal{H} \otimes \Lambda_{\rm nov}^*, \mathfrak{n}_1^{\mathfrak{j}\cdot b})$ is an isomorphism of complexes, and induces an isomorphism $(\mathfrak{j}_1^b)_* : HF^*\big((L, b); \Lambda_{\rm nov}^*\big) \to HF^*\big((L, \mathfrak{j} \cdot b); \Lambda_{\rm nov}^*\big)$. As $\mathfrak{j}$ is homotopic to the identity, this $(\mathfrak{j}_1^b)_*$ is independent of the choice of $\mathfrak{j}$ for fixed $b, \mathfrak{j} \cdot b$. Thus, Floer cohomology $HF^*\big((L, b); \Lambda_{\rm nov}^*\big)$ depends up to canonical isomorphism only on $G_{\mathcal{H},\mathfrak{n}} \cdot b \in \mathcal{M}_{\mathcal{H},\mathfrak{n}}$, rather than on $b \in \widehat{\mathcal{M}}_{\mathcal{H},\mathfrak{n}}$.

Now let $(\mathcal{H} \otimes \Lambda_{\rm nov}^0, \mathfrak{n})$ and $(\mathcal{H} \otimes \Lambda_{\rm nov}^0, \tilde{\mathfrak{n}})$ be two possible outcomes in Corollary 11.4. Then the corollary gives a gapped filtered A_∞ isomorphism $\mathfrak{j} : (\mathcal{H} \otimes \Lambda_{\rm nov}^0, \mathfrak{n}) \to (\mathcal{H} \otimes \Lambda_{\rm nov}^0, \tilde{\mathfrak{n}})$, unique up to homotopy. For $b \in \widehat{\mathcal{M}}_{\mathcal{H},\mathfrak{n}}$, define $\mathfrak{j} \cdot b$ as above. Then the same proof shows that $\mathfrak{j} \cdot b$ is a bounding cochain for $(\mathcal{H} \otimes \Lambda_{\rm nov}^0, \tilde{\mathfrak{n}})$. This defines a map $\mathfrak{j}\cdot : \widehat{\mathcal{M}}_{\mathcal{H},\mathfrak{n}} \to \widehat{\mathcal{M}}_{\mathcal{H},\tilde{\mathfrak{n}}}$. It is a 1-1 correspondence, with inverse $(\mathfrak{j}^{-1})\cdot$, and it intertwines the actions of $G_{\mathcal{H},\mathfrak{n}}, G_{\mathcal{H},\tilde{\mathfrak{n}}}$ on $\widehat{\mathcal{M}}_{\mathcal{H},\mathfrak{n}}, \widehat{\mathcal{M}}_{\mathcal{H},\tilde{\mathfrak{n}}}$, and thus induces a 1-1 correspondence $\mathfrak{j}_* : \mathcal{M}_{\mathcal{H},\mathfrak{n}} \to \mathcal{M}_{\mathcal{H},\tilde{\mathfrak{n}}}$.

As $\mathfrak{j}$ is unique up to homotopy, this $\mathfrak{j}_*$ is independent of the choice of $\mathfrak{j}$, for fixed $\mathfrak{n}, \tilde{\mathfrak{n}}$. Defining $\mathfrak{j}_1^b$ as in (146), the same proofs show $\mathfrak{j}_1^b : \big(\mathcal{H}\otimes\Lambda_{\rm nov}^*, \mathfrak{n}_1^b\big) \to \big(\mathcal{H}\otimes\Lambda_{\rm nov}^*, \tilde{\mathfrak{n}}_1^{\mathfrak{j}\cdot b}\big)$ is an isomorphism of complexes, and induces an isomorphism $(\mathfrak{j}_1^b)_* : HF^*\big((L, b); \Lambda_{\rm nov}^*\big) \to HF^*\big((L, \mathfrak{j} \cdot b); \Lambda_{\rm nov}^*\big)$, which is independent of the choice of $\mathfrak{j}$ for fixed $\mathfrak{n}, \tilde{\mathfrak{n}}, b, \mathfrak{j} \cdot b$. We can also use Theorem 11.2(b) to check that, given three choices $\mathfrak{n}, \tilde{\mathfrak{n}}, \tilde{\tilde{\mathfrak{n}}}$, the corresponding isomorphisms $(\mathfrak{j}_1^b)_*$ form commutative triangles.

This implies that the moduli space of bounding cochains $\mathcal{M}_{\mathcal{H},\mathfrak{n}}$ is *independent of choice of* $\mathfrak{n}$ *up to canonical bijection*, and that under these bijections, Lagrangian Floer cohomology $HF^*\big((L, b); \Lambda_{\rm nov}^*\big)$, regarded as depending on $G_{\mathcal{H},\mathfrak{n}} \cdot b \in \mathcal{M}_{\mathcal{H},\mathfrak{n}}$, is also *independent of the choice of* $\mathfrak{n}$ *up to canonical isomorphism.* So by Corollary 11.4, in this sense, the moduli space $\mathcal{M}_{\mathcal{H},\mathfrak{n}}$ and associated Floer cohomology groups $HF^*\big((L, b); \Lambda_{\rm nov}^*\big)$ *depend only on* (M, ω), $\iota : L \to M$ *and its relative spin structure, and the indices* $\eta_{(p_-,p_+)}$, *and are independent of all other choices.*

In Remark 11.3 we showed that if $(\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda_{\rm nov}^0, \mathfrak{m}), (\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda_{\rm nov}^0, \tilde{\mathfrak{m}})$ are constructed in Definition 11.1 using different indices $\eta_{(p_-,p_+)}, \tilde{\eta}_{(p_-,p_+)}$, but otherwise exactly the same choices, then we can construct $\boldsymbol{\Xi} : (\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda_{\rm nov}^0, \mathfrak{m}) \to (\mathbb{Q}\mathcal{X}\hat{\otimes}\Lambda_{\rm nov}^0, \tilde{\mathfrak{m}})$ which is *almost* a strict gapped filtered A_∞ isomorphism. In the same way, if $(\mathcal{H} \otimes \Lambda_{\rm nov}^0, \mathfrak{n}), (\tilde{\mathcal{H}} \otimes \Lambda_{\rm nov}^0, \tilde{\mathfrak{n}})$ are constructed in Corollary 11.4 using different choices of $\eta_{(p_-,p_+)}, \tilde{\eta}_{(p_-,p_+)}$, but otherwise exactly the same choices, then we can construct $\boldsymbol{\Xi} : (\mathcal{H} \otimes \Lambda_{\rm nov}^0, \mathfrak{n}) \to (\tilde{\mathcal{H}} \otimes \Lambda_{\rm nov}^0, \tilde{\mathfrak{n}})$, which is almost a strict gapped filtered A_∞ isomorphism, but does not satisfy all of Definition 3.14(i).

Then $\boldsymbol{\Xi}_1 : \mathcal{H} \otimes \Lambda_{\rm nov}^0 \to \tilde{\mathcal{H}} \otimes \Lambda_{\rm nov}^0$ takes bounding cochains to bounding cochains, so $\boldsymbol{\Xi}_1 : \widehat{\mathcal{M}}_{\mathcal{H},\mathfrak{n}} \to \widehat{\mathcal{M}}_{\tilde{\mathcal{H}},\tilde{\mathfrak{n}}}$ is a bijection which induces a bijection $(\boldsymbol{\Xi}_1)_* : \mathcal{M}_{\mathcal{H},\mathfrak{n}} \to \mathcal{M}_{\tilde{\mathcal{H}},\tilde{\mathfrak{n}}}$. If $b \in \widehat{\mathcal{M}}_{\mathcal{H},\mathfrak{n}}$, so that $\boldsymbol{\Xi}_1(b) \in \mathcal{M}_{\tilde{\mathcal{H}},\tilde{\mathfrak{n}}}$, then $\boldsymbol{\Xi}_1 : \big(\mathcal{H}\otimes\Lambda^*_{\rm nov},\mathfrak{n}_1^b\big)\rightarrow\big(\tilde{\mathcal{H}}\otimes\Lambda^*_{\rm nov},\tilde{\mathfrak{n}}_1^{\boldsymbol{\Xi}_1(b)}\big)$ is an isomorphism of complexes, and induces an isomorphism $(\boldsymbol{\Xi}_1)_* : HF^*\big((L,b);\Lambda^*_{\rm nov}\big)\rightarrow HF^*\big((L,\boldsymbol{\Xi}_1(b));\Lambda^*_{\rm nov}\big)$. Thus, in the same sense as above, $\mathcal{M}_{\mathcal{H},\mathfrak{n}}$ and $HF^*\big((L,b);\Lambda^*_{\rm nov}\big)$ are also *independent of the choice of indices* $\eta_{(p_-,p_+)}$.

We state our conclusions as:

Theorem 13.4. *In Definitions* 13.1 *and* 13.3, *the moduli space of bounding cochains $\mathcal{M}_{\mathcal{H},\mathfrak{n}}$ depends up to canonical bijection only on (M,ω), $\iota : L\rightarrow M$, and its relative spin structure, and the Floer cohomology groups $HF^*\big((L,b);\Lambda^*_{\rm nov}\big)$ also depend as a $\Lambda^*_{\rm nov}$-algebra up to canonical isomorphism only on (M,ω), $\iota : L\rightarrow M$ and its relative spin structure, and the canonical bijection equivalence class of the point $G_{\mathcal{H},\mathfrak{n}}\cdot b\in\mathcal{M}_{\mathcal{H},\mathfrak{n}}$. They are independent in this sense of all other choices, including the almost complex structure J, $\mathcal{G},\mathcal{X},\mathfrak{m},\mathcal{H},\mathfrak{n}$, and $\lambda_{(p_-,p_+)},\eta_{(p_-,p_+)},o_{(p_-,p_+)}$ for $(p_-,p_+)\in R$.*

13.2. The Floer cohomology of two Lagrangians. Now let (M,ω) be a compact symplectic manifold and $\iota_0 : L_0\rightarrow M$, $\iota_1 : L_1\rightarrow M$ be compact immersed Lagrangians in (M,ω) with only transverse double self-intersections, which intersect transversely in finitely many points $\iota_0(L_0)\cap\iota_1(L_1)$ in M, that are not self-intersection points of L_0 or L_1. Let $(\mathbb{Q}\mathcal{X}_0\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m}^0),(\mathbb{Q}\mathcal{X}_1\hat{\otimes}\Lambda^0_{\rm nov},\mathfrak{m}^1)$ be gapped filtered A_∞ algebras in Definition 11.1 for $\iota_0 : L_0\rightarrow M$, $\iota_1 : L_1\rightarrow M$, constructed using almost complex structures J_0,J_1, and let $(\mathcal{H}_0\otimes\Lambda^0_{\rm nov},\mathfrak{n}^0),(\mathcal{H}_1\otimes\Lambda^0_{\rm nov},\mathfrak{n}^1)$ be the corresponding gapped filtered A_∞ algebras in Corollary 11.4. Let b_0,b_1 be bounding cochains for $(\mathcal{H}_0\otimes\Lambda^0_{\rm nov},\mathfrak{n}^0),(\mathcal{H}_1\otimes\Lambda^0_{\rm nov},\mathfrak{n}^1)$ respectively.

Then following Fukaya et al. [**8**, §3.7], one can define Lagrangian Floer cohomology $HF^*\big((L_0,b_0),(L_1,b_1);\Lambda^*_{\rm nov}\big)$ for the pair of immersed Lagrangians L_0,L_1. Doing this in the immersed rather than the embedded case raises no new issues that we have not already dealt with above. In fact, as we explain below, for immersed Lagrangians one can easily recover Floer cohomology for two Lagrangians L_0,L_1 from the Floer cohomology for one Lagrangian $L_0\amalg L_1$ in §13.1. Therefore on this issue we will simply quote the conclusions of [**8**] with brief explanations.

Write $CF(L_0,L_1;\Lambda^*_{\rm nov})$ for the free $\Lambda^*_{\rm nov}$-module with basis $\iota_0(L_0)\cap\iota_1(L_1)$, where each $p\in\iota_0(L_0)\cap\iota_1(L_1)$ is graded in a similar way to the $\eta_{(p_-,p_+)}$ in §4.3. Then by choosing a smooth family $J_t : t\in[0,1]$ of almost complex structures on M compatible with ω interpolating between J_0 and J_1, and considering [**8**, §3.7.4] moduli spaces $\overline{\mathcal{M}}_{k_1,k_0}\big(L^1,L^0;[\ell_p,w_1],[\ell_p,w_2]\big)$ of stable maps of holomorphic discs into M with boundary in $\iota_0(L_0)\cup\iota_1(L_1)$, which are holomorphic w.r.t. the family $J_t : t\in[0,1]$ in a certain sense, one can give $CF(L_0,L_1;\Lambda^*_{\rm nov})$ the structure of a *gapped filtered A_∞ bimodule* over $(\mathbb{Q}\mathcal{X}_0\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m}^0),(\mathbb{Q}\mathcal{X}_1\hat{\otimes}\Lambda^0_{\mathrm{nov}},\mathfrak{m}^1)$.

Passing to canonical models, one can also give $CF(L_0,L_1;\Lambda^*_{\mathrm{nov}})$ the structure of a gapped filtered A_∞ bimodule over $(\mathcal{H}_0\otimes\Lambda^0_{\mathrm{nov}},\mathfrak{n}^0),(\mathcal{H}_1\otimes\Lambda^0_{\mathrm{nov}},\mathfrak{n}^1)$, [**8**, Th. F, §1.2]. This bimodule structure is independent of the choice of bounding cochains. But once we choose bounding cochains b_0,b_1 for $(\mathcal{H}_0\otimes\Lambda^0_{\mathrm{nov}},\mathfrak{n}^0),(\mathcal{H}_1\otimes\Lambda^0_{\mathrm{nov}},\mathfrak{n}^1)$, we can define a differential δ^{b_0,b_1} on $CF(L_0,L_1;\Lambda^*_{\mathrm{nov}})$, so that $\bigl(CF(L_0,L_1;\Lambda^*_{\mathrm{nov}}),\delta^{b_0,b_1}\bigr)$ is a complex. We then define $HF^*\bigl((L_0,b_0),(L_1,b_1);\Lambda^*_{\mathrm{nov}}\bigr)$ to be the cohomology of $\bigl(CF(L_0,L_1;\Lambda^*_{\mathrm{nov}}),\delta^{b_0,b_1}\bigr)$, graded in the same way as (144).

In this way we obtain an analogue of Theorem 13.4:

Theorem 13.5. *In the situation above, $HF^*\bigl((L_0,b_0),(L_1,b_1);\Lambda^*_{\mathrm{nov}}\bigr)$ depends as a Λ^*_{nov}-module up to canonical isomorphism only on (M,ω), $\iota_0:L_0\to M$, $\iota_1:L_1\to M$ and their relative spin structures, and the canonical bijection equivalence classes of the points $G_{\mathcal{H}_0,\mathfrak{n}^0}\cdot b_0\in\mathcal{M}_{\mathcal{H}_0,\mathfrak{n}^0}$ and $G_{\mathcal{H}_1,\mathfrak{n}^1}\cdot b_1\in\mathcal{M}_{\mathcal{H}_1,\mathfrak{n}^1}$.*

Actually, if we take J^0,J^1 and J^t for $t\in[0,1]$ to be some fixed almost complex structure J, the definition of Floer cohomology $HF^*\bigl((L_0,b_0),(L_1,b_1);\Lambda^*_{\mathrm{nov}}\bigr)$ for two Lagrangians is implicit in our definition of Floer cohomology $HF^*\bigl((L,b);\Lambda^*_{\mathrm{nov}}\bigr)$ for one immersed Lagrangian in §13.1. Take $L=L_0\amalg L_1$ with immersion $\iota=\iota_0\amalg\iota_1:L\to M$. Then bounding cochains b_0,b_1 for L_0,L_1 give a bounding cochain b for L, and there is a canonical isomorphism

$$\begin{aligned}&HF^*\bigl((L,b);\Lambda^*_{\mathrm{nov}}\bigr)\cong HF^*\bigl((L_0,b_0);\Lambda^*_{\mathrm{nov}}\bigr)\oplus HF^*\bigl((L_1,b_1);\Lambda^*_{\mathrm{nov}}\bigr)\oplus\\ &\quad HF^*\bigl((L_0,b_0),(L_1,b_1);\Lambda^*_{\mathrm{nov}}\bigr)\oplus HF^*\bigl((L_1,b_1),(L_0,b_0);\Lambda^*_{\mathrm{nov}}\bigr).\end{aligned}\tag{147}$$

Thus, Floer cohomology for two Lagrangians L_0,L_1 is just a sector of Floer cohomology for one Lagrangian $L_0\amalg L_1$, and one can deduce Theorem 13.5 from Theorem 13.4 with little effort. This works only for immersed Lagrangians, since even if L_0,L_1 are embedded, $L_0\amalg L_1$ is immersed unless $\iota_0(L_0)\cap\iota_1(L_1)=\emptyset$.

Although it is not covered in [**8**], it follows from the framework of Fukaya [**7**] that if L_0,L_1,L_2 are immersed Lagrangians in (M,ω) with only transverse double self-intersections, which intersect pairwise transversely as above, with no triple self-intersections, and b_0,b_1,b_2 are bounding cochains for L_0,L_1,L_2, then we can define a Λ^*_{nov}-bilinear product

$$\begin{aligned}\bullet_{012}:HF^*\bigl((L_0,b_0),(L_1,b_1);\Lambda^*_{\mathrm{nov}}\bigr)\times HF^*\bigl((L_1,b_1),(L_2,b_2);\Lambda^*_{\mathrm{nov}}\bigr)&\\ \longrightarrow HF^*\bigl((L_0,b_0),(L_2,b_2);\Lambda^*_{\mathrm{nov}}\bigr).&\end{aligned}\tag{148}$$

This is basically composition of morphisms between objects (L_0,b_0), (L_1,b_1) and (L_2,b_2) of the derived Fukaya category of (M,ω).

As in (147), $HF^*\big((L_i,b_i),(L_j,b_j);\Lambda^*_{\rm nov}\big)$ for $i,j=0,1,2$ are all sectors of the one-Lagrangian Floer cohomology $HF^*\big((L,b);\Lambda^*_{\rm nov}\big)$ for $L=L_0\amalg L_1\amalg L_2$, and then $\bullet_{012}$ in (148) is just the product $\bullet$ on $HF^*\big((L,b);\Lambda^*_{\rm nov}\big)$ in Definition 13.1 restricted to these sectors. For four such Lagrangians $L_0,\ldots,L_3$, associativity of $\bullet$ for $L=L_0\amalg\cdots\amalg L_3$ gives the associativity property

$$\bullet_{023}\circ\big(\bullet_{012}\times{\rm id}_{HF^*(L_2,L_3)}\big)=\bullet_{013}\circ\big({\rm id}_{HF^*(L_0,L_1)}\times\bullet_{123}\big).$$

When we work over $\Lambda_{\rm nov}$ rather than $\Lambda^0_{\rm nov}$, Lagrangian Floer cohomology has very important invariance properties under Hamiltonian isotopy, most of which is proved by Fukaya et al. [**8**, Th. G, §1.2] in the embedded case:

Theorem 13.6. *Let (M,ω) be a compact symplectic manifold, and $\psi_t:t\in[0,1]$ be a smooth 1-parameter family of Hamiltonian equivalent symplectomorphisms of (M,ω), with $\psi_0={\rm id}_M$. Then:*

(a) *Let $\iota_0:L_0\to M$ be a compact immersed Lagrangian in (M,ω) and $\iota_1:L_1\to M$ be the image of $\iota_0:L_0\to M$ under ψ_1, that is, $L_1=L_0$ and $\iota_1=\psi_1\circ\iota_0$. Let $(\mathcal{H}_0\otimes\Lambda^0_{\rm nov},\mathfrak{n}^0),(\mathcal{H}_1\otimes\Lambda^0_{\rm nov},\mathfrak{n}^1)$ be gapped filtered A_∞ algebras in Corollary 11.4 for L_0,L_1. Then using $\psi_t:t\in[0,1]$ we can define a gapped filtered A_∞ isomorphism $\mathbf{\Psi}:(\mathcal{H}_0\otimes\Lambda^0_{\rm nov},\mathfrak{n}^0)\to(\mathcal{H}_1\otimes\Lambda^0_{\rm nov},\mathfrak{n}^1)$, unique up to homotopy. This induces a unique bijection $\mathbf{\Psi}_*:\mathcal{M}_{\mathcal{H}_0,\mathfrak{n}^0}\to\mathcal{M}_{\mathcal{H}_1,\mathfrak{n}^1}$.*

(b) *In (a), if L_0,L_1 intersect transversely in M, then whenever $b_0\in\widehat{\mathcal{M}}_{\mathcal{H}_0,\mathfrak{n}^0}$ and $b_1\in\widehat{\mathcal{M}}_{\mathcal{H}_1,\mathfrak{n}^1}$ with $\mathbf{\Psi}_*(G_{\mathcal{H}_0,\mathfrak{n}^0}\cdot b_0)=G_{\mathcal{H}_1,\mathfrak{n}^1}\cdot b_1$, there is a canonical isomorphism*

$$HF^*\big((L_0,b_0);\Lambda_{\rm nov}\big)\cong HF^*\big((L_0,b_0),(L_1,b_1);\Lambda_{\rm nov}\big).\tag{149}$$

(c) *In (a), if $\iota_2:L_2\to M$ is another compact immersed Lagrangian in (M,ω) which intersects L_0,L_1 transversely, with $(\mathcal{H}_2\otimes\Lambda^0_{\rm nov},\mathfrak{n}^2)$ in Corollary 11.4, and $b_0\in\widehat{\mathcal{M}}_{\mathcal{H}_0,\mathfrak{n}^0},b_1\in\widehat{\mathcal{M}}_{\mathcal{H}_1,\mathfrak{n}^1}$ and $b_2\in\widehat{\mathcal{M}}_{\mathcal{H}_2,\mathfrak{n}^2}$ with $\mathbf{\Psi}_*(G_{\mathcal{H}_0,\mathfrak{n}^0}\cdot b_0)=G_{\mathcal{H}_1,\mathfrak{n}^1}\cdot b_1$, there is a canonical isomorphism*

$$HF^*\big((L_0,b_0),(L_2,b_2);\Lambda_{\rm nov}\big)\cong HF^*\big((L_1,b_1),(L_2,b_2);\Lambda_{\rm nov}\big).\tag{150}$$

Here part (a) is immediate from §13.1, since ψ_1 is an isomorphism from $M,\omega,\iota_0:L_0\to M$ to $M,\omega,\iota_1:L_1\to M$. The nontrivial statements are (b),(c). They are proved using the homotopy theory of A_∞ bimodules over two A_∞ algebras (two Lagrangians), as in Fukaya et al. [**8**, §5], rather than the homotopy theory of one A_∞ algebra (one Lagrangian), as in [**8**, §4] and §3 above.

Remark 13.7. **(i)** Equations (149) and (150) do *not* hold in general for Floer cohomology over $\Lambda^0_{\rm nov}$. From $HF^*\big((L_0,b_0),(L_1,b_1);\Lambda^0_{\rm nov}\big)$ we can recover the $\mathbb{Q}$-vector space with basis $\iota_0(L_0)\cap\iota_1(L_1)$. Thus, if (150) held over $\Lambda^0_{\rm nov}$ it would force $\bigl|\iota_0(L_0)\cap\iota_2(L_2)\bigr| = \bigl|\iota_1(L_1)\cap\iota_2(L_2)\bigr|$, which is false in general.

(ii) In the embedded case, it is well known that Theorem 13.6 has important consequences in symplectic geometry. Using (b) one can deduce the *Arnold Conjecture* for compact monotone symplectic manifolds.

(iii) The only place where we use compactness of M is to ensure that moduli spaces of J-holomorphic curves $\mathcal{M}_{k+1}(\alpha,\beta,J)$ are compact. If M is *noncompact* but J has suitable convexity properties at infinity which ensure compactness of $\mathcal{M}_{k+1}(\alpha,\beta,J)$, then Lagrangian Floer cohomology is well-defined and Theorem 13.6 holds. This can be done for cotangent bundles T^*L and $\mathbb{C}^n$, for instance.

By taking $M=T^*L$ for L a compact n-manifold, and L_0 to be the zero section of T^*L, part (b) implies another conjecture of Arnold on cotangent bundles.

Taking $M=\mathbb{C}^n$, if $\iota_0: L_0\to\mathbb{C}^n$ is a compact immersed Lagrangian, then by choosing ψ_1 to be a large translation in $\mathbb{C}^n$ we can arrange that $\iota_0(L_0)\cap\iota_1(L_1)=\emptyset$. Thus $CF(L_0,L_1;\Lambda_{\rm nov})=\{0\}$, so $HF^*\bigl((L_0,b_0),(L_1,b_1);\Lambda_{\rm nov}\bigr)=\{0\}$, and (b) gives $HF^*\bigl((L_0,b_0);\Lambda_{\rm nov}\bigr)=\{0\}$ for any bounding cochain b_0 for L_0.

13.3. Floer cohomology for graded Lagrangians in Calabi–Yau n-folds. As in §12, suppose (M,J,ω,Ω) is a Calabi–Yau n-fold and $(\iota: L\to M,\phi)$ an immersed graded Lagrangian with only transverse double self-intersections. Choose a relative spin structure for $\iota: L\to M$. Theorem 12.3 constructs gapped filtered A_∞ algebras $(\mathbb{Q}\mathcal{X}\hat\otimes\Lambda^0_{\rm CY},\mathfrak{m})$ and $(\mathcal{H}\otimes\Lambda^0_{\rm CY},\mathfrak{n})$. We can then go through the whole of §13.1 and §13.2 using graded Lagrangians, and working over the Calabi–Yau Novikov rings $\Lambda^0_{\rm CY},\Lambda_{\rm CY}$ rather than $\Lambda^0_{\rm nov},\Lambda_{\rm nov}$.

Use $\Lambda^*_{\rm CY}$ to mean $\Lambda^0_{\rm CY}$ or $\Lambda_{\rm CY}$. Write triples (L,ϕ,b) as a shorthand for an immersed graded Lagrangian $(\iota: L\to M,\phi)$ together with a bounding cochain b for $(\mathcal{H}\otimes\Lambda^0_{\rm CY},\mathfrak{n})$ in Theorem 12.3(c). Then we may define *Lagrangian Floer cohomology groups* $HF^*\bigl((L,\phi,b);\Lambda^*_{\rm CY}\bigr)$ for one graded Lagrangian as in §13.1, and $HF^*\bigl((L_0,\phi_0,b_0),(L_1,\phi_1,b_1);\Lambda^*_{\rm CY}\bigr)$ for two graded Lagrangians as in §13.2.

Theorem 13.8. *The analogues of Theorems* 13.4, 13.5 *and* 13.6 *hold for Lagrangian Floer cohomology of immersed graded Lagrangians in Calabi–Yau n-folds, over the Novikov rings $\Lambda^0_{\rm CY},\Lambda_{\rm CY}$.*

As in Remark 13.2 $HF^k\bigl((L,b);\Lambda^*_{\rm nov}\bigr)\cong HF^{k+2d}\bigl((L,b);\Lambda^*_{\rm nov}\bigr)$ for d in $\mathbb{Z}$, so one should regard $HF^*\bigl((L,b);\Lambda^*_{\rm nov}\bigr)$ as graded over $\mathbb{Z}_2$ rather than $\mathbb{Z}$. In contrast, $HF^*\bigl((L,\phi,b);\Lambda^*_{\rm CY}\bigr)$ really is graded over $\mathbb{Z}$, and this makes Floer cohomology for graded Lagrangians a more powerful tool, as Seidel [**19**] points out.

In particular, we can give useful criteria for existence and uniqueness of bounding cochains. Since $\Lambda^0_{\rm CY}$ is graded of degree 0, a bounding cochain b for $(\mathcal{H}\otimes\Lambda^0_{\rm CY},\mathfrak{n})$ lies in $b\in F^\lambda(\mathcal{H}^0\otimes\Lambda^0_{\rm CY})$ for some $\lambda>0$ and must satisfy $\sum_{k\geqslant 0}\mathfrak{n}_k(b,\dots,b)=0$ in $\mathcal{H}^1\otimes\Lambda^0_{\rm CY}$. But (138) gives

$$\begin{aligned}\mathcal{H}^0&=H_{n-1}(L;\mathbb{Q})\oplus\textstyle\bigoplus_{(p_-,p_+)\in R:\,\eta_{(p_-,p_+)}=1}\mathbb{Q}(p_-,p_+),\\ \mathcal{H}^1&=H_{n-2}(L;\mathbb{Q})\oplus\textstyle\bigoplus_{(p_-,p_+)\in R:\,\eta_{(p_-,p_+)}=2}\mathbb{Q}(p_-,p_+).\end{aligned}\tag{151}$$

Thus we deduce:

Proposition 13.9. *Suppose (M,J,ω,Ω) is a Calabi–Yau n-fold, $(\iota:L\rightarrow M,\phi)$ is an immersed graded Lagrangian with only transverse double self-intersections, and $(\mathcal{H}\otimes\Lambda^0_{\rm CY},\mathfrak{n})$ is as in Theorem* 12.3(c). *Then:*

- **(a)** *If $b_{n-2}(L)=0$ and $\eta_{(p_-,p_+)}\neq 2$ for all $(p_-,p_+)\in R$, then every $b\in F^\lambda(\mathcal{H}^0\otimes\Lambda^0_{\rm CY})$ for $\lambda>0$ is a bounding cochain; and*
- **(b)** *if $b_{n-1}(L)=0$ and $\eta_{(p_-,p_+)}\neq 1$ for all $(p_-,p_+)\in R$, then 0 is the only possible bounding cochain.*

Since $\iota:L\rightarrow M$ has a relative spin structure, L is oriented, so $b_{n-1}(L)=0$ in (b) is equivalent to $b^1(L)=0$, which is a sufficient condition for an immersed Lagrangian $\iota:L\rightarrow M$ to admit a grading ϕ. As in Remark 13.7(iii), we can also apply the theory to *noncompact* Calabi–Yau manifolds (M,J,ω,Ω), provided J is convex at infinity. For example, $M=\mathbb{C}^n$ with the Euclidean J,ω,Ω will do.

In the noncompact case we may suppose (M,ω) is an exact symplectic manifold, that is, $\omega=\mathrm{d}\xi$ for some 1-form ξ on M. If $\iota:L\rightarrow M$ is an immersed Lagrangian then $\iota^*(\xi)$ is a closed 1-form on L, and we call L *exact* if $\iota^*(\xi)$ is exact. If L is exact, then there can be no nonconstant holomorphic discs in M whose boundaries lie in $\iota(L)$ and lift continuously to L, as Stokes' Theorem shows that their area would be zero. This implies that the component of $\mathfrak{n}_0$ in $H_{n-2}(L;\mathbb{Q})\otimes\Lambda^0_{\rm CY}$ is zero. If also $\eta_{(p_-,p_+)}\neq 2$ for all $(p_-,p_+)\in R$ then $\mathfrak{n}_0=0$, so 0 is a bounding cochain, giving:

Proposition 13.10. *Let (M,J,ω,Ω) be a noncompact, exact Calabi–Yau n-fold, with J convex at infinity, $(\iota:L\rightarrow M,\phi)$ an exact immersed graded Lagrangian with only transverse double self-intersections, and $\eta_{(p_-,p_+)}\neq 2$ for all $(p_-,p_+)\in R$. Then 0 is a bounding cochain for $(\mathcal{H}\otimes\Lambda^0_{\rm CY},\mathfrak{n})$ in Theorem* 12.3(c).

Now let $(\iota:L\rightarrow M,\phi)$ be a compact immersed graded Lagrangian in $\mathbb{C}^n$. Propositions 13.9(a) and 13.10 give two sufficient conditions for 0 to be a bounding cochain for L. Then $HF^*\bigl((L,\phi,0);\Lambda_{\rm CY}\bigr)$ is well-defined, and Remark 13.7(iii) shows that $HF^*\bigl((L,\phi,0);\Lambda_{\rm CY}\bigr)=\{0\}$. But $HF^*\bigl((L,\phi,0);\Lambda_{\rm CY}\bigr)$ is the cohomology of the complex $(\mathcal{H}\otimes\Lambda_{\rm CY},\mathfrak{n}_1)$.

To have zero cohomology imposes constraints upon the ranks over Λ_{CY} of the graded pieces of a free Λ_{CY}-complex. For instance, we have:

Corollary 13.11. *Let $(\iota : L \to M, \phi)$ be a compact, immersed, graded Lagrangian in $\mathbb{C}^n$, with transverse double self-intersections. Suppose that $\eta_{(p_-,p_+)} \neq 2$ for all $(p_-, p_+) \in R$, and either $b_{n-2}(L) = 0$ or L is exact. Then $\dim \mathcal{H}^d \leqslant \dim \mathcal{H}^{d-1} + \dim \mathcal{H}^{d+1}$ for all $d \in \mathbb{Z}$, with $\mathcal{H}^d$ given in* (138).

Here is an example.

Example 13.12. Define a curve in $\mathbb{C}$ by $C = \{s + it : s, t \in \mathbb{R}, t^2 = s^2 - s^4\}$. This is sketched in Figure 13.1. It is an immersed circle in $\mathbb{R}^2$, the shape of an ∞ sign, with one self-intersection point at 0. For $n \geqslant 1$, define

$$L_n = \big\{(\lambda x_1, \ldots, \lambda x_n) : \lambda \in C,\ x_1, \ldots, x_n \in \mathbb{R},\ x_1^2 + \cdots + x_n^2 = 1\big\}.$$

It is easy to see that L_n is the image of an immersed Lagrangian sphere $\iota : \mathcal{S}^n \to \mathbb{C}^n$, which has one transverse self-intersection point at $0 \in \mathbb{C}^n$ with $\iota(p_-) = \iota(p_+) = 0$, where $p_\pm = (\pm 1, 0, \ldots, 0) \in \mathcal{S}^n$. Note that L_n is $\mathrm{SO}(n)$-invariant, and we can choose ι to be equivariant with respect to the actions of $\mathrm{SO}(n)$ on $\mathcal{S}^n$ fixing $p_\pm$, and on $\mathbb{C}^n$. The tangent spaces to $\iota(\mathcal{S}^n)$ at the self-intersection point are

$$\begin{aligned} \mathrm{d}\iota(T_{p_-}\mathcal{S}^n) &= \big\{(\mathrm{e}^{-i\pi/4}x_1, \ldots, \mathrm{e}^{-i\pi/4}x_n) : x_1, \ldots, x_n \in \mathbb{R}\big\}, \\ \mathrm{d}\iota(T_{p_+}\mathcal{S}^n) &= \big\{(\mathrm{e}^{i\pi/4}x_1, \ldots, \mathrm{e}^{i\pi/4}x_n) : x_1, \ldots, x_n \in \mathbb{R}\big\}. \end{aligned} \tag{152}$$

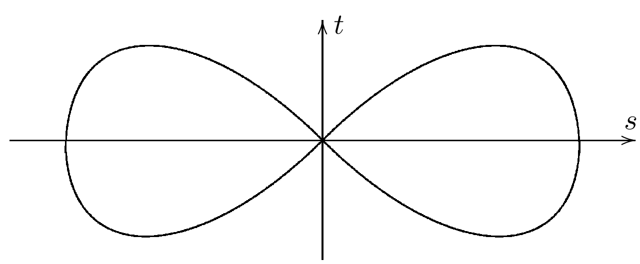


Figure 13.1. The curve C in $\mathbb{C}$

We shall calculate the index $\eta_{(p_-,p_+)}$ using Proposition 12.2. Despite the comparison between (140) and (152), we are not free to put $\phi_-^j = -\frac{\pi}{4}$ and $\phi_+^j = \frac{\pi}{4}$, since (152) only determines the $\phi_\pm^j$ up to addition of $\pi\mathbb{Z}$. We have to choose a framing $\phi : \mathcal{S}^n \to \mathbb{R}$ for $\iota : \mathcal{S}^n \to \mathbb{C}^n$, and choose the $\phi_\pm^j$ to satisfy $\phi_\pm^1 + \cdots + \phi_\pm^n = \phi(p_\pm)$.

Consider $p : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathcal{S}^n$ defined by $p(u) = (\sin u, \cos u, 0, \ldots, 0)$. Then $p(\pm\frac{\pi}{2}) = p_\pm$, and $\iota \circ p(u) = (\lambda(u), 0, \ldots, 0)$, where $\lambda : [-\frac{\pi}{2}, \frac{\pi}{2}] \to C$

sweeps out the right hand lobe $s \geqslant 0$ of C in the anticlockwise direction. Calculation shows that for $u \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we have

$$\mathrm{d}\iota(T_{p(u)}\mathcal{S}^n) = \big\{ (\tfrac{\mathrm{d}\lambda}{\mathrm{d}u}(u)x_1, \lambda(u)x_2, \lambda(u)x_n) : x_1, \dots, x_n \in \mathbb{R} \big\}.$$

From Figure 13.1 we see that $\arg \frac{\mathrm{d}\lambda}{\mathrm{d}u}(u)$ increases continuously from $-\frac{\pi}{4}$ to $\frac{5\pi}{4}$ and $\arg \lambda(u)$ increases continuously from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$ over $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Therefore $\iota : \mathcal{S}^n \to \mathbb{C}^n$ has a framing $\phi : \mathcal{S}^n \to \mathbb{R}$ with $\phi(p_-) = -\frac{n\pi}{4}$ and $\phi(p_+) = \frac{n\pi}{4} + \pi$, and in Proposition 12.2 we may take $\phi_-^j = -\frac{\pi}{4}$ for $j = 1, \dots, n$, $\phi_+^1 = \frac{5\pi}{4}$, and $\phi_+^j = \frac{\pi}{4}$ for $j = 2, \dots, n$. Hence $\big[\frac{\phi_+^j - \phi_-^j}{\pi}\big]$ is 1 for $j = 1$ and 0 for $j = 2, \dots, n$, and (141) gives $\eta_{(p_-,p_+)} = n + 1$, and similarly $\eta_{(p_+,p_-)} = -1$. Thus (138) gives $\mathcal{H}^d = \mathbb{Q}$ if $d = -2, -1, n-1, n$, and $\mathcal{H}^d = 0$ otherwise.

When $n > 2$, Proposition 13.9 implies that 0 is the unique bounding cochain for $\iota : \mathcal{S}^n \to \mathbb{C}^n$. When $n = 2$ Proposition 13.9(a) does not apply, but this is an exact Lagrangian, so Propositions 13.9(b) and 13.10 show that 0 is the unique bounding cochain for $\iota : \mathcal{S}^2 \to \mathbb{C}^2$. Thus as above $HF^*\big((\mathcal{S}^n, \phi, 0); \Lambda_{\mathrm{CY}}\big)$ is well-defined, and zero. Corollary 13.11 holds.

If (M, J, ω, Ω) is a compact Calabi–Yau n-fold and $p \in M$, then by shrinking the example above by a homothety and locally identifying $\mathbb{C}^n$ near 0 with M near p using Darboux' Theorem, we can construct Lagrangian immersions $\iota : \mathcal{S}^n \to M$. The same arguments then prove:

Proposition 13.13. *Let (M, J, ω, Ω) be a compact Calabi–Yau n-fold for $n > 1$, and $p \in M$. Then there exists an immersed, graded Lagrangian $(\iota : \mathcal{S}^n \to M, \phi)$ with exactly one transverse double self-intersection point at $p = \iota(p_-) = \iota(p_+)$, with $\eta_{(p_-,p_+)} = n+1$. It has unique bounding cochain* 0, *and $HF^*\big((\mathcal{S}^n, \phi, 0); \Lambda_{\mathrm{CY}}\big) = \{0\}$.*

Thus there are many immersed Lagrangians with unobstructed Floer cohomology, but which are zero objects in the derived immersed Fukaya category.

13.4. Local Hamiltonian equivalence of immersed Lagrangians.

For immersed Lagrangians, there are two different notions of Hamiltonian equivalence.

Definition 13.14. Let (M, ω) be a symplectic manifold, and $\iota : L \to M$, $\iota' : L' \to M$ be compact, immersed Lagrangians in M. Then

(i) We say that $\iota : L \to M$, $\iota' : L' \to M$ are *globally Hamiltonian equivalent* if there exists a diffeomorphism $h : L \to L'$ and a smooth 1-parameter family $\psi_t : t \in [0,1]$ of Hamiltonian equivalent symplectomorphisms of (M, ω) with $\psi_0 = \mathrm{id}_M$, such that $\psi_1 \circ \iota \equiv \iota' \circ h$.

(ii) We say that $\iota : L \to M$, $\iota' : L' \to M$ are *locally Hamiltonian equivalent* if there exists a diffeomorphism $h : L \to L'$ and a smooth 1-parameter family $\iota_t : t \in [0,1]$ of Lagrangian immersions $\iota_t : L \to M$, such that $\iota_0 = \iota$ and $\iota_1 = \iota' \circ h$, and for each $t \in [0,1]$ the 1-form $\mathrm{d}\iota_t^*\big(\frac{\mathrm{d}\iota_t}{\mathrm{d}t} \cdot \iota_t^*(\omega)\big)$ on L is *exact*.

Here $\frac{\mathrm{d}\iota_t}{\mathrm{d}t}, \iota_t^*(\omega)$ and $\frac{\mathrm{d}\iota_t}{\mathrm{d}t} \cdot \iota_t^*(\omega)$ are sections of the vector bundles $\iota_t^*(TM), \iota_t^*(\Lambda^2 T^*M), \iota_t^*(T^*M)$ over L, respectively, $\mathrm{d}\iota_t : TL \to \iota_t^*(TM)$ is the derivative of ι_t, and $\mathrm{d}\iota_t^* : \iota_t^*(T^*M) \to T^*L$ the dual map. It follows from ι_t a Lagrangian immersion for $t \in [0,1]$ that $\mathrm{d}\iota_t^*\big(\frac{\mathrm{d}\iota_t}{\mathrm{d}t} \cdot \iota_t^*(\omega)\big)$ is a closed 1-form.

By setting $\iota_t = \psi_t \circ \iota$, we see that global implies local Hamiltonian equivalence. For embedded Lagrangians, if the $\iota_t : L \to M$ are embeddings for all $t \in [0,1]$ then we can find a family $\psi_t : t \in [0,1]$ as in (i) such that $\iota_t = \psi_t \circ \iota$, so that local implies global Hamiltonian equivalence. Thus, *for embedded Lagrangians, global and local Hamiltonian equivalence is the same*. But for immersed Lagrangians, local Hamiltonian equivalence can slide sheets of L over each other, change the number of self-intersection points, and so on, but global Hamiltonian equivalence cannot. Hence, *for immersed Lagrangians, local Hamiltonian equivalence is weaker than global Hamiltonian equivalence.*

Theorem 13.6 shows that Floer cohomology over Λ_{nov} has strong invariance properties under global Hamiltonian equivalence. So it makes sense to ask:

Question 13.15. Does Floer cohomology $HF^*\big((L_0, b_0); \Lambda_{\mathrm{nov}}\big)$ and $HF^*\big((L_0, b_0), (L_1, b_1); \Lambda_{\mathrm{nov}}\big)$ have any useful invariance properties under (possibly restricted classes of) *local* Hamiltonian equivalences of $\iota_0 : L_0 \to M$ and $\iota_1 : L_1 \to M$?

For arbitrary local Hamiltonian equivalences, the answer to this must be no. The *Lagrangian h-principle*, due to Gromov [**10**, p. 60-61] and Lees [**16**], states that two Lagrangian immersions $\iota_0 : L \to M$, $\iota_1 : L \to M$ are homotopic through (possibly *exact*) Lagrangian immersions $\iota_t : L \to M$ for $t \in [0,1]$ if and only if ι_0, ι_1 are homotopic in a weaker sense, that is, $(\iota_0, \mathrm{d}\iota_0), (\iota_1, \mathrm{d}\iota_1)$ should be homotopic through pairs $(\iota, \tilde{\iota})$, where $\iota : L \to M$ is smooth and $\tilde{\iota} : TL \to TM$ is a bundle map covering ι which embeds TL as a bundle of Lagrangian subspaces in TM.

Thus, the Lagrangian h-principle implies that two immersed Lagrangians are locally Hamiltonian equivalent (at least when either $M = \mathbb{C}^n$, so that [**10**, p. 60-61] applies, or $b^1(L) = 0$, so that [**16**, Th. 1] applies, and probably more generally) if and only if they are homotopic in a weak sense which can be well understood using homotopy theory. But Floer cohomology detects 'quantum' information not visible to classical algebraic topology — this is its whole point. So arbitrary local Hamiltonian equivalence is too coarse an equivalence relation to preserve Floer cohomology.

However, it could still be true that Floer cohomology over $\Lambda_{\rm nov}$ is in some sense invariant under some special class of local Hamiltonian equivalences more general than global Hamiltonian equivalences. For example, in Theorem 13.6(c), $\iota_0 \amalg \iota_2 : L_0 \amalg L_2 \to M$ and $\iota_1 \amalg \iota_2 : L_1 \amalg L_2 \to M$ are immersed Lagrangians which are locally Hamiltonian equivalent but generally not globally so — for instance, if $\bigl|\iota_0(L_0) \cap \iota_2(L_2)\bigr| \neq \bigl|\iota_1(L_1) \cap \iota_2(L_2)\bigr|$ then $L_0 \amalg L_2$ and $L_1 \amalg L_2$ have different numbers of self-intersection points, and cannot be globally Hamiltonian equivalent. But (147) and Theorem 13.6(c) imply that there is a canonical isomorphism

$$HF^*\bigl((L_0 \amalg L_2, b_0 \amalg b_2); \Lambda_{\rm nov}\bigr) \cong HF^*\bigl((L_1 \amalg L_2, b_1 \amalg b_2); \Lambda_{\rm nov}\bigr).$$

Another possibility: in the Calabi–Yau, graded Lagrangian case, Proposition 13.9 suggests that only self-intersections with $\eta_{(p_-,p_+)} = 1$ or 2 are relevant to existence of bounding cochains. So we could consider only local Hamiltonian equivalences through immersions $\iota_t : L \to M$ which have no self-intersections with $\eta_{(p_-,p_+)} = 1$ or 2, and perhaps these will preserve Floer cohomology over $\Lambda_{\rm CY}$.

We shall now describe a mechanism for how the moduli spaces of bounding cochains $\mathcal{M}_{\mathcal{H},\mathfrak{n}}$ can change under local Hamiltonian equivalence.

Example 13.16. Let (M,ω) be a compact symplectic $2n$-manifold, L a compact n-manifold, and $\iota_t : L \to M$ for $t \in [0,1]$ a smooth family of Lagrangian immersions, which have only transverse double self-intersections for all $t \in [0,1]$. This implies that the number of self-intersections of $\iota_t : L \to M$ is independent of t. Therefore we can choose a smooth family of diffeomorphisms $\delta_t : M \to M$ with $\delta_0 = \mathrm{id}_M$, such that $\iota_t = \delta_t \circ \iota_0$. So δ_t^{-1} identifies $(M,\omega), \iota_t : L \to M$ with $(M, \delta_t^*(\omega)), \iota_0 : L \to M$. That is, we can work with a fixed immersion $\iota_0 : L \to M$, but a 1-parameter family of symplectic forms $\delta_t^*(\omega)$ on M for $t \in [0,1]$.

Let $t > 0$ be small. Then ω and $\delta_t^*(\omega)$ are C^0 close as 2-forms on M. Dimension calculations show that we can choose an almost complex structure J_0 on M compatible with both ω and $\delta_t^*(\omega)$. Write $J_t = (\delta_t)_*(J_0)$, so that J_t is compatible with ω as J_0 is compatible with $\delta_t^*(\omega)$. Then δ_t identifies $M, \iota_0 : L \to M, J_0$ with $M, \iota_t : L \to M, J_t$. Thus, δ_t takes J_0-holomorphic curves in M with boundary in $\iota_0(L)$ to J_t-holomorphic curves in M with boundary in $\iota_t(L)$. However, δ_t need not preserve the areas of the curves computed using ω.

Let $(\mathbb{Q}\mathcal{X}_0 \hat\otimes \Lambda^0_{\rm nov}, \mathfrak{m}^0)$, $(\mathcal{H}_0 \otimes \Lambda^0_{\rm nov}, \mathfrak{n}^0)$ be the gapped filtered A_∞ algebras in Theorem 11.2 and Corollary 11.4, associated to (M,ω) and $\iota_0 :$

$L \to M$ with almost complex structure J_0. Let $(\mathbb{Q}\mathcal{X}_t \hat{\otimes} \Lambda^0_{\mathrm{nov}}, \mathfrak{m}^t), (\mathcal{H}_t \otimes \Lambda^0_{\mathrm{nov}}, \mathfrak{n}^t)$ be the corresponding gapped filtered A_∞ algebras associated to (M, ω) and $\iota_t : L \to M$ with almost complex structure J_t, where *the choices made to construct* $\mathcal{X}^t, \mathfrak{m}^t, \mathcal{H}_t, \mathfrak{n}^t$ *are the images under* δ_t *of the choices made to construct* $\mathcal{X}^0, \mathfrak{m}^0, \mathcal{H}_0, \mathfrak{n}^0$. That is, we have $\mathcal{X}_t = \{\delta_t \circ f : f \in \mathcal{X}_0\}$, and then δ_t induces isomorphisms

$$\begin{aligned} \overline{\mathcal{M}}_{k+1}(\alpha, \beta, J_0, f_1, \ldots, f_k) \\ \cong \overline{\mathcal{M}}_{k+1}(\alpha, (\delta_t)_*(\beta), J_t, \delta_t \circ f_1, \ldots, \delta_t \circ f_k), \end{aligned} \tag{153}$$

and we choose all orientations and perturbation data compatible with these.

The difference between $(\mathbb{Q}\mathcal{X}_0 \hat{\otimes} \Lambda^0_{\mathrm{nov}}, \mathfrak{m}^0), (\mathcal{H}_0 \otimes \Lambda^0_{\mathrm{nov}}, \mathfrak{n}^0)$ and $(\mathbb{Q}\mathcal{X}_t \hat{\otimes} \Lambda^0_{\mathrm{nov}}, \mathfrak{m}^t)$, $(\mathcal{H}_t \otimes \Lambda^0_{\mathrm{nov}}, \mathfrak{n}^t)$ is that δ_t changes the areas of J_0- and J_t-holomorphic curves, and this changes the coefficients λ in the multilinear maps $\mathfrak{m}^{\lambda,\mu}_k, \mathfrak{n}^{\lambda,\mu}_k$ which make up $\mathfrak{m}^0, \mathfrak{n}^0, \mathfrak{m}^t, \mathfrak{n}^t$. The changes in areas of curves can be expressed like this: there exist constants $c_{(p_-,p_+)} \in \mathbb{R}$ for all $(p_-, p_+) \in R$, with $c_{(p_-,p_+)} + c_{(p_+,p_-)} = 0$, such that if $\overline{\mathcal{M}}_{k+1}(\alpha, \beta, J_0, f_1, \ldots, f_k) \neq \emptyset$ then

$$(\delta_t)_*(\beta) \cdot [\omega]_{M,\iota_t(L)} = \beta \cdot [\omega]_{M,\iota_0(L)} + \textstyle\sum_{i \in I} c_{\alpha(i)}, \tag{154}$$

where $[\omega]_{M,\iota_0(L)}$ and $[\omega]_{M,\iota_t(L)}$ are the classes of ω in $H^2(M, \iota_0(L); \mathbb{R})$ and $H^2(M, \iota_t(L); \mathbb{R})$.

By (138) we have $\mathcal{H}_0 = \mathcal{H}_t = H_*(L; \mathbb{Q}) \oplus \bigoplus_{(p_-,p_+) \in R} \mathbb{Q}(p_-, p_+)$. Using similar ideas to Remark 11.3, define a Λ_{nov}-linear map $\hat{\Xi}_t : \mathcal{H}_0 \otimes \Lambda_{\mathrm{nov}} \to \mathcal{H}_t \otimes \Lambda_{\mathrm{nov}}$ to be the identity on $H_*(L; \mathbb{Q})$ and to satisfy $\hat{\Xi}_t(p_-, p_+) = T^{-c_{(p_-,p_+)}}(p_-, p_+)$, where T is the formal variable in Λ_{nov} from §3.4. Then using (153)–(154) we see that $\mathfrak{m}^t_k\bigl(\hat{\Xi}_t(h_1), \ldots, \hat{\Xi}_t(h_k)\bigr) = \hat{\Xi}_t \circ \mathfrak{m}^0_k(h_1, \ldots, h_k)$ for all $h_1, \ldots, h_k \in \mathcal{H}_0 \otimes \Lambda_{\mathrm{nov}}$.

Thus, as in Remark 11.3, it is *nearly* true that setting $\boldsymbol{\Xi}_1 = \hat{\Xi}_t$ and $\boldsymbol{\Xi}_k = 0$ for $k \neq 1$ defines a strict gapped filtered A_∞ isomorphism $\boldsymbol{\Xi} : (\mathcal{H}_0 \otimes \Lambda^0_{\mathrm{nov}}, \mathfrak{n}^0) \to (\mathcal{H}_t \otimes \Lambda^0_{\mathrm{nov}}, \mathfrak{n}^t)$. The problem is that if $c_{(p_-,p_+)} > 0$ for some $(p_-, p_+) \in R$ then $(p_-, p_+) \in \mathcal{H}_0 \otimes \Lambda^0_{\mathrm{nov}}$ but $\hat{\Xi}_t(p_-, p_+) = T^{-c_{(p_-,p_+)}}(p_-, p_+) \notin \mathcal{H}_t \otimes \Lambda^0_{\mathrm{nov}}$, so $\hat{\Xi}_t$ does not map $\mathcal{H}_0 \otimes \Lambda^0_{\mathrm{nov}} \to \mathcal{H}_t \otimes \Lambda^0_{\mathrm{nov}} \subset \mathcal{H}_t \otimes \Lambda_{\mathrm{nov}}$.

However, if $b \in \mathcal{H}_0 \otimes \Lambda^0_{\mathrm{nov}}$ is a bounding cochain for $(\mathcal{H}_0 \otimes \Lambda^0_{\mathrm{nov}}, \mathfrak{n}^0)$, and $\hat{\Xi}_t(b)$ lies in $F^\lambda(\mathcal{H}_t \otimes \Lambda_{\mathrm{nov}})$ for some $\lambda > 0$, then $\hat{\Xi}_t(b)$ is a bounding cochain for $(\mathcal{H}_t \otimes \Lambda^0_{\mathrm{nov}}, \mathfrak{n}^t)$. Also $\hat{\Xi}_t$ is an isomorphism of complexes $(\mathcal{H}_0 \otimes \Lambda_{\mathrm{nov}}, \mathfrak{n}^{0,b}_1) \to (\mathcal{H}_t \otimes \Lambda_{\mathrm{nov}}, \mathfrak{n}^{t,b}_1)$, and so induces an isomorphism of Floer cohomology over Λ_{nov} (though not over Λ^0_{nov}):

$$(\hat{\Xi}_t)_* : HF^*(\iota_0 : L \to M, b; \Lambda_{\mathrm{nov}}) \longrightarrow HF^*(\iota_t : L \to M, \hat{\Xi}_t(b); \Lambda_{\mathrm{nov}}).$$

We have discovered a kind of *wall-crossing phenomenon*. When $t \in [0, \epsilon)$ for some $\epsilon > 0$ we can map bounding cochains b for $\iota_0 : L \to M$

to bounding cochains $\hat{\Xi}_t(b)$ for $\iota_t : L \to M$, and this map induces canonical isomorphisms on Lagrangian Floer cohomology. We have $\hat{\Xi}_t(b) \in F^{\lambda(t)}(\mathcal{H}_t \otimes \Lambda_{\rm nov})$, where we take $\lambda(t)$ as large as possible. For $\hat{\Xi}_t(b)$ to be a bounding cochain we need $\lambda(t) > 0$. However, it may happen that at $t = \epsilon$ we have $\lambda(\epsilon) = 0$, and for $t > \epsilon$ we have $\lambda(t) > 0$. Then at $t = \epsilon$ we cross a 'wall' where the bounding cochain for $\iota_0 : L \to M$ no longer corresponds to any bounding cochain for $\iota_t : L \to M$ when $t \geqslant \epsilon$.

This example suggests the following conjectural picture:

Conjecture 13.17. *Suppose that (M, ω) is a compact symplectic manifold, and that $\iota_t : L \to M$ for $t \in [0,1]$ is a smooth 1-parameter family of compact Lagrangian immersions satisfying the exactness condition of Definition 13.14(ii). Let $S \subseteq [0,1]$ be the open subset of $t \in [0,1]$ for which $\iota_t : L \to M$ has only transverse double self-intersections. Suppose for simplicity that L is oriented and spin; this induces relative spin structures for $\iota_t : L \to M$ for all $t \in [0,1]$, as in §5.1. Then for all $t \in S$, we have the moduli space of bounding cochains $\mathcal{M}_{\mathcal{H}_t,\mathfrak{n}^t}$ for $\iota_t : L \to M$, which is independent of choices up to canonical bijection by Theorem 13.4.*

We conjecture that for all $s, t \in S$ there should exist open subsets $O_{s,t} \subseteq \mathcal{M}_{\mathcal{H}_s,\mathfrak{n}^s}$ and homeomorphisms $\Phi_{s,t} : O_{s,t} \to O_{t,s}$ with $\Phi_{t,s} = \Phi_{s,t}^{-1}$, and whenever $G_{\mathcal{H}_s,\mathfrak{n}^s} \cdot b_s \in O_{s,t}$, $G_{\mathcal{H}_t,\mathfrak{n}^t} \cdot b_t \in O_{t,s}$ with $\Phi_{s,t}(G_{\mathcal{H}_s,\mathfrak{n}^s} \cdot b_s) = G_{\mathcal{H}_t,\mathfrak{n}^t} \cdot b_t$, there should exist canonical isomorphisms

$$\begin{aligned} &HF^*(\iota_s : L \to M, b_s; \Lambda_{\rm nov}) \cong HF^*(\iota_t : L \to M, b_t; \Lambda_{\rm nov}), \\ &HF^*\big((\iota_s : L \to M, b_s), (L', b'); \Lambda_{\rm nov}\big) \\ &\qquad\qquad \cong HF^*\big((\iota_t : L \to M, b_t), (L', b'); \Lambda_{\rm nov}\big), \end{aligned}$$

for any compact immersed Lagrangian $\iota' : L' \to M$ with transverse double self-intersections intersecting $\iota_s(L), \iota_t(L)$ transversely, and bounding cochain b'.

Furthermore, for any $G_{\mathcal{H}_s,\mathfrak{n}^s} \cdot b_s \in \mathcal{M}_{\mathcal{H}_s,\mathfrak{n}^s}$ the set $T_s = \{t \in S : G_{\mathcal{H}_s,\mathfrak{n}^s} \cdot b_s \in O_{s,t}\}$ is an open subset of S containing s, and at the boundary of T_s in S, a wall-crossing phenomenon like that in Example 13.16 occurs.

13.5. Immersed Lagrangians and embedded Legendrians. We now develop the ideas of §13.4 further in the context of *contact geometry* and *Legendrian submanifolds*. Let (M, ω) be a compact symplectic $2n$-manifold, and suppose $[\omega] \in H^2(M; \mathbb{R})$ lies in the image of $H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{R})$. Then there exists a principal U(1)-bundle $P \to M$ with first Chern class $c_1(P) = 2\pi[\omega]$, and a connection A on P with curvature $2\pi\omega$. Write the U(1) action on P as $(\mathrm{e}^{\sqrt{-1}\theta}, p) \mapsto \mathrm{e}^{\sqrt{-1}\theta} \cdot p$, and let $v \in C^\infty(TP)$ be the vector field of the U(1)-action, so that $\mathrm{e}^{\sqrt{-1}\theta}$ acts as $\exp(\theta v) : P \to P$. Write $\pi : P \to M$ for the natural projection whose fibres are U(1)-orbits $\mathrm{U}(1) \cdot p$ for $p \in P$. Let γ be the 1-form of the connection on P, so that $\gamma \in C^\infty(T^*P)$ is U(1)-invariant with $v \cdot \gamma \equiv 1$ and $\mathrm{d}\gamma \equiv \pi^*(2\pi\omega)$.

Then P has the structure of a *contact* $(2n+1)$*-manifold*, with contact 1-form γ and Reeb vector field v. An immersed n-manifold $\tilde\iota : L \to P$ is called a *Legendrian submanifold* if $\tilde\iota^*(\gamma) \equiv 0$. If $\tilde\iota : L \to P$ is Legendrian then $\pi \circ \tilde\iota : L \to P$ is a Lagrangian immersion. Conversely, if $\iota : L \to M$ is a Lagrangian immersion, then $\iota^*(P) \to L$ is a U(1)-bundle with a flat U(1)-connection, and there exists a Legendrian immersion $\tilde\iota : L \to P$ with $\iota = \pi \circ \tilde\iota$ if and only if this flat U(1)-connection has a constant section, that is, if it is trivial. Since flat U(1)-connections are classified by morphisms $H^1(L;\mathbb{Z}) \to \mathrm{U}(1)$, a sufficient condition for an immersed Lagrangian $\iota : L \to M$ to lift to an immersed Legendrian $\tilde\iota : L \to P$ is that $H^1(L;\mathbb{Z}) = \{0\}$.

If $\tilde\iota : L \to P$ is an embedding we identify L with $\tilde\iota(L) \subset P$ and regard L as a subset of P, with $\gamma|_L \equiv 0$. Generic Legendrians in P are embedded. If $L \subset P$ is an embedded Legendrian then $\pi = \pi|_L : L \to M$ is an immersed Lagrangian, which in general is *not* embedded.

We call two Legendrian immersions $\tilde\iota : L \to P$, $\tilde\iota' : L' \to P$ *immersed Legendrian isotopic* if there exists a diffeomorphism $h : L \to L'$ and a smooth 1-parameter family $\tilde\iota_t : t \in [0,1]$ of Legendrian immersions $\tilde\iota_t : L \to P$, such that $\tilde\iota_0 = \tilde\iota$ and $\tilde\iota_1 = \tilde\iota' \circ h$. If $\tilde\iota, \tilde\iota'$ are embeddings, we call $\tilde\iota : L \to P$, $\tilde\iota' : L' \to P$ *embedded Legendrian isotopic* if there exist $\tilde\iota_t : t \in [0,1]$ as above with each $\tilde\iota_t : L \to P$ an embedding. Clearly, embedded Legendrian isotopic implies immersed Legendrian isotopic.

If $\tilde\iota : L \to P$, $\tilde\iota' : L' \to P$ are Legendrian immersions and $h : L \to L'$, $\tilde\iota_t : t \in [0,1]$ is an immersed Legendrian isotopy between them, then $\pi \circ \tilde\iota : L \to M$, $\pi \circ \tilde\iota' : L' \to M$ are Lagrangian immersions, and $h : L \to L'$, $\pi \circ \tilde\iota_t : t \in [0,1]$ is a local Hamiltonian equivalence between them, in the sense of Definition 13.14(ii). Conversely, if $\iota : L \to M$, $\iota' : L' \to M$ are locally Hamiltonian equivalent Lagrangian immersions, then there exists a Legendrian lift $\tilde\iota : L \to P$ with $\iota \equiv \pi \circ \tilde\iota$ if and only if there exists a Legendrian lift $\tilde\iota' : L' \to P$ with $\iota' \equiv \pi \circ \tilde\iota'$, and then $h, \iota_t : t \in [0,1]$ in Definition 13.14(ii) lift to an immersed Legendrian isotopy $h, \tilde\iota_t : t \in [0,1]$ between $\tilde\iota : L \to P$ and $\tilde\iota' : L' \to P$. So *local Hamiltonian equivalence in M corresponds exactly to immersed Legendrian isotopy in P.*

Now embedded Legendrian isotopies are a special class of immersed Legendrian isotopies, and so project to a special class of local Hamiltonian equivalences. Question 13.15 asked whether Floer cohomology is invariant under any special classes of local Hamiltonian equivalences. So it makes sense to ask:

Question 13.18. In the situation above, let $L_0, L_1 \subset P$ be compact embedded Legendrians. Suppose that the Lagrangian immersions $\pi : L_0 \to M$, $\pi : L_1 \to M$ have only transverse double self-intersections. Is Floer cohomology $HF^*\big((\pi : L_0 \to M, b_0); \Lambda_{\rm nov}\big), HF^*\big((\pi : L_0 \to M, b_0), (\pi : L_1 \to M, b_1); \Lambda_{\rm nov}\big)$ preserved under embedded Legendrian isotopies of L_0, L_1?

The authors expect the problem to be better behaved if we work over a smaller Novikov ring $\Lambda_{\rm nov}^{\mathbb{Z}}$. Suppose $L \subset P$ is a compact embedded Legendrian, and $\pi : L \to M$ has only transverse double points. Define R as in §4.1. If $(p_-, p_+) \in R$ then $p_-, p_+ \in L$ with $p_- \neq p_+$ and $\pi(p_-) = \pi(p_+)$ in M. Thus p_-, p_+ are distinct points in the same U(1)-orbit, and $p_+ = {\rm e}^{\sqrt{-1}\,\theta} \cdot p_-$ for some unique $\theta \in (0, 2\pi)$. Define $a_{(p_-,p_+)} = \frac{\theta}{2\pi}$. Then $a_{(p_-,p_+)} \in (0,1)$, and $a_{(p_-,p_+)} + a_{(p_+,p_-)} = 1$.

The areas of J-holomorphic curves in M with boundaries in $\pi(L)$ have an integrality property involving the $a_{(p_-,p_+)}$ for $(p_-, p_+) \in R$. We can express it like this: if $\overline{\mathcal{M}}_{k+1}(\alpha, \beta, J) \neq \emptyset$ and $[\omega]_{M,\pi(L)}$ is the class of ω in $H^2(M, \pi(L); \mathbb{R})$ then

$$\beta \cdot [\omega]_{M,\pi(L)} - \textstyle\sum_{i \in I} a_{\alpha(i)} \in \mathbb{Z}. \tag{155}$$

To prove (155), suppose $[\Sigma, \vec{z}, u, l, \bar{u}] \in \overline{\mathcal{M}}_{k+1}(\alpha, \beta, J)$, and for simplicity take $\Sigma \cong D^2$ nonsingular. Then $\bar{u} : \mathcal{S}^1 \setminus \{\zeta_i : i \in I\} \to L$ is smooth, with $(\lim_{\theta \uparrow 0} \bar{u}(e^{\sqrt{-1}\,\theta}\zeta_i), \lim_{\theta \downarrow 0} \bar{u}(e^{\sqrt{-1}\,\theta}\zeta_i)) = \alpha(i)$ in R, for all $i \in I$.

Modify this $\bar{u}$ to a piecewise smooth map $\tilde{u} : \mathcal{S}^1 \to P$ by inserting at each ζ_i for $i \in I$, the line segment $[0, 2\pi a_{(p_-,p_+)}] \to P$ mapping $\theta \mapsto {\rm e}^{\sqrt{-1}\,\theta} \cdot p_-$, where $\alpha(i) = (p_-, p_+)$. Then $\int_{\mathcal{S}^1} \tilde{u}^*(\gamma) = 2\pi \sum_{i \in I} a_{\alpha(i)}$, since $\gamma|_L \equiv 0$ and $v \cdot \gamma \equiv 1$. Now consider the U(1)-bundle $u^*(P) \to \Sigma$. It has a connection $u^*(\gamma)$ with curvature $2\pi u^*(\omega)$, and we have in effect constructed a section $\tilde{u}$ of $u^*(P)|_{\partial\Sigma}$ with $\int_{\partial\Sigma} \tilde{u}^*(\gamma) = 2\pi \sum_{i \in I} a_{\alpha(i)}$. But $\int_\Sigma 2\pi\, u^*(\omega) = \int_{\partial\Sigma} \tilde{u}^*(\gamma) + 2\pi c_1\big(u^*(P); \tilde{u}\big)$, where $c_1\big(u^*(P); \tilde{u}\big) \in \mathbb{Z} \cong H^2(\Sigma, \partial\Sigma; \mathbb{Z})$ is the first Chern class of the U(1)-bundle $u^*(P) \to \Sigma$ relative to the trivialization of $u^*(P)|_{\partial\Sigma}$ induced by $\tilde{u}$. Putting all this together gives (155).

By analogy with (13)–(16), define *Novikov rings*

$$\begin{aligned}
\Lambda_{\rm nov}^{\mathbb{Z}} &= \big\{\textstyle\sum_{i=0}^\infty a_i T^{\lambda_i} e^{\mu_i} : a_i \in \mathbb{Q},\ \lambda_i \in \mathbb{Z},\ \mu_i \in \mathbb{Z},\ \lim_{i\to\infty} \lambda_i = \infty\big\},\\
\Lambda_{\rm nov}^{\mathbb{N}} &= \big\{\textstyle\sum_{i=0}^\infty a_i T^{\lambda_i} e^{\mu_i} : a_i \in \mathbb{Q},\ \lambda_i \in \mathbb{N},\ \mu_i \in \mathbb{Z},\ \lim_{i\to\infty} \lambda_i = \infty\big\},\\
\Lambda_{\rm CY}^{\mathbb{Z}} &= \big\{\textstyle\sum_{i=0}^\infty a_i T^{\lambda_i} : a_i \in \mathbb{Q},\ \lambda_i \in \mathbb{Z},\ \lim_{i\to\infty} \lambda_i = \infty\big\},\\
\Lambda_{\rm CY}^{\mathbb{N}} &= \big\{\textstyle\sum_{i=0}^\infty a_i T^{\lambda_i} : a_i \in \mathbb{Q},\ \lambda_i \in \mathbb{N},\ \lim_{i\to\infty} \lambda_i = \infty\big\},
\end{aligned}$$

where $\mathbb{N} = \{0, 1, 2, \ldots\} \subset \mathbb{Z}$. Then in the situation of §11, having constructed $\mathcal{X}$ define $\widetilde{\mathbb{Q}\mathcal{X}}$ to be the $\mathbb{Q}$-vector space with basis f for $f \in \mathcal{X}$ with $f : \Delta_a \to L$, and $T^{a_{(p_-,p_+)}} f$ for $f \in \mathcal{X}$ with $f : \Delta_a \to$

$\{(p_-,p_+)\}\subset R$. Similarly, modifying (138), define a $\mathbb{Q}$-vector space $\tilde{\mathcal{H}}=\bigoplus_{d\in\mathbb{Z}}\tilde{\mathcal{H}}^d$ by

$$\tilde{\mathcal{H}}^d=H_{n-d-1}(L;\mathbb{Q})\oplus\bigoplus\nolimits_{\substack{(p_-,p_+)\in R:\\ d=\eta_{(p_-,p_+)}-1}}\mathbb{Q}\cdot T^{a_{(p_-,p_+)}}(p_-,p_+).$$

We can then go through §7–§13 using $\Lambda_{\rm nov}^{\mathbb{Z}},\Lambda_{\rm nov}^{\mathbb{N}},\Lambda_{\rm CY}^{\mathbb{Z}},\Lambda_{\rm CY}^{\mathbb{N}}$ in place of $\Lambda_{\rm nov},\Lambda_{\rm nov}^0,\Lambda_{\rm CY},\Lambda_{\rm CY}^0$, and $\widetilde{\mathbb{Q}\mathcal{X}}\hat\otimes\Lambda_{\rm nov}^{\mathbb{N}},\tilde{\mathcal{H}}\otimes\Lambda_{\rm nov}^{\mathbb{N}},\tilde{\mathcal{H}}\otimes\Lambda_{\rm nov}^{\mathbb{Z}},\widetilde{\mathbb{Q}\mathcal{X}}\hat\otimes\Lambda_{\rm CY}^{\mathbb{N}},\tilde{\mathcal{H}}\otimes\Lambda_{\rm CY}^{\mathbb{N}},\tilde{\mathcal{H}}\otimes\Lambda_{\rm CY}^{\mathbb{Z}}$ in place of $\mathbb{Q}\mathcal{X}\hat\otimes\Lambda_{\rm nov}^0,\mathcal{H}\otimes\Lambda_{\rm nov}^0,\mathcal{H}\otimes\Lambda_{\rm nov},\mathbb{Q}\mathcal{X}\hat\otimes\Lambda_{\rm CY}^0,\mathcal{H}\otimes\Lambda_{\rm CY}^0,\mathcal{H}\otimes\Lambda_{\rm CY}$, respectively. The integrality condition (155) and the definitions of $\widetilde{\mathbb{Q}\mathcal{X}},\tilde{\mathcal{H}}$ ensure we can choose $\mathfrak{m}_k$ to map $\bigl(\widetilde{\mathbb{Q}\mathcal{X}}\hat\otimes\Lambda_{\rm nov}^{\mathbb{N}}\bigr)^{\times^k}\to\widetilde{\mathbb{Q}\mathcal{X}}\hat\otimes\Lambda_{\rm nov}^{\mathbb{N}}$, and similarly for $\mathfrak{n}_k$. That is, only powers T^l or $T^{l+a_{(p_-,p_+)}}$ for $l\in\mathbb{N}$ and $(p_-,p_+)\in R$ occur in $\widetilde{\mathbb{Q}\mathcal{X}}\hat\otimes\Lambda_{\rm nov}^{\mathbb{N}}$, and in the terms $T^\lambda e^\mu\mathfrak{m}_k^{\lambda,\mu}$ in $\mathfrak{m}_k$, the only allowed values for $\lambda\in\mathbb{R}$ are those which take possible total powers of T in $\bigl(\widetilde{\mathbb{Q}\mathcal{X}}\hat\otimes\Lambda_{\rm nov}^{\mathbb{N}}\bigr)^{\times^k}$ to possible powers of T in $\widetilde{\mathbb{Q}\mathcal{X}}\hat\otimes\Lambda_{\rm nov}^{\mathbb{N}}$.

Thus, in §11 we construct gapped filtered A_∞ algebras $(\widetilde{\mathbb{Q}\mathcal{X}}\hat\otimes\Lambda_{\rm nov}^{\mathbb{N}},\mathfrak{m})$ and $(\tilde{\mathcal{H}}\otimes\Lambda_{\rm nov}^{\mathbb{N}},\mathfrak{n})$ over $\Lambda_{\rm nov}^{\mathbb{N}}$, and in the graded case of §12 we construct $(\widetilde{\mathbb{Q}\mathcal{X}}\hat\otimes\Lambda_{\rm CY}^{\mathbb{N}},\mathfrak{m})$ and $(\tilde{\mathcal{H}}\otimes\Lambda_{\rm CY}^{\mathbb{N}},\mathfrak{n})$ over $\Lambda_{\rm CY}^{\mathbb{N}}$. Then as in §13.1–§13.3, we define Lagrangian Floer cohomology $HF^*\bigl((L,b);\Lambda_{\rm nov}^{\mathbb{N}}\bigr),HF^*\bigl((L,b);\Lambda_{\rm nov}^{\mathbb{Z}}\bigr)$, over $\Lambda_{\rm nov}^{\mathbb{N}}$ or $\Lambda_{\rm nov}^{\mathbb{Z}}$, and similarly for two Lagrangians, and for graded Lagrangians over $\Lambda_{\rm CY}^{\mathbb{N}},\Lambda_{\rm CY}^{\mathbb{Z}}$. Several of the definitions of gapped filtered A_∞ algebras, morphisms, etc. require minor modification to allow for inclusion of factors $T^{a_{(p_-,p_+)}}$ in $\widetilde{\mathbb{Q}\mathcal{X}},\tilde{\mathcal{H}}$.

We can now make our most important point. Consider the wall-crossing phenomenon described in Example 13.16. This occurs when, for a family of immersed Lagrangians $\iota_t:L\to M$ for $t\in[0,1]$, we have a family of bounding cochains $b_t\in F^{\lambda(t)}(\mathcal{H}_t\otimes\Lambda_{\rm nov})$, where $\lambda(t)>0$ is necessary for b_t to be a bounding cochain. If $\lambda(\epsilon)=0$ then at $t=\epsilon$ we cross a 'wall' where b_t ceases to be a bounding cochain.

Now if $\iota_t=\pi\circ\tilde\iota_t$ for a smooth family of Legendrian embeddings $\tilde\iota_t:L\to M$, then the only allowed powers of T in bounding cochains $b(t)$ are T^l for $l=1,2,\ldots$ and $T^{l+a_{(p_-,p_+)}(t)}$ for $l=0,1,\ldots$, where $a_{(p_-,p_+)}(t)\in(0,1)$. Thus, the leading power of T in b_t could only deform continuously to zero at $t=\epsilon$ if $a_{(p_-,p_+)}(t)\to0$ as $t\to\epsilon$. But $a_{(p_-,p_+)}(\epsilon)=0$ implies that $\tilde\iota_\epsilon(p_-)=\tilde\iota_\epsilon(p_+)$, that is, $\tilde\iota_\epsilon:L\to P$ is an immersion, but not an embedding.

This shows that *the wall-crossing phenomenon in Example* 13.16 *cannot happen for bounding cochains for* $(\tilde{\mathcal{H}}\otimes\Lambda_{\rm nov}^{\mathbb{N}},\mathfrak{n})$ *under embedded Legendrian isotopy.* If, as Conjecture 13.17 claims, this is the only mechanism by which Floer cohomology changes under local Hamiltonian equivalence, then Floer cohomology over $\Lambda_{\rm nov}^{\mathbb{Z}}$ should be unchanged under embedded Lagrangian isotopy. So we conjecture:

Conjecture 13.19. *In the situation above, suppose that $\tilde\iota_t : L \to P$ for $t \in [0,1]$ is a smooth 1-parameter family of Legendrian embeddings with L compact, oriented, and spin, and that $\pi \circ \tilde\iota_0 : L \to M$ and $\pi \circ \tilde\iota_1 : L \to M$ have only transverse double self-intersections. Then there should exist a canonical bijection $\Psi : \mathcal{M}_{\mathcal{H}_0,\mathfrak{n}^0} \to \mathcal{M}_{\mathcal{H}_1,\mathfrak{n}^1}$ between the moduli spaces of bounding cochains for $\pi \circ \tilde\iota_0 : L \to M$ and $\pi \circ \tilde\iota_1 : L \to M$. Let $b_0 \in \widehat{\mathcal{M}}_{\mathcal{H}_0,\mathfrak{n}^0}$ and $b_1 \in \widehat{\mathcal{M}}_{\mathcal{H}_1,\mathfrak{n}^1}$ with $\Psi(G_{\mathcal{H}_0,\mathfrak{n}^0} \cdot b_0) = G_{\mathcal{H}_1,\mathfrak{n}^1} \cdot b_1$, and suppose L_2 is a compact embedded Legendrian in P, such that $\pi : L_2 \to M$ has only transverse double self-intersections, and b_2 is a bounding cochain for $\pi : L_2 \to M$. Then there are canonical isomorphisms*

$$HF^*\big((\pi\circ\tilde\iota_0 : L \to M, b_0);\Lambda^{\mathbb{Z}}_{\mathrm{nov}}\big) \cong HF^*\big((\pi\circ\tilde\iota_1 : L \to M, b_1);\Lambda^{\mathbb{Z}}_{\mathrm{nov}}\big),$$
$$HF^*\big((\pi\circ\tilde\iota_0 : L \to M, b_0),(\pi : L_2 \to M, b_2);\Lambda^{\mathbb{Z}}_{\mathrm{nov}}\big) \cong$$
$$HF^*\big((\pi\circ\tilde\iota_1 : L \to M, b_1),(\pi : L_2 \to M, b_2);\Lambda^{\mathbb{Z}}_{\mathrm{nov}}\big).$$

This conjecture suggests there should exist a theory of *Legendrian Floer cohomology* for embedded Legendrians in contact manifolds P which are U(1)-bundles over symplectic manifolds (M,ω). This should clearly be related to the theory of *Legendrian contact homology*, which was described informally by Eliashberg, Givental and Hofer [**4**, §2.8], and by Chekanov [**3**] for Legendrian knots in $\mathbb{R}^3$, and has been developed rigorously by Ekholm, Etnyre and Sullivan [**5**, **6**], for embedded Legendrians L in $\mathbb{R}^{2n+1}$ and in $M \times \mathbb{R}$ for (M,ω) an exact symplectic manifold.

In particular, for (M,ω) exact one can compare our $HF_*(L,b;\Lambda^{\mathbb{Z}}_{\mathrm{nov}})$ for embedded Legendrians in $M \times \mathcal{S}^1$, and Ekholm et al.'s $HC_*(L,J)$ for embedded Legendrians L in $M \times \mathbb{R}$, [**6**]. It seems that $HC_*(L,J)$ should be a sector of $HF_*(L,b;\Lambda^{\mathbb{Z}}_{\mathrm{nov}})$, but not the whole thing, since $HC_*(L;J)$ is the homology of a complex involving $H_1(L;\mathbb{Z})$ and the set of double points of $\pi(L)$ in M, but $HF_*(L,b;\Lambda^{\mathbb{Z}}_{\mathrm{nov}})$ is the cohomology of a complex involving all of $H_*(L;\mathbb{Q})$ and R, which has two points $(p_-,p_+),(p_+,p_-)$ for each double point p of $\pi(L)$ in M. We hope our conjecture will lead to progress in Legendrian contact homology.

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