Notes and questions to aid A-level Mathematics revision

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1 Introduction

There are some students who find the first year's study at UCL and many other universities difficult. Of course there can be many reasons for this but these notes aim to help overcome problems that may arise from:

1. Difficulties students have carrying out algebraic manipulation accurately and quickly. The best remedy for this is practice and more practice.

2. Difficulties which arise from a lack of knowledge, or more likely application of the students' knowledge in new areas.

If you work through this document and do the questions you will hopefully find yourself better prepared for starting your course at UCL. You are likely to find that the pace of a university mathematics course is greater that is the case at school. New ideas and results will be introduced and used in rapid sucession. You will be better able to cope with the difficulties this can lead to if you are totally familiar with the material you have covered at school. It is also worth starting to prepare now for the fact that in the mathematics exams at UCL you are not allowed the use of a calculator, a book of tables or a formula sheet!

The material here concentrates on some of the knowledge and techniques required for the first term M14A methods course although you will also find it helpful for the applied mathematics course M13A. It does not cover everything you will need to know.

Please note carefully:

1. There is a core A-level syllabus which *all* students completing a single maths A-level may be expected to know. Much of the material below is in the core A-Level syllabus but some is on the edges of the core and you may not be familiar with it especially those of you with a single mathematics A-level. A little material is definitely not in the core but is presented here anyway because it follows naturally from other material.

2. The entrance requirements at UCL do not require two A-levels in mathematics. Indeed to insist on such qualifications could well prevent many students from studying maths at university. At UCL a great deal of effort is expended by the lecturers in the first year to make the course accessible to those with a single maths A-level. The off-core and further mathematics material is covered. However it is often covered briefly and it is in your interest to spend some time familiarising yourself with it now. These notes will help you in that task also. Over 60% of our students do in fact have Further Mathematics A-level.

3. It is hoped, but not guaranteed, that there will be additional lectures covering some of the material in these notes during the first few weeks of the course at UCL. So, should you find that these notes do not help you prepare adequately, extra support may be available then.

4. Be prepared to have to use your current A-level notes and texts to help answer some of the questions.

5. Few answers are given in these notes. Remember to check an integration all you need to do is differentiate and its good practice too. To check a division, multiply out. To check a partial fraction, put it together again etc.

Finally any comments on the relevance and difficulty of these exercise and examples will be gratefully received and will help enormously in improving this document for students in years to come. Look out for mistakes. I know there are a few in there.

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2 Algebraic Manipulation

2.1 Completing the square

You should know how to *complete the square* and be able to write

$$ax^2 + bx + c = a(x + b/2a)^2 + c - b^2/4a.$$

Qu. 2.A Complete the square for

a) $x^2 + 2x + 1$, b) $3x^2 + 4x - 5$ c) $2 - 3x - x^2$, d)$c + dx - 4x^2$.

2.2 Surds

You should know how to manipulate *surds*

Example

$$\frac{\sqrt{2}+1}{\sqrt{2}-1} = \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\left(\frac{\sqrt{2}+1}{\sqrt{2}+1}\right) = \frac{(\sqrt{2}+1)^2}{2-1},$$

as $(u + 1)(u - 1) = u^2 - 1$, with $u = \sqrt{2}$, so

$$\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{2+2\sqrt{2}+1}{1} = (3 + 2\sqrt{2})$$

Qu. 2.B Show

a) $\dfrac{1}{(1-\sqrt{2})^2} - \dfrac{1}{(1+\sqrt{2})^2} = 4\sqrt{2}$, b) $\dfrac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}} = 5 + 2\sqrt{6}$,

c) $\ln(\sqrt{2} - 1) = -\ln(\sqrt{2} + 1)$, d) $\dfrac{x}{\sqrt{y}+\sqrt{x}} + \dfrac{x}{\sqrt{y}-\sqrt{x}} = \dfrac{2x\sqrt{y}}{y-x}$.

Here $\ln x$ is the natural logarithm, or logarithm to base e of x. In addition we may write e^x or $\exp(x)$ to mean e raised to the power x

3 Trigonometrical Formulae

3.1 Elementary Formulae

You should know

1. The definitions

$$\operatorname{cosec}\theta = 1/\sin\theta$$
$$\sec\theta = 1/\cos\theta$$
$$\cot\theta = 1/\tan\theta$$

2. Pythagoras' theorem

$$\sin^2\theta + \cos^2\theta = 1 \qquad (\blacktriangledown)$$
$$\sec^2\theta = \tan^2\theta + 1$$
$$\operatorname{cosec}^2\theta = \cot^2\theta + 1.$$

(*How do we get these last two from the first* ▼*?*)

3. The **compound angle formulae**

$$\sin(a+b) = \sin a\cos b + \cos a\sin b, \qquad (\blacktriangle)$$
$$\sin(a-b) = \sin a\cos b - \cos a\sin b,$$
$$\cos(a+b) = \cos a\cos b - \sin a\sin b,$$
$$\cos(a-b) = \cos a\cos b + \sin a\sin b,$$
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b},$$
$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a\tan b},$$

(*Can you figure out how to derive these starting from just the first one* ▲*?. Can you prove* ▲*?. What assumptions do you start from?*)

4. The **double angle formulae**

$$\sin 2a = 2\sin a\cos a,$$
$$\cos 2a = \cos^2 a - \sin^2 a = 1 - 2\sin^2 a = 2\cos^2 a - 1,$$
$$\tan 2a = \frac{2\tan a}{1 - \tan^2 a}.$$

(*Can you derive these from the compound angle formulae?*)

In addition you should know the sine, cosine and tangent of the angles 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, $2\pi/3$, $3\pi/4$, etc and from them be able to evaluate the cosecant, secant and cotangent. Also you should be able to use the symmetries in the graphs of the trigonometric functions to find values for arguments outside the range 0 to $\pi/2$, including negative values of the argument. Notice radians are now the preferred measure of angle.

Example Show $\frac{1}{\tan a+\cot a} = \sin a\cos a = \frac{\sin 2a}{2}$.

$$\frac{1}{\tan a+\cot a} = \frac{1}{\frac{\sin a}{\cos a}+\frac{\cos a}{\sin a}} = \frac{\sin a\cos a}{\sin^2 a+\cos^2 a} = \sin a\cos a$$

and

$$\frac{1}{2}\sin 2a = \frac{1}{2}2\sin a\cos a = \sin a\cos a.$$

Qu. 3.A

1) If $\sin\theta = 1/4$, what is $\sin(\pi-\theta)$, $\sin(\pi+\theta)$ and $\sin(2\pi-\theta)$.

2) If $\tan\theta = 0.2$, write down $\cot(\pi-\theta)$, $\cot(3\pi-\theta)$ and $\cot(-\theta)$.

3) Write in surd form $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$ and $\cot\theta$ when $\theta = 5\pi/6$, $\theta = 2\pi/3$ and $\theta = 7\pi/4$.

4) Find θ in the range 0 to 2π if $\sin\theta = -1/2$ and $\tan\theta = 1/\sqrt{3}$.

5) Show

a) $\cot\theta-\tan\theta = 2\cot 2\theta$, b) $\operatorname{cosec} 2\theta-\cot 2\theta = \cot\theta$,

c) $\dfrac{\sin\theta}{1+\cos\theta} = \tan(\theta/2)$, d) $\dfrac{\sin 2\theta+\cos 2\theta+1}{\sin 2\theta-\cos 2\theta+1} = \cot\theta$.

6) If $\sin a = 1/\sqrt{10}$ and $\sin b = 1/\sqrt{5}$ show $\sin(a+b) = 1/\sqrt{2}$.

7) Show

$$\frac{\sec\theta+\operatorname{cosec}\theta}{\tan\theta+\cot\theta} = \frac{\tan\theta-\cot\theta}{\sec\theta-\operatorname{cosec}\theta}.$$

8) Show

a) $\sec^2\theta+\operatorname{cosec}^2\theta = 4\operatorname{cosec}^2 2\theta$, b) $\tan\theta+\cot\theta = \dfrac{2}{\sin 2\theta}$,

c) $\sin(a+b)\sin(a-b) = \sin^2 a-\sin^2 b$.

3.2 Using these results

You should know that the expression $a\cos\theta + b\sin\theta$ may be written in the form $R\cos(\theta-\alpha)$ or $R\sin(\theta-\beta)$ for positive R, although α and β may be of either sign. If

$$R\cos(\theta-\alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha = a\cos\theta + b\sin\theta,$$

then

$$R\cos\alpha = a, \qquad R\sin\alpha = b,$$

so that squaring

$$R^2 = a^2+b^2, \quad \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}, \quad \tan\alpha = \frac{b}{a}$$

Example Find the maximum and minimum value of $2\cos\theta + 3\sin\theta$

$$2\cos\theta + 3\sin\theta = R\cos(\theta-\alpha)$$

where

$$R^2 = 2^2+3^2 = 13, \quad \tan\alpha = 3/2$$

The maximum value occurs where $(\theta-\alpha) = 0$ so for $\theta = \arctan(3/2)$. The maximum is $R = \sqrt{13}$. Note I am happy to leave the answer in this form and not obtain an approximate value of θ or R by using a calculator. You will not be allowed to use calculators in nearly all your exams at UCL.

Qu. 3.B

1) Show $\cos\theta + \sin\theta = \sqrt{2}\cos(\theta - \pi/4)$ and find the solutions in the range $-\pi$ to π to the equation $\cos\theta + \sin\theta = 1$ together with the maximum and minimum values of the expression.

2) Find the range of values in 0 to 2π for which a) $2\sin\theta + \cos\theta$ is positive and b) $4\cos\theta - 3\sin\theta$ is negative.

3.3 More formulae

You should know the formulae

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right),$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right),$$

(*How do you derive these from the compound angle formulae?*)

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4 Series

4.1 Geometric Series

You should know The formula for the sum to n terms of the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = a\frac{1-r^n}{1-r},$$

and that if $|r| < 1$, the sum to infinity is $a/(1-r)$

Example

1. The series 2, 2/3, 2/9, 2/27, etc has first term $a = 2$ and common ratio $r = 1/3$ which has $|r| < 1$ and the series has a sum equal to $2/(1-1/3) = 6/(3-1) = 3$.

2. The series
$$\sin 2\alpha - \sin 2\alpha\cos 2\alpha + \sin 2\alpha\cos^2 2\alpha + \cdots,$$
has a first term $a = \sin 2\alpha$ and a common ratio $(-\cos 2\alpha)$ and so a sum to infinity of $\sin 2\alpha/(1+\cos 2\alpha) = 2\sin\alpha\cos\alpha/2\cos^2\alpha = \tan\alpha$ if $|\cos\alpha| < 1$.

Qu. 4.A

1) Find the sum to infinity, when it exists, of

a) 5, 10, 20, 40, $\ldots$, b) 1/2, 1/4, 1/8, 1/16, $\ldots$, c) 1, .1, .01, 0.001, $\ldots$,
d) a, a/r, a/r^2, $a/r^3, \ldots$, e) x, x^2/y, x^3/y^2, $x^4/y^3, \ldots$.

2) Find the sum to infinity of

$$1 + \frac{x}{1+x} + \frac{x^2}{(1+x)^2} + \cdots,$$

and determine the set of values of x for which the result holds.

3) Find the set of values of θ for which the series

$$1 + 2\cos^2\theta + 4\cos^4\theta + 8\cos^6\theta + \cdots$$

has a sum to infinity and show that for these values of θ the sum is $-\sec 2\theta$.

4.2 The Binomial Expansion

You should know that, if n is a positive integer,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \cdots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}x^r + \cdots + x^n.$$

and that

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} = {}^nC_r = \binom{n}{r}.$$

Example

1. $(1+4x)^4 = 1 + (4)(4x) + (4.3)/(2)(4x)^2 + (4.3.2)/(3.2)(4x)^3 + (4.3.2.1)/(4.3.2.1)(4x)^4 = 1 + 16x + 96x^2 + 256x^3 + 256x^4$
2. To find the term independent of x in the expansion of $\left(x^2 - \frac{2}{x}\right)^6$, note that this is $\left(x^2 + \frac{-2}{x}\right)^6$ with a general term in its expansion $\binom{6}{r}(x^2)^{6-r}\left(\frac{-2}{x}\right)^r = \binom{6}{r}x^{12-3r}(-2)^r$ so that the contribution independent of x has $r = 4$ and is ${}^6C_4(-2)^4 = 6! \times 16/4!2! = 6.5.8 = 240$.

Qu. 4.B

1) Find the coefficient of x^3 in $(1-2x)^5$.
2) Show
$$(2x-3y)^5 = 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5.$$
3) Find the coefficient of x^3 in $(3-x)^{10}$ and of y^4 in $(2-3y)^7$.
4) Simplify, using the binomial theorem $(2x-3)^3 - (2x+3)^4$.
5) What is the term independent of x in the expansion of $\left(x - \frac{3}{x^2}\right)^{15}$?
6) Find the first three terms of $(1-3x+x^2)^8$—write it as $(1-3x(1-x/3))^8$ and use the binomial theorem.
7) Find the coefficient of x^5 in $(1+x+x^2)^4$.
8) Find the coefficient of x^2 in $(2+2x+x^2)^n$.

You *may* also know: If n is not a postive integer, and need not even be an integer, then if $|x| < 1$,
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \cdots .$$
Also
$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots ,$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots ,$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots ,$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^{(n+1)} x^n}{n!} + \cdots \qquad |x| < 1.$$

5 Partial Fractions

5.1 Division of polynomials

You should know How to divide one polynomial $P(x)$ by a second $Q(x)$ to find the quotient and remainder.

$$\frac{P(x)}{Q(x)} = \text{quotient} + \frac{\text{remainder}}{Q(x)}.$$

Example

$$\frac{x^4 - 2x^2 + 3x - 6}{x^2 - 4x + 3} = x^2 + 4x + 11 + \frac{35x - 39}{x^2 - 4x + 3}.$$

$$\begin{array}{r l}
 & x^2 + 4x + 11 \qquad \text{(quotient)} \\
x^2 - 4x + 3 \,| & \overline{x^4 \qquad\quad - 2x^2 + 3x - \quad 6} \\
 & x^4 - 4x^3 + 3x^2 \\
 & \overline{\quad 4x^3 - 5x^2 + \ 3x} \\
 & \quad 4x^3 - 16x^2 + 12x \\
 & \qquad\quad \overline{11x^2 - 9x - 6} \\
 & \qquad\quad 11x^2 - 44x + 33 \\
 & \qquad\qquad \overline{35x - 39} \qquad \text{(remainder)}.
\end{array}$$

So the quotient is $x^2 + 4x + 11$ and the remainder is $35x - 39$ and we have the result.

Qu. 5.A Find the quotient and remainder for

a) $(x^3 - x^2 - 5x + 2)/(x + 2)$,
b) $(x^4 - 2x^2 + 3x - 6)/(x^2 - 4x + 3)$,
c) $(x^5 + x^4 + 3x^3 + 5x^2 + 2x + 8)/(x^2 - x + 2)$,
d) $(x^4 - 3x^2 + 7)/(x + 3)$,
e) $(3x^5 - 5x^4 + x^2 + 1)/(x^3 + 1)$,
f) $(x^3 - x^2 - 4)/(x^2 - 1)$,
g) $(2x^5 - 3x^2 + 1)/(x^2 + 2x)$,

5.2 Partial Fractions

You should know How to express a polynomial $\frac{P(x)}{Q(x)}$ as a sum of *partial fractions* if the denominator $Q(x)$ can factorise.

1. For every linear factor $(ax+b)$ of $Q(x)$ there will be a partial fraction of the form $\frac{A}{ax+b}$.
2. For every repeated linear factor $(ax+b)^2$, there will be two terms in the partial fraction expression: $\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$.
3. For every quadratic factor in $Q(x)$ of the form ax^2+bx+c there will be a contribution to the partial fraction expression of the form $\frac{Ax+B}{ax^2+bx+c}$

Can you work out what to do if factors are repeated more than once, or for repeated quadratic factors or for factors of degree higher than 2?

Once a partial fraction representation of the correct form, with unknown coefficients A, B, $C\ldots$ has been chosen, as above, then one brings all the terms together to a single term simply by adding the fractions in the usual way. Comparing the coefficients of similar powers of x in the numerator of this single term and in $P(x)$ one then obtains sufficient linear equations in the unknowns A, B, $C\ldots$ to enable them to be found uniquely. Alternative methods such as the cover-up rule may also be used.

Example

$$\frac{2}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{x^2-1} = \frac{(A+B)x+(B-A)}{x^2-1},$$

recognising the factors of the denominator and so choosing the form of the partial fraction representation. Next, comparing coefficients of x and 1 we have

$$A+B=0, \qquad B-A=2, \qquad \text{so} \qquad A=-1, \quad B=1,$$

and

$$\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}.$$

You should also know that if the degree of $P(x)$ is greater to equal to the degree of $Q(x)$, one should divide $Q(x)$ into $P(x)$ to obtain a quotient and a remainder $R(x)$ and then write $\frac{R(x)}{Q(x)}$ in partial fractions

Example So

$$\frac{4x^3+16x^2-15x+13}{(x+2)(2x-1)^2} = 1 + \frac{12x^2-8x+11}{(x+2)(2x-1)^2},$$

where we have divided $4x^3+16x^2-15x+13$ by $(x+2)(2x-1)^2 = 4x^3+4x^2-7x+2$ to get the quotient 1 and remainder $2x^2-8x+11$. Now we write

$$\frac{2x^2-8x+11}{(x+2)(2x-1)^2} = \frac{A}{x+2} + \frac{B}{2x-1} + \frac{C}{(2x-1)^2}$$

and proceed to find $A=3$, $B=0$ and $C=4$. So

$$\frac{4x^3+16x^2-15x+13}{(x+2)(2x-1)^2} = 1 + \frac{3}{x+2} + \frac{4}{(2x-1)^2}.$$

Note that if we had to differentiate this function several times, or to integrate it, it is much easier if the function is in its partial fraction form.

Qu. 5.B Express the following in partial fractions:

a) $\frac{x+2}{x^2-1}$, b) $\frac{4-3x}{(1-2x)(2+x)}$, c) $\frac{1}{(1-2x)(1-3x)}$,

d) $\frac{2}{(x+7)(x+9)}$, e) $\frac{2x+5}{(x+2)(x+3)}$, f) $\frac{x^2+10x+6}{x^2+2x-8}$,

g) $\frac{x^3}{(x+1)(x+2)}$, h) $\frac{3x}{(1-x)^2(1+x^2)}$, i) $\frac{x^3+x^2+2}{x^3-x^2+x-1}$.

6 Differentiation

6.1 Elementary results and their use

You should know the following table

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\ln x$	$1/x$
$\exp(x)$	$\exp(x)$

and how to use them in combination with the *product rule*, *quotient rule*, and *chain rule* to evaluate derivatives of combinations of these.

$y(x)$	$y'(x)$
$u(x)v(x)$	$u'(x)v(x) + u(x)v'(x)$ the product rule
$u(x)/v(x)$	$(v(x)u'(x) - v'(x)u(x))/v^2(x)$ the quotient rule
$f(g(x))$	$f'(g(x))g'(x)$ the chain rule

where a $'$ indicates differentiation.
Using these rules we obtain the the table

$f(x)$	$f'(x)$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$

which you should know.

Example

1) If $y = \sin^3 x$, then we may use the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

and $y = u^3$, $u = \sin x$, so that $\frac{dy}{du} = 3u^2 = 3\sin^2 x$ and $\frac{du}{dx} = \cos x$ so that $\frac{dy}{dx} = 3\sin^2 x \cos x$.

2) The chain rule can be used more than once to evaluate $\frac{dy}{dx}$ if $y = \exp(\cos(x^2))$ for example. Write $y = y(u(v(x)))$ with $y = \exp(u)$, $u = \cos(v)$ and $v = x^2$. Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx} = \exp(u)(-\sin v)(2x) = -2x(\sin x^2)\exp(\cos(x^2)).$$

3) If

$$y(x) = \frac{x^2 \ln(x)}{x + \sin(\exp(\cos x))}$$

then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

where $u = x^2 \ln(x)$, $v = x + \sin(\exp(\cos x))$,

$$\frac{du}{dx} = 2x\ln(x) + x^2(1/x) = x(2\ln x + 1),$$

$$\frac{dv}{dx} = 1 + (-\sin x)\exp(\cos x)\cos(\exp(\cos x)),$$

using the quotient rule, the product rule to differentiate $x^2 \ln(x)$ and the chain rule (twice !) to differentiate $\sin(\exp(\cos x))$.

4) If $y = \sec x$, then $y = \frac{1}{\cos x}$ so that

$$\frac{dy}{dx} = (-\sin x)\frac{-1}{(\cos x)^2} = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x \tan x.$$

Qu. 6.A

1) Differentiate

a) $y = x^3 + \cos x - \ln x + 4$, b) $y = 7x^4$, c) $y = xe^x$

d) $y = \dfrac{\sin x}{x^2}$, e) $y = \dfrac{\ln(5x)}{x^2}$, f) $y = (2x^3 - 1)\sin x$,

g) $y = e^x \sin x$, h) $y = x^5 \ln x + \cos x$. i) $y = \ln(x^3 + \sin x)$,

j) $y = x^{-3/2}$, k) $y = 2x^2(x+1) + 2$, l) $y = \dfrac{3x^4 - x}{x^3}$,

m) $y = \dfrac{x\cos x + \sin x}{x^2}$, n) $y = \dfrac{3x-1}{\sqrt{x^2+1}}$, o) $y = \tan^4(2x)$

p) $y = \dfrac{1}{\sqrt{4-x^2}}$, q) $y = \dfrac{2}{\sqrt{1-4x}}$, r) $y = \operatorname{cosec}\dfrac{1}{x}$,

s) $y = \dfrac{\tan 2x}{2x}$, t) $y = \operatorname{cosec}^2 \dfrac{x}{4}$, u) $y = \sec x \tan x$,

v) $y = \cos^3(\sqrt{x})$, w) $y = \dfrac{\tan x}{1-x}$, x) $y = \dfrac{\tan 3x}{x^3+1}$.

2) Show

$$\frac{d}{dx}(\tan x - x) = \tan^2 x.$$

6.2 Maxima and minima

You should know how to use differentiation to find *local maxima* and *minima* and *points of inflection*, the definition of these terms and how to use such techniques in curve sketching.

Qu. 6.B

1) Given $y = \frac{\sin x - \cos x}{\sin x + \cos x}$, show $dy/dx = 1 + y^2$. When is $d^2y/dx^2 = 0$?

2) Show $f(x) = x^3 - x^2 + x - 1$ is never decreasing.

3) A curve is given by $x = \ln(1+t)$, $y = e^{t^2}$ for $t > -1$. Find dy/dx and d^2y/dx^2 in terms of t. Show that the curve has only one turning point and that this must be a minimum.

6.3 A warning

Do not confuse the two expressions $\sin^{-1} x$ and $(\sin x)^{-1}$. The first is another way of writing $\arcsin x$ in just the same way as you may write the inverse of $f(x)$ as $f^{-1}(x)$. The second is a shorthand for $\frac{1}{\sin x}$. The confusion arises as we do often write $\sin^2 x$ for $(\sin x)^2$.

6.4 Differentiation of inverse functions

You should know how to differentiate inverse functions, using the fact

$$\frac{dy}{dx} = 1/\frac{dx}{dy}.$$

Results you should know are

$f(x)$	$f'(x)$
$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$
$\cos^{-1}\frac{x}{a}$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2+x^2}$

The first two seem to imply that $\sin^{-1} x + \cos^{-1} x$*=0. Why is this not, in fact, being implied? The answer lies in considering constants of integration.*

Example If $y = \sin^{-1}\left(\frac{x}{a}\right)$ then $x = a\sin y$ and differentiating with respect to y, we find $\frac{dx}{dy} = a\cos y$. Now write $a\cos y = a\sqrt{1-\sin^2 y}$, if $\cos y \geq 0$, so that $\frac{dx}{dy} = a\sqrt{1-x^2/a^2} = \sqrt{a^2-x^2}$ and so $\frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}$

Qu. 6.C

1) Derive the results in the table above.

2) Show

$$\frac{1}{a}\frac{d}{dx}\sec^{-1}\frac{x}{a} = -\frac{1}{a}\frac{d}{dx}\operatorname{cosec}^{-1}\frac{x}{a} = \frac{1}{x\sqrt{x^2-a^2}}$$

7 Integration

7.1 Elementary Integration

You should know That integration is the inverse of differentiation. One should therefore be able to recognise integrals that may be done directly, or almost directly from the tables of derivatives above. It cannot be overstressed how much success in integration relies on a thorough familiarity with the results of differentiating simple functions. Also of importance is a familiarity with the forms of derivatives that arise when the chain, product and quotient rules are used to differentiate combinations of these simple functions.

Qu. 7.A Write down the values of

a) $\displaystyle\int_a^b x^{10}\,dx$, b) $\displaystyle\int_0^1 x^{10}\,dx$, c) $\displaystyle\int_1^2 x^n\,dx$, d) $\displaystyle\int_2^3 \frac{1}{x}\,dx$,

e) $\displaystyle\int_0^{\pi/2} \cos x\,dx$, f) $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx$, g) $\displaystyle\int_0^{\pi/4} \sec x\tan x\,dx$, h) $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$,

i) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$, j) $\displaystyle\int_a^b x+\cos x\,dx$.

Qu. 7.B Evaluate:

a) $\displaystyle\int_0^1 (3+e^x)(2+e^{-x})\,dx$, b) $\displaystyle\int_1^4 \left(\frac{3}{x}-\sqrt{x}\right)\,dx$, c) $\displaystyle\int_0^{\pi/6} \sin 3x\,dx$,

d) $\displaystyle\int_1^3 \frac{dx}{2x-1}$, e) $\displaystyle\int_1^2 \frac{x^4-1}{x^3}\,dx$, f) $\displaystyle\int_1^8 \sqrt[3]{x}+\frac{1}{2\sqrt[3]{x}}\,dx$

Qu. 7.C This technique is very useful: Use trigonometrical formulae to express $\sin^2 x$ and $\cos^2 x$ in terms of $\cos 2x$ and $\tan^2 x$ in terms of $\sec^2 x$ or similar methods to integrate

a) $2\cos^2 x$, b) $3\sin^2 x$, c) $\cos^2 3x$, d) $\sin^2(x/2)$,

e) $\sin(2x)\cos(2x)$, f) $\tan^2 x$, g) $\tan^2(x/2)$, h) $-4\cos^4 3x$.

7.2 Integration by elementary substitution

You should know that do the integral $I = \int f(x)dx$ it is sometimes useful to make a substitution and introduce a new variable $u = g(x)$, so that $du/dx = g'(x)$ and $dx/du = 1/g'(x)$, say. Written in terms of u,

$$I = \int f(x)dx = \int F(u)\frac{dx}{du}du, \qquad \text{where} \qquad f(x) = F(u).$$

The new integrand must be written in terms of u by eliminating x in favour of u. In some cases and for the correct choice of substitution $u = g(x)$, this new form of the integral may be easier to do than the first. If the integral has limits then the new form becomes

$$I = \int_{x_0}^{x_1} f(x)dx = \int_{u_0}^{u_1} F(u)\frac{dx}{du}du \qquad \text{where} \qquad u_0 = g(x_0) \quad u_1 = g(x_1).$$

You should be able to recognise when to use a trigonometric substitution, suggested by the table in section 6.4.
A good attitude to integration is to try any substitution that comes into your head. It may work or it may not but in either case you have learnt something about the problem you face. Practice and then more practice will make it easier to spot the substitution that works.

Example

1) We may evaluate the integral $\int(7-2x)^4\,dx$ by using the binomial theorem to expand out the bracket and integrate term by term. Alternatively we may make the substitution $u = (7-2x)$ so that $du = -2dx$ and the integral becomes $-\int u^4/2\,du = -u^5/10 = -(7-2x)^5/10$. You should aim at being able to do integrals of this type immediately, without explicitly using the substitution. See section 7.3 below.

2) In the integral $\int \frac{\cos\sqrt{x}}{\sqrt{x}}$, we make the substitution $\sqrt{x} = u$—after all we know the integral of $\cos u$ so get rid of the square root which is worrying us. Now $x = u^2$ so $dx/du = 2u$. The integral becomes

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}} = \int \frac{\cos u}{u} 2u\, du = 2\int \cos u\, du = 2\sin u + c = 2\sin\sqrt{x} + c.$$

3) Consider now $I = \int_0^1 \frac{dx}{\sqrt{1+2x-x^2}}$. Here it is not clear how to proceed, until we complete the square and write $1+2x-x^2 = 2-(x-1)^2$ so that $I = \int_0^1 \frac{dx}{\sqrt{2-(x-1)^2}}$. This is similar to the form $\frac{1}{\sqrt{a^2-u^2}}$ which we know is the derivative of $\sin^{-1}\frac{u}{a}$. This helps us choose the correct trigonometric substitution and write $(x-1) = \sqrt{2}\sin u$. Thus $2-(x-1)^2 = 2-2\sin^2 u = 2\cos^2 u$ and $\frac{dx}{du} = \sqrt{2}\cos u$ the limits $x = 0$ and $x = 1$ become $u = \sin^{-1}(-1/\sqrt{2}) = -\pi/4$ and $u = 0$ so that

$$I = \int_{-\pi/4}^{0} \frac{\sqrt{2}\cos u\, du}{\sqrt{2}\cos u} = \frac{\pi}{4}$$

Qu. 7.D Use the given substitution to show:

a) $\displaystyle\int \frac{6}{(2x+1)^2}\, dx = c - \frac{3}{2x+1}, \qquad (u = 2x+1)$

b) $\displaystyle\int 2x\sqrt{3x+4}\, dx = c + 4(3x+4)^{5/2}/45 - 16(3x+4)^{3/2}/27, \qquad (u = 3x+4)$

c) $\displaystyle\int_0^2 \sqrt{4-x^2}\, dx = \int_0^{\pi/2} 4\cos^2\theta\, d\theta = \pi, \qquad (x = 2\sin\theta)$

d) $\displaystyle\int_0^2 x\sqrt{4-x^2}\, dx = 8/3, \qquad (u = 4-x^2)$

e) $\displaystyle\int_0^{\pi/2} \sin^3\theta\cos^2\theta\, d\theta = 2/15, \qquad (u = \cos\theta)$

Qu. 7.E Differentiate $\ln(x + \sqrt{x^2+k})$ with respect to x, where ln represents the logarithm to base e. Hence find

$$\text{a)} \int \frac{dx}{\sqrt{x^2+5}}, \qquad \text{b)} \int \frac{dx}{\sqrt{(x+2)^2+5}}, \qquad \text{c)} \int \frac{dx}{\sqrt{x^2+6x+2}}.$$

Qu. 7.F

1) Integrate

a) $\int 2x\sqrt{5x+1}\,dx$, b) $\int_1^2 x(2-x)^7\,dx$, c) $\int (3x+2)^3\,dx$,

d) $\int \sqrt{4-x}\,dx$, e) $\int \frac{dx}{\sqrt{1-2x}}$, f) $\int \frac{dx}{(\frac{1}{2}x+\frac{1}{3})^3}$

g) $\int \sin^3 x\cos^2 x\,dx$, h) $\int \frac{\cos x}{\sin^2 x}\,dx$, i) $\int 2x\sqrt{x^2+2}\,dx$

j) $\int (2x^2+x+1)^2(4x+1)\,dx$ k) $\int \frac{4x+1}{2x^2+x+1}\,dx$, l) $\int (x^3-3x)^2(x^2-1)\,dx$

2) If $t=\tan\frac{1}{2}\theta$, show that $dt=\frac{1}{2}(1+t^2)d\theta$ and hence do the integrals

a) $\int \frac{d\theta}{1-\cos\theta}$, b) $\int \frac{d\theta}{1+\sin\theta}$,

c) $\int \frac{d\theta}{5+4\cos\theta}$, d) $\int \frac{d\theta}{5-4\cos\theta}$.

3) Use the substitution $t=\tan\frac{1}{2}x$ to show these results, which you should learn

$f(x)$	$\int f(x)\,dx$
$\operatorname{cosec} x$	$\ln\lvert\tan\frac{1}{2}x\rvert + c = \ln\lvert\operatorname{cosec} x - \cot x\rvert + c$
$\sec x$	$\ln\lvert\tan(\frac{1}{4}\pi+\frac{1}{2}x)\rvert + c = \ln\lvert\sec x - \tan x\rvert + c$

You will have to do some algebraic manipulation of trigonometric quantities to show the equivalence of the last two forms.

4) Use the results of section 3.2 to evaluate

$$\int \frac{dx}{\sin x+\sqrt{3}\cos x}, \qquad \int \frac{dx}{\sqrt{2}\cos x+\sqrt{3}\sin x}.$$

7.3 Integration by recognition

After doing many integrals using substitution one becomes able to do integrals by immediately recognising they have a special form. For example since

$$\frac{d}{dx}\left[(\phi(x))^{n+1}\right] = (n+1)\phi'(x)[\phi(x)]^n,$$

from the chain rule, then we have

$$\int \phi'(x)[\phi(x)]^n\,dx = \frac{\phi(x)^{(n+1)}}{n+1} + c$$

if $n \neq -1$. The case $n = -1$ is taken care of by the observation

$$\frac{d}{dx}\left[\ln(\phi(x))\right] = \frac{\phi'(x)}{\phi(x)},$$

again from the chain rule, so

$$\int \frac{\phi'(x)}{\phi(x)} = \ln|\phi(x)| + c.$$

Similarly

$$\int \phi'(x)\exp(\phi(x))\,dx = \exp(\phi(x)) + c$$

Note ϕ is the greek letter *phi* and is often used instead of f for a function.

These rules give rise to the results

$f(x)$	$\int f(x)\,dx$
$\tan x$	$\ln\lvert\sec x\rvert + c$
$\cot x$	$\ln\lvert\sin x\rvert + c$

which you should derive and learn.

Example

1)

$$\int x\exp(x^2)\,dx = \frac{1}{2}\int 2x\exp(x^2)\,dx = \frac{1}{2}\exp(x^2) + c.$$

In practice one would soon learn to miss out the middle step in similar examples.

2)

$$\int \frac{x}{4x^2+7}\,dx = \frac{1}{8}\int \frac{8x}{4x^2+7}\,dx = \frac{1}{8}\ln|4x^2+7| + c.$$

Again practice would enable one to practically quote the result

3)

$$\int x^2(3x^3+12)^3\,dx = \frac{(3x^3+12)^4}{36} + c,$$

missing out that step.

Qu. 7.G Try these:

a) $\displaystyle\int \sec^2 x\tan^2 x\;dx,$ b) $\displaystyle\int \sec^2 x\tan^n x\;dx,$ c) $\displaystyle\int x^2(8+x^3)^5\;dx,$

d) $\displaystyle\int x^2(8+x^3)^n\;dx,$ e) $\displaystyle\int xe^{(1+x^2)}\;dx,$ f) $\displaystyle\int xf'(1+x^2)\;dx,$

g) $\displaystyle\int \frac{2ax+b}{ax^2+bx+c}\,dx,$ h) $\displaystyle\int \frac{\sin x}{\cos^2 x}\,dx,$ i) $\displaystyle\int \frac{\cos x}{\sin^2 x}\,dx.$

j) $\displaystyle\int \frac{\exp(x)+\exp(-x)}{\exp(x)-\exp(-x)}\,dx$ k) $\displaystyle\int \frac{2ax+b}{(ax^2+bx+c)^3}\,dx$ l) $\displaystyle\int (2ax+b)(ax^2+bx+c)^4\,dx$

m) $\displaystyle\int (\exp(x)+a)^n\exp(x)\,dx$ n) $\displaystyle\int \frac{\frac{1}{1+x^2}}{\tan^{-1}x}\,dx$ o) $\displaystyle\int \frac{dx}{x\ln(x)},$

p) $\displaystyle\int \frac{xdx}{\sqrt{1+x^2}},$ q) $\displaystyle\int \frac{dx}{\exp(x)+\exp(-x)},$ r) $\displaystyle\int \frac{dx}{2\sqrt{x}\sqrt{1-x}},$

s) $\displaystyle\int nx^{n-1}\cos x^n, dx,$ t) $\displaystyle\int \frac{1}{x}\cos(\ln(x))\,dx,$ u) $\displaystyle\int \frac{3x^2}{1+x^6}\,dx.$

7.4 Integrating rational functions

You should know From above

$$\int \frac{2ax+b}{ax^2+bx+c}dx = \ln|ax^2+bx+c| + \text{constant}.$$

Integrals of the form

$$\int \frac{1}{ax^2+bx+c}dx$$

may be approached by completing the square. If the quadratic has two real roots then proceed to use partial fractions and then integrate. Alternatively, if the roots are not real then partial fractions will not work but a trigonometric substitution may.

Integrals of the form

$$\int \frac{px+q}{ax^2+bx+c}$$

may be tackled by writing the integrand as

$$\frac{p}{2a}\frac{2ax+b}{ax^2+bx+c} + \frac{q-pb/2a}{ax^2+bx+c}$$

and both of these may be tackled using the techniques above.
Using these methods and partial frcations integrals of many rational functions of x can be obtained.

Example The partial fraction representation of

$$\frac{9x+9}{(x-3)(x^2+9)} = \frac{2}{x-3} + \frac{3-2x}{x^2+9},$$

so its integral is

$$2\ln|x-3| + 9\tan^{-1}\frac{x}{3} - \ln|x^2+9| + c = 9\tan^{-1}\frac{x}{3} + \ln\left|\frac{(x-3)^2}{x^2+9}\right| + c.$$

Qu. 7.H

1) Show

$$\int_0^1 \frac{x^2+7x+2}{(1+x^2)(2-x)}dx = \frac{11}{2}\ln 2 - \frac{\pi}{4}.$$

2) Show

$$\int \frac{x}{x^2+4x+5}dx = \frac{1}{2}\ln|x^2+4x+5| - 2\tan^{-1}(x+2) + c.$$

7.5 Integration by parts

You should know Integrating a rearranged form of the formula giving the derivative of a product $u(x)v(x)$,

$$u(x)\frac{dv}{dx} = u(x)v(x) - v(x)\frac{du}{dx}, \tag{1}$$

gives the formula for *integration by parts*

$$\int u(x)\frac{dv}{dx}\, dx = [u(x)v(x)] - \int v(x)\frac{du}{dx}\, dx \tag{2}$$

You should also know how to use this formula.

Example

1. Choosing $u = x^2$ and $dv/dx = e^x$, so that $du/dx = 2x$ and $v = e^x$, gives

$$\begin{aligned}\int x^2 e^x &= [x^2 e^x] - \int 2xe^x\, dx \\ &= x^2e^x - [2e^x] + \int 2e^x\, dx \\ &= x^2e^x - 2e^x + 2e^x + c \\ &= e^x(x^2 - 2x + 2) + c\end{aligned}$$

   where we have used integration by parts a second time with $u = 2x$, $dv/dx = e^x$ and $du/dx = 2$, $v = e^x$.

2. Sometimes you have to be a little ingenious in the choice of u and v. Here we choose $du/dx = 1$, $v = \ln x$ so that $u = x$, $dv/dx = 1/x$.

$$\begin{aligned}\int \ln x\, dx &= [x \ln x] - \int x.(1/x)\, dx \\ &= x(\ln x - 1) + c.\end{aligned}$$

Qu. 7.I Integration by parts will work for these:

a) $\displaystyle\int x\sin x\,dx$, b) $\displaystyle\int \ln x\,dx$, c) $\displaystyle\int x^2\cos x\,dx$,

d) $\displaystyle\int \sin^{-1} x\,dx$, e) $\displaystyle\int \tan^{-1} x\,dx$, f) $\displaystyle\int x\ln x\,dx$,

g) $\displaystyle\int x\tan^{-1} x\,dx$, h) $\displaystyle\int x\sec x\tan x\,dx$, i) $\displaystyle\int x\sec^2 x\,dx$

j) $\displaystyle\int x\exp(x)\,dx$, k) $\displaystyle\int x^3\exp(x)\,dx$, l) $\displaystyle\int x\sin x\cos x\,dx$.

Qu. 7.J These need more ingenuity:

a) $\displaystyle\int e^{2x}\cos 3x\,dx$, b) $\displaystyle\int e^{ax}\cos bx\,dx$, c) $\displaystyle\int \sqrt{a^2-x^2}\,dx$.

7.6 A mixture of integrals

Of course a good integrator needs no clue as to which of the above techniques to apply to an integral and is even able to use a selection of techniques, one after the other if necessary to succeed.

Qu. 7.K Try these:

a) $\displaystyle\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$, b) $\displaystyle\int x^3(x^4-3)^5\,dx$, c) $\displaystyle\int \tan x\,dx$,

d) $\displaystyle\int xe^{x^2}\,dx$, e) $\displaystyle\int \sin x(1+\cos^2 x)\,dx$, f) $\displaystyle\int x\sin 2x\,dx$,

g) $\displaystyle\int \frac{x^2}{\sqrt{1+x^3}}\,dx$, h) $\displaystyle\int (3+2x)^3\,dx$. i) $\displaystyle\int_2^3 \frac{dx}{(x-1)\sqrt{x^2-2x}}$,

j) $\displaystyle\int \sqrt{a^2-x^2}\,dx$, k) $\displaystyle\int_0^\pi x\sin^2 x\,dx$, l) $\displaystyle\int x^2\sin^{-1} x\,dx$,

m) $\displaystyle\int \operatorname{cosec} 2x\,dx$, n) $\displaystyle\int \frac{dx}{\cos^2 x-\sin^2 x}$, o) $\displaystyle\int \frac{\cot x}{\log\sin x}\,dx$.