Solving DSGE models with perturbation methods and a change of variables

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Abstract

This paper explores the application of the changes of variables technique to solve the stochastic neoclassical growth model. We use the method of Judd [2003. Perturbation methods with nonlinear changes of variables. Mimeo, Hoover Institution] to change variables in the computed policy functions that characterize the behavior of the economy. We report how the optimal change of variables reduces the average absolute Euler equation errors of the solution of the model by a factor of three. We also demonstrate how changes of variables correct for variations in the volatility of the economy even if we work with first-order policy functions and how we can keep a linear representation of the laws of motion of the model if we use a nearly optimal transformation. We discuss how to apply our results to estimate dynamic equilibrium economies.
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1. Introduction

This paper explores the application of the changes of variables technique to solve the stochastic neoclassical growth model. In an important recent contribution, Judd (2003) has provided formulas to apply changes of variables to the solutions of dynamic equilibrium economies obtained through the use of perturbation techniques. Standard perturbation methods provide a Taylor expansion of the policy functions that characterize the equilibrium of the economy in terms of the state variables of the model and a perturbation parameter. Judd's derivations allow moving from this Taylor expansion to any other series in terms of nonlinear transformations of the state variables without the need to recompute the whole solution. This alternative approximation can be more accurate than the first because the change of variables induces nonlinearities that can help to track the true, but unknown, policy functions.

Judd's results are important for several reasons. First, we often want to solve problems with a large number of state variables. Solving these models is difficult and costly. If we use perturbation methods, the number of derivatives required to compute the parameters of the Taylor expansion of the solution quickly explodes as we increase the order of the approximation (see Judd and Guu, 1997). One alternative strategy is to obtain a low-order expansion (a quadratic or even just a linear one) of the solution and move to a more accurate representation using a change of variables. Computing this low-order expansion and the change of variables is relatively inexpensive. Consequently, if we are able to select an adequate transformation, we can increase the accuracy of the solution enough to use our computation for quantitative analysis despite the high dimensionality of the state space.

Second, researchers frequently exploit approximated solutions to understand the analytics of dynamic equilibrium models. See, for example, the classical treatment of Hall (1971) or the recent work of Woodford (2003). Because of tractability considerations, those analyses limit themselves to first-order expansions. Clearly the usefulness of this approach depends on the quality of the approximation. Linear solutions, however, may perform poorly outside a small region around which we linearize. More importantly, as pointed out by Benhabib et al. (2001), linear solutions may even lead to incorrect findings regarding the existence and uniqueness of equilibrium. An optimal change of variables can address both increase the accuracy of the solution and avoid misleading local results while maintaining analytic tractability.

Third, a change of variables can be useful to estimate dynamic equilibrium economies using a likelihood approach. In general, we do not know how to directly evaluate the likelihood function of these economies. Consequently, we need to either linearize the model and use the Kalman filter or to resort to simulation methods like the particle filter that are expensive to implement (see Fernández-Villaverde and Rubio-Ramírez, 2004). A change of variables may deliver a representation of the economy suitable for efficient estimation while capturing some of the nonlinearities of the model.

Motivated by the arguments above, this paper applies Judd's methodology to the stochastic neoclassical growth model with leisure, the workhorse of dynamic macroeconomics. First, we derive a simple close-form relation between the parameters of the linear and the loglinear solution of the model. We extend this approach to a more general class of changes of variables: those that generate a policy function with a power function structure.

Second, we study the effects of that last particular class of changes of variable on the size of the Euler equation errors, i.e., those errors that appear in the optimality conditions of the agents because of the use of approximated solutions instead of the exact ones. We search for the optimal change of variable inside the class of power functions and report results for a benchmark calibration of the model and for alternative parameterizations. In that way, we study the performance of the procedure both for a nearly linear case (the benchmark calibration) and for more nonlinear cases (for example those with higher variance of the productivity shock). We find that, for the benchmark calibration, we reduce the average absolute Euler equation errors by a factor of three. This reduction makes the new approximated solution of the model competitive to much more involved nonlinear methods, such as finite elements or value function iteration, and to second-order perturbations.

Third, sensitivity analysis reveals how the change of variables corrects for movements in the exogenous variance of the economy through changes in the optimal values of the parameters of the transformation. This is true even for a first-order approximation to the policy function. In comparison, a standard linear (or loglinear) solution can only provide a certainty equivalent approximation. This property is a key advantage of our approach.

Fourth, sensitivity analysis also suggests that a particular change of variables that preserves the linearity of the solution is roughly optimal. We propose the use of this approximation because of two reasons. First, because as explained before, such linearity allows researchers to use the well understood toolbox of linear systems while achieving a level of accuracy not possible with the standard linearization. Second, because it facilitates taking the model to the data. In addition to being linear, we show how this quasi-optimal approximated solution also implies that the disturbances to the model are normally distributed. Consequently we can write the economy in a state-space form and use the Kalman filter to perform likelihood based inference.

Finally, we extend some of our results to the change of variables in second-order approximations. We show how a change of variables noticeably increases the accuracy of second-order approximations, even if those are already quite accurate when applied to the stochastic neoclassical growth model (see Aruoba et al., 2005, for details). Thus, our findings suggest that a second-order approximation with a change of variables is a highly attractive solution algorithm in terms of accuracy and simplicity.

The rest of the paper is organized as follows. Section 2 presents the canonical stochastic neoclassical growth model. Section 3 outlines how we solve for the standard linear representation of the model policy functions in levels. Section 4 discusses changes of variables in general terms. Section 5 derives the relation

between the linear and loglinear solution of the model. Section 6 explores the optimal change of variables within a flexible class of functions and reports a sensitivity analysis exercise. Section 7 discusses the use of the change of variables for estimation. Section 8 extends some of our results to second-order approximations. Section 9 concludes.

2. The stochastic neoclassical growth model

As mentioned above, we want to explore how the approximated solutions of the stochastic neoclassical growth model with leisure respond to nonlinear changes of variables. Three reason lead us to use this model. First, its popularity (directly or with small changes) to address a large number of questions (see Cooley, 1995) makes it a natural laboratory to explore the potential of Judd's contribution. Second, because of this importance any analytical result regarding the (approximated) solution of the model is of interest in itself. Finally, since this model is nearly linear for a benchmark calibration, its use provides a particularly difficult laboratory for the change of variables approach: it bounds tightly the improvements we can obtain. We find important advantages of the technique even for this case. Consequently, the improvements coming from the change of variables approach are likely to be higher for more nonlinear economies.[1]

Since the model is well known (see Cooley and Prescott, 1995), we provide only the minimum exposition required to fix notation. There is a representative agent in the economy, whose preferences over stochastic sequences of consumption c_t and leisure $1 - l_t$ are representable by the utility function:

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t),$$

where $\beta \in (0, 1)$ is the discount factor, E_0 is the conditional expectation operator and $U(\cdot, \cdot)$ satisfies the usual technical conditions.

There is one good in the economy, produced according to the aggregate production function $y_t = \mathrm{e}^{z_t} k_t^{\alpha} l_t^{1-\alpha}$ where k_t is the aggregate capital stock, l_t is the aggregate labor input, and z_t is a stochastic process representing random technological progress. The technology follows a first-order process $z_t = \rho z_{t-1} + \varepsilon_t$ with $|\rho| < 1$ and $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$. Capital evolves according to the law of motion $k_{t+1} = (1 - \delta)k_t + i_t$ where δ is the depreciation rate and the economy must satisfy the resource constraint $y_t = c_t + i_t$.

Since both welfare theorems hold in this economy, we can solve directly for the social planner's problem. We maximize the utility of the household subject to the production function, the evolution of the stochastic process, the law of motion for

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[1]The special case of the neoclassical growth model with log utility function and total depreciation is not very informative regarding the usefulness of a change of variables: we already know that the exact solution of that case is loglinear. We want to evaluate the performance of those changes of variables in the general situation where we do not have an analytical solution.

capital, the resource constraint and some initial conditions for capital and the stochastic process.

The solution to this problem is fully characterized by the equilibrium conditions:

$$U_c(t) = \beta E_t\{U_c(t+1)(1 + \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} - \delta)\},$$
$$U_l(t) = -U_c(t)(1-\alpha)e^{z_t} k_t^{\alpha} l_t^{-\alpha},$$
$$c_t + k_{t+1} = e^{z_t} k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t,$$
$$z_t = \rho z_{t-1} + \varepsilon_t$$

given some initial capital and some initial value for the stochastic process for technology. The first equation is the standard Euler equation that relates current and future marginal utilities from consumption, the second one is the static first-order condition between labor and consumption, and the last two equations are the resource constraint of the economy and the law of motion of technology.

We solve for the equilibrium of this economy by finding two policy functions for next period's capital $k'(\cdot,\cdot,\cdot)$, and labor $l(\cdot,\cdot,\cdot)$, which deliver the optimal choice of these controls as functions of the two state variables, capital and technology level, and the standard deviation of the innovation to the productivity level σ. Using the budget constraint, $c(\cdot,\cdot,\cdot)$ is a function of $k(\cdot,\cdot,\cdot)$ and $l(\cdot,\cdot,\cdot)$. Note that we include the perturbation parameter as one explicit variable in the policy functions. That notation is convenient for the next section.

3. Solving the model using a perturbation approach

The system of equations listed above does not have a known analytical solution and we need to use a numerical method to solve it. In a series of seminal papers, Judd and coauthors (see Judd and Guu, 1992, 1997 and the textbook exposition in Judd, 1998, among others) have proposed to build a Taylor series expansion of the policy functions $k'(\cdot,\cdot,\cdot)$, and $l(\cdot,\cdot,\cdot)$ around the deterministic steady state where $z = 0$ and $\sigma = 0$.

If k_0 is equal to the value of the capital stock in the deterministic steady state and the policy function for labor is smooth, we can use a Taylor series to approximate this policy function around $(k_0, 0, 0)$ with the form

$$l_p(k, z, \sigma) \simeq \sum_{i,j,m} \frac{1}{(i+j+m)!} \frac{\partial^{i+j+m} l_p(k, z, \sigma)}{\partial k^i \partial z^j \partial \sigma^m}\bigg|_{k_0,0,0} (k-k_0)^i z^j \sigma^m,$$

where the subscript p stands for perturbation approximation. The policy function for next period capital will have an analogous representation.

The key idea of the perturbation methods is to set the parameter σ equal to zero (since in this case the model can be solved analytically) and to exploit implicit-function theorems to pin down the unknown coefficients $\frac{\partial^{i+j+m} l_p(k,z,\sigma)}{\partial k^i \partial z^j \partial \sigma^m}|_{k_0,0,0}$ in a recursive fashion.

To do so, the first step is to linearize the equilibrium conditions of the model around the deterministic steady state, i.e. when $\sigma = 0$. Then, we substitute capital and labor by the values given by the first-order expansion of the policy function and we solve for the unknown coefficients. The next step is to find the second-order expansion of the equilibrium conditions, again around the deterministic steady state, plug in the quadratic approximation of the policy function (evaluated with the first-order coefficients found before) and solve for the unknown second-order coefficients. We can iterate on this procedure as many times as desired to get an approximation of arbitrary order. For a more detailed explanation of these steps we refer the reader to Judd and Guu (1992).

Since perturbations only deliver an asymptotically locally correct expression for the policy functions, the accuracy achieved by the method may be poor either away from the deterministic steady state or when the order of the expansion is low.

Several routes have been proposed to correct for these problems. One, by Collard and Juillard (2001), uses bias correction to find the approximation of the solution around a more suitable point that the deterministic steady state. The second one, suggested by Judd (2003), changes the variables in terms of which we express the computed solution of the model. We review Judd's proposal in next section.

4. The change of variables

The first-order perturbation solution to the stochastic neoclassical growth model can be written as $f(x) \simeq f(a) + (x - a)f'(a)$ where $x = (k, z)$ are the variables of the expansion, $a = (k_0, 0)$ is the deterministic steady-state value of those variables, and $f(x) = (k'_\mathrm{p}(k, z), l_\mathrm{p}(k, z))$ is the unknown policy function of the model. Since the first-order perturbation solution does not depend on σ, we drop it in this section to save on notation. We could easily consider higher order perturbation solutions, but we concentrate in the first-order approximations because of expositional reasons.

Let us now transform both the domain and the range of $f(x)$. Thus, to express some nonlinear function of $f(x)$, $h(f(x)) : \Re^2 \to \Re^2$ as a polynomial in some transformation of x, $Y(x) : \Re^2 \to \Re^2$, we can use the Taylor series of $g(y) = h(f(X(y)))$ around $b = Y(a)$, where $X(y)$ is the inverse of $Y(x)$.

Using the chain rule, Judd (2003) shows that

$$g(y) = h(f(X(y))) = g(b) + G(b)(Y(x) - b), \tag{1}$$

where $G(b)$ is a 2×2 matrix, whose i, j entry is

$$G_{ij} = \sum_m \frac{\partial h^i}{\partial f^m} \sum_n \frac{\partial f^m}{\partial X^n} \frac{\partial X^n}{\partial y^j}.$$

From this expression we see that if we have computed the values of partial derivatives of f, it is straightforward to find the value of G.

In the next two sections we obtain particular values for this general transformation, one for moving between a solution in levels and in logs and a second for a more general class of power functions.

5. A particular case: the loglinearization

Since the exact solution of the stochastic neoclassical growth model in the case of log utility and $\delta = 1$ is loglinear, many practitioners have favored the loglinearization of the equilibrium conditions of the model over linearization in levels.[2] This practice generates the natural question of finding the relation between the coefficients on both representations. We use (1) to get a simple closed-form answer to this question.

A first-order perturbation produces an approximated policy function in levels of the form[3]

$$(k' - k_0) = a_1(k - k_0) + b_1 z,$$
$$(l - l_0) = c_1(k - k_0) + d_1 z,$$

where k and z are the current states of the economy, l_0 is steady-state value for labor, and where, for convenience, we have dropped the subscript p where no ambiguity exists. As mentioned before, all the coefficients on σ are zero in the first-order perturbation.

Analogously a loglinear approximation of the policy function will take the form

$$\log k' - \log k_0 = a_2(\log k - \log k_0) + b_2 z,$$
$$\log l - \log l_0 = c_2(\log k - \log k_0) + d_2 z,$$

or in equivalent notation:

$$\widehat{k'} = a_2 \widehat{k} + b_2 z,$$
$$\widehat{l} = c_2 \widehat{k} + d_2 z,$$

where $\widehat{x} = \log x - \log x_0$ is the percentage deviation of the variable x with respect to its steady state.

How do we go from one approximation to the second one? First, we follow Judd's (2003) notation and write the linear system in levels as

$$k'_p(k, z, \sigma) = f^1(k, z, \sigma) = f^1(k_0, 0, 0) + f^1_1(k_0, 0, 0)(k - k_0) + f^1_2(k_0, 0, 0)z,$$
$$l_p(k, z, \sigma) = f^2(k, z, \sigma) = f^2(k_0, 0, 0) + f^2_1(k_0, 0, 0)(k - k_0) + f^2_2(k_0, 0, 0)z,$$

where

$f^1(k_0, 0, 0) = k_0$	$f^1_1(k_0, 0, 0) = a_1$	$f^1_2(k_0, 0, 0) = b_1$
$f^2(k_0, 0, 0) = l_0$	$f^2_1(k_0, 0, 0) = c_1$	$f^2_2(k_0, 0, 0) = d_1$

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[2]The wisdom of this practice is disputable. See for example, Dotsey and Mao (1992) and Aruoba et al. (2005) for two papers that find that linearization performs better than loglinearization.

[3]See Uhlig (1999) for details. Remember that this solution is the same as the one generated by a linear quadratic approximation of the utility function (Kydland and Prescott, 1982), the eigenvalue decomposition (Blanchard and Kahn, 1980; King et al., 2002), generalized Schur decomposition (Klein, 2000) or the QZ decomposition (Sims, 2002b) among others. Subject to applicability, all methods find the same policy functions since the linear space approximating a nonlinear space is unique.

Second, we propose the changes of variables:

$$
\begin{array}{lll}
h^1 = \log f^1 & Y^1(x) = \log x^1 & X^1 = \exp y^1 \\
h^2 = \log f^2 & Y^2 = x^2 & X^2 = y^2
\end{array}
$$

Judd's (2003) formulas for this particular example imply

$$
\begin{pmatrix} \log k'(\log k, z) \\ \log l(\log k, z) \end{pmatrix} = g(\log k, z) = \begin{pmatrix} \log f^1(k_0, 0, 0) \\ \log f^2(k_0, 0, 0) \end{pmatrix}
$$
$$
+ \begin{pmatrix} \dfrac{1}{k_0} f_1^1(k_0,0,0)k_0 & \dfrac{1}{k_0} f_2^1(k_0,0,0) \\ \dfrac{1}{l_0} f_1^2(k_0,0,0)k_0 & \dfrac{1}{l_0} f_2^2(k_0,0,0) \end{pmatrix} \begin{pmatrix} \log k - \log k_0 \\ z \end{pmatrix},
$$

and thus

$$
\log k' - \log k_0 = f_1^1(k_0,0,0)(\log k - \log k_0) + \frac{1}{k_0} f_2^1(k_0,0,0)z,
$$
$$
\log l - \log l_0 = \frac{k_0}{l_0} f_1^2(k_0,0,0)(\log k - \log k_0) + \frac{1}{l_0} f_2^2(k_0,0,0)z.
$$

By equating coefficients we obtain a simple closed-form relation between the parameters of both representations:[4]

$$
\begin{array}{lll}
a_2 = a_1 & \qquad & b_2 = \frac{1}{k_0} b_1 \\[4pt]
c_2 = \frac{k_0}{l_0} c_1 & \qquad & d_2 = \frac{1}{l_0} d_1
\end{array}
$$

Note that we have not used any assumption on the utility or production functions except that they satisfy the general technical conditions of the neoclassical growth

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[4]An alternative heuristic argument that delivers the same result is as follows. Take the system

$$
(k' - k_0) = a_1(k - k_0) + b_1 z,
$$
$$
(l - l_0) = c_1(k - k_0) + d_1 z.
$$

and divide on both sides by the steady-state value of the control variable:

$$
\frac{k' - k_0}{k_0} = a_1 \frac{k - k_0}{k_0} + \frac{1}{k_0} b_1 z,
$$
$$
\frac{l - l_0}{l_0} = c_1 \frac{k - k_0}{k_0} + \frac{1}{l_0} d_1 z.
$$

Since $\frac{x' - x_0}{x_0} \simeq \log x - \log x_0$, we get the same relation that the one presented in the paper. Our argument is more general and does not depend on an additional approximation.

model. Also, moving from one coefficient set to the other one is an operation that only involves k_0 and l_0, values that we need to find anyway to compute the linearized version in levels. Therefore, once you have the linear solution, obtaining the loglinear one is immediate.

6. The optimal change of variables

In the last section, we showed how to find a loglinear approximation to the solution of the neoclassical growth model directly from its linear representation. Now, we generalize our result to encompass the relationship between any power function approximation and the linear coefficients of the policy function. Also, we search for the optimal change of variable inside this class of power functions and we report how the Euler equation errors improve with respect to the linear representation.

6.1. A power function transformation

Before, we argued that some practitioners have defended the use of loglinearizations to capture some of the nonlinearities in the data. This practice can be pushed one step ahead. We can generalize the log function into a general class of power function of the form

$$k'_p(k, z; \gamma, \zeta, \mu)^\gamma - k_0^\gamma = a_3(k^\zeta - k_0^\zeta) + b_3 z,$$
$$l_p(k, z; \gamma, \zeta, \mu)^\mu - l_0^\mu = c_3(k^\zeta - k_0^\zeta) + d_3 z.$$

This class of functions is attractive because it provides a lot of flexibility in shapes with only three free parameters, while including the log transformation as the limit case when the coefficients γ, ζ, and μ tend to zero. A similar power function with only two parameters is proposed by Judd (2003) in a growth model without leisure and stochastic perturbations. His finding of notable improvements in the accuracy of the solution when he optimally selects the value of these parameters is suggestive of the advantages of using this parametric family.

The changes of variables for this family of functions are given by

$h^1 = (f^1)^\gamma$	$Y^1 = (x^1)^\zeta$	$X_1 = (y^1)^{\frac{1}{\zeta}}$
$h^2 = (f^1)^\mu$	$Y_2 = x^2$	$X_2 = y^2$

Following the same reasoning than in the previous section, we derive a form for the system in term of the original coefficients:

$$k'_p(k, z; \gamma, \zeta, \mu)^\gamma - k_0^\gamma = \frac{\gamma}{\zeta} k_0^{\gamma-\zeta} a_1(k^\zeta - k_0^\zeta) + \gamma k_0^{\gamma-1} b_1 z,$$
$$l_p(k, z; \gamma, \zeta, \mu)^\mu - l_0^\mu = \frac{\mu}{\zeta} l_0^{\mu-1} k_0^{1-\zeta} c_1(k^\zeta - k_0^\zeta) + \mu l_0^{\mu-1} d_1 z.$$

Therefore, the relation of between the new and the old coefficients is again simple to compute:

$$a_3 = \tfrac{\gamma}{\zeta} k_0^{\gamma-\zeta} a_1 \qquad\qquad b_3 = \gamma k_0^{\gamma-1} b_1$$
$$c_3 = \tfrac{\mu}{\zeta} l_0^{\mu-1} k_0^{1-\zeta} c_1 \qquad\qquad d_3 = \mu l_0^{\mu-1} d_1$$

6.2. Searching for the optimal transformation

One difference between the transformation from linear into a loglinear solution and the current transformation into a power law is that we have three free parameters γ, ζ, and μ. How do we select optimal values for those?

A reasonable criterion is to select them in order to improve the accuracy of the solution of the model. This desideratum raises a question. How do we measure this accuracy, conditional on the fact that we do not know the true solution of the model?

Judd and Guu (1992) solves this problem evaluating the normalized Euler equation errors (see also Santos, 2000; Den Haan and Marcet, 1994). Note that in our model intertemporal optimality implies that

$$U_c(t) = \beta E_t \{ U_c(t+1)(1 + \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} - \delta) \}. \tag{2}$$

Since the solutions are not exact, condition (2) will not hold exactly when evaluated using the computed decision rules. Instead, for any choice of γ, ζ, and μ and its associated policy rules $k'_p(\cdot, \cdot; \gamma, \zeta, \mu)$ and $l_p(\cdot, \cdot; \gamma, \zeta, \mu)$, we can define the normalized absolute value Euler equation errors as

$$EE(k, z; \Xi) = \left| 1 - \frac{(U_c^{-1}(E_t U_c(t+1)\beta R'(k, z; \Xi), l_p(k, z; \Xi)))}{c_p(k, z; \Xi)} \right|, \tag{3}$$

where $\Xi = (\gamma, \zeta, \mu)$, $R'(k, z; \Xi) = (1 + \alpha e^{z_{t+1}} k'(k, z; \Xi)^{\alpha-1} l'(k, z; \Xi)^{1-\alpha} - \delta)$ is the gross return on capital next period, and where we have used the static optimality condition to substitute labor by its optimal choice.

This expression evaluates the (unit free) error in the Euler equation measured as a fraction of $c_p(\cdot, \cdot; \Xi)$ as a function of the current states k, and z, and the change of variables defined by γ, ζ, and μ. Judd and Guu (1997) interpret this function as the mistake, in dollars, incurred by each dollar spent. For example, $EE(k, z; \Xi) = 0.01$ means that the consumer makes a 1 dollar mistake for each 100 dollar spent.

Our criterion is to select values of Ξ that minimize the Euler error function. But here we face another problem. This function depends not only on the parameters Ξ, but also on the values of the state variables. How do we eliminate that dependency? Do we minimize the Euler equation error at one particular point of the state space like the deterministic steady state? Do we minimize instead some weighted mean of it?

A first choice could be the criterion $\min_\Xi \int EE(k, z; \Xi) \, d\Phi$ where Φ is the stationary distribution of k and z. This choice is intuitive. We would weight each Euler equation error by the percentage of the time that the economy spends in that

particular point: we would want to nail down the Euler equation errors for those parts of the stationary distribution where most of the action would happen, while we would care less about accuracy in those points not frequently visited. The difficulty of the criterion is that we do not know that true stationary distribution (at least for capital) before we find the policy functions of the model.

To avoid this difficulty, we could solve a fixed-point problem where we get some approximation of the model, compute a stationary distribution, resolve the minimization problem, find the new stationary distribution, and continue until convergence. This approach faces two problems. First, it would be expensive in computing time. Second, there is no known theory result that guarantees the convergence of such iteration to the right set of values for the parameters of the transformation.

Given the difficulties of the previous approach, we follow a simpler strategy. Inspired by collocation schemes in projection methods, we minimize the Euler equation errors over a grid of points of k and z:

$$\min_{\Xi} SEE(\Xi) = \min_{\Xi} \sum_{k,z} EE(k, z; \Xi). \tag{4}$$

Setting this grid around the deterministic steady state (big enough to be representative of the stationary distribution of the economy but not too wide to avoid minimization over regions infrequently visited) may achieve our desire goal of improving the accuracy of the solution.[5]

Inspection of (4) reveals that this problem depends on the values of the structural parameters of the model, i.e. those describing preferences and technology. Consequently, we need to take a stand on those before we can report any result. We will calibrate of the model to match basic observations of the US economy following the common strategy in macroeconomics and, later, perform sensitivity analysis. Hence, we evaluate how much accuracy we win in a 'real life' situation by applying our change of variables and how confident we are in our results.

First we select as our utility function the CRRA form

$$\frac{(c_t^\theta (1 - l_t)^{1-\theta})^{1-\tau}}{1 - \tau},$$

where τ determines the elasticity of intertemporal substitution and θ controls labor supply. Then we pick the benchmark calibration values as follows. The discount factor $\beta = 0.99$ matches an annual interest rate of 4 percent. The parameter that governs risk aversion $\tau = 2$ is a common choice in the literature. $\theta = 0.36$ matches

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[5]When judging this scheme we need to remember that we do not look for the absolute best change of variables (although certainly finding it would be nice) but for an improvement in the accuracy of the solution that can be easily found. This point can be important in models with dozens of state variables, where a multivariate maximization may be very costly. We can use a simulation procedure like a Markov chain Monte Carlo method to find an improvement 'sufficiently good'. Those approaches are easy to code and get close to a global maximum (although not exactly at it) fairly quickly. In this paper, however, we can solve the problem using a Newton algorithm implemented in a *Mathematica* notebook that is available online at the following URL http://www.econ.upenn.edu/~jesusfv

the microeconomic evidence of labor supply to 0.31 of available time in the deterministic steady state. We set $\alpha = 0.4$ to match labor share of national income (after the adjustments to National Income and Product Accounts suggested by Cooley and Prescott, 1995). The depreciation rate $\delta = 0.0196$ fixes the investment/output ratio and $\rho = 0.95$ and $\sigma = 0.007$ follow the stochastic properties of the Solow residual of the US economy. Table 1 summarizes the discussion. Later, we discuss the sensitivity of our results to change in parameter values.

Also, we need to define the set of points for k and z over which we are going to sum the normalized absolute value Euler equation errors. We define a grid of 21 capital points covering ± 30 percent above and below the steady-state capital and a grid of 21 productivity points using Tauchen (1986) procedure. Using highly accurate nonlinear methods, we computed that our grid covers over 99 percent of the stationary distribution of the economy.

We report our main findings in Table 2. The last entry of the second row is the value of $SEE(1, 1, 1)$, i.e., for the linear approximation for the Benchmark calibration. The third row is the solution to problem (4).

We highlight several points. First, the parameters γ and ζ are both close to one suggesting that the nonlinearities in capital are of little importance (although not totally absent, since $SEE(\Xi)$ suffers when we impose that both parameters are equal to one). That result is confirmed by inspecting the nearly linear policy functions for capital found by very accurate but expensive nonlinear methods (see the results reported in Aruoba et al., 2005). Second, μ is far away from zero. The interpretation is that the labor supply function is much more nonlinear in capital and allowing the policy function to capture that behavior increases the accuracy of the solution.

The last column of the Table 2 shows that the optimal change of variables improves the average absolute value Euler equation error by a factor of around three. This means that, for each mistake of three dollars made using the linear approximation, the consumer would only have made a one dollar mistake using the

Table 1
Calibrated parameters

Parameter	β	τ	θ	α	δ	ρ	σ
Value	0.99	2.0	0.36	0.4	0.02	0.95	0.007

Table 2
Euler equation errors

γ	ζ	μ	$SEE(\Xi)$
1	1	1	0.0856279
0.986534	0.991673	2.47856	0.0279944

optimal change of variable, a sizeable (but not dramatic) improvement in accuracy. This same result is shown in Fig. 1, where we plot the decimal log of the absolute Euler equation errors at $z = 0$ for the ordinary linear solution and the optimal change of variable. The use of decimal logs eases the reading of the graph. A value of -3 in the vertical axis represents an error of 1 dollar out of each 1000 dollars, a value of -4 an error of 1 dollar out of each 10,000 dollars and so on. In Fig. 1, we observe that when only one dimension is considered, the optimal change of variable can improve Euler equation errors from around 1 dollar every 10,000 dollars to 1 dollar every 1,000,000 dollars (an even more in some points).

We explore how the optimal change of variable compares with other more conventional nonlinear methods in terms of accuracy. In Fig. 2, we compare the Euler equation errors generated by the optimal change of variables with the Euler errors implied by different linear and nonlinear solution methods (see Aruoba et al., 2005, for details about the other methods). The graph shows how the optimal change of variable solution pushes a first-order approximation to the accuracy level delivered by the finite elements method for most of the interval. If we consider the computational and coding costs of implementing the finite elements method, this result provides strong evidence in favor of using a combination of linear solution and optimal change of variable as a solution method for macroeconomic models. Also, the new approximation is roughly comparable in terms of accuracy with a second-order approximation of the policy function defended by Sims (2002a) and Schmitt-Grohé and Uribe (2004).

We finish by pointing out how the change of variables might have the additional advantage of suffering less often from the problem of explosive behavior of simulations present in the second and higher order perturbations. This explosive behavior appears because of the presence of higher order terms in the perturbation solution. Sometimes, the productivity shocks are so big that the high powers of the
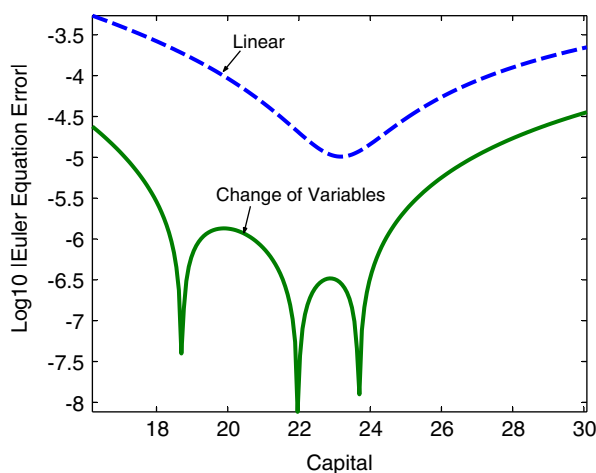

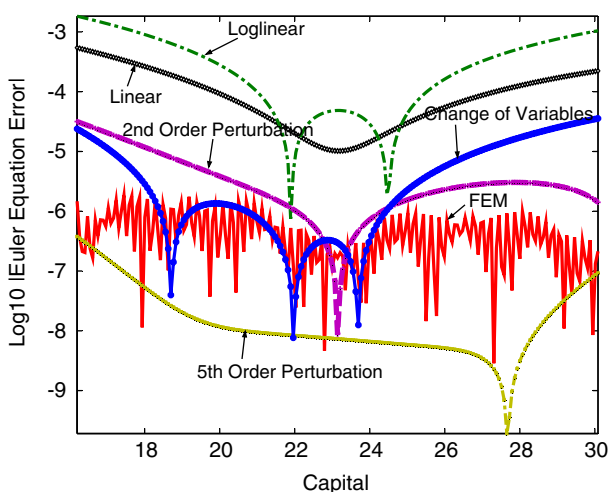Fig. 1. Euler equation errors at $z = 0$, $\tau = 2/\sigma = 0.007$.

Fig. 2. Euler equation errors at $z = 0, \tau = 2/\sigma = 0.007$.

solution send the path of capital outside the stable manifold. For example, Aruoba et al. (2005) find that, for a model like the one studied in this paper, up to 4 percent of the simulations led to explosive paths when they use a fifth-order approximation. There are two solutions to this problem. First, those explosive simulation can be eliminated from the computation. However, there is a certain degree of arbitrariness in the definition of when a path becomes explosive. Second, Kim et al. (2003) propose an algorithm that always produces stationary second-order accurate dynamics whenever the first-order dynamics are stable.

The change of variables suffers less from the problem of explosive paths because it raises the state variables at lower powers. In our application, the powers in the policy function for capital (the one exposed to explosive behavior) vary from 0.97 to 1.24. These lower powers stop most explosive paths. An example can be seen in Fig. 3. We draw two simulated time series of capital for our model with $\sigma = 0.56$ (the higher variance of productivity shocks needed to show more clearly the presence of explosive paths). The first line plots the series of capital induced by a fifth-order perturbation. The second line plots the series of capital for the optimal change of variables. From this figure we can see how for the first 150 periods, both series are close. However, around period 175 they separate and, after a few periods, the simulation from the fifth-order explodes while the simulation from the optimal change of variables, despite also showing an increase in the amount of capital, stays within the stable manifold.

6.3. Sensitivity analysis

How does the solution to (4) depend on the chosen calibration? Are the optimal parameters values in Ξ robust to changes in the values of the structural parameters of the economy?
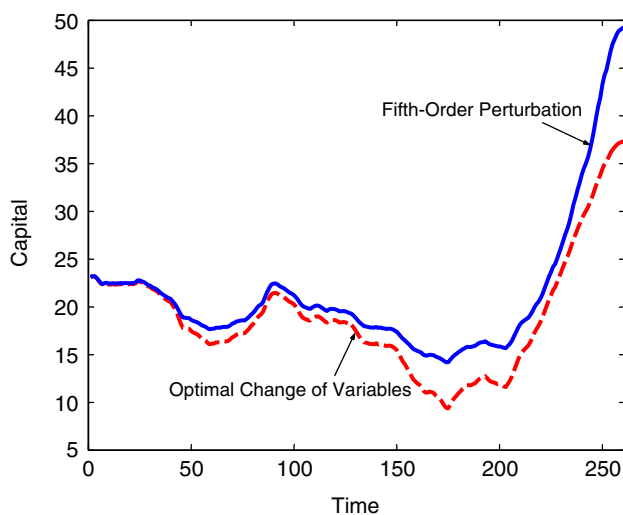
Fig. 3. Time series of capital.

Table 3
Optimal parameters for different αs

α	γ	ζ	μ
0.2	1.05064	1.06714	0.90284
0.4	0.98653	0.99167	2.47856
0.6	0.97704	0.97734	2.47796
0.8	0.94991	0.94889	2.47811

We analyzed in detail how the optimal change of variables depends on the different parameter values. We report only a sample of our findings with the most interesting results. We discuss the effect of changes in the share of capital α (Table 3), in the parameter governing the intertemporal elasticity of substitution τ (Table 4), and in the standard deviation of the technology shock σ (Table 5).

In Table 3 we see how γ and ζ stay close to 1 although some variation in their values is induced by the changes in the capital share. On the other hand, μ changes from 0.9 to 2.47. Interestingly, all of this change happens when we move from $\alpha = 0.2$ to $\alpha = 0.4$. This drastic change is due to the fact that the steepness of the marginal productivity of capital changes rapidly for values of α around this range.

Table 4 shows the results are more robust with respect to changes in τ. This finding is important because it documents that the change of variables has a difficult time to capture the effects of large risk aversion.

Table 5 shows the effects of changing σ from its benchmark case of 0.07 to 0.014, 0.028, and 0.056. Here we observe more action: γ and ζ move a bit and μ goes all the way down from 2.48 to 0.78. Higher variance of the productivity shock increases the

Table 4
Optimal parameters for different τs

τ	γ	ζ	μ
4	1.02752	1.03275	2.47922
10	1.04207	1.04671	2.47830
50	1.01138	1.01638	2.47820

Table 5
Optimal parameters for different σs

σ	γ	ζ	μ
0.014	0.98140	0.98766	2.47753
0.028	1.04804	1.05265	1.73209
0.056	1.23753	1.22394	0.77869

precautionary motive for consumers and induces stronger nonlinearities in the computed policy functions of the model.

The variation in optimal parameter values is a significant result. It shows how the change of variables allows for a correction based on the level of uncertainty existing in the economy. This correction cannot be achieved by linearization approach since that strategy produces a certainty equivalent approximation. The ability to adapt to different volatilities while keeping a first-order structure in the solution is a key advantage of the change of variables procedure.

We conclude from our sensitivity analysis that the choices of optimal parameter values of γ, ζ, and μ are stable across very different parameterizations and that the only relevant change is for the value of μ when we vary capital share or the variance of the technology shock.

6.4. A linear representation of the solution

Our previous findings motivate the following observation. Since γ and ζ are roughly equal across different parameterizations, we can confine ourselves to the two parameter transformation:

$$k'^{\gamma} - k_0^{\gamma} = a_3(k^{\gamma} - k_0^{\gamma}) + b_3 z,$$
$$l^{\mu} - l_0^{\mu} = c_3(k^{\gamma} - k_0^{\gamma}) + d_3 z,$$

while keeping nearly all the same level of accuracy obtained by the more general case.

To illustrate our point we plot the Euler Equation error for this restricted optimal case in Fig. 4 with $\gamma = 1.11498$ and $\mu = 0.948448$. Comparing with the optimal change we can appreciate how we keep nearly all the increment in accuracy. More formally the $SEE(\Xi)$ moves now to 0.0420616.
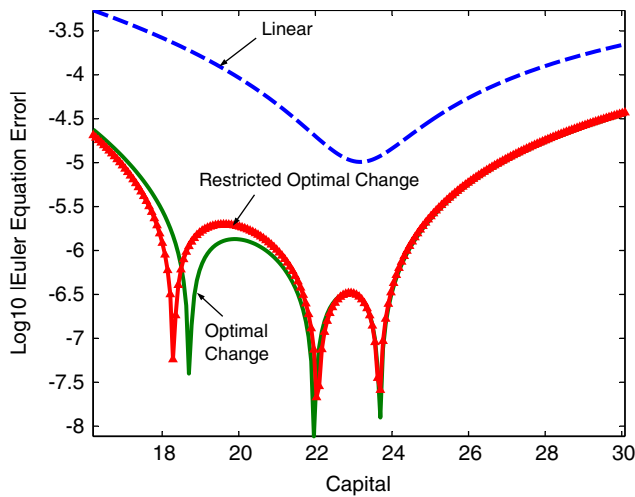
Fig. 4. Euler equation errors at $z = 0, \tau = 2/\sigma = 0.007$.

The proposed two parameter transformation is very convenient because if we define $\widehat{k} = k^\gamma - k_0^\gamma$ and $\widehat{l} = l^\mu - l_0^\mu$ we can rewrite the equations as

$$\widehat{k'} = a_3\widehat{k} + b_3 z,$$
$$\widehat{l} = c_3\widehat{k} + d_3 z$$

producing a linear system of difference equations.

This implies that we can study the neoclassical growth model (or a similar dynamic equilibrium economy) in the following way. First, we find the equilibrium conditions of the economy. Second, we linearize them (or loglinearize if it is easier) and solve the resulting problem using standard methods. Third, we transform the variables (leaving the parameters of the transformation undetermined until the final numerical analysis) and study the qualitative properties of the new system using standard techniques. If the transformation captures some part of the nonlinearities of the problem we may be able to avoid the problems singled out by Benhabib et al. (2001). Fourth, if we want to study the quantitative behavior of the economy we pick values for the parameters according to (4) and simulate the economy using the transformed system.

7. Changes of variables for estimation

Often, dynamic equilibrium models can be written in a state-space representation, with a transition equation that determines the law of motion for the states and a measurement equation that relates states and observables. It is well known that models with this representation are easily estimated using the Kalman Filter (see Harvey, 1989).

Consequently, a common practice to take the stochastic neoclassical growth model to the data has been to linearize the equilibrium conditions in levels or in logs around the steady state to get a transition and a measurement equation, use the Kalman recursion to evaluate the implied likelihood, and then, either to maximize it (in a classical perspective) or to draw from the posterior (in a Bayesian approach).

In this subsection we propose a variation of this strategy. Since in our computations we find that γ and ζ are roughly equal, we propose to use

$$\widehat{k}' = a_3\widehat{k} + b_3 z \tag{5}$$

in the transition equation and (possibly)

$$\widehat{l} = c_3\widehat{k} + d_3 z \tag{6}$$

in the measurement equation, where $\widehat{k} = k^\gamma - k_0^\gamma$ and $\widehat{l} = l^\mu - l_0^\mu$, instead of the usual linear or loglinear relations. Since both equations are still linear in the transformed variables, the conditional distributions of the variables is normal and we can use the Kalman Filter to evaluate the likelihood. The only difference now will be that, in addition to depend on the structural parameters of the model, the likelihood will also be a function of the parameters γ and μ.

This approach allows us to keep a linear representation suitable for efficient estimation while capturing an important part of the nonlinearities of the model and avoiding the use of expensive simulation methods like a particle filter.

To illustrate the potentiality of our proposal, we proceed as follows. First, we generate 200 observation of 'artificial' data for hours worked, $l^T = \{l_t\}_{t=1}^{200}$, from the stochastic neoclassical growth model that we presented above. We solved the model using the finite elements method as described in Aruoba et al. (2005) to generate an artificial sample with nonlinearities. Second, we compute the likelihood function implied by the approximated solution (5) and (6), where γ and ζ are chosen to minimize the problem (4). For comparison purposes, we compute the likelihood implied by the loglinear solution. We evaluate both likelihood functions at the same parameter values that we use to generate the 'artificial' data for hours worked.

A simple procedure to compare both solutions is to implement a likelihood ratio test. This test will tells which of the two models, the one represented by (5) and (6) or the usual loglinear model, fits the data better.

We compute the likelihood ratio test for two different points in the parameter space. First, we use the parameter values described in our benchmark calibration. Second, we also consider a more extreme case where $\tau = 50$ and $\sigma = 0.035$, while we maintain the value of the rest of the parameters to the values reported in Table 1. We recognize that the extreme case (high risk aversion/high variance) is not realistic, but it allows us to easily consider a very nonlinear case.[6]

The likelihood ratio is defined by

$$\widehat{\lambda} = \frac{L^R(l^T|\Theta, \gamma = 1, \mu = 1)}{L^U(l^T|\Theta, \gamma, \mu)},$$

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[6]We thank the editor, Kenneth Judd, for suggesting this extreme case to us.

Table 6
Likelihood ratio test

Case	Benchmark	Extreme
$-2\log\widehat{\lambda}$	3.52	163.06

where, $\Theta = (\beta, \sigma, \tau, \theta, \alpha, \delta, \rho)$, $L^{U}(l^{T}|\Theta, \gamma, \mu)$ stands for the likelihood of the 'unrestricted model', where γ and ζ are chosen to minimize (4), and $L^{R}(l^{T}|\Theta, \gamma = 1, \mu = 1)$ stands for the likelihood of the 'restricted model', where γ and ζ are restricted to be equal to one. Under the null hypothesis that $\gamma = 1$ and $\mu = 1$, $-2\log\widehat{\lambda}$ is distributed as a χ^2 with two degrees of freedom (corresponding to the two restrictions). Table 6 reports the values of $-2\log\widehat{\lambda}$ for the two different choices of the parameter values.

In the benchmark case, the value of the statistic is 3.52. The statistic clearly indicates that the optimal change of variables fits the data better. However, the improvement in fit is not dramatic and the statistic is only significant at the 20 percent level. The result is not surprising, since in the benchmark case the model is almost linear.

In the extreme case, the results are overwhelming in favor of the solution described by (5) and (6). The improvement in fit is so high that the value of the statistic is 163.06, with significance at any tabulated value. This example shows how, when the data is generated from a very nonlinear model, the change of variables approach is able to capture those linearities and fits the data much better than the typical loglinear model.

The large differences in likelihoods corroborate a finding by Fernández-Villaverde et al. (2005). The authors document how second-order approximation errors in the policy function, which almost always are ignored by researchers when estimating dynamic equilibrium models, have first-order effects on the likelihood function. The authors show that the approximated likelihood function converges at the same rate as the approximated policy function. However, the error in the approximated likelihood function gets compounded with the size of the sample. Therefore, period by period, small errors in the policy function accumulate at the same rate at which the sample size grows.

The results of Fernández-Villaverde et al. (2005) can also be interpreted as suggesting the need for highly accurate solution methods for dynamic equilibrium models if we want to estimate them. Since the change of variables substantially reduces the approximation errors in the policy function for a trivial computational cost, the method in this paper presents itself as a very attractive alternative for applied researchers.

8. The change of variables for the second-order approximation

Following Judd and Guu (1992, 1997), recent work by Schmitt-Grohé and Uribe (2004) and by Sims (2002a) has popularized the use of second-order approximations to the solution of dynamic equilibrium models. Consequently, it is of interest to show how the change of variables applies to this second-order approximations.

The second-order perturbation solution to the stochastic neoclassical growth model can be written as $f(x) \simeq f(a) + (x - a)f'(a) + \frac{1}{2}(x - a)f''(a)(x - a)'$ where $x = (k, z, \sigma)$ are the variables of the expansion, $a = (\bar{k}_0, 0, 0)$ is the deterministic steady-state value of those variables, and $f(x) = (k'_p(k, z, \sigma), l_p(k, z, \sigma))$ is the unknown policy function of the model. Note that, since it this case the policy functions depend on σ, we add this parameter to the state variables of the system.

As before, we transform both the domain and the range of $f(x)$. Thus, to express some nonlinear function of $f(x)$, $h(f(x)) : \Re^2 \to \Re^2$ as a polynomial in some transformation of x, $Y(x) : \Re^3 \to \Re^3$, we can use the Taylor series of $g(y) = h(f(X(y)))$ around $b = Y(a)$, where $X(y)$ is the inverse of $Y(x)$.

Using the chain rule, it can be shown that

$$g^i(y) = h^i(f(X(y))) = g^i(b) + G^i(b)(Y(x) - b) + \frac{1}{2}(Y(x) - b)'H^i(b)(Y(x) - b)$$

for $i = 1, 2$ where $G^i(b)$ is a 1×3 matrix, whose j entry is

$$G^i_j = \frac{\partial h^i}{\partial f^i} \frac{\partial f^i}{\partial X^i} \frac{\partial X^i}{\partial y^j}$$

and $H^i(b)$ is a 3×3 matrix, whose s, j entry is

$$H^i_{sj} = \frac{\partial G^i_s}{\partial y^j}$$

for $i = 1, 2$ and $s, j = 1, 2, 3$. As for the linear case, if we have computed the values of f', then it is straightforward to find the value of G.

8.1. A power function transformation

Now, we apply our previous derivations to the second-order perturbation of the neoclassical growth model:

$$k'_p(k, z; \gamma, \zeta, \mu) - k_0 = a_1(k - k_0) + b_1 z + c_1(k - k_0)^2 + d_1 z^2 + e_1 \sigma^2 + f_1(k - k_0)z,$$

$$l_p(k, z; \gamma, \zeta, \mu) - l_0 = a_2(k - k_0) + b_2 z + c_2(k - k_0)^2 + d_2 z^2 + e_2 \sigma^2 + f_2(k - k_0)z.$$

Following our previous framework for the linear approximation, we consider the power transformation for the second-order perturbation of the class:

$$k'_p(k, z; \gamma, \zeta, \mu)^\gamma - k_0^\gamma = a_3(k^\zeta - k_0^\zeta) + b_3 z + c_3(k^\zeta - k_0^\zeta)^2 + d_3 z^2 + e_3 \sigma^2 + f_3(k^\zeta - k_0^\zeta)z,$$

$$l_p(k, z; \gamma, \zeta, \mu)^\mu - l_0^\mu = a_4(k^\zeta - k_0^\zeta) + b_4 z + c_4(k^\zeta - k_0^\zeta)^2 + d_4 z^2 + e_4 \sigma^2 + f_4(k^\zeta - k_0^\zeta)z.$$

The changes of variables for this family of functions are given by $h^1 = (f^1)^\gamma$ and $h^2 = (f^1)^\mu$, while

$Y^1 = (x^1)^\zeta$	$Y_2 = x^2$	$Y_2 = x^2$
$X_1 = (y^1)^{\frac{1}{\zeta}}$	$X_2 = y^2$	$X_3 = y^3$

We derive a form for the system in term of the original coefficients:

$$k'_{\mathrm{p}}(k, z; \gamma, \zeta, \mu)^\gamma - k_0^\gamma = \frac{\gamma}{\zeta} k_0^{\gamma-\zeta} a_1 (k^\zeta - k_0^\zeta) + \gamma k_0^{\gamma-1} b_1 z,$$

$$l_{\mathrm{p}}(k, z; \gamma, \zeta, \mu)^\mu - l_0^\mu = \frac{\mu}{\zeta} l_0^{\mu-1} k_0^{1-\zeta} c_1 (k^\zeta - k_0^\zeta) + \mu l_0^{\mu-1} d_1 z.$$

Therefore, the relation of between the new and the old coefficients is simple to compute:

$a_3 = \frac{\gamma}{\zeta} k_0^{\gamma-\zeta} a_1$	$a_4 = \frac{\mu}{\zeta} l_0^{\mu-1} k_0^{1-\zeta} a_2$
$b_3 = \gamma k_0^{\gamma-1} b_1$	$b_4 = \mu l_0^{\mu-1} b_2$
$c_3 = \frac{\gamma}{\zeta^2} k_0^{\gamma-2\zeta} (c_1 k_0 + a_1 (1 + a_1(\gamma - 1) - \zeta))$	$c_4 = \frac{\mu}{\zeta^2} k_0^{1-2\zeta} l_0^{\mu-2} (c_1 k_0 l_0 + a_2$ $(l_0(1 - \zeta) + k_0 a_2(\mu - 1)))$
$d_3 = \gamma k_0^{-2+\gamma} (d_1 k_0 + b_1^2(\gamma - 1))$	$d_4 = \mu l_0^{-2+\mu} (d_2 l_0 + b_2^2(\mu - 1))$
$e_3 = e_1 k_0^{\gamma-1} \gamma$	$e_4 = e_2 l_0^{\mu-1} \mu$
$f_3 = \frac{\gamma}{\zeta} k_0^{\gamma-\zeta-1} (f_1 k_0 + a_1 b_1(\gamma - 1))$	$f_4 = \frac{\mu}{\zeta} k_0^{1-\zeta} l_0^{-2+\mu}$ $(f_2 l_0 + a_2 b_2(\mu - 1))$

8.2. Results

As in the linear case, we determine the coefficients γ, ζ, and μ in order to minimize the sum of the Euler Errors, as defined by problem (4). Table 7 presents the results from such exercise.

We highlight two findings. First, the change of variables reduces the sum of Euler Errors by a 38 percent, a substantial improvement in terms of accuracy given the trivial computation cost of the change of variables. Second, this increase in accuracy is on top of the already good performance in terms of accuracy of the second-order perturbation when applied to the neoclassical growth model as documented by Aruoba et al. (2005). Consequently, our results suggest that a second-order approximation with a change of

Table 7
Euler equation errors

γ	ζ	μ	$SEE(\Xi)$
1	1	1	0.00044651
0.518336	0.521921	4.04106	0.00027822

variables is a highly attractive solution procedure for dynamic equilibrium models both in terms of accuracy and simplicity of computation.

Unfortunately, the second-order approximation implies the loss of linearity of the system, even with a restricted optimal transformation. This lack of linearity prevent us for using the Kalman filter for estimation purposes. Fernández-Villaverde and Rubio-Ramírez (2004) show how to apply the particle filter to estimate a nonlinear model similar to the one described here.

9. Concluding remarks

This paper has explored the effects of changes of variables in the policy function of the stochastic neoclassical growth model as first proposed by Judd (2003). We have shown how this change of variables helps to obtain a more accurate solution to the model both for analytical and empirical applications.

The procedure proposed is conceptually straightforward and simple to implement, yet powerful enough to substantially increase the quality of our solution to the neoclassical growth model. For our benchmark calibration the average Euler equation error is divided by three and the new policy function has a performance comparable with the policy functions generated by fully nonlinear methods. In addition, within the class of power functions considered in the paper, the optimal change of variables allows us to keep a linear structure of the model. This finding is useful for analytical and estimation purposes.

Several questions remain to be explored further. What is the optimal class of parametric families to use in the changes of variables? Are those optimal families robust across different dynamic equilibrium models? How big is the increment in accuracy in other types of models of interest to macroeconomists? How much do we gain in accuracy of our estimates by using a transformed linear state space representation of the model? We plan to address these issues in our future research.

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