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Further Results on Stability of Singular Time Delay Systems in the Sense of Non-Lyapunov: A New Delay Dependent Conditions

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Abstract In this paper, we consider the problem of finite-time stability of a class of linear singular continuous time delay systems. By using Lyapunov-like functional with time-delay, new delay-dependent stability condition has been derived in terms of matrix inequality such that the system under consideration is regular, impulse free and finite time stable. In the proposed stability criterion, Drazin inverse of a singular matrix is used.

Keywords: *singular time delayed systems, finite time stability, delay dependent conditions*

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1. Introduction

It was noticed that the characteristics of the dynamic and static state should be considered at the same time for some systems. Singular systems (also referred to as degenerate, descriptor, generalized, differential-algebraic or semi-state systems) are systems whose dynamic is governed by the complexity of algebraic and differential equations. Recently, many researchers have paid much attention to singular systems and they have accomplished numerous interesting conclusions. The complex nature of singular systems generates many difficulties in the analytical and numerical solution of such systems, particularly, when there is a need for their control.

Recently, the singular systems have been one of the major research fields of control theory. During the past three decades singular systems have attracted much attention due to the comprehensive applications in economics, as the Leontief dynamic model, in electrical applications using the theory described in [1], in mechanical models as in [2], etc. Singular systems in control theory have been initially discussed in [3] and [4].

The investigation of time delay systems has been carried out over many years. Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc.

It has been observed that variety of singular systems is characterized by the phenomena of time delay. Such systems are called singular differential systems with time delay. These systems have many special characteristics. In order to mathematically describe those systems more accurately, and to control them more effectively, the effort must be put to investigate this specific class of the singular systems. In this article, the new approach to the stability of the singular time delay systems is presented.

2. Stability Concepts

Classical stability concepts (e.g. Lyapunov stability, BIBO stability) are deal with systems operating over an infinite interval of time. These concepts require that state variables be bounded, whereby the values of the bounds are not prescribed. However, often asymptotic stability is not enough for practical applications, because there are some cases where large values of the state are not acceptable. For example, for the chemical processes (temperature, humidity, pressure. ..), rockets, airplanes and space vehicles, it is expected that their state variables will have been controlled within certain bounds over a finite time interval. In these cases, we need to check that these unacceptable values are not attained by the state. For this purposes, the concepts of the finite-time stability (FTS) and practical stability has been used. A system is said to be FTS if, once a time interval is fixed, its state does not exceed some bounds during this time interval.

Realizing this fact, numerous definitions of the so-called technical and practical stability have been introduced. Generally speaking, these definitions are essentially based on the predefined boundaries for the perturbation of initial conditions and the allowable perturbation of the system response. In the engineering applications of control systems, this fact becomes very important and sometimes crucial for the purpose of characterizing in advance in a quantitative manner including possible deviations of the system response. Thus, the analysis of these particular bound properties of the solutions is an important step, which precedes the design of control signals, when finite time or practical stability control is taken into account.

In this article the singular time delay systems have been considered. The various notations of stability over a finite time interval for continual-time systems and constant set trajectory bounds were introduced in [5,6,7].

3. Review of Previous Results

In the following short overview a few important results for the continuous linear systems in the area of the non-Lyapunov stability have been presented.

3.1. Finite Time Stability – Singular Systems Layout

In the context of practical stability for linear continuous singular systems, various results were presented in [8]. The further extensions of these results were presented in [9] for both regular and irregular singular systems. Namely, these papers have examined some practically important boundedness and associated unboundedness properties of the response of linear singular systems. For the first time the finite time stability of singular systems operating under perturbing forces has been investigated in papers [10,11,12] using the Coppel's inequality and matrix measure approach. The concept of finite time and practical stability applied to the class of time varying singular systems were presented in [13].

Necessary and sufficient conditions for the linear singular systems stability operating over a finite time interval have been derived in [14].

Furthermore, the reciprocal problem of the instability of the same class of systems has been solved in [15].

The initial work, where the Bellman-Gronwall approach was applied in the study of linear singular systems, was presented in [16]. The modified extension of the practical stability to general singular systems was introduced in [17]. A novel approach in time domain, based on the fundamental matrix of singular systems, has been applied in [18].

In the [19], the practical stabilization and controllability of singular systems have been examined using the modified extension approach. The finite-time control of linear singular systems with parametric uncertainness and disturbances was analyzed in the [20].

3.2. Finite Time Stability – Time Delay Systems

In the context of finite or practical stability for a particular class of the nonlinear singularly perturbed multiple time delay systems various results were presented in [21]. The authors presented stability definitions based on [6,7] adopted for time delay systems. Various results for finite time and practical stability for linear continuous time delay systems were first introduced in [22,23,24]. In these papers some basic results from the finite time and practical stability were extended to the particular class of linear continuous time delay systems. The matrix measure approach was applied to the analysis of the practical and finite time stability of linear time delayed systems in [10] and [25,26].

Based on the Coppel's inequality, introducing the matrix measure approach the work provided simple delay-dependent sufficient conditions of the practical and finite time stability with no need for time delay fundamental matrix calculations. The collection of all previous results and contributions were presented in [27,28] with the overall comments and modified Bellman–Gronwall approach.

Finally, the modified Bellman–Gronwall principle has been extended to the particular class of continuous non-autonomous time delayed systems operating over the finite time interval, [29].

3.3. Finite Time Stability – Singular Time Delay Systems

In [30] the idea of practical stability with time delays in terms of two measurements has been introduced. The work represented the first attempts to apply the non-Lyapunov concept to this specific class of control systems. The paper also shows the difficulty to calculate the time derivatives along the systems trajectory using the classical aggregate (Lyapunov) function for singular time delayed system.

4. Notations and Preliminaries

In this article, we have presented a novel approach to stability of singular time delay systems. The results have been directly expressed in terms of matrices E, A_0 and A_1 naturally occurring in the system model. In this approach there is no need to introduce any canonical form in the statement of the theorems. The geometric theory of consistency leads to the natural class of positive definite quadratic forms on the subspace containing all solutions. This fact makes the construction of the Lyapunov and non- Lyapunov stability theory possible even for the *linear continuous singular time-delay systems* (LCSTDS). Moreover, the attractive property is equivalent to the existence of symmetric positive definite solutions of a weak form of the Lyapunov matrix equation, incorporating conditions which refer to the boundedness of solutions.

Another approach is based on a classical theory mostly used in deriving sufficient delay independent conditions of the finite time stability. In the former case a new definition has been introduced based on the attractivity properties of the system solution which can be treated as analogous to the quasi-contractive stability as in [6,7].

The following notation has been used:

$\mathbb{R}$	Real vector space
$\mathbb{C}$	Complex vector space
I	Identity matrix
F^T	Transpose of matrix F
$F > 0$	Positive definite matrix
$F \geq 0$	Positive semi definite matrix
$\Re(F)$	Range of matrix F

$\aleph(F)$ Null space (kernel) of matrix F
$\lambda(F)$ Eigenvalue of matrix F
$\|F\|$ Euclidean matrix norm of F

The general expression of singular control systems with time delay can be written in its differential form as:

$$E(t)\dot{\mathbf{x}}(t)=\mathbf{f}\left(t,\mathbf{x}(t),\mathbf{x}(t-\tau),\mathbf{u}(t)\right),\quad t\geq 0 \qquad (1)$$
$$\mathbf{x}(t)=\boldsymbol{\varphi}(t),\ -\tau\leq t\leq 0$$

where $\mathbf{x}(t)\in\mathbb{R}^n$ is a state vector, $\mathbf{u}(t)\in\mathbb{R}^m$ is a control vector, $E(t)\in\mathbb{R}^{n\times n}$ is a singular matrix, $\boldsymbol{\varphi}\in\mathcal{C}=\left([-\tau,0],\mathbb{R}^n\right)$ is an admissible initial state functional, $\mathcal{C}=\mathcal{C}\left([-\tau,0],\ \mathbb{R}^n\right)$ is the Banach space of continuous functions mapping the interval [-τ, 0] into $\mathbb{R}^n$ with topology of uniform convergence.

The vector function satisfies:

$$\mathbf{f}(\cdot):\Im\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n \qquad (2)$$

and it is assumed to be smooth enough to assure the existence and uniqueness of solutions over a time interval:

$$\Im=\left[t_0,\ (t_0+T)\right[\ \in\mathbb{R}^+ \qquad (3)$$

as well as the continuous dependence of the solutions denoted by $\mathbf{x}(t,\ t_0,\mathbf{x}_0)$ with respect to t and the initial data.

Quantity T may be either a positive real number or the symbol $+\infty$, so that the finite time stability and practical stability can be treated simultaneously, respectively.

In general, it is not required that

$$\mathbf{f}(t,\ \mathbf{0},\ \mathbf{0})\equiv\mathbf{0} \qquad (4)$$

for an autonomous system, which means that the origin of the state space is not necessarily required to be an equilibrium state.

Let $\mathbb{R}^n$ denote the state space of a system given by (1) and $\|(\cdot)\|$ the Euclidean norm.

Let $V:\Im\times\mathbb{R}^n\to\mathbb{R}$, be the tentative aggregate function, so that $V(t,\mathbf{x}(t))$ is bounded and for which $\|\mathbf{x}(t)\|$ is also bounded.

Define the Eulerian derivative of $V(t,\mathbf{x}(t))$ along the trajectory of the system (2), with

$$\dot{V}(t,\mathbf{x}(t))=\frac{\partial V(t,\mathbf{x}(t))}{\partial t}+\left[grad\,V(t,\mathbf{x}(t))\right]^T\mathbf{f}(\cdot). \qquad (5)$$

For time-invariant sets it is assumed: $\mathcal{S}_{(\ )}$ is a bounded open set.

The closure and boundary of $\mathcal{S}_{(\ )}$ are denoted by $\bar{\mathcal{S}}_{(\ )}$ and $\partial\mathcal{S}_{(\ )}$, respectively, so: $\partial\mathcal{S}_{(\ )}=\bar{\mathcal{S}}_{(\ )}\setminus\mathcal{S}_{(\ )}$.

Let $\mathcal{S}_\beta$ be a given set of all allowable states of the system $\forall t\in\Im$.

Set $\mathcal{S}_\alpha$, $\mathcal{S}_\alpha\subset\mathcal{S}_\beta$ denotes the set of all allowable initial states and $\mathcal{S}_\varepsilon$ corresponds with the set of allowable disturbances.

Sets $\mathcal{S}_\alpha$, $\mathcal{S}_\beta$ are connected and *a priori* known. $\lambda(\ )$ denotes the eigenvalues of matrix ( ).

$\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues, respectively. $\sigma_{(\ )}$ denotes the matrix singular values.

5. Some Necessary Definitions

Consider a linear continuous singular system with state delay, described by

$$E\,\dot{\mathbf{x}}(t)=A_0\mathbf{x}(t)+A_1\mathbf{x}(t-\tau) \qquad (6)$$

with a known compatible vector valued function of the initial conditions

$$\mathbf{x}(t)=\boldsymbol{\varphi}(t),\ -\tau\leq t\leq 0 \qquad (6a)$$

where A_0 and A_1 are the constant matrices of appropriate dimensions.

Moreover, we shall assume that $rank\ E=r<n$.

Definition 1. Matrix pair (E,A_0) is said to be regular if $\det(sE-A_0)$ is not identically zero, [31].

Definition 2. The matrix pair (E,A_0) is said to be impulsive free if $degree\ \det(sE-A_0)=rank\ E$, [31].

The linear continuous singular time delay system (1) may have an impulsive solution. However, the regularity and the absence of impulses of the matrix pair (E, A_0) ensure the existence and uniqueness of an impulse-free solution of the system. The existence of such solutions is defined in the following Lemma.

Lemma 1. Suppose that the matrix pair (E,A_0) is regular and impulsive free, then the solution to (6.a) exists and is impulse-free and unique on [0, ∞], [31].

As a necessity for the system stability investigation there is a need for establishing a proper stability definition. Therefore, the following definition can be established.

Definition 3. LCSTDS (6) is said to be regular and impulsive free, if the matrix pair (E,A_0) is regular and impulsive free. (b) LCSTDS (6) is said to be stable, if for any $\varepsilon>0$ there exists a scalar $\delta(\varepsilon)>0$ such that, for any compatible initial conditions $\boldsymbol{\varphi}(t)$, $\sup\limits_{-\tau\leq t\leq 0}\|\boldsymbol{\varphi}(t)\|\leq\delta(\varepsilon)$ the solution $\mathbf{x}(t)$ of system (2) satisfies $\|\mathbf{x}(t)\|\leq\varepsilon,\ \forall t\geq 0$ .Moreover, if $\lim\limits_{t\to\infty}\|\mathbf{x}(t)\|\to 0$, the system is said to be asymptotically stable, [31].

In the further analysis the following case has been considered: the subspace of consistent initial conditions for singular time delay and singular non-delay systems coincides.

Remark 1. The singularity of matrix E will ensure that solutions of (6) exist only for special choices of $\boldsymbol{\varphi}(t) \in \mathcal{W}^*_{cont.}$, $\forall t \in [-\tau, 0]$.

In [32] the subspace of $\mathcal{W}^*_k$ of consistent initial is shown to be the limit of the nested subspace algorithm

$$\begin{aligned} \mathcal{W}^*_{k,0} &= \mathbb{R}^n \\ &\vdots \\ \mathcal{W}^*_{k,(j+1)} &= A_0^{-1}\left(E\mathcal{W}^*_{k,(j)}\right)_{A_1=0}, \quad j \geq 0 \end{aligned} \tag{7}$$

Moreover, if $\boldsymbol{\varphi}(t) \in \mathcal{W}^*_k$, $\forall t \in [-\tau, 0]$ then $\mathbf{x}(t) \in \mathcal{W}^*_k$, $\forall t \geq 0$ and $(\lambda E - A_0)_{A_1=0}$ is invertible for some $\lambda \in \mathbb{C}$ (condition for uniqueness), then $\mathcal{W}^*_k \cap \aleph(E) = \{0\}$.

Remark 2. Note that here $G = E^T R E \geq 0$, where $R = R^T > 0$ is arbitrarily matrix.

Note, also, that $\mathcal{W}^*_k \cap \aleph(E) = \{0\}$ implies that:

$$\|\mathbf{x}(k)\|_{E^T R E} = \sqrt{\mathbf{x}^T(k) E^T R E \mathbf{x}(k)} \tag{8}$$

is norm on $\mathcal{W}^*_k$.

Remark 3. We will also need the following *definitions* of the smallest, respectively the largest eigenvalues of matrix $R = R^T$, with respect to subspace of consistent initial conditions $\mathcal{W}^*_k$ and matrix G

Proposition 1. If $\mathbf{x}^T(t) R \mathbf{x}(t)$ is quadratic form on $\mathbb{R}^n$, then it follows that there exist numbers $\lambda_{\min}(R, G, \mathcal{W}^*_k)$ and $\lambda_{\max}(R, G, \mathcal{W}^*_k)$ satisfying $-\infty \leq \lambda_{\min}(R, G, \mathcal{W}^*_k) \leq \lambda_{\max}(R, G, \mathcal{W}^*_k) \leq +\infty$ such that:

$$\begin{aligned} \lambda_{\min}(R, G, \mathcal{W}^*_k) &\leq \frac{\mathbf{x}^T(t) R \mathbf{x}(t)}{\mathbf{x}^T(t) G \mathbf{x}(t)} \\ &\leq \lambda_{\max}(R, G, \mathcal{W}^*_k), \\ &\forall \mathbf{x}(t) \in \mathcal{W}_k \setminus \{\mathbf{0}\} \end{aligned} \tag{9}$$

with matrix $R = R^T$ and corresponding eigenvalues:

$$\lambda_{\min}(R, G, \mathcal{W}^*_k) = = \min\left\{ \begin{array}{c} \mathbf{x}^T(k) R \mathbf{x}(k): \ \mathbf{x}(k) \in \mathcal{W}^*_k \setminus \{\mathbf{0}\}, \\ \mathbf{x}^T(k) G \mathbf{x}(k) = 1 \end{array} \right\} \tag{10}$$

$$\lambda_{\max}(R, G, \mathcal{W}^*_k) = = \max\left\{ \begin{array}{c} \mathbf{x}^T(t) R \mathbf{x}(t): \ \mathbf{x}(t) \in \mathcal{W}^*_k \setminus \{0\}, \\ \mathbf{x}^T(t) G \mathbf{x}(t) = 1 \end{array} \right\} \tag{11}$$

Note that $\lambda_{\min}(\cdot) > 0$ if $R = R^T > 0$.

Before starting our results, the concept of finite-time stability for the time-delay system (1) is introduced. This concept can be formalized through the following definition.

Definition 4. Singular time delayed system (6) is regular, impulsive free and finite-time stable with respect to $\left\{\mathcal{S}_\alpha, \mathcal{S}_\beta, T, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if $\forall \boldsymbol{\varphi}(t) \in \mathcal{W}^*_{cont.}$, $\forall t \in [-\tau, 0]$ satisfying

$$\sup_{t \in [-\tau, 0]} \boldsymbol{\varphi}^T(t) E^T E \boldsymbol{\varphi}(t) \leq \alpha \tag{12}$$

implies

$$\mathbf{x}^T(t) E^T E \mathbf{x}(t) < \beta, \quad \forall t \in \Im \tag{13}$$

where $\mathcal{W}^*_k$ is a subspace of consistent initial conditions, [31].

6. Some Previous Results

In the further analysis the following case has been considered: the subspace of consistent initial conditions for singular time delay and singular non-delay systems coincides.

Theorem 1. Singular time delayed system (6) is *finite time stable* in the sense of *Definition* 5, with respect to $\left\{t_0, \Im, \mathcal{S}_\alpha, \mathcal{S}_\beta, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if there exists a positive real number q, $q > 1$, [33], such that:

$$\begin{aligned} &\|\mathbf{x}(t+\vartheta)\|^2 < q^2 \|\mathbf{x}(t)\|^2, \ \vartheta \in [-\tau, 0] \\ &\forall t \in \Im, \ \mathbf{x}(t) \in \mathcal{W}^*_k, \ \forall \mathbf{x}(t) \in \mathcal{S}_\beta \end{aligned} \tag{14}$$

and if the following condition is satisfied:

$$e^{\bar{\lambda}_{\max}(\Xi)(t-t_0)} < \frac{\beta}{\alpha}, \quad \forall t \in \Im \tag{15}$$

where

$$\bar{\lambda}_{\max}(\Xi) = = \bar{\lambda}_{\max}\left(\begin{array}{c} \mathbf{x}^T(t) \left(\begin{array}{c} A_0^T E + E^T A_0 + \\ E^T A_1 \left(I - E^T E\right)^{-1} A_1^T E \\ +q^2 I \end{array} \right) \mathbf{x}(t) \\ \mathbf{x}(t) \in \mathcal{W}^*_k, \ \mathbf{x}^T(t) E^T E \mathbf{x}(t) = 1 \end{array} \right) \tag{16}$$

[34].

Theorem 2. Singular time delayed system (6) is regular, impulse free and finite-time stable with respect to $\left\{\mathcal{S}_\alpha, \mathcal{S}_\beta, T, \|(\cdot)\|^2\right\}$, $\alpha < \beta$ if there exists a positive real number q such that the following conditions are satisfied:

$$\boldsymbol{\varphi}(t) \in \mathcal{W}^*_{k.}, \quad \forall t \in [-\tau, 0] \tag{17}$$

$$\mathbf{x}^T(t-\theta)\mathbf{x}(t-\vartheta) \le q\mathbf{x}^T(t)\mathbf{x}(t), \quad \vartheta \in [-\tau, 0], \quad \forall t \in [0, T] \tag{18}$$

$$A_0^T E + E^T A_0 + qI = E^T \Phi E \tag{19}$$

$$\left(1+\tau\cdot\lambda_{\max}(Q)\right)e^{\lambda_{\max}(\Xi)t} < \frac{\beta}{\alpha}, \quad \forall t \in [0, T] \tag{20}$$

$$\Xi = \Phi + A_1\left(I + E^T E\right)^{-1} A_1^T + Q \tag{21}$$

$$Q = Q^T > 0, \quad \Phi = \left(A_0^T E + E^T A_0\right) \tag{22}$$

where $\mathcal{W}_k^*$ is a subspace of consistent initial conditions, [31].

Lemma 2. For any real constant $\wp > 0$ and any symmetric, positive definite matrix $\Xi = \Xi^T > 0$ the following condition is satisfied:

$$-2\mathbf{u}^T(t)\mathbf{v}(t) \le \wp\mathbf{u}^T(t)\Xi^{-1}\mathbf{u}(t) + \frac{1}{\wp}\mathbf{v}^T(t)\Xi\mathbf{v}(t) \tag{23}$$

[33,35].

Theorem 3. Time delayed system (6) is regular, impulse free and finite time stable with respect to $\left\{\mathcal{S}_\alpha, \mathcal{S}_\beta, T, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if there exist positive real numbers $\wp$ and q, such that the following condition is satisfied $\forall\boldsymbol{\varphi}(t) \in \mathcal{W}_k^*$:

$$\mathbf{x}^T(t-\vartheta)\mathbf{x}(t-\vartheta) < q\mathbf{x}^T(t)E^T E\mathbf{x}(t), \quad \vartheta \in [-2\tau, 0], \quad \forall t \in [0, T] \tag{24}$$

$$e^{\lambda_{\max}(\Pi)T} < \frac{\beta}{\alpha} \tag{25}$$

$$\left(A_0 + A_1\right)^T E + E^T\left(A_0 + A_1\right) = E^T \Phi E \tag{26}$$

$$\Pi = \Phi + \tau\wp A_1\left(A_0 A_0^T + A_1 A_1^T\right)A_1^T + \frac{2q\tau}{\wp}I \tag{27}$$

where $\mathcal{W}_k^*$ is a subspace of consistent initial conditions, [31,36].

Theorem 4. Singular time delayed system (6) is regular, impulse free and finite time stable with respect to $\left\{\mathcal{S}_\alpha, \mathcal{S}_\beta, T, R\right\}$, $\alpha < \beta$ if there exist a positive scalar $\wp$ and two positive definite matrices $\Pi \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{n\times n}$, and matrix P such that the following conditions hold:

$$PE = E^T P^T \ge 0 \tag{28}$$

$$PE = E^T \Pi E \tag{29}$$

$$\Xi = \begin{bmatrix} A_0^T P^T + PA_0 + E^T QE - \wp EP & PA_1 \\ A_1^T P^T & -E^T QE \end{bmatrix} \le 0 \tag{30}$$

and:

$$\lambda_{\max}(\Pi) + \tau\lambda_{\max}(Q) < \frac{\beta}{\alpha}e^{-\wp T}\lambda_{\min}(\Pi). \text{ [31]} \tag{31}$$

7. Main Result

Theorem 5. A singular time delayed system (6) is regular, impulse free and finite-time stable with respect to $\left\{\mathcal{S}_\alpha, \mathcal{S}_\beta, T, \|(\cdot)\|^2\right\}$, $\alpha < \beta$ if the following conditions are satisfied:

$$\mathbf{x}^T(t-\tau)\left(E^T E\left(E^T E\right)\left(E^T E\right)^D - I\right)\mathbf{x}(t-\tau) \ge 0, \quad \forall\mathbf{x}(t-\tau) \in \mathcal{W}_k \setminus \{0\} \tag{32}$$

$$\Pi = \left[I + E^T E\left(I - \left(E^T E\right)\left(E^T E\right)^D\right)\right] > 0 \tag{33}$$

$$e^{\bar{\lambda}_{\max}(\Xi)t}(1+\tau) < \frac{\beta}{\alpha}, \quad \forall t \in \mathfrak{I} \tag{34}$$

where $(\ )^D$ is *Drazin* inverse of matrix $(\ )$ and

$$\bar{\lambda}_{\max}(\Xi) = \max\begin{bmatrix} \mathbf{x}^T(t)\Xi\mathbf{x}(t): \\ \mathbf{x}^T(t)E^T E\mathbf{x}(t) = 1 \end{bmatrix} \tag{35}$$

$$\Xi = A_0^T E + E^T A_0 + E^T E + E^T A_1 \Pi^{-1} A_1^T E \tag{36}$$

Proof. A tentative aggregation function is defined as:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)E^T E\mathbf{x}(t) + \int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)E^T E\mathbf{x}(\vartheta)d\vartheta \tag{37}$$

Total derivative $\dot{V}(t, \mathbf{x}(t))$ along the trajectories of the system is:

$$\begin{aligned}\dot{V}(t, \mathbf{x}(t)) &= \frac{d}{dt}\left(\mathbf{x}^T(t)E^T E\mathbf{x}(t)\right) + \frac{d}{dt}\int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)E^T E\mathbf{x}(\vartheta)d\vartheta \\ &= \mathbf{x}^T(t)\left(A_0^T E + E^T A_0\right)\mathbf{x}(t) \\ &+ 2\mathbf{x}^T(t)E^T A_1\mathbf{x}(t-\tau) \\ &+ \mathbf{x}^T(t)E^T E\mathbf{x}(t) - \mathbf{x}^T(t-\tau)E^T E\mathbf{x}(t-\tau)\end{aligned} \tag{1}$$

Based on the known inequality [1], with a particular choice (33), and using (32) it is clear that (38) is reduced to

$$\begin{aligned}\dot{V}(t, \mathbf{x}(t)) &= \mathbf{x}^T(t)\left(A_0^T E + E^T A_0 + E^T E\right)\mathbf{x}(t) \\ &+ \mathbf{x}^T(t)E^T A_1 \Pi^{-1} A_1^T E\mathbf{x}(t) \\ &+ \mathbf{x}^T(t-\tau)\Pi\mathbf{x}(t-\tau) \\ &- \mathbf{x}^T(t-\tau)E^T E\mathbf{x}(t-\tau)\end{aligned} \tag{39}$$

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[1] $2\mathbf{u}^T(t)\mathbf{v}(t-\tau) \le \begin{pmatrix}\mathbf{u}^T(t)\Pi^{-1}\mathbf{u}(t) \\ +\mathbf{v}^T(t-\tau)\Pi\mathbf{v}(t-\tau)\end{pmatrix},$

$\Pi = \Pi^T > 0$

or

$$\dot{V}\left(t,\mathbf{x}(t)\right)\leq\mathbf{x}^{T}(t)\Xi\mathbf{x}(t) -\mathbf{x}^{T}(t-\tau)\left(E^{T}EE^{T}E\left(E^{T}E\right)^{D}-I\right)\mathbf{x}(t-\tau) < \mathbf{x}^{T}(t)\Xi\mathbf{x}(t) \quad (40)$$

Further

$$\frac{\dot{V}\left(t,\mathbf{x}(t)\right)}{\mathbf{x}^{T}(t)E^{T}E\,\mathbf{x}(t)} < \frac{\mathbf{x}^{T}(t)\Xi\mathbf{x}(t)}{\mathbf{x}^{T}(t)E^{T}E\mathbf{x}(t)} \leq \max\left\{\frac{\mathbf{x}^{T}(t)\Xi\mathbf{x}(t)}{\mathbf{x}^{T}(t)E^{T}E\mathbf{x}(t)}\right\} = \max\left\{\begin{matrix}\mathbf{x}^{T}(t)\Xi\mathbf{x}(t): \\ \mathbf{x}^{T}(t)E^{T}E\mathbf{x}(t)=1\end{matrix}\right\} = \bar{\lambda}_{\max}(\Xi) \quad (41)$$

Preceding equation allows us to write

$$\dot{V}\left(\mathbf{x}(t)\right) < \bar{\lambda}_{\max}(\Xi)\mathbf{x}^{T}(t)E^{T}E\,\mathbf{x}(t) \leq \bar{\lambda}_{\max}(\Xi)\mathbf{x}^{T}(t)E^{T}E\,\mathbf{x}(t) + \bar{\lambda}_{\max}(\Xi)\int_{t-\tau}^{t}\mathbf{x}^{T}(\vartheta)E^{T}E\,\mathbf{x}(\vartheta)\,d\vartheta = \bar{\lambda}_{\max}(\Xi)V\left(\mathbf{x}(t)\right) \quad (42)$$

Integrating (42) from 0 to $t\leq T$, leads to

$$V\left(\mathbf{x}(t)\right) < e^{\bar{\lambda}_{\max}(\Xi)\,t}V\left(\mathbf{x}(0)\right) \quad (43)$$

Then

$$V\left(\mathbf{x}(0)\right)=\mathbf{x}^{T}(0)E^{T}E\,\mathbf{x}(0) + \int_{-\tau}^{0}\mathbf{x}^{T}(\vartheta)E^{T}E\,\mathbf{x}(\vartheta)\,d\vartheta \leq \alpha+\alpha\cdot\tau=\alpha(1+\tau) \quad (44)$$

based on *Definition* 4.

So one can write

$$V\left(\mathbf{x}(t)\right) < e^{\bar{\lambda}_{\max}(\Xi)t}V\left(\mathbf{x}(0)\right) < \alpha\cdot e^{\bar{\lambda}_{\max}(\Xi)t}(1+\tau) \quad (45)$$

On the other hand

$$\mathbf{x}^{T}(t)E^{T}E\,\mathbf{x}(t) < \mathbf{x}^{T}(t)E^{T}E\,\mathbf{x}(t) + \int_{t-\tau}^{t}\mathbf{x}^{T}(\vartheta)E^{T}E\,\mathbf{x}(\vartheta)\,d\vartheta = V\left(\mathbf{x}(t)\right) < \alpha\cdot e^{\bar{\lambda}_{\max}(\Xi)t}(1+\tau) \quad (46)$$

Using, finally (35), it is easy to show

$$\mathbf{x}^{T}(t)E^{T}E\,\mathbf{x}(t) < \beta, \quad \forall t\in\mathfrak{I} \quad (47)$$

what completes the proof.

8. Conclusion

Generally, this paper extends some of the basic results in the area of the non-Lyapunov stability to the class of LCSTDS. Furthermore, part of this result is a geometric counterpart of the algebraic theory in [1] supplemented with appropriate criteria to cover the need for system stability in the presence of actual time delay terms. A novel sufficient delay-dependent criterion for the finite-time stability of linear continuous singular time-delay system has been presented.

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References

[1] Campbell, S. L., “*Singular Systems of Differential Equations*”, Pitman, Marshfield, MA, 1980.

[2] Müller, P.C., “Stability of Linear Mechanical Systems with Holonomic Constraints”, *Applied Mechanics Review*, 46 (11). 160-164. 1993.

[3] Campbell, S.L., Meyer, C.D., Rose, N.J. “Application of Drazin Inverse to Linear Systems of Differential Equations”, *SIAM J. Appl. Math.*, 31. 411-425. 1976.

[4] Luenberger, D.G., “Dynamic Equations in Descriptor Form”, *IEEE Trans. Automat. Cont*., 22 (3). 312-321. 1977.

[5] La Salle, Lefschet, S., “*Stability by Lyapunov’s Direct Method*”, Academic Press, New York, 1961.

[6] Weiss, L., Infante, E.F., “On the Stability of Systems Defined over a Finite Time Interval”, *Proc. National Acad. Sci.*, 54. 44-48. 1965.

[7] Weiss, L., Infante, E.F., “Finite Time Stability under Perturbing Forces and on Product Spaces”, *IEEE Trans. Automat. Cont.*, 12. 54-59. 1967.

[8] Debeljkovic, D., Owens, H., “On practical stability of singular systems”, *Proc. MELECON Conf. 85*, Madrid, Spain, 103-105. 1985.

[9] Debeljkovic, D.Lj., Bajic, V.B., Gajic, Z., Petrovic, B., “Boundedness and existence of solutions of regular and irregular singular systems", Publications of the Faculty of Electrical Eng, Belgrade, Automatic Control, 1. 69-78. 1993.

[10] Debeljkovic, D.Lj., Lazarevic, M.P., Koruga, Đ., Tomasevic, S., “Finite time stability of singular systems operating under perturbing forces: Matrix measure approach”, *Proc.* AMSE Conference, Melbourne Australia, 447-450. October 29-31. 1997.

[11] Kablar, N.A., Debeljkovic, D.Lj., “Non - Lyapunov stability of linear singular systems: Matrix measure approach”, MNTS - Mathematical Theory of Networks and Systems, Presented lecture, also in Proc. of Ext. Abstracts, TM7, Padova, Italy, July 6-10, 1998.

[12] Kablar, N.A., Debeljkovic, D.Lj., “Non - Lyapunov stability of linear singular systems: Matrix measure approach”, Preprints 5th IFAC *Symposium on Low Cost Automation*, Shenyang, China, September 8-10, TS13. 16-20, 1998.

[13] Kablar, N.A., Debeljkovic, D.Lj., “Finite time stability of time varying singular systems”, *Proc*. IEEE CDC 98, Florida, USA, pp.3831-3836, December 10-12, 1998.

[14] Debeljkovic, D.Lj., Kablar, N.A., “On necessary and sufficient conditions of linear singular systems stability operating on finite time interval”, *Proc.* XII CBA, Uberlandia, Brazil, IV. 1241-1246. September 14 -18. 1998.

[15] Kablar, N.A., Debeljkovic, D.Lj., “Finite time instability of time varying linear singular systems”, *Proc. IEEE ACC* 99, San Diego, USA, 1796-1800. June 2-4. 1999.

[16] Debeljkovic, D.Lj., Kablar, N.A., “Finite time stability of linear singular systems: Bellman-Gronwall approach”, *Proc. ACC* 99, San Diego, USA, 1803-1806. June 2-4. 1999.

[17] Yang, C.Y., Zhang, Q.L., Lin, Y.P., "Practical Stability of Descriptor Systems", *Dynamics of Continuous, Discrete and Impulsive Systems*, Series B: 12.b. 44-57. 2005.

[18] Nie, Y.Y., Debeljkovic, D.Lj., "Non–Lyapunov Stability of Linear Singular Systems: A Quite new Approach in Time Domain", *Dynamics of Continuous, Discrete and Impulsive Systems,* Vol.11, Series A: Math. Analysis, No. 5-6, pp.751-760., 2004.

[19] Yang, C.Y., Zhang, Q.L., Zhou, L., "Practical Stabilization and Controllability of Descriptor Systems", *International Journal of Information and System Science*, 1 (3-4). 455-466. 2005.

[20] Jun-E, F., Zhen, W., Jia-Bing, S., "Finite-Time Control of Linear Singular Systems with Parametric Uncertainties and Disturbances", *Acta Automatica Sinica*, 31(4). 634-637. 2005

[21] Feng, H.H., Hunsarg, J., "Stabilization of Nonlinear Singularly Perturbed Multiple Time Delay Systems by Dither", Trans. ASME *J. of Dynamic Systems, Measurement and Control*, 118 (3). 177-181. 1996.

[22] Debeljkovic, D.Lj., Nenadic, Z..Lj., Milinkovic, S.A., Jovanovic, M.B., "On Practical and Finite-Time Stability of Time-Delay Systems", *Proc. ECC 97*, Brussels, Belgium, 307-311. July 2-6. 1997.

[23] Debeljkovic, D.Lj., Nenadic, Z.Lj., Milinkovic, S.A., Jovanovic, M.B. "On the Stability of Linear Systems with Delayed State Defined over Finite Time Interval", Proc. IEEE CDC 97, San Diego, California, USA, 2771-2772. December 21-23. 1997.

[24] Nenadic, Z.Lj., Debeljkovic, D.Lj., Milinkovic, S.A., "On Practical Stability of Time Delay Systems", *Proc. IEEE American Control Conference*, Albuquerque, USA, 3235-3235. June 4-6. 1997.

[25] Debeljkovic, D.Lj., Koruga, Đ., Milinkovic, S.A., Jovanovic, M. B., Jacic, Lj. "Further Results on Non-Lyapunov Stability of Time Delay Systems", *Proc. MELECON* 98, Tel-Aviv, Israel, 1. 509-512. May 18-20. 1998.

[26] Debeljkovic, D.Lj., Lazarevic, M.P., Milinkovic, S.A., Jovanovic, M.B., "Finite Time Stability Analysis of Linear Time Delay Systems: Bellman-Gronwall Approach", *Proc*. 1st *IFAC Workshop on Linear Time Delay Systems,* Grenoble, France, 171-175. July 6-7. 1998.

[27] Debeljkovic, D.Lj., Lazarevic, M.P., Nenadic, Z.Lj., Milinkovic, S.A., "Finite Time Stability of Time Delay Systems" *IMA J. Math. Control and Info.,* 16 (3). 101-109. 1999.

[28] Debeljkovic, D.Lj., Visnjic, N.S., Pjescic, M., "The Stability of Linear Continuous Singular Systems over the Finite Time Interval: An Overview", *International Journal of Information & System Science*, 4 (4). 560-584. 2008.

[29] Debeljkovic, D.Lj., Lazarevic, M.P., Koruga, Đ., Milinkovic, S.A., Jovanovic, M.B., "Further results on the stability of linear nonautonomous systems with delayed state defined over finite time interval", *Proc. IEEE ACC* 2000, Chicago, Illinois, USA, 1450-1451. June 28-30. 2000.

[30] Yang, C., Zhang, Q., Lin, Y., Zhou, L., "Practical stability of descriptor systems with time-delay in terms of two measurements", *Journal of the Franklin Institute*, 343 (6). 635-646. 2006.

[31] Debeljkovic, D.Lj, Stojanovic, S.B., Aleksendric, M.S., "Stability of Singular Time Delay Systems in the Sense of Non-Lyapunov: Classical and Modern Approach", *Hemijska Industrija*, 61 (5). 193-202. 2013.

[32] Owens, D.H., Debeljkovic, D.Lj., "Consistency and Lyapunov Stability of Linear Descriptor Systems: a Geometric Analysis", *IMA Journal of Math. Control and Information*, 2. 139-151. 1985.

[33] Su, J.H., "Further results on the robust stability of linear systems with single time delay", *Systems & Control Letters* 23 (5). 375-379. 1994.

[34] Debeljkovic, D.Lj., Buzurovic, I.M., Nestorovic, T., Popov, D., "A New Approach to Stability of Singular Time Delay Systems in the sense of Non-Lyapunov: Delay Independent Conditions", *Proc.* 2011 IEEE ICCA (International Conference on Control and Automation, Santiago (Chile), December 19-21. 312-317. 2011.

[35] Xu, S. Dooren, P.V., Stefan, R., Lam, J., "Robust Stability and Stabilization for Singular Systems with State Delay and Parameter Uncertainty", *IEEE Trans. Automat. Control*, 47 (7). 1122-1128. 2002.

[36] Debeljkovic, D.Lj., Buzurovic, I.M., Nestorovic, T., Popov, D.,"On Finite and Practical Stability of Time Delayed Systems: Lyapunov-Krassovski Approach: Delay Dependent Criteria", *Proc.* The 23 rd, Chinese Control and Decision Conference CCDC 2011, Mianyang, (China), 23-25. 331-337. 2011.