RESONANCE EXPANSIONS OF PROPAGATORS IN THE PRESENCE OF POTENTIAL BARRIERS

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1. Introduction and statement of results

The purpose of this note is to present an expansion of a semi-classical propagating state in terms of resonances. We consider energy levels at which there exists a potential barrier separating the interaction region from a non-trapping region near infinity. This assumption allows a significant strengthening of the results of Burq and the third author [3, Theorem 1]. It is motivated by the recent work of the first author on the spectral shift function in the presence of barriers [14].

Our results are applicable to the semi-classical Schrödinger equation for long range "black box" perturbations [18] – see Sect.2 for a review of definitions, and for the barrier assumption in the general setting. We denote by $\mathrm{Res}(P(h))$ the set of resonances of $P(h)$, that is the set of poles of the meromorphic continuation of $R(z,h) = (z - P(h))^{-1}$ from $\mathrm{Im}\, z > 0$ to the lower half-plane.

A typical operator to keep in mind is $P(h) = -h^2\Delta + V(x)$, where $|V(x)| \leq C|x|^{-\epsilon}$, $\epsilon > 0$, and V is analytic in a complex conic neighborhood of infinity. In that case the barrier assumption [14] takes the following simple form: let $p(x,\xi) = \xi^2 + V(x)$ be the symbol of $P(h)$. We then say

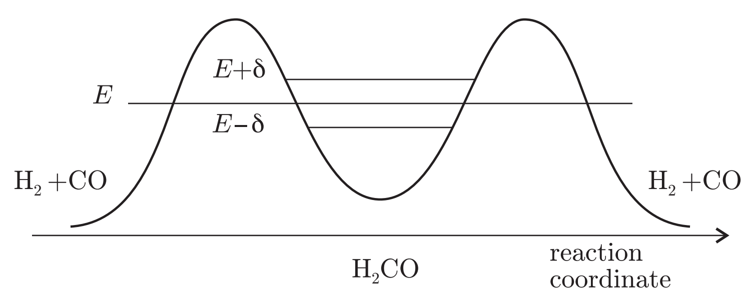

Figure 1. A classical example from molecular dynamics: a cross-section of the energy surface of formaldehyde, H_2CO. Considered as a resonant state its lifetime is very long at the energy E due to the strong barrier. That is not surprising considering the well known properties of formaldehyde. Unimolecular dissociation is possible only for states excited to the top level of the barrier and that is not covered by our theory.

that a barrier exists at an energy E, if there exist closed sets, $\Sigma^e(E)$, $\Sigma^i(E)$, such that

$$\begin{aligned} &p^{-1}(E) = \Sigma^e(E) \cup \Sigma^i(E) \,, \quad \Sigma^e(E) \cap \Sigma^i(E) = \emptyset \,, \quad \Sigma^i(E) \Subset T^*\mathbb{R}^n \,, \\ &(y,\eta) \in \Sigma^e(E) \quad \Longrightarrow \quad |\exp(tH_p)(y,\eta)| \longrightarrow \infty \ \text{ as } t \longrightarrow \pm\infty. \end{aligned} \tag{1.1}$$

Here $H_p = \sum_{j=1}^n \partial_{\xi_j} p \partial_{x_j} - \partial_{x_j} p \partial_{\xi_j}$ is the Hamilton vector field of p whose integral lines describe the classical motion on the energy surface $p^{-1}(E)$, and $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ is the natural projection – see Fig.1 for an example.

To motivate our result let us recall the basic consequence of the spectral theorem: if $\psi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R})$ is supported near E and the spectrum of $P^\sharp(h)$ is discrete near E (for instance, $P^\sharp(h) = -h^2\Delta + V^\sharp(x)$, $\liminf_{|x|\to\infty} V^\sharp(x) > E$), then

$$e^{-itP^\sharp(h)/h}\psi(P^\sharp(h)) = \sum_{z \in \text{Spec }(P^\sharp(h))} e^{-itz/h} \text{Res}(\bullet - P^\sharp(h))^{-1}, z)\psi(z) \,, \tag{1.2}$$

$\text{Res}(\bullet - (P^\sharp(h))^{-1}, z)$ denotes the residue of the resolvent of $P^\sharp(h)$ at z, that is, the spectral projection at z. In the presence of barriers, just as in the example shown in Fig.1, we do not have a discrete spectrum near E even though the local classical picture is the same as in the case of an infinite well. Hence, in the semi-classical limit, we expect a result similar to (1.2) to hold, once we localize to the region isolated by the barrier. The resolvent has to be replaced by its meromorphic continuation through the continuous spectrum, $R(z, h)$. Its poles, the *resonances*, are the analogues of eigenvalues in (1.2) and they represent the states just as eigenvalues did.

Motivated by this discussion we can state our main

Theorem. *Let $P(h)$ be an operator satisfying the general assumptions of Sect.2 (for instance, a Schrödinger operator). Assume in addition that* (1.1) *holds and let $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$. We decompose χ as $\chi = \chi_1 + \chi_2$, where*

$$\pi(\Sigma^e(E)) \cap \operatorname{supp}\chi_1 = \emptyset \,, \quad \pi(\Sigma^i(E)) \cap \operatorname{supp}\chi_2 = \emptyset \,, \quad \pi : T^*\mathbb{R}^n \to \mathbb{R}^n \,. \tag{1.3}$$

Let $\psi \in \mathcal{C}^\infty_{\rm c}((0,\infty))$ be supported in a small neighborhood of E. Then for any fixed $\delta > 0$ and $0 < h < h_0$ there exists $\delta < c(h) < 2\delta$ such that for any $C_1 > 0$ there exists $C_2 > 0$ such that

$$\begin{aligned} \chi e^{-itP(h)/h}\chi\psi(P(h)) = &\sum_{z\in\Omega(h)\cap\text{Res }(P(h))} \chi_1 \text{Res}(e^{-it\bullet/h}R(\bullet,h),z)\chi_1\psi(P(h)) \\ &+ \chi_2\mathcal{O}_{\mathcal{H}\to\mathcal{H}}(\langle (t-C_2)_+/h\rangle^{-\infty})\chi_2 + \mathcal{O}_{\mathcal{H}\to\mathcal{H}}(h^\infty) \,, \\ \Omega(h) = (a - c(h), b + c(h)) - i[0, C_1 h] \,, &\quad \text{convex hull}(\operatorname{supp}\psi) = [a,b] \,. \end{aligned} \tag{1.4}$$

Here $\text{Res}(f(\bullet), z)$ *denotes the residue of a meromorphic family of operators, f, at z, and $\langle\bullet\rangle = (1 + |\bullet|^2)^{\frac12}$.*

Finding χ_1 and χ_2 with the desired properties is clear for Schrödinger operators and is possible in general in view of Lemma 4.1 below.

In general, the function $c(h)$ depends on the distribution of resonances: roughly speaking we cannot "cut" through a dense cloud of resonances. In a different context of resonance expansions for the modular surface [4, Theorem 1] there is also, currently at least, a need for some non-explicit grouping of terms – see also the remark at the end of Sect.5.

The proof of the theorem follows from standard facts about semi-classical propagation, and from more recent results of Martinez [12], Nakamura [15], Burq-Zworski [3], and Stefanov [23]. As a byproduct we obtain a result on the approximation of clusters of resonant states by clusters of eigenfunctions – see Proposition 3.3 and also [23] for closely related results. We also show that for times which are exponentially large in $1/h$ the eigenvalues and eigenfunctions are a good approximations for resonances when the barriers are present – see Proposition 5.1.

As was pointed out to us by Christian Gérard a result implying (1.4) in a special case involving a separation condition (not unlike the separation condition in [27]) was proved by Gérard-Martinez in [6] with an argument based on the Helffer-Sjöstrand theory of resonances. It is quite possible that their method combined with the more recent resonance counting techniques (see [18]) gives an alternative proof of our result.

Our result agrees with the basic intuition that the real part of a resonance corresponds to the rest energy or frequency, and the imaginary part to the rate of decay – see [31]. In many aspects of wave mechanics we want to achieve a *state specificity* given by expansions of signals into normal modes (which are real, such as eigenvalues of self-adjoint operators), or quasi-normal modes (which have imaginary parts, corresponding to decay, as in the case of resonances). The specific modes can then be identified with isolated states of the system. Examples and references related to chemistry can be found in [16] and to gravitational waves in [10]. In both physical situations the trapping is typically weaker than in the presence of barriers.

It is an interesting open problem to give a general dynamical definition of resonances. We recall that in odd dimensions the Lax-Phillips theory [11] provides an elegant dynamical definition – see [20] for a concise and self-contained presentation of a generalized Lax-Phillips theory. However that abstract definition does not provide concrete information of the type given in Theorem above.

Finally we remark that time dependent theories of resonances were investigated recently by Merkli-Sigal [13] and Soffer-Weinstein [21] (see also [9] and [29] for earlier results). The difference here lies in considering many resonances at high energies and not a time dependent theory of *a single* resonance obtained by perturbing an embedded eigenvalue. Our motivation comes from semi-classical molecular dynamics (see [4],[23], and [27] for other recent mathematical results) rather than from perturbations of non-linear Schrödinger equations – see [28] for a recent study and references.

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2. Assumptions on the operator

To avoid the analysis of specific aspects of obstacle, potential, or metric scattering we work in the "black box" formalism introduced in [19] and generalized further in [18]. Sjöstrand's paper [18] contains a review of the theory on which our work is based.

The operator we study acts on $\mathcal{H}$, a complex Hilbert space with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)) \,,$$

where $R_0 > 0$ is fixed and $B(x, R) = \{y \in \mathbb{R}^n : |x - y| < R\}$.

The corresponding orthogonal projections are denoted by $u|_{B(0,R_0)}$ and $u|_{\mathbb{R}^n \setminus B(0,R_0)}$ or by $1\!\!1_{B(0,R_0)}u$ and $1\!\!1_{\mathbb{R}^n \setminus B(0,R_0)}u$ respectively, where $u \in \mathcal{H}$.

We work in the semi-classical setting and for each $h \in (0, h_0]$, we have

$$P(h) : \mathcal{H} \longrightarrow \mathcal{H}$$

with the domain $\mathcal{D}$, independent of h, and satisfying

$$1\!\!1_{\mathbb{R}^n \setminus B(0,R_0)}\mathcal{D} = H^2(\mathbb{R}^n \setminus B(0, R_0))$$

uniformly with respect to h (see [18] for a precise meaning of this statement).

We also assume that

$$\mathbb{1}_{B(0,R_0)}(P(h)+i)^{-1} : \mathcal{H} \longrightarrow \mathcal{H}_{R_0} \quad \text{is compact}\,, \tag{2.1}$$

that there exists $C > 0$ such that

$$P(h) \geq -C\,, \tag{2.2}$$

and that

$$\mathbb{1}_{\mathbb{R}^n\setminus B(0,R_0)} P(h) u = Q(h)(u|_{\mathbb{R}^n\setminus B(0,R_0)})\,, \quad \text{for } u \in \mathcal{D}\,, \tag{2.3}$$

where $Q(h)$ is a formally self-adjoint operator on $L^2(\mathbb{R}^n)$ given by

$$Q(h)v = \sum_{|\alpha|\leq 2} a_\alpha(x;h)(hD_x)^\alpha v \quad \text{for } v \in C_0^\infty(\mathbb{R}^n) \tag{2.4}$$

such that $a_\alpha(x;h) = a_\alpha(x)$ is independent of h for $|\alpha| = 2$, $a_\alpha(x;h) \in C_b^\infty(\mathbb{R}^n)$ are uniformly bounded with respect to h, here $C_b^\infty(\mathbb{R}^n)$ denotes the space of C^∞ functions on $\mathbb{R}^n$ with bounded derivatives of all orders,

$$\sum_{|\alpha|=2} a_\alpha(x;h)\xi^\alpha \geq (1/c)|\xi|^2, \;\; \forall \xi \in \mathbb{R}^n\,,$$

for some constant $c > 0$, $\sum_{|\alpha|\leq 2} a_\alpha(x;h)\xi^\alpha \longrightarrow \xi^2$ uniformly with respect to h as $|x| \to \infty$.

The meromorphic continuation of the resolvent,

$$(P(h)-z)^{-1} \;:\; \mathcal{H}_{\rm comp} \longrightarrow \mathcal{D}_{\rm loc}\,,$$

is guaranteed by the following analyticity assumption: there exist $\theta \in [0,\pi)$, $\epsilon > 0$ and $R \geq R_0$ such that the coefficients $a_\alpha(x;h)$ of $Q(h)$ extend holomorphically in x to

$$\{r\omega : \omega \in \mathbb{C}^n, \operatorname{dist}(\omega, \mathbf{S}^n) < \epsilon, r \in \mathbb{C}, |r| > R, \arg r \in [-\epsilon, \theta_0 + \epsilon)\}$$

with $\sum_{|\alpha|\leq 2} a_\alpha(x;h)\xi^\alpha \longrightarrow \xi^2$ uniformly with respect to h as $|x| \to \infty$ remains valid in this larger set of x's.

We use $P(h)$ to construct a self-adjoint operator $P^\sharp(h)$ on

$$\mathcal{H}^\sharp = \mathcal{H}_{R_0} \oplus L^2(M \setminus B(0,R_0))$$

as in [19] where $M = (\mathbb{R}/R\mathbb{Z})^n$ for some $R \gg R_0$. Let $N(P^\sharp(h), I)$ denote the number of eigenvalues of $P^\sharp(h)$ in the interval I, we assume

$$N(P^\sharp(h), [-\lambda,\lambda]) = \mathcal{O}((\lambda/h^2)^{n/2}), \;\; \text{for } \lambda \geq 1\,. \tag{2.5}$$

Following [19] we could also make a more general assumption, replacing n by some $n^\sharp \geq n$.

Under the above assumptions on $P(h)$, the resonances close to the real axis can be defined by the method of complex scaling (see [18] and references given there). They coincide with the poles of the meromorphic continuation of the resolvent $(P(h)-z)^{-1}$ from $\operatorname{Im} z > 0$ to a conic neighborhood of the positive half axis in the lower half plane. The set of resonances of $P(h)$ will be denoted by $\operatorname{Res}P(h)$ and we include them with their multiplicity.

The spectral assumption (2.5) implies (in a non-trivial way – see [18] and references given there) a bound on the number of resonances: let $\Omega \Subset \{z : \operatorname{Im} z \leq 0\,, \operatorname{Re} z > 0\}$, then

$$\#\,\Omega \cap \operatorname{Res}(P(h)) \leq C h^{-n}\,, \tag{2.6}$$

where # denotes the number of elements counted according to their multiplicities.

As mentioned above, the basic tool for the study of resonances is the method of complex scaling – see [18] and [19] for the theory in our context and references. The operator $P(h)$ is deformed into a non-self-adjoint operator

$$P_\theta(h) \,:\, \mathcal{D}_\theta \longrightarrow \mathcal{H}_\theta\,.$$

For z with $\arg z > -2\theta$, $P_\theta(h) - z$ is a Fredholm operator, and the corresponding spectrum is discrete and coincides with $\mathrm{Res}(P(h))$. We remark that θ can be taken h dependent. We also recall that the complex scaling method guarantees that

$$\chi(P-z)^{-1}\chi = \chi(P_\theta - z)^{-1}\chi\,, \tag{2.7}$$

for any χ supported in a compact set unaffected by the complex deformation.

Notational Convention. In the context of "black box" perturbations, $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ tacitly means that $f \equiv \mathrm{const}$ in $B(0,R_0)$ with an obvious multiplicative action on $\mathcal{H}_{R_0}$.

The *barrier assumption* in the general setting is given in terms of the (semi-classical) principal symbol $q(x,\xi)$ of $Q(h)$ and it takes the following form:

$$\begin{gathered} q^{-1}(E) = \Sigma^e(E) \cup \Sigma^i(E)\,, \quad \Sigma^e(E)\cap\Sigma^i(E) = \emptyset\,,\\ \Sigma^\bullet(E) \text{ are closed in } T^*\mathbb{R}^n\,, \quad \Sigma^i(E) \Subset T^*_{B(0,R_0)}\mathbb{R}^n\,,\\ (y,\eta)\in\Sigma^e(E) \implies |\exp(tH_q)(y,\eta)| \longrightarrow \infty \text{ as } t\longrightarrow\infty. \end{gathered} \tag{2.8}$$

Here, as before, $H_q = \sum_{j=1}^n \partial_{\xi_j} q \partial_{x_j} - \partial_{x_j} q \partial_{\xi_j}$ is the Hamilton vector field of q.

3. Preliminaries

In this section we will review results which are needed in the proof of the main theorem. The first result is a quantitative version of the absence-of-resonances result of Martinez [12]:

Proposition 3.1. *Suppose that $Q(h)$ satisfies the general assumptions following* (2.4)*. Denoting by q the principal symbol of $Q(h)$, assume in addition that for $(y,\eta)\in q^{-1}(E)$ we have the non-trapping condition:*

$$|\exp(tH_q)(y,\eta))| \longrightarrow \infty \ \text{ as } t \longrightarrow \infty..$$

Then there exists $\delta > 0$ such that for all $M > 0$

$$\mathrm{Res}\, Q(h) \cap ([E-\delta, E+\delta] - i[0, Mh\log(1/h)]) = \emptyset\,, \quad h < h_0 = h_0(M)\,,$$

where $\mathrm{Res}\, Q(h)$ denotes the set of resonances of $Q(h)$. In addition, if $Q_\theta(h)$ is the scaled operator above, with

$$\theta = \widetilde{M} h \log(1/h)\,, \quad M \ll \widetilde{M}\,, \tag{3.1}$$

then then for every M there exists $h_0 = h_0(M)$ such that

$$\begin{gathered} \|(Q_\theta(h) - z)^{-1}\|_{L^2(\Gamma_\theta)\to L^2(\Gamma_\theta)} \leq C\exp(C|\operatorname{Im} z|/h)/h\,,\\ z\in[E-\delta,E+\delta] - i[0, Mh\log(1/h)]\,, \quad h<h_0\,. \end{gathered} \tag{3.2}$$

Remark: We will only use the h dependent θ in conjuction with (2.7). Hence, unless explicitely stated θ will be small and fixed.

Proof. To see this we recall that the weighted estimates of [12] provide a seemingly weaker bound on the scaled resolvent with θ satisfying (3.1):

$$\begin{gathered} \|(Q_\theta(h) - z)^{-1}\|_{L^2(\Gamma_\theta)\to L^2(\Gamma_\theta)} \leq Ch^{-CM}\,,\\ z\in[E-\delta,E+\delta] - i[0, Mh\log(1/h)]\,, \quad h<h_0(M)\,, \end{gathered}$$

which gives the bound (3.2) for $\operatorname{Im} z = Mh\log(1/h)$. For $\operatorname{Im} z = 0$ we get the bound from the following lemma which is an adaptation of the non-trapping limiting absorption principle – see [30] for the most general version and references:

Lemma 3.1. *Suppose that $Q(h)$ satisfies the assumptions of Proposition 3.1. Then for $z \in [E-\delta, E+\delta] \subset \mathbb{R}$, and $\theta \gg h\log(1/h)$, we have*

$$\|(Q_\theta(h) - z)^{-1}\|_{L^2(\Gamma_\theta)\to L^2(\Gamma_\theta)} \le C/h\,. \tag{3.3}$$

A local and semi-classical adaptation of the three line theorem in the spirit of [26] gives (3.2). In fact, for any $\gamma_1 > \gamma_2 > \delta$, we can construct a holomorphic function $f(z,h)$ with the following properties:

$$\begin{gathered} |f(z,h)| \le C \quad \text{for} \quad z \in [E-\gamma_1, E+\gamma_1] - i[0, Mh\log(1/h)]\,, \\ |f(z,h)| \ge 1 \quad \text{for} \quad z \in [E-\delta, E+\delta] - i[0, Mh\log(1/h)]\,, \\ |f(z,h)| \le h^{CM} \quad \text{for} \quad z \in ([E-\gamma_1, E+\gamma_1] \setminus [E-\gamma_2, E+\gamma_2]) - i[0, Mh\log(1/h)]\,. \end{gathered}$$

We can then apply the maximum principle in $[E-\gamma_1, E+\gamma_1] - i[0, Mh\log(1/h)]$ to the subharmonic function

$$\log \|(Q_\theta(h) - z)^{-1}\|_{L^2(\Gamma_\theta)\to L^2(\Gamma_\theta)} + \log|f(z,h)| - C\frac{\operatorname{Im} z}{h}\,,$$

proving the estimate (3.2). □

For the reader's convenience we also recall the standard elliptic semi-classical estimate. If Ω and $\widetilde{\Omega}$ are open sets and $\overline{\Omega} \Subset \widetilde{\Omega}$, then for differential operators, $B(h) = \sum_{|\alpha|\le m} b_\alpha(x)(hD_x)^\alpha$ which are classically elliptic, $\sum_{|\alpha|=n} b_\alpha(x)\xi^\alpha \neq 0$ for $\xi \neq 0$, we have

$$\sum_{|\alpha|\le m} \|(hD_x)^\alpha u\|_{L^2(\Omega)} \le C\left(\|u\|_{L^2(\widetilde{\Omega})} + \|B(h)u\|_{L^2(\widetilde{\Omega})}\right)\,. \tag{3.4}$$

The basic tunneling estimate for semi-classical elliptic operators is given in

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set such that $\overline{\Omega} \Subset \mathbb{R}^n$, and $\pi^{-1}(\overline{\Omega}) \cap q^{-1}(E) = \emptyset$. For any open $\widetilde{\Omega} \supset \overline{\Omega}$, and for z in a small complex neighborhood of E, there exist $\delta_0 > 0$, $C > 0$, such that*

$$\|u\|_{L^2(\Omega)} \le Ce^{-\delta_0/h}\|u\|_{L^2(\widetilde{\Omega})} + C\|(Q(h)-z)u\|_{L^2(\widetilde{\Omega})}. \tag{3.5}$$

Proof. We use the idea of [15]. Let $f, g \in C_0^\infty(\widetilde{\Omega})$ such that $0 \le f(x), g(x) \le 1$, $f(x) = 1$ on Ω, and $g(x) = 1$ on $\operatorname{supp} f$. We also arrange, as we may, that

$$\pi^{-1}(\operatorname{supp} g) \cap q^{-1}(E) = \emptyset\,.$$

Then by the sharp Gårding inequality we have, in the sense of operators,

$$g\left|e^{\delta f/h}(Q(h)-E)e^{-\delta f/h}\right|^2 g \ge c^2 g^2$$

with some $c > 0$, provided $\delta > 0$ is sufficiently small. The same inequality, with perhaps a smaller $c > 0$, holds if E is replaced by a complex z close enough to E. This implies

$$\begin{aligned} \|ge^{\delta f/h}u\| &\le c^{-1}\|e^{\delta f/h}(Q(h)-z)gu\| \\ &\le c^{-1}\|e^{\delta f/h}g(Q(h)-z)u\| + c^{-1}\|e^{\delta f/h}[Q(h), g]u\| \end{aligned}$$

for $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Since $f = 0$ on $\operatorname{supp}[Q(h), g]$, (3.4) gives

$$\begin{aligned}\|e^{\delta f/h}[Q(h), g]u\| &\leq C\big(\|hD_x u\|_{L^2(\tilde{\Omega})} + \|u\|_{L^2(\tilde{\Omega})}\big) \\ &\leq C'\|u\|_{L^2(\tilde{\Omega})} + C'\|(Q(h) - z)u\|_{L^2(\tilde{\Omega})}.\end{aligned}$$

Combining these, we obtain

$$e^{\delta/h}\|u\|_{L^2(\Omega)} \leq C'c^{-1}\|u\|_{L^2(\tilde{\Omega})} + c^{-1}(e^{\delta/h} + C')\|(Q(h) - z)u\|_{L^2(\tilde{\Omega})}$$

and the claim follows immediately. □

The next general result is quite technical but it is worthwhile to present it separately. It is adapted from [23] and [25].

Proposition 3.3. *Let θ be small, positive, and fixed. Suppose that there exists $e^{-C/h} \leq S(h) = \mathcal{O}(h^\infty)$ such that*

$$\begin{aligned}&\operatorname{Res}(P(h)) \cap \big([E - \epsilon, E + \epsilon] - i[S(h), h^{-2n-2}S(h)]\big) = \emptyset \\ &\|(P_\theta(h) - z)^{-1}\| \leq \frac{C}{S(h)}, \quad z \in [E - \epsilon, E + \epsilon] + iS(h).\end{aligned} \tag{3.6}$$

Then there exist $a_j(h) < b_j(h) < a_{j+1}(h)$, $j = 1, \cdots, J(h) = \mathcal{O}(h^{-n})$ such that

(3.7)

$$\begin{gathered}\operatorname{Res}(P(h)) \cap ([E - \epsilon, E + \epsilon] - i[0, S(h)]) \subset \bigcup_{j=1}^{J(h)} \Omega_j(h), \quad \Omega_j(h) = [a_j(h), b_j(h)] - i[0, S(h)], \\ \operatorname{width}(\Omega_j(h)) \leq Ch^{-n}\omega(h), \quad \operatorname{dist}\{\Omega_j(h), \operatorname{Res} P(h) \setminus \Omega_j(h)\} \geq 4\omega(h),\end{gathered}$$

where $\omega(h) = h^{-(5n+1)/2}S(h)$, and we have

$$\begin{gathered}\|(P_\theta(h) - z)^{-1}\| \leq \frac{C}{S(h)}, \qquad z \in \partial\widetilde{\Omega}_j(h), \\ \widetilde{\Omega}_j(h) = [a_j(h) - \omega(h), b_j(h) + \omega(h)] + i[-h^{-n}S(h), S(h)].\end{gathered} \tag{3.8}$$

Remark. By refining some estimates of Burq [2] it is shown in [25, Sect.3.2,(3.20)] that the second assumption in (3.6) is always satisfied with $S(h) > \exp(-h^{-\frac{1}{3}})$. In our special case we can take exponentially small $S(h)$ – see Lemma 4.7.

Proof. The existence of the decomposition into cluster follows easily from the upper bound on the number of resonances (2.6). The estimate (3.8) is essentially proven in [25, Proposition 3.2] for the restriction of P_θ onto the space $\Pi_j^\theta\mathcal{H}$, where Π_j^θ is the projection on resonant states in the cluster $\Omega_j(h)$. The same proof applies for P_θ in the whole space $\mathcal{H}$ by noticing that if we define the neighborhood $\widehat{\Omega}_j(h)$ of $\widetilde{\Omega}_j(h)$ by

$$\widehat{\Omega}_j(h) \stackrel{\text{def}}{=} [a_j(h) - 2w(h), b_j(h) + 2w(h)] + i[-S(h)h^{-2n-2}, S(h)],$$

then there are no resonances in $\widehat{\Omega}_j(h) \setminus \widetilde{\Omega}_j(h)$. For the convenience of the reader, we will give the proof. Let $z_{jk}(h)$, $k = 1, \ldots, K_j(h) = \mathcal{O}(h^{-n})$, be the resonances in $\widetilde{\Omega}_j(h)$, each one repeated according to its multiplicity. Set

$$\tilde{z}_{jk}(h) \stackrel{\text{def}}{=} \bar{z}_{jk}(h) + 2iS(h), \quad k = 1 \ldots K_j(h),$$

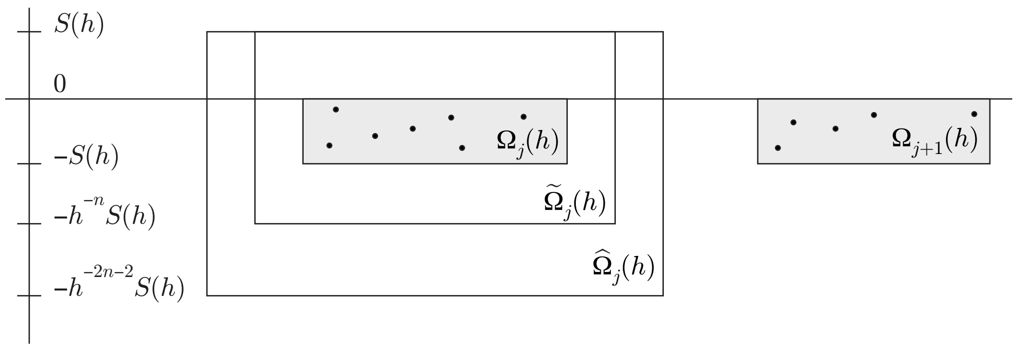

FIGURE 2. Different regions needed in the statement and proof of Proposition 3.3.

where the bar denotes complex conjugate. Then z_{jk} and $\tilde{z}_{jk}$ are symmetric about the line $\operatorname{Im} z = S(h)$ and on that line we have $\|(z - P_\theta(h))^{-1}\| \leq C/S(h)$ by our assumption (3.6). Set

$$G_j(z,h) \stackrel{\text{def}}{=} \frac{(z - z_{j1}) \dots (z - z_{jK_j})}{(z - \tilde{z}_{j1}) \dots (z - \tilde{z}_{jK_j})}.$$

We observe first that

$$|G_j(z,h)| \leq 1 \quad \text{for} \quad \operatorname{Im} z \leq S(h). \tag{3.9}$$

The function $F_j(z,h) \stackrel{\text{def}}{=} G_j(z,h)(z - P_\theta(h))^{-1}$ is holomorphic in $\hat{\Omega}_j(h)$. We now apply the "semiclassical maximum principle" [26] in the form presented in [23, Lemma 1]:

Lemma 3.2. *Let $0 < h < 1$, and $a(h) < b(h)$. Suppose that $F(z,h)$ is a holomorphic function of z defined in a neighborhood of*

$$\Omega(h) = [a(h) - 5w(h), b(h) + 5w(h)] + i[-S_-(h), S_+(h)h^{-n-\epsilon}],$$

where $0 < S_-(h) \leq S_+(h) \leq w(h)h^{3n/2+2\epsilon}$, $\epsilon > 0$, and $w(h) \to 0$ as $h \to 0$. If $F(z,h)$ satisfies

$$|F(z,h)| \leq Ae^{Ah^{-n}\log(1/h)} \quad \textit{in } \Omega(h),$$
$$|F(z,h)| \leq M(h) \quad \textit{on} \quad [a(h) - 5w(h), b(h) + 5w(h)] - iS_-(h)\,, \quad M(h) \to \infty\,,\ h \to 0\,,$$

then there exists $h_1 = h_1(S_-, S_+, A, \epsilon)$ such that

$$|F(z,h)| \leq 2e^3 M(h)\,,$$

for $z \in \widetilde{\Omega} = [a(h) - w(h), b(h) + w(h)] + i[-S_-(h), S_+(h)]\,, \quad 0 < h < h_1\,.$

We apply this to to the function F_j in the extended domain $\widehat{\Omega}_j(h) \supset \widetilde{\Omega}_j(h)$. Note that the closest resonance outside this region stays at distance at least $g(h) = 2w(h) \geq e^{-C/h}$ by (3.7), thus $\log(1/g(h)) \leq C/h$. By the a priori exponential estimate of the resolvent outside $g(h)$–neighborhood of the resonances [26], we get $\|F_j(z,h)\| = O(\exp(Ch^{-n-1}))$ on the boundary of the extended domain. By the maximum principle, this is true inside it as well. Now we are in position to apply [23, Lemma 1]. Since $\|F_j(z,h)\| \leq 1/S(h)$ on the upper part of $\widehat{\Omega}_j$, we deduce that for h small enough,

$$\|G_j(z)(z - P_\theta(h))^{-1}\| \leq C/S(h), \quad \forall z \in \widetilde{\Omega}_j(h). \tag{3.10}$$

Next step is to show that

$$1/C \leq |G_j(z,h)| \quad \text{on } \partial\widetilde{\Omega}_j(h). \tag{3.11}$$

It is enough to estimate $(z-\tilde{z}_{jk})/(z-z_{jk})$ on $\partial\widetilde{\Omega}_j(h)$. Observe first that $|z_{jk}-\tilde{z}_{jk}| \leq 4S(h)$, $\forall k$. The distance from each z_{jk} from the three sides $\operatorname{Im} z = -S(h)h^{-n}$, $\operatorname{Re} z = a_j - w$, $\operatorname{Re} z = b_j + w$ of $\widetilde{\Omega}_j$ is bounded below by $S(h)h^{-n}/2$ for $h \ll 1$. Therefore,

$$\left|\frac{z-\tilde{z}_{jk}}{z-z_{jk}} - 1\right| = \left|\frac{z_{jk}-\tilde{z}_{jk}}{z-z_{jk}}\right| \leq \frac{4S(h)}{S(h)h^{-n}/2} = 8h^n, \quad \forall z \in \partial\widetilde{\Omega}_j(h) \setminus \{\operatorname{Im} z = S(h)\}.$$

This yields

$$\left|\frac{z-\tilde{z}_{jk}}{z-z_{jk}}\right| \leq (1+8h^n), \quad \forall z \in \partial\widetilde{\Omega}(h) \setminus \{\operatorname{Im} z = S(h)\}. \tag{3.12}$$

On the fourth side $\operatorname{Im} z = S(h)$ of $\partial\widetilde{\Omega}_j(h)$ we have $|(z-\tilde{z}_{jk})/(z-z_{jk})| = 1$, thus (3.12) is trivially true there. Since $(1+x)^{1/x} < e$, $0 < x < \infty$, we get

$$|1/G_j(z,h)| \leq (1+8h^n)^{Ch^{-n}} \leq e^{8C}.$$

This proves (3.11). Estimates (3.10) and (3.11) together imply the proof of the proposition. □

Finally we recall the propagation result valid for all perturbations but for very large times only [3]:

Proposition 3.4. *Suppose that $P(h)$ satisfies the general assumptions above and let $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ be equal to one on a neighborhood of $B(0,R_0)$. Let $\psi \in \mathcal{C}^\infty_c((0,\infty))$ and let* chsupp $\psi = [a,b]$. *There exists $0 < \delta < c(h) < 2\delta$ such that for every $M > M_0$ there exists $L = L(M)$, and we have*

$$\begin{aligned}\chi e^{-itP(h)/h}\chi\psi(P(h)) = & \sum_{z\in\Omega(h)\cap\operatorname{Res}(P(h))} \chi \operatorname{Res}(e^{-it\bullet/h}R(\bullet,h),z)\chi\psi(P(h)) \\ & + \mathcal{O}_{\mathcal{H}\to\mathcal{H}}(h^\infty)\,, \quad \text{for } \ t > h^{-L}\,, \\ \Omega(h) = & \ (a-c(h), b+c(h)) - i[0,h^M)\,,\end{aligned}$$

and where $\operatorname{Res}(f(\bullet),z)$ *denotes the residue of a meromorphic family of operators, f, at z.*

4. Some estimates

In this section we assume that the *barrier assumption* (2.8) holds and start by showing that the x-projections of $\Sigma^e(E)$ and $\Sigma^i(E)$ do not intersect. This will allow us to use cut-off functions depending on x only:

Lemma 4.1. *The assumption* (2.8) *implies that*

$$\pi(\Sigma^e(E)) \cap \pi(\Sigma^i(E)) = \emptyset\,. \tag{4.1}$$

Proof. We write the principal symbol $q(x,\xi)$ of $Q(h)$ in the form

$$q(x,\xi) = A(x)\xi\cdot\xi + b(x)\cdot\xi + c(x)\,.$$

Then

$$q(x,\xi) = \left|A^{1/2}(x)\xi + \frac{1}{2}A^{-1/2}(x)b(x)\right|^2 + \tilde{V}(x), \quad \tilde{V}(x) \stackrel{\text{def}}{=} c(x) - \frac{1}{4}A(x)^{-1}b(x)\cdot b(x). \tag{4.2}$$

We easily see that $\tilde{V}^{-1}(-\infty, E] = \pi(\Sigma^e(E)) \cup \pi(\Sigma^i(E))$ and we claim that $\pi(\Sigma^e(E)) \cap \pi(\Sigma^i(E)) = \emptyset$. In fact, assume that $x_0 \in \pi(\Sigma^e(E)) \cap \pi(\Sigma^i(E))$. Then there exist ξ^e and ξ^i such that $(x_0, \xi^e) \in \Sigma^e(E)$ and $(x_0, \xi^i) \in \Sigma^i(E)$. Therefore, by (4.2),

$$\left| A^{1/2}(x_0)\xi^e + \frac{1}{2}A^{-1/2}(x_0)b(x_0) \right| = \left| A^{1/2}(x_0)\xi^i + \frac{1}{2}A^{-1/2}(x_0)b(x_0) \right| = \left(E - \tilde{V}(x_0) \right)^{1/2}.$$

Since the ellipsoid $\left|A^{1/2}(x_0)\xi + \frac{1}{2}A^{-1/2}(x_0)b(x_0)\right| = (E - \tilde{V}(x_0))^{1/2}$ is connected, we can connect ξ^e and ξ^i by a continuous curve $\xi(t)$ on that ellipsoid and this shows that $(x_0, \xi(t))$ is a continuous curve connecting $\Sigma^e(E)$ and $\Sigma^i(E)$, contrary to our assumption (2.8). ☐

As in [14] we introduce two reference operators: $P^\sharp(h)$ with a discrete spectrum near E, and $Q^\sharp(h)$ with a non-trapping classical flow:

$$\begin{gathered} P(h) = \chi_1 P^\sharp(h) + \chi_2 Q^\sharp(h)\,, \\ \chi_1 + \chi_2 = 1\,, \quad \pi(\Sigma_i(E)) \subset \operatorname{supp}\chi_1\,, \;\; \pi(\Sigma_e(E)) \subset \operatorname{supp}\chi_2\,. \end{gathered} \tag{4.3}$$

We can choose $P^\sharp(h)$ to be of the form

$$P^\sharp(h) = P(h) + V^\sharp(x)\,, \quad V^\sharp \geq 0\,, \quad V^\sharp(x) = \begin{cases} 0 & \text{for } x \text{ near } \pi\left(\Sigma^i(E)\right) \\ E + 2\delta & \text{for } x \text{ near } \pi\left(\Sigma^e(E)\right), \end{cases}$$

with $\operatorname{supp} V^\sharp \cap \operatorname{supp}\chi_1 = \emptyset$.

The complex scaling on $P(h)$, giving the operator, $P_\theta(h)$ are always performed outside of the barrier, that is, in a slightly informal notation,

$$P_\theta(h)1\!\!1_U = P(h)1\!\!1_U\,, \quad \text{where } U \text{ is a neighbourhood of } \complement\Sigma^{\mathrm{e}}(E). \tag{4.4}$$

The next lemma is an easy consequence of Proposition 3.1. In greater generality it has been established before in [7].

Lemma 4.2. *Suppose that $Q(h)$ satisfies the assumptions of Proposition 3.1. Then for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\psi \in \mathcal{C}_c^\infty((E-\epsilon, E+\epsilon))$ we have*

$$\chi e^{-itQ(h)/h}\psi(Q(h))\chi = \mathcal{O}(\langle (t-C)_+/h\rangle^{-\infty}) \;:\; L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)\,, \tag{4.5}$$

where $\langle \bullet \rangle = (1 + |\bullet|^2)^{\frac{1}{2}}$, and $\bullet_+$ denotes the positive part of a function.

Proof. Let us write

$$R_\pm(z,h) = (Q(h) - z)^{-1}\,, \quad \text{analytic for } \;\pm \operatorname{Im} z > 0\;,$$

using the same notation for the meromorphic continuation. The spectral projection is then given by Stone's formula:

$$dE_\lambda = (2\pi i)^{-1}(R_-(\lambda) - R_+(\lambda))\,,$$

and the left hand side of (4.5) can be rewritten as

$$\chi e^{-itQ(h)/h}\psi(Q(h))\chi = \frac{1}{2\pi i}\int_0^\infty e^{-it\lambda/h}\chi(R_-(\lambda,h) - R_+(\lambda,h))\chi\psi(\lambda)d\lambda\,.$$

If we fix M then for $h < h_0(M)$ we have a resonance free region

$$\Omega(h) = [E-\epsilon, E+\epsilon] - i[0, Mh\log(1/h)]\,,$$

and the bounds on the resolvents:

$$\chi R_\pm(z,h)\chi = \mathcal{O}(1)e^{C|\operatorname{Im} z|/h}/h \;:\; \mathcal{H} \longrightarrow \mathcal{H}\,, \quad z \in \Omega(h)\,,$$

as given in Proposition 3.1 (using (2.7)). If $\tilde{\psi} \in \mathcal{C}_{\rm c}^\infty(\mathbb{C})$ is an almost analytic extension of ψ, Green's formula gives

$$\chi e^{-itQ(h)/h}\psi(Q(h))\chi = \frac{1}{2\pi i}\int_{\operatorname{Im} z = -Mh\log(1/h)} e^{-itz/h}\chi(R_-(z,h) - R_+(z,h))\chi\tilde{\psi}(z)dz$$
$$+ \frac{1}{\pi}\iint_{-Mh\log(1/h)<\operatorname{Im} z<0} e^{-itz/h}\chi(R_-(z,h) - R_+(z,h))\chi\bar{\partial}_z\tilde{\psi}(z)dm(z)\,.$$

For t large the norm of the first term on the right hand side is bounded by $h^{(t-C)M} = \mathcal{O}((h/t)^M)$. Using $\bar{\partial}_z\tilde{\psi}(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$, we see that the norm of the second term is bounded by

$$\int_0^1 e^{-st/h}e^{Cs/h}\mathcal{O}(s^\infty)ds = \mathcal{O}((h/t)^\infty)\,, \quad t > 2C\,.$$

Since for all times the propagator is bounded this proves the lemma. □

The tunneling estimates of [15] provide the following

Lemma 4.3. *Suppose that* $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$ *and that* $\operatorname{supp}\chi \cap \pi(\Sigma^e(E) \cup \Sigma^i(E)) = \emptyset$. *Then*

$$\chi\psi(P(h))\,,\ \chi\psi(P^\sharp(h)) = \mathcal{O}(e^{-\delta/h})\,, \quad \delta > 0\,.$$

Proof. For the operator $P^\sharp(h)$, Proposition 3.2 and the spectral theorem imply the estimate in the lemma. For $P(h)$ we proceed as follows. Let $I = [e_-, e_+] \subset \mathbb{R}$, $\operatorname{supp}\psi \subset I$, and suppose that $\chi \in \mathcal{C}_{\rm c}^\infty(\Omega)$, and

$$\Omega \cap \bigcup_{e_- - \epsilon < E < e_+ + \epsilon} \pi(\Sigma^e(E) \cup \Sigma^i(E)) = \emptyset, \qquad \Omega \Subset \mathbb{R}^n\,, \quad \epsilon > 0\,.$$

Then by Proposition 3.2 (see also [15, Theorem 2.6]) we have, for $\operatorname{Im} z \neq 0$,

$$\|\chi(P(h) - z)^{-1}u\| \leq e^{-\delta/h}\|(P(h) - z)^{-1}u\| + C\|u\| \leq (C + e^{-\delta/h}/|\operatorname{Im} z|)\|u\|\,.$$

Hence,

$$\|\chi(P(h) - z)^{-1}\| \leq C_1$$

if Re $z \leq e_+ + \epsilon$, $|\text{Im } z| \geq e^{-\delta/h}$ (here $(P(h) - z)^{-1}$ denotes the resolvent holomorphic in $\mathbb{C} \setminus \mathbb{R}$). Let $\tilde{\psi} \in C_c^\infty(\mathbb{C})$ be an almost analytic continuation of ψ, and let

$$\Omega = [e_- - \epsilon, e_+ + \epsilon] + i[-c_5e^{-\delta/h}, c_5e^{-\delta/h}].$$

We use the formula:

$$\psi(P(h)) = \frac{1}{\pi}\int_\Omega \bar{\partial}\tilde{\psi}(z)(P(h) - z)^{-1}dm(z) - \frac{1}{2\pi i}\int_{\partial\Omega}\tilde{\psi}(z)(P(h) - z)^{-1}dz.$$

Then using the resolvent identity,

$$\chi((P(h)-z)^{-1} - (P(h)-\bar{z})^{-1})\chi = 2i\operatorname{Im} z\chi(P(h)-z)^{-1}(P(h)-\bar{z})^{-1}\chi = 2i\operatorname{Im} z|\chi(P(h)-z)^{-1}|^2\,,$$

we have

$$\begin{aligned}\chi\psi(P(h))\chi &= -\frac{1}{2\pi i}\int_{\partial\Omega}\tilde{\psi}(z)\chi(P(h) - z)^{-1}\chi dz + \mathcal{O}(e^{-\delta/h}) \\ &= -\frac{1}{\pi}\int_{e_- - \epsilon + ic_5e^{-\delta/h}}^{e_+ + \epsilon + ic_5e^{-\delta/h}} (\text{Im } z)\left|\chi(P(h) - z)^{-1}\right|^2 dz + \mathcal{O}(e^{-\delta/h}) \\ &= \mathcal{O}(e^{-\delta/h})\,.\end{aligned}$$

This proves the lemma once we change χ and ψ in the proof:

$$\chi\psi(P(h))(\chi\psi(P(h)))^* = \chi|\psi|^2(P(h))\chi\,.$$

□

The next lemma relates different propagators:

Lemma 4.4. *With the notation of* (4.3) *and with* $\psi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}_+)$ *supported near* E*, there exists* $\delta > 0$ *for which*

(4.6)
$$\chi_1 \exp(-itP(h)/h)\psi(P(h))\chi_1 = \chi_1 \exp(-itP^\sharp(h)/h)\chi_1\psi(P^\sharp(h)) + \mathcal{O}(1+|t|)e^{-\delta/h} + \mathcal{O}(h^\infty)\,,$$
$$\chi_2 \exp(-itP(h)/h)\psi(P(h))\chi_2 = \chi_2 \exp(-itQ^\sharp(h)/h)\chi_2\psi(Q^\sharp(h)) + \mathcal{O}(1+|t|)e^{-\delta/h} + \mathcal{O}(h^\infty)\,,$$
$$\chi_1 \exp(-itP(h)/h)\psi(P(h))\chi_2 = \mathcal{O}(1+|t|)e^{-\delta/h} + \mathcal{O}(h^\infty)\,,$$

Proof. We will first prove the last estimate in (4.6) under a more general assumption that
$$\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 \cap (\pi(\Sigma^e(E) \cup \Sigma^i(E))) = \emptyset\,, \quad \operatorname{supp} D\chi_1 \cap (\pi(\Sigma^e(E) \cup \Sigma^i(E))) = \emptyset\,.$$
Let χ_1' have the same properties as χ_1 above and $\chi_1\chi_1' = \chi_1$. By Lemma 4.3 we have
$$\chi_1\psi(P(h))\chi_2 = \chi_1'\psi(P(h))\chi_1\chi_2 + \chi_1'[\chi_1, \psi(P(h))]\chi_2 = \mathcal{O}(h^\infty).$$
Also,
$$(ih\partial_t - P(h))\chi_1 \exp(-itP(h)/h)\psi(P(h))\chi_2 = [P(h), \chi_1]\exp(-itP(h)/h)\psi(P(h))\chi_2 = \mathcal{O}(e^{-\delta/h})\,,$$
since by the assumption on the support of $D\chi_1$ we have
$$[P(h), \chi_1]\psi(P(h)) = \mathcal{O}(e^{-\delta/h})\,.$$
Now, an application of the Duhamel formula gives the last claim in (4.6):
$$\chi_1 \exp(-itP(h)/h)\psi(P(h))\chi_2 = \mathcal{O}((1+|t|)e^{-\delta/h}) + \mathcal{O}(h^\infty)\,.$$

The remaining estimates are proved similarly. We first observe that in view of what we have just proved we can modify χ_1 and χ_2 and assume that $\chi_1 P(h) = \chi_1 P^\sharp(h)$ and that $\chi_2 Q(h) = \chi_2 Q^\sharp(h)$. Then
$$\begin{aligned}&(ih\partial_t - P(h))\chi_1\left(\exp(-itP(h)/h)\psi(P(h))\chi_1 - \psi(P^\sharp(h))\exp(-itP^\sharp(h)/h)\chi_1\right)\\ &= -\,[P(h), \chi_1]\left(\exp(-itP(h)/h)\psi(P(h))\chi_1 - \exp(-itP^\sharp(h)/h)\psi(P^\sharp(h))\chi_1\right)\,,\end{aligned}$$
since $\chi_1(P(h) - P^\sharp(h)) = 0$. As in the previous argument we see that the right hand side is $\mathcal{O}(e^{-\delta/h})$. Since the initial data is again $\mathcal{O}(h^\infty)$ we obtain the first estimate in (4.6). The second one follows similarly. □

Lemma 4.5. *Suppose that* $z \in \operatorname{Res}\,(P(h))$*,* $\operatorname{Re} z \in [E-\delta, E+\delta]$*, and that* $(P_\theta(h) - z)u = 0$*,* $u \in \mathcal{H}_\theta$*,* $\|u\|_{\mathcal{H}_\theta} = 1$*. Then there exists* $\delta_0 > 0$ *such that*
$$|\operatorname{Im} z| \leq e^{-\delta_0/2h} \quad \textit{or} \quad \|\chi_1 u\|_{\mathcal{H}} \leq e^{-\delta_0/2h}\,.$$

Proof. We follow the usual method of relating the resonance width to tunneling estimates and write
$$\begin{aligned}-2i\operatorname{Im} z\|\chi_1 u\|^2_{\mathcal{H}} &= \langle (P-z)\chi_1 u, \chi_1 u\rangle_{\mathcal{H}} - \langle \chi_1 u, (P-z)\chi_1 u\rangle_{\mathcal{H}}\\ &= \langle [P, \chi_1]u, \chi_1 u\rangle_{\mathcal{H}} - \langle \chi_1 u, [P, \chi_1]u\rangle_{\mathcal{H}}\\ &= 2i\operatorname{Im}\langle [P, \chi_1]u, \chi_1 u\rangle_{\mathcal{H}}\,.\end{aligned}$$
By Proposition 3.2 and the semi-classical elliptic estimate (3.4), we now have
$$|\operatorname{Im} z|\|\chi_1 u\|_{\mathcal{H}} \leq e^{-\delta_0/h}\|u\|_{\mathcal{H}_\theta}\,,$$
which completes the proof. □

Lemma 4.6. *In the notation of Lemma 4.5 assume that*

$$\|\chi_1 u\|_{\mathcal{H}} = \mathcal{O}(h^\infty)\,. \tag{4.7}$$

Then for any M there exists $h_0 = h_0(M)$, such that

$$|\operatorname{Im} z| > Mh\log(1/h)\,, \quad \text{if } h < h(M).$$

Proof. We observe that $Q^\sharp(h)$ defined in the beginning of this section satisfies the assumptions of Proposition 3.1 and that we can find $\tilde\chi_1 \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$ such that $\chi_1 \equiv 1$ on $\operatorname{supp}\tilde\chi_1$, and

$$Q^\sharp_\theta(h)(1-\tilde\chi_1) = P_\theta(h)(1-\tilde\chi_1)\,.$$

Suppose now that $(P_\theta(h)-z)u = 0$ with $z \in [E-\delta, E+\delta] - i[0, Mh\log(1/h)]$, and that θ satisfies (3.1). (We note that $\chi_1 u$ is independent of θ due to (4.4).) We then have

$$(1-\tilde\chi_1)u = (Q^\sharp_\theta(h)-z)^{-1}(P_\theta(h)-z)(1-\tilde\chi_1)u = (Q^\sharp_\theta(h)-z)^{-1}[P_\theta(h),\tilde\chi_1]u\,.$$

The semi-classical elliptic estimate (3.4) and the assumption (4.7) show that $\|[P_\theta(h),\tilde\chi_1]u\|_{\mathcal{H}} = \mathcal{O}(h^\infty)$, and hence

$$1 = \|u\|_{\mathcal{H}_\theta} = \|(1-\tilde\chi_1)u\|_{\mathcal{H}_\theta} + \mathcal{O}(h^\infty) = \|(Q^\sharp_\theta(h)-z)^{-1}\|_{L^2(\Gamma_\theta)\to L^2(\Gamma_\theta)}\mathcal{O}(h^\infty) + \mathcal{O}(h^\infty)\,.$$

Proposition 3.1 shows that the right hand side is $\mathcal{O}(h^\infty)$ which gives a contradiction. □

Lemma 4.7. *Suppose that $P_\theta(h) = \chi P(h) + (1-\chi)Q_\theta(h)$, $\theta \gg h\log(1/h)$, where $Q(h)$ satisfies the assumptions of Proposition 3.1 and (4.4) holds. Then there exists $\delta_0 > 0$ such that for any $S(h)$ satisfying*

$$e^{-\delta_0/h} < S(h) < Ch\,,$$

we have

$$\|(P_\theta(h)-z)^{-1}\| \leq \frac{C}{S(h)}\,, \quad z \in [E-\epsilon, E+\epsilon] + iS(h)\,. \tag{4.8}$$

Proof. Let us choose $\phi_1, \phi_2 \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$\pi(\Sigma^i(E)) \subset \{\phi_1 = 1\}\,, \quad \pi(\Sigma^e(E)) \subset \{\phi_2 = 1\}\,, \quad \phi_1^2 + \phi_2^2 = 1\,.$$

In addition we arrange that $\phi_1 P_\theta(h) = \phi_1 P(h)$ and $\phi_2 P_\theta = \phi_2 Q_\theta(h)$, where $Q(h)$ is non-trapping. Then Proposition 3.2 shows that

$$\|[\phi_j, P_\theta(h)]u\| = \mathcal{O}(e^{-\delta_1/h})\|u\| + \mathcal{O}(h)\|(P_\theta(h)-z)u\|\,, \tag{4.9}$$

and we have

$$\begin{aligned}\|(P_\theta(h)-z)u\|^2 &= \sum_{j=1,2}\|\phi_j(P_\theta(h)-z)u\|^2\\ &\geq \sum_{j=1,2}\|(P_\theta(h)-z)\phi_j u\|^2+\sum_{j=1,2}\|[\phi_j,P_\theta(h)]u\|^2\\ &\quad -2\sum_{j=1,2}\|(P_\theta(h)-z)\phi_j u\|\,\|[\phi_j,P_\theta(h)]u\|\\ &\geq \sum_{j=1,2}\|(P_\theta(h)-z)\phi_j u\|^2-\sum_{j=1,2}\|[\phi_j,P_\theta(h)]u\|^2\\ &\quad -2\sum_{j=1,2}\|\phi_j(P_\theta(h)-z)u\|\,\|[\phi_j,P_\theta(h)]u\|\\ &\geq \sum_{j=1,2}\|(P_\theta(h)-z)\phi_j u\|^2-\sum_{j=1,2}\|[\phi_j,P_\theta(h)]u\|^2\\ &\quad -2\|(P_\theta(h)-z)u\|\Big(\sum_{j=1,2}\|[\phi_j,P_\theta(h)]u\|^2\Big)^{\frac12}.\end{aligned}$$

Using (4.9), the resolvent estimate on $(P(h)-z)^{-1}$, and the estimate of Lemma 3.1 on $(Q_\theta(h)-z)^{-1}$ (note that $\operatorname{Im} z>0$), we obtain

$$\begin{aligned}\|(P_\theta(h)-z)u\|^2 &\geq \operatorname{Im} z\|\phi_1 u\|^2+Ch\|\phi_2 u\|^2-Ce^{-2\delta_1/h}\|u\|^2-\mathcal{O}(h^2)\|(P_\theta(h)-z)u\|^2\\ &\quad -2\|(P_\theta(h)-z)u\|(Ce^{-\delta_1/h}\|u\|+\mathcal{O}(h)\|(P_\theta(h)-z)u\|)\\ &\geq S(h)\|u\|^2-\mathcal{O}(h)\|(P_\theta(h)-z)u\|^2,\end{aligned}$$

proving the estimate. □

5. Semi-classical expansions

We will show that the resonances of $P(h)$ are close to the eigenvalues of $P^\sharp(h)$ and that this correspondence will give an approximate agreement of propagators up to exponentially large times, $|t|\leq e^{1/Ch}$.

In this section θ will be small, positive, and fixed.

From Lemmas 4.5, 4.6, and 4.7 we see that the assumptions (3.6) of Proposition 3.3 are satisfied with $S(h)=\exp(-1/Ch)$ with any $C\geq C_0$, where $C_0>0$ depends on $P(h)$ and more precisely, on the constant δ_0 in the tunneling estimate of Proposition 3.2. Since the number of eigenvalues of $P^\sharp(h)$ is bounded by Ch^{-n} we can modify our cluster decompositions (3.7) so that it holds with $\operatorname{Res}(P(h))$ replaced by $\operatorname{Res}(P(h))\cup\operatorname{Spec}(P^\sharp(h))$.

Let us denote by

$$\Pi_j^\theta(h)=\frac{1}{2\pi i}\oint_{\partial\widetilde\Omega_j(h)}(z-P_\theta(h))^{-1}dz \quad :\mathcal{H}_{\mathrm{comp}}\to\mathcal{H}_{\mathrm{loc}}$$

the spectral projector of $P_\theta(h)$ related to the resonances in $\Omega_j(h)$. Here we assume that $\partial\widetilde\Omega_j(h)$ is a positively oriented contour. Similarly we define the spectral projections of $P^\sharp(h)$:

$$\Pi_j^\sharp(h)=\frac{1}{2\pi i}\oint_{\partial\widetilde\Omega_j(h)}(z-P^\sharp(h))^{-1}dz\,.$$

Assuming that the scaling is performed as in Lemma 4.7, the operator $\chi_1 \Pi_j^\theta(h) \chi_1$ is acting on $\mathcal{H}$, since $\chi_1(P(h)-z)^{-1}\chi_1 = \chi_1(P_\theta(h)-z)^{-1}\chi_1$. We also note that $\chi_1\Pi_j^\theta\chi_1$ is independent of θ. Hence the following statement makes sense:

Proposition 5.1. *Under the assumptions above, for C_0 large enough,*

$$\chi_1 \Pi_j^\theta(h)\chi_1 = \chi_1\Pi_j^\sharp(h)\chi_1 + \mathcal{O}_{\mathcal{H}\to\mathcal{H}}(e^{-\delta_1/h}) \,, \quad \forall j = 1, \ldots, J(h), \tag{5.1}$$

where $\delta_1 > 0$ depends on $P(h)$ and E only. In addition, for $|t| \leq e^{1/Ch}$ we have

$$\sum_{z\in\Omega_j(h)} \chi_1 \mathrm{Res}(e^{-it\bullet/h}R(\bullet,h),z)\chi_1 = \sum_{z\in\Omega_j(h)} e^{-itz/h}\chi_1\Pi_z^\sharp(h)\chi_1 + \mathcal{O}_{\mathcal{H}\to\mathcal{H}}(e^{-\delta_1/h}) \,, \tag{5.2}$$

where $\Pi_z^\sharp(h)$ is the spectral projection for $z \in \mathrm{Spec}(P^\sharp(h))$.

Proof. Note first that $\chi_1(z-P^\sharp(h))^{-1}\chi_1 = \chi_1(z-P_\theta^\sharp(h))^{-1}\chi_1$, where $P_\theta^\sharp(h) = P_\theta + V^\sharp$ is the complex scaled version of $P^\sharp(h)$. Notice that the scaled potential $V_\theta^\sharp(x)$ coincides with $V^\sharp$ because $V^\sharp = \text{const.}$ for large $|x|$ where the scaling is performed. We therefore get by the resolvent identity

$$\chi_1\left((z-P^\sharp(h))^{-1} - (z-P_\theta(h))^{-1}\right)\chi_1 = \chi_1(z-P_\theta^\sharp(h))^{-1}V^\sharp(z-P_\theta(h))^{-1}\chi_1.$$

Next,

$$\chi_1\left(\Pi_j^\theta(h) - \Pi_j^\sharp(h)\right)\chi_1 = -\frac{1}{2\pi i}\oint_{\partial\widetilde{\Omega}_j(h)} \chi_1(z-P_\theta^\sharp(h))^{-1}V^\sharp(z-P_\theta(h))^{-1}\chi_1\, dz. \tag{5.3}$$

We claim that with some $S_1(h) = \mathcal{O}(e^{-1/C_1h})$ we have

$$\left\|V^\sharp(z-P_\theta(h))^{-1}\chi_1\right\| = S_1(h)\left\|(z-P_\theta(h))^{-1}\chi_1\right\|, \quad \text{for } z \in \partial\widetilde{\Omega}_j(h). \tag{5.4}$$

The idea is to use Proposition 3.2 to gain exponential decay inside the classically forbidden region away from $\mathrm{supp}\,\chi_1$, and then to use the fact that on $\mathrm{supp}\,V^\sharp$, the operator $P(h)$ is non-trapping so that we can extend this estimate from the classically forbidden region to infinity. Note first that for $z \in \partial\widetilde{\Omega}_j(h)$, we have,

$$|\,\mathrm{Im}\, z| \leq e^{-\delta_0/h}\,, \quad \mathrm{Re}\, z \in [E-\epsilon-\mathcal{O}(h^\infty), E+\epsilon+\mathcal{O}(h^\infty)]\,.$$

Therefore, the operator $P(h)$ satisfies the assumptions of Proposition 3.2 for such z in any compact in $\mathbb{R}^n \setminus \left(\pi(\Sigma^e(E))\cup\pi(\Sigma^i(E))\right)$ for $0<\epsilon\ll 1$ and the resulting estimate will be uniform for such z and in particular indepenednt of the choice of the exponential function $S(h)$. Recall that $\mathrm{supp}\,V^\sharp \cap \mathrm{supp}\,\chi_1 = \emptyset$ and therefore, $P_\theta(h)V^\sharp = Q_\theta^\sharp(h)V^\sharp$. This implies

$$\begin{aligned} V^\sharp(z-P_\theta(h))^{-1}\chi_1 &= (z-Q_\theta^\sharp(h))^{-1}(z-Q_\theta^\sharp(h))V^\sharp(z-P_\theta(h))^{-1}\chi_1 \\ &= -(z-Q_\theta^\sharp(h))^{-1}[Q^\sharp(h),V^\sharp](z-P_\theta(h))^{-1}\chi_1 \\ &= -(z-Q_\theta^\sharp(h))^{-1}[Q^\sharp(h),V^\sharp](z-P(h))^{-1}\chi_1. \end{aligned} \tag{5.5}$$

Proposition 3.1 (the part about the lack of resonances), Lemma 3.1, and Lemma 3.2 show that

$$\|(z-Q_\theta^\sharp(h))^{-1}\| \leq Ch^{-P}\,, \tag{5.6}$$

with some $P>0$ for z we consider here. On the other hand, Proposition 3.2 and (3.4) imply

$$\left\|[Q^\sharp(h),V^\sharp](z-P(h))^{-1}\chi_1\right\| \leq e^{-1/Ch}\left\|(z-P_\theta(h))^{-1}\chi_1\right\|.$$

This, (5.5), and (5.6) prove (5.4).

From (5.3), (5.4) and Proposition 3.3 we now get the following estimate:

$$\left\|\chi_1\left(\Pi_j^\theta(h)-\Pi_j^\sharp(h)\right)\chi_1\right\| \leq C\frac{S_1(h)}{S(h)}\oint_{\partial\widetilde{\Omega}_j(h)}\left\|\chi_1(z-P_\theta^\sharp(h))^{-1}\tilde{\chi}_2\right\| |dz|, \tag{5.7}$$

where $\tilde{\chi}_2$ is such that $\tilde{\chi}_2 V^\sharp = V^\sharp$ and $\operatorname{supp} \tilde{\chi}_2 \cap \operatorname{supp} \chi_1 = \emptyset$.

We estimate the integrand above similarly to (5.4). To this end, choose first an operator $\tilde{Q}^\sharp_\theta(h) = Q^\sharp_\theta(h) + W(x)$, where the potential $W \geq 0$ is chosen so that $\tilde{\chi}_2 W = 0$ and $\tilde{Q}^\sharp_\theta(h) - z$ is elliptic as an h-ΨDO including at the infinite points of $T^*\mathbb{R}^n$ for z as in the proposition. In particular, the real part of the principal symbol of $\tilde{Q}^\sharp_\theta(h) - z$ is positive everywhere, bounded from below by $C(1+|\xi|^2)$. The ellipticity implies that $(z - \tilde{Q}^\sharp_\theta(h))^{-1}$ exists and is $\mathcal{O}(1)$. Using the fact that $\tilde{\chi}_2 \tilde{Q}^\sharp_\theta(h) = \tilde{\chi}_2 P^\sharp_\theta(h)$, we now write, similarly to (5.5),

$$\begin{aligned} \chi_1 (z - P^\sharp_\theta(h))^{-1} \tilde{\chi}_2 &= \chi_1 (z - P^\sharp_\theta(h))^{-1} \tilde{\chi}_2 (z - \tilde{Q}^\sharp_\theta(h))(z - \tilde{Q}^\sharp_\theta(h))^{-1} \\ &= -\chi_1 (z - P^\sharp_\theta(h))^{-1} [\tilde{\chi}_2, Q^\sharp(h)](z - \tilde{Q}^\sharp_\theta(h))^{-1} \\ &= -\chi_1 (z - P^\sharp(h))^{-1} [\tilde{\chi}_2, Q^\sharp(h)](z - \tilde{Q}^\sharp_\theta(h))^{-1}. \end{aligned}$$

Arguing as in the proof of (5.4), we get

$$\left\| \chi_1 (z - P^\sharp_\theta(h))^{-1} \tilde{\chi}_2 \right\| \leq C \left\| \chi_1 (z - P^\sharp(h))^{-1} [\tilde{\chi}_2, Q^\sharp(h)] \right\| \leq \frac{S_2(h)}{h^{-n} S(h)},$$

where again $S_2(h) = \mathcal{O}(e^{-1/C_2 h})$ in view of Proposition 3.2. This, (5.7), and the spectral theorem imply

$$\left\| \chi_1 \left(\Pi^\theta_j(h) - \Pi^\sharp_j(h) \right) \chi_1 \right\| \leq \frac{S_1(h)}{S(h)} |\partial \widetilde{\Omega}_j(h)| \frac{S_2(h)}{S(h)} \leq C h^{-\frac{7n+1}{2}} \frac{S_1(h) S_2(h)}{S(h)}. \tag{5.8}$$

Therefore, if we go back and choose $S(h) \geq S_1(h)$, we get from (5.8),

$$\left\| \chi_1 \left(\Pi^\theta_j(h) - \Pi^\sharp_j(h) \right) \chi_1 \right\| \leq C h^{-\frac{7n+1}{2}} S_2(h) = \mathcal{O}(e^{-1/C_2' h}), \quad \forall C_2' > C_2. \tag{5.9}$$

To obtain (5.2) we proceed similarly. We now have a factor of $e^{-itz/h}$ in the integrands but it remains bounded on $\partial \widetilde{\Omega}_j(h)$ for $|t| \leq e^{1/Ch}$ for some constant C.

We would like to note that in the case of the Schrödinger operator, the constants C_0 and δ_1, and also δ_0 below, can be related to the Agmon distance between $\pi(\Sigma^i(E))$ and $\pi(\Sigma^e(E))$ and similar interpretation can also be given in the general case. □

The first consequence is worth stating separately:

Corollary. *For $\epsilon > 0$ small enough, there exists a one-to-one correspondence*

$$F : \operatorname{Res}\,(P(h)) \cap ([E-\epsilon, E+\epsilon] - i[0, Ch]) \longrightarrow \operatorname{Spec}(P^\sharp(h)) \cap [E - \epsilon - \alpha_1(h), E + \epsilon + \alpha_2(h)],$$

$\alpha_1(h), \alpha_2(h) = \mathcal{O}(e^{-1/Ch})$, *(with multiple eigenvalues and resonances considered as separate points) satisfying*

$$|F(z) - z| \leq C e^{-\delta_0/h},$$

for some $\delta_0 > 0$.

Remark. This result can also be deduced directly from the analysis of the scattering phase in [14] (assuming that the perturbation decays sufficiently fast at infinity, $|V(x)| \leq C|x|^{-n-\epsilon}$ in the case of a potential) and from the global Breit-Wigner approximation of [1] and [17]. Since this could be of independent interest we review this connection in the appendix.

Proof of Theorem. We first observe that Proposition 3.4 shows that we only need to prove the theorem for $t < h^{-L}$.

Next, we comment on the position of the energy cut-off, $\psi(P(h))$: in Proposition 3.4 and in our main statement it appears on the left, and in Lemmas 4.2 and 4.4 inside the spatial cut-offs. In general, however, if $\psi_1 = 1$ on the support of ψ_2 then

$$\psi_1(P(h))\chi\psi_2(P(h)) = \chi\psi_2(P(h)) + [\psi_1(P(h)), \chi]\psi_2(P(h)) = \chi\psi_2(P(h)) + \mathcal{O}(h^\infty)\,,$$

by the usual semi-classical calculus and a localization argument in the black box case – see [3, Lemma 3.2]. Hence, introducing additional cut-offs needed in applications of Lemmas 4.2 and 4.4 will produce admissible errors.

Our theorem will now follow if we show that for $t < h^{-M}$

$$\begin{aligned}\chi e^{-itP(h)/h}\chi\psi(P(h)) = {} & \chi_1 e^{-itP^\sharp(h)/h}\chi_1\psi(P^\sharp(h)) \\ & + \chi_2\mathcal{O}_{\mathcal{H}\to\mathcal{H}}(\langle (t-C)_+/h\rangle^{-\infty})\chi_2 + \mathcal{O}_{\mathcal{H}\to\mathcal{H}}(h^\infty)\,,\end{aligned}$$

and

$$\begin{aligned}\chi_1 e^{-itP^\sharp(h)/h}\chi_1\psi(P^\sharp(h)) = {} & \sum_{z\in\Omega(h)\cap\mathrm{Res}\,(P(h))} \chi_1 \mathrm{Res}(e^{-it\bullet/h}R(\bullet,h),z)\chi_1\psi(P(h)) \\ & + \mathcal{O}_{\mathcal{H}\to\mathcal{H}}(h^\infty)\,,\end{aligned} \tag{5.10}$$

where $\Omega(h)$ is as in Proposition 3.4. The first statement is immediate from Lemmas 4.4 and 4.2. To obtain the second one we first note that

$$\chi_1\Pi_z^\sharp(h)\chi_1\psi(P(h)) = \chi_1\Pi_z^\sharp(h)\psi(P^\sharp(h))\chi_1 + \mathcal{O}(h^\infty) = \mathcal{O}(h^\infty)\,,$$

if $z \notin \operatorname{supp}\psi + [-\delta, \delta]$, again as in [3, Lemma 3.2]. Hence

$$\chi_1 e^{-itP^\sharp(h)/h}\chi_1\psi(P(h)) = \sum_{\Omega_j(h)\subset\Omega(h)}\sum_{z\in\Omega_j(h)} e^{-itz/h}\chi_1\Pi_j^\sharp(h)\chi_1\psi(P(h)) + \mathcal{O}(h^\infty)\,.$$

The definition of $c(h)$ (and hence of $\Omega(h)$) in [3, Sect.2] shows that we either have $\Omega_j(h) \subset \Omega(h)$ or $\Omega_j(h) \cap \Omega(h) = \emptyset$. Hence Proposition 5.1 implies (5.10) completing the proof.

□

Remark. For a large number of scattering systems (including Schrödinger operators) Burq [2] showed that the resonances always satisfy $|\operatorname{Im} z| \geq e^{-\gamma/h}$ for some $\gamma > 0$. In that case the method of proof of [23, Theorem 2] shows that $c(h)$ can be chosen exponentially close to δ:

$$\delta < c(h) < \delta + e^{-\gamma_1/h}\,.$$

Appendix

In this appendix we give a proof of Corollary presented in Sect.5 based on the relation between the scattering phase and resonances provided by a global Breit-Wigner approximation.

The two results below are quoted directly from [14] and [1] respectively. The first one relates the behavior of the scattering phase to the distribution of eigenvalues of the reference operator given in (4.3). We now need, in addition to the assumption of Sect.2, a stronger decay of the coefficients which guarantees the existence of the scattering phase:

$$|a_\alpha(x,h)| \leq C\langle x\rangle^{-n-\epsilon}\,, \quad \epsilon > 0\,. \tag{A.1}$$

With this in place we have

Proposition A.1. *Let $N(P^\sharp(h), I)$ be the number of eigenvalues of $P^\sharp(h)$ in $I \subset \mathbb{R}$. If* (2.8) *holds then there exists $\beta > 0$, such that for $\lambda \in [E-\epsilon, E+\epsilon]$ we have*

$$\begin{aligned} &N(P^\sharp(h), [\lambda-\delta+e^{-\beta/h}, \lambda+\delta-e^{-\beta/h}]) \leq \sigma(\lambda+\delta, h) - \sigma(\lambda-\delta, h) + \mathcal{O}(\delta)h^{-n} \\ &\quad \leq N(P^\sharp(h), [\lambda-\delta-e^{-\beta/h}, \lambda+\delta+e^{-\beta/h}])\,, \quad e^{-\beta/h} < \delta < 1/C\,. \end{aligned} \tag{A.2}$$

Strictly speaking the results of [14] are stated only for Schrödinger operators but using Proposition 3.2 there is no difficulty in extending the proof to the general case.

The relation between the resonances and σ comes in the form of the *Breit-Wigner approximation*. A global form of it was established by Petkov-Zworski [17] and generalized by Bruneau-Petkov [1]. It is closely related to the local trace formula of Sjöstrand [18].

Proposition A.2. *Under the general assumptions of Sect.2 and* (A.1) *(but no barrier assumption) we have the following local expansion of the derivative of the scattering phase:*

$$\sigma'(\lambda, h) = -\frac{1}{\pi} \sum_{z \in \operatorname{Res}(P(z))\,,\ |\lambda - z| < \epsilon} \frac{\operatorname{Im} z}{|z-\lambda|^2} + \mathcal{O}(h^{-n})\,. \tag{A.3}$$

We remark that the actual Breit-Wigner approximation formulated in [17] takes only resonances with $|\lambda - z| < Ch$ in the sum but that requires additional assumptions. Proposition A.2 follows from factorization of the scattering determinant in [17] and from the meromorphic continuation of the scattering phase in [1].

We now have

Corollary A. *Suppose that* (A.2) *holds and that the resonances in $[E-\epsilon_1, E+\epsilon_1] - i[0, Ch]$ satisfy*

$$|\operatorname{Im} z| > e^{-\gamma_1/h} \implies |\operatorname{Im} z| > e^{-\gamma_2/h}\,, \quad \gamma_1 > \beta > \gamma_2\,,$$

where β is as in (A.2). *Then for any $\epsilon < \epsilon_1$ and any $\gamma > 0$ there exists a one-to-one correspondence*

$$F : \operatorname{Res}(P(h)) \cap ([E-\epsilon, E+\epsilon] - i[0, e^{-\gamma h}]) \longrightarrow \operatorname{Spec}(P^\sharp(h)) \cap [E - \epsilon - \alpha_1(h), E+\epsilon+\alpha_2(h)]\,,$$

$\alpha_1(h), \alpha_2(h) = \mathcal{O}(e^{-1/Ch})$, *(with multiple eigenvalues and resonances considered as separate points) satisfying*

$$|F(z) - z| \leq Ce^{-\gamma_0/h}\,,$$

for some $\gamma_0 > 0$.

Proof. Let us fix γ satisfying $\gamma_2 < \gamma < \beta$. The bound on the number of resonances and eigenvalues of $P^\sharp(h)$ shows that we can group them as in (3.7) with $S(h) = e^{-\gamma_2/h}$ and $\omega(h) = e^{-\gamma/h}$. We will now apply (A.3): suppose that

$$\operatorname{dist}(\lambda, \Omega_k(h)) = \min_j \operatorname{dist}(\lambda, \Omega_j(h))\,,$$

and write

$$\sigma(\lambda+\delta, h) - \sigma(\lambda+\delta, h) = \frac{1}{\pi}\int_{\lambda-\delta}^{\lambda+\delta} \sum_{z \in \Omega_k(h)} \frac{|\operatorname{Im} z|}{|z-t|^2} dt + \frac{1}{\pi}\int_{\lambda-\delta}^{\lambda+\delta} \sum_{z \notin \Omega_k(h)} \frac{|\operatorname{Im} z|}{|z-t|^2} dt + \mathcal{O}(\delta)(h^{-n})\,,$$

where the sums are over resonances in $|\lambda - z| < \epsilon$. If $\delta \ll e^{-\gamma/h}$ then for $z \notin \Omega_k(h)$

$$\frac{|\operatorname{Im} z|}{|z-t|^2} \leq \max\{e^{\gamma_2/h}, e^{\gamma/h}\} = e^{\gamma/h}\,.$$

Hence for $\delta \ll e^{-\gamma/h}$ we have

$$\sigma(\lambda+\delta, h) - \sigma(\lambda+\delta, h) = \frac{1}{\pi}\int_{\lambda-\delta}^{\lambda+\delta} \sum_{z\in\Omega_k(h)} \frac{|\operatorname{Im} z|}{|z-t|^2} dt + \mathcal{O}(\delta)(e^{\gamma/h}) \,.$$

If $\delta = e^{-\gamma_2/h}$, $\gamma < \gamma_2 < \gamma_1$, then this gives (A.2) with $N(P^\sharp, I)$ replaced by $\sharp\mathrm{Res}(P(h)) \cap (I - i[0, e^{-\delta_2/h}])$ and β by some $\beta' > 0$.

To see that we can take any $\gamma > 0$ in the domain of F we observe that by a modification of the argument above, any resonances with $|\operatorname{Im} z| < e^{-\gamma/h}$ would contribute to a lower bound for the variation of the scattering phase, and hence by (A.2) to the number of eigenvalues of $P^\sharp(h)$. But that number has already been achieved by the resonances with $|\operatorname{Im} z| < e^{-\gamma_2/h}$. □

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