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Version 1

Sotirios Persidis

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ALIVE BOOKS®

MATHEMATICAL HANDBOOK

Part A

Version 1

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Cover picture: A mathematician at work - trying to find order in chaos.

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CONTENTS

Preface

This is a MATHEMATICAL HANDBOOK. It has been designed to cover the main needs of everyone who uses mathematics: mathematicians, physicists, engineers and other professionals. Most of all, it has been written with the students, and all those who strive to learn, in mind.

The previous objective imposes severe constraints on the material of the book and its presentation. The subject matter must cover an extremely broad area of mathematical knowledge, from elementary to advanced. It must be clear, concise and reliable. Its presentation must appeal to the user. Its typography, the arrangement of the material and its coloring should make it attractive to the eye. The use of the book must be easy and require as little time as possible, conveying the maximum information. All of these concerns have been taken into account as much as possible. The chapters contain well-separated content, from the more simple and basic mathematical areas to medium and more advanced concepts. The contents and the index help the user find what he wants. At the same time, the book is kept as small as possible.

This is also a new kind of book, the first in the series of Alive Books®. We live in the 21st century, and computers and Internet are changing the world. How will these factors influence the authoring and publishing of scientific books? This book gives a temporary answer to this question. The MATHEMATICAL HANDBOOK has both printed and electronic parts linked and working together. The icons of the printed part A tell the user what related information is contained in the electronic part C and how to access it. Thus, a book of 340 printed pages offers more than 1000 pages of expanding mathematical knowledge, while the printed part remains the solid and unchanged base. This is only the beginning. The MATHEMATICAL HANDBOOK will continue to be written after its publication and will grow in size!

For this book, I had the privilege of receiving suggestions from my colleagues E. Economou, N. Kylafis, St. Trachanas, K. Tsiganos and many others. I thank them very much. In all cases, however, the responsibility for any faults is mine. For all that has been accomplished, the user will make the final judgment.

Sotirios Persidis
Athens, September 2007

Structure and Use of the Book

Structure of the book

This is an Alive Book®. It is the first publication of a new and extended concept of a book and has four parts, A and B in printed form, C and D in electronic form:

Part A is the present book. It contains the main material related to the cover title.

Part B is the smaller accompanying book that contains a summary of Part A.

Part C contains *additions*, i.e. files that can be obtained using the *icon numbers*.

Part D contains material in many forms related to the subject in a broader sense.

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The icons used in Part A indicate where additional material is available and the type of addition. The colors indicate the level of difficulty: Green Exa for elementary level, blue Exa for medium level, red Exa for advanced level. The three letters inside each icon indicate the content of the addition as follows:

App **App**lication: An application of a theory or a method.

Cal **Cal**culation: Calculation of an integral or an expression.

Exa **Exa**mple: A specific example of a case or a method.

Ext **Ext**ension: More of the same or related material.

Inf **Inf**ormation: General related information.

Pro **Pro**of: Proof of a theorem, a statement or a formula.

Tab **Tab**le: Numerical table(s) of data needed in calculations.

The **The**ory or **The**orem: More theory or theorems or rigorous conditions.

Zoom of a drawing or picture.

Each icon represents an *addition* and has a naturally assigned, unique four-digit *icon number*, made of the three-digit number of the page and the one-digit number expressing the top-down order of the icon on the page. For example, the icons in page 92 have icon numbers 0921 and 0922, while the icons in page 251 have icon numbers 2511, 2512, 2513. The icon numbers are used to obtain the corresponding additions. To access an addition, go to **www.ifoes.org**, click on **English** and on **Mathematical Handbook** and follow the directions. You can also have a permanent icon on your computer desktop to facilitate the communication with parts C and D.

In part A, the following symbols are also used:

➊➋➌➍➎➏ Different cases or methods explained previously.

► Important points, cases or statements.

Errors

A scientific book is a very complicated work, and errors may escape efforts to detect and avoid them. Although this book has been checked thoroughly, we have set up a mechanism for corrections and improvements. To minimize the possibility of an error, we have assumed that each formula or mathematical statement is correct if at least one of the following applies:

(a) It is found in or verified by two (usually three or more) independent references.

(b) It is found in or verified by one reference and verified by well known mathematical software (mainly Mathematica or Maple).

(c) It is derived by hand in all its details.

In addition to the above, we have set up a mechanism for corrections and improvements *after the publication*:

(a) All errors found after publication are posted immediately on the Internet (part D) and are readily available to all users.

(b) If you find an error (even an insignificant one), or if you seriously suspect there is an error, please report it to us by email to **mathbook@ifoes.org**. We will examine it immediately. If you are right, you will have helped considerably.

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Important changes or additions made to part A are available to registered users at **www.ifoes.org** > **English** > **Mathematical Handbook**.

Questions, Proposals

Serious questions and proposals can be submitted by email to **mathbook@ifoes.org**. We will try to answer your question or evaluate your proposal, but, please bear in mind that our time in this world is finite.

Main References

The books and other resources used for writing this handbook exceed 200. Some of them have been well recognized as the most reliable and authoritative printed references of pure and applied mathematical knowledge that humanity has accumulated in its history. Most of them are works from the last 50 years, a few are older than 50 years and two are more than 100 years old. A comprehensive list of these references is given in part D. Here, we give only the most widely used collective works:

Abramowitz, M., and Stegun, I., *Handbook of Mathematical Functions*, Dover, New York, 1968.

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1 CONSTANTS

1.1 Mathematical Constants

$\sqrt{2}$ = 1.41421 35623 73095 04880 16887 2421 … **Cal**

$\sqrt{3}$ = 1.73205 08075 68877 29352 74463 4151 …

$\sqrt{5}$ = 2.23606 79774 99789 69640 91736 6873 …

$\sqrt{6}$ = 2.44948 97427 83178 09819 72840 7471 …

$\sqrt{7}$ = 2.64575 13110 64590 59050 16157 5364 …

$\sqrt{8}$ = 2.82842 71247 46190 09760 33774 4842 …

$\sqrt{10}$ = 3.16227 76601 68379 33199 88935 4443 …

$\sqrt[3]{2}$ = 1.25992 10498 94873 16476 72106 0728 …

$\sqrt[3]{3}$ = 1.44224 95703 07408 38232 16383 1078 …

$\sqrt[5]{2}$ = 1.14869 83549 97035 00679 86269 4678 …

$\sqrt[5]{3}$ = 1.24573 09396 15517 32596 66803 3664 …

$\frac{1}{2}(1 + \sqrt{5})$ = 1.61803 39887 49894 84820 45868 3436 … [Golden ratio]

π = 3.14159 26535 89793 23846 26433 8328 … **Cal**

π^2 = 9.86960 44010 89358 61883 44909 9988 …

π^{-1} = 0.31830 98861 83790 67153 77675 26745 …

$\sqrt{\pi}$ = 1.77245 38509 05516 02729 81674 8334 …

e = 2.71828 18284 59045 23536 02874 7135 … [natural base of logaritms] **Inf**

e^2 = 7.38905 60989 30650 22723 04274 6058 …

$\sqrt{e}$ = 1.64872 12707 00128 14684 86507 8781 …

e^{π} = 23.14069 26327 79269 00572 90863 679 …

π^{e} = 22.45915 77183 61045 47342 71522 045 …

e^{e} = 15.15426 22414 79264 18976 04302 726 …

Inf

$\gamma = 0.57721\ 56649\ 01532\ 86060\ 65120\ 90082\ \ldots$ [Euler constant]

$e^{\gamma} = 1.78107\ 24179\ 90197\ 98523\ 65041\ 0311\ \ldots$

$G = 0.91596\ 55941\ 77219\ 01505\ 46035\ 14932\ \ldots$ [Catalan constant]

$K = 2.68545\ 20010\ 65306\ 44530\ 97148\ 3548\ \ldots$ [Khinchin constant]

$A = 1.28242\ 71291\ 00622\ 63687\ 53425\ 6886\ \ldots$ [Glaisher constant]

$\mu = 1.45136\ 92348\ 83381\ 05028\ 39684\ 8589\ \ldots$ [Soldner constant]

$\log_{10} 2 = 0.30102\ 99956\ 63981\ 19521\ 37388\ 94724\ \ldots$

$\log_{10} 3 = 0.47712\ 12547\ 19662\ 43729\ 50279\ 03255\ \ldots$

$\log_{10} \pi = 0.49714\ 98726\ 94133\ 85435\ 12682\ 88291\ \ldots$

$\log_{10} e = 0.43429\ 44819\ 03251\ 82765\ 11289\ 18917\ \ldots$

$\log_e 2 = \ln 2 = 0.69314\ 71805\ 59945\ 30941\ 72321\ 21458\ \ldots$

$\log_e 3 = \ln 3 = 1.09861\ 22886\ 68109\ 69139\ 52452\ 3692\ \ldots$

$\log_e 10 = \ln 10 = 2.30258\ 50929\ 94045\ 68401\ 79914\ 5468\ \ldots$

$\log_e \pi = \ln \pi = 1.14472\ 98858\ 49400\ 17414\ 34273\ 5135\ \ldots$

$\log_e \gamma = \ln \gamma = -0.54953\ 93129\ 81644\ 82233\ 76617\ 68803\ \ldots$

$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} = 1.77245\ 38509\ 05516\ 02729\ 81674\ 8334\ \ldots$ [$\Gamma(x)$ = gamma function]

$\Gamma(\tfrac{1}{3}) = 2.67893\ 85347\ 07747\ 63365\ 56929\ 4097\ \ldots$

$\Gamma(\tfrac{1}{4}) = 3.62560\ 99082\ 21908\ 31193\ 06851\ 5587\ \ldots$

1 radian $= 180°/\pi = 57.29577\ 95130\ 82320\ 87679\ 81548\ 141\ \ldots°$

$1° = \pi/180$ radians $= 0.01745\ 32925\ 19943\ 29576\ 92369\ 07685\ \ldots$ radians

1.2 Physical Constants

Inf

Alpha particle mass	$m_\alpha = 6.6446565 \times 10^{-27}$ kg $= 4.001506179149$ u
Atomic mass	$m_u = 1.66053886 \times 10^{-27}$ kg
Avogadro number	$N_A = 6.0221415 \times 10^{23}$ mol^{-1}
Bohr magneton	$\mu_B = 9.27400949 \times 10^{-24}$ J T^{-1} $= 5.788381804 \times 10^{-5}$ eV T^{-1}

Bohr radius	$a_B = 0.5291772108 \times 10^{-10}$ m
Boltzmann constant	$k_B = 1.3806505 \times 10^{-23}$ J K^{-1} $= 8.617343 \times 10^{-5}$ eV K^{-1}
Characteristic impendance of vacuum	$Z_0 = 376.730313461\ \Omega$
Classical electron radius	$r_e = 2.817940325 \times 10^{-15}$ m
Compton wavelength	$\lambda_C = 2.426310238 \times 10^{-12}$ m
Conductance quantum	$G_0 = 7.748091733 \times 10^{-5}$ s
Dielectric constant of vacuum	$\varepsilon_0 = 8.854187817 \times 10^{-12}$ F m^{-1}
Electron charge to mass quotient	$e/m_e = -1.75882012 \times 10^{11}$ C kg^{-1}
Electron magnetic moment	$\mu_e = -1.0011596521859\mu_B$ $= -9.28476412 \times 10^{-24}$ J T^{-1}
Electron mass	$m_e = 9.1093826 \times 10^{-31}$ kg $= 5.4857990945 \times 10^{-4}$ u
Faraday constant	$F = 96485.3383$ C mol^{-1}
Fermi coupling constant	$G_F/(\hbar c)^3 = 1.16639 \times 10^{-5}$ GeV^{-2}
Fine-structure constant	$\alpha = 1/137.03599911$ $= 7.297352568 \times 10^{-3}$
Gravitational constant	$G = 6.6742 \times 10^{-11}$ m^3 kg^{-1} s^{-2}
Hartree energy	$E_h = 4.35974417 \times 10^{-18}$ J $= 27.2113845$ eV
Josephson constant	$K_J = 483597.879 \times 10^9$ Hz V^{-1}
Magnetic flux quantum	$\Phi_0 = 2.06783372 \times 10^{-15}$ Wb
Magnetic susceptibility of vacuum	$\mu_0 = 4\pi \times 10^{-7}$ NA^{-2} (exact)
Molar gas constant	$R = 8.314472$ J mol^{-1} K^{-1}
Muon mass	$m_\mu = 1.88353140 \times 10^{-28}$ kg $= 0.1134289264$ u
Neutron Compton wavelength	$\lambda_{Cn} = 1.3195909067 \times 10^{-15}$ m
Neutron-electron mass ratio	$m_n/m_e = 1838.6836598$

Neutron magnetic moment	$\mu_n = -1.91304273\mu_N$ $= -0.96623645 \times 10^{-26}$ J T^{-1}
Neutron mass	$m_n = 1.67492728 \times 10^{-27}$ kg $= 1.00866491560$ u
Nuclear magneton	$\mu_N = 5.05078343 \times 10^{-27}$ J T^{-1} $= 3.152451259 \times 10^{-8}$ eV T^{-1}
Planck constant	$h = 6.6260693 \times 10^{-34}$ J s $= 4.13566743 \times 10^{-15}$ eV s
	$\hbar = h/2\pi = 1.05457168 \times 10^{-34}$ J s $= 6.58211915 \times 10^{-16}$ eV s
Planck length	$l_P = 1.61624 \times 10^{-35}$ m
Planck mass	$m_P = 2.17645 \times 10^{-8}$ kg
Planck temperature	$T_P = 1.41679 \times 10^{32}$ K
Planck time	$t_P = 5.39121 \times 10^{-44}$ s
Proton charge	$e = 1.60217653 \times 10^{-19}$ C
Proton Compton wavelength	$\lambda_{Cp} = 1.3214098555 \times 10^{-15}$ m
Proton-electron mass ratio	$m_p/m_e = 1836.15267261$
Proton magnetic moment	$\mu_p = 2.792847351\mu_N$ $= 1.41060671 \times 10^{-26}$ J T^{-1}
Proton mass	$m_p = 1.67262171 \times 10^{-27}$ kg $= 1.00727646688$ u
Rydberg constant	$R_\infty = 10973731.568525$ m^{-1}
Speed of light (in vacuum)	$c = 299\,792\,458$ m s^{-1} (exact)
Stefan-Boltzmann constant	$\sigma = 5.670400 \times 10^{-8}$ W m^{-2} K^{-4}
Thomson cross section	$\sigma_e = 0.665245856 \times 10^{-28}$ m^2
Wien displacement law constant	$b = \lambda_{\max}T = 2.8977685 \times 10^{-3}$ m K

2 ALGEBRA

2.1 Identities

$(x-y)^2 = x^2 - 2xy + y^2$

$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

$x^2 - y^2 = (x-y)(x+y)$

$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

$x^4 - y^4 = (x-y)(x+y)(x^2 + y^2)$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

$x^4 + y^4 = (x^2 - \sqrt{2}\,xy + y^2)(x^2 + \sqrt{2}\,xy + y^2)$

$(x-y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

$(x-y)^6 = x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$

$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

$x^5 - y^5 = (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$

$x^5 + y^5 = (x+y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$

$x^6 - y^6 = (x-y)(x+y)(x^2 - xy + y^2)(x^2 + xy + y^2)$

$x^3 + x^2y + xy^2 + y^3 = (x+y)(x^2 + y^2)$

$x^4 + x^2y^2 + y^4 = (x^2 - xy + y^2)\,(x^2 + xy + y^2)$

$x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5 = (x+y)(x^2 - xy + y^2)(x^2 + xy + y^2)$

$(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

$(x+y+z)^3 = x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3y^2z + 3yz^2 + 3z^2x + 3zx^2 + 6xyz$

$(x+y+z+w)^2 = x^2 + y^2 + z^2 + w^2 + 2xy + 2xz + 2xw + 2yz + 2yw + 2zw$

For a positive integer n, we have

$x^{2n} - y^{2n} = (x^2 - y^2)(x^{2n-2} + x^{2n-4}y^2 + x^{2n-6}y^4 + \cdots + x^2y^{2n-4} + y^{2n-2})$

```
\P{x^(2*n)-y^(2*n)}, x^2-2*x*y*cos(k*π/n)+y^2
```

Pro

$x^{2n} + y^{2n} =$ \P{x^(2*n)+y^(2*n)}, x^2+2*x*y*cos((2*k+1)*π/(2*n))+y^2

Pro

$x^{2n+1} - y^{2n+1} = (x - y)(x^{2n} + x^{2n-1}y + x^{2n-2}y^2 + \cdots + y^{2n})$

\P{x^(2*n+1)-y^(2*n+1)}, x^2-2*x*y*cos(2*k*π/(2*n+1))+y^2

$x^{2n+1} + y^{2n+1} = (x + y)(x^{2n} - x^{2n-1}y + x^{2n-2}y^2 - \cdots + y^{2n})$

\P{x^(2*n+1)+y^(2*n+1)}, x^2+2*x*y*cos(2*k*π/(2*n+1))+y^2

2.2 Binomial Formula

For $n = 1, 2, 3, \ldots$ we have the *binomial formula*

\S{(x+y)^n}, x^n+n*x^(n-1)*y+(n*(n-1)/(2!))*x^(n-2)*y^2

or \S{(x+y)^n}, x^n+(n;1)*x^(n-1)*y+(n;2)*x^(n-2)*y^2+...

The *binomial coefficients* are defined by the relation

(n;k)=n!/(k!*(n-k)!)

Ext

where n and k are integers with $0 \le k \le n$, $0! = 1$ and $n! = 1 \cdot 2 \cdot 3 \cdot 4 \ldots n$.

Properties of the binomial coefficients

(n;0)=(n;n)=1, (n;1)=(n;n-1)=n

(n;k)=(n;n-k)=(-1)^k*(k-n-1;k)

(n;k+1)=(n-k)*(n;k)/(k+1), (n+1;k), (n+1;k+1) (n;0)+(n;1)+(n;2)+...+(n;n)=2^n

(n;0)-(n;1)+(n;2)-...+(-1)^n*(n;n)=0

Pro

(n;n)+(n+1;n)+(n+2;n)+...+(m;n)=(m+1;n+1)

(m;n)-(m;n+1)+(m;n+2)-...+(-1)^(m+n)*(m;m)=(m-1;n-1)

(n;0)+(n;2)+(n;4)+...=(n;1)+(n;3)+(n;5)+...=2^(n-1)

(n;0)^2+(n;1)^2+(n;2)^2+...+(n;n)^2=(2*n;n)

(m;0)*(n;p)+(m;1)*(n;p-1)+...+(m;p)*(n;0)=(m+n;p)

Pro

1*(n;1)+2*(n;2)+3*(n;3)+...+n*(n;n)=n*2^(n-1)

1*(n;1)-2*(n;2)+3*(n;3)-...+(-1)^(n-1)*n*(n;n)=0

Table of binomial coefficients

Inf

n \ *k*	0	1	2	3	4	5
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

2.3 Complex Numbers

If a and b are real numbers, and $i = \sqrt{-1}$ is the *imaginary unit*, then $z = a + bi$ is a *complex number*. The *complex conjugate* of $a + bi$ is $a - bi$. The real and the imaginary part of $a + bi$ are $\text{Re}(z) = a$ and $\text{Im}(z) = b$, respectively.

Complex plane

The complex number $x + yi$ is represented by a point P with abscissa x and ordinate y. **Exa**

If r=(x^2+y^2)^(1/2) is the length of the vector **OP,** then the *polar form* of the complex number is

$$x + yi = r(\cos\theta + i\sin\theta)$$

where r is the *absolute value* or *magnitude* and θ is the *angle* or *argument* of the complex number.

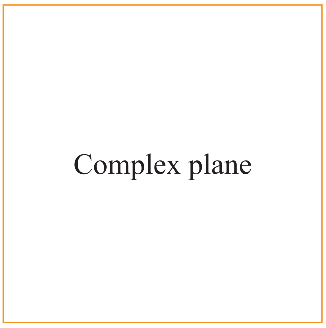

Fig. 2-1

Simple calculations

$$(a + bi) + (c + di) = a + c + (b + d)i$$

$$(a + bi) - (c + di) = a - c + (b - d)i$$

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

(a+b*i)/(c+d*i)

$$[r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

(r_1*(cosθ_1+i*sinθ_1))/(r_2*(cosθ_2+i*sinθ_2))

De Moivre's theorem

If p is a real number, then

$$[r(\cos\theta + i\sin\theta)]^p = r^p(\cos p\theta + i\sin p\theta)$$

Roots

Inf

If n is a positive integer, then the roots of order n of a complex number are

(r*(cosθ+i*sinθ))^(1/n)=

r^(1/n)*(cos((θ+2*k*π)/n)+i*sin((θ+2*k*π)/n))

App

where $k = 0, 1, 2, 3, \ldots, n - 1$.

2.4 Powers and Logarithms

Powers

For real numbers a, b, p, q we have the following relations (under the condition that each term is meaningful and the denominators are different from zero):

$a^p \cdot a^q = a^{p+q}$, $a^p / a^q = a^{p-q}$,

$(a^p)^q = a^{pq}$, a^(p/q)

$a^0 = 1 \quad (a \neq 0)$, a^(-p)=1/a^p ,

$(ab)^p = a^p b^p$, (a/b)^p=a^p/b^p

(a*b)^(1/p)=a^(1/p)*b^(1/p) (a/b)^(1/p)=a^(1/p)/b^(1/p)

Exa

Logarithms

Let c be a positive number, different than 1. If $c^x = A$ (A a positive number), then the exponent x is called *logarithm* of A with respect to *base* c and we write $x = \log_c A$.

If $c = 10$, we have the *decimal* logarithms. If $c = e = 2.71828\ldots$, we have the *natural* logarithms and we use ln instead of $\log_e$.

$\log_c c^x = x$ $\log_c (A \cdot B) = \log_c A + \log_c B$

$(\log_c b)(\log_b c) = 1$ log_c(A/B)=log_cA+log_cB

$\log_c c = 1, \quad \log_c 1 = 0$ $\log_c A^p = p \log_c A$

Change of base

log_cA=log_bA/log_bc

$$\ln A = \ln 10 \cdot \log_{10} A = 2.302585\ldots \cdot \log_{10} A$$

$$\log_{10} A = \log_{10} e \cdot \ln A = 0.434294\ldots \cdot \ln A$$

Complex numbers

$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$

sinθ=(e^(i*θ)-e^(-i*θ))/(2*i)   cosθ=(e^(i*θ)+e^(-i*θ))/2

$e^{i(\theta+2k\pi)} = e^{i\theta}$ (k = integer, periodicity)

$a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$

with r=(a^2+b^2)^(1/2), $\cos\theta = a/r$, $\sin\theta = b/r$

$(re^{i\theta})(qe^{i\varphi}) = rqe^{i(\theta+\varphi)}$ (r*e^(i*θ))/(q*e^(i*φ)), (r/q)*e^(i(θ-φ))

$(re^{i\theta})^p = r^p e^{ip\theta}$ (de Moivre's theorem)

$\ln(re^{i\theta}) = \ln r + i\theta + 2k\pi i$ (k = integer)

2.5 Matrices and Determinants

A *matrix* of order $m \times n$ is an array (usually two dimensional)

A=mat(a_(11),a_(12),...,a_(1n);
a_(21),a_(22),...,a_(2n);
...,...,...,...;
a_(m1),a_(m2),...,a_(mn))

of constant or variable elements a_{ik} arranged in m rows and n columns. A *square* matrix has $m = n$. A *unit* matrix **I** is a square matrix with $a_{ik} = \delta_{ik}$, where $\delta_{ik} = 1$ for $i = k$ and zero otherwise.

Sum of two matrices $\mathbf{A} = [a_{ik}]$, $\mathbf{B} = [b_{ik}]$ of the same order $m \times n$

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = [c_{ik}] \qquad \text{where } c_{ik} = a_{ik} + b_{ik}$$

Product of two matrices $\mathbf{A} = [a_{ik}]$ of order $m \times n$, $\mathbf{B} = [b_{ik}]$ of order $n \times r$

$\mathbf{C} = \mathbf{AB} = [c_{ik}]$ where c_(ik)=\S[j=1,n]a_(ij)*b_(jk)

Two square matrices **A** and **B**, both of order $n \times n$, are *inverse* to each other iff **AB** = **BA** = **I**.

The *determinant* D of a square matrix $\mathbf{A} = [a_{ik}]$ of order $n \times n$ is

D=det(a_(ik))=det(a_(11),a_(12),...,a_(1n);
a_(21),a_(22),...,a_(2n);
...,...,...,...;
a_(n1),a_(n2),...,a_(nn))

Ext

where the sum contains all the possible $n!$ terms $a_{1\alpha}a_{2\beta}a_{3\gamma}\cdots a_{n\chi}$, with $\alpha\beta\gamma\cdots\chi$ an *ordered* set of 1, 2, 3, …, n, and r is the number of interchanges needed to obtain $\alpha\beta\gamma\cdots\chi$ from $123\cdots n$.

For $n = 2$, $\quad D = a_{11}a_{22} - a_{12}a_{21}$

For $n = 3$, $\quad D = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$

2.6 Roots of Algebraic Equations

Linear equation (first-degree equation)

$$ax + b = 0$$

If $a \neq 0$, there is *one* root, x=-b/a .

If $a = 0$ and $b \neq 0$, there is no root.

If $a = 0$ and $b = 0$, every number is a root.

Quadratic equation (second-degree equation)

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

Discriminant: $\quad D = b^2 - 4ac$

Roots

x=(-b\+(b^2-4*a*c)^(1/2))/(2*a)

Pro

If a, b, c are real, then:

(i) If $D > 0$, we have two real and unequal roots.

(ii) If $D = 0$, we have two real and equal roots.

(iii) If $D < 0$, we have two complex conjugate roots.

Relations between roots

Sum of roots: x_1+x_2=-b/a

Product of roots: x_1*x_2=c/a

Cubic equation (third-degree equation) The

$$x^3 + ax^2 + bx + c = 0$$

Let p=(3*b-a^2)/9, q=(9*a*b-27*c-2*a^3)/54

A=(q+(p^3+q^2)^(1/2))^(1/3), B=(q-(p^3+q^2)^(1/2))^(1/3)

Roots

x_1=A+B-a/3;
x_(2,3)=-(A+B)/2-a/3\+3^(1/2)*(A-B)*i/2

Exa

If a, b, c are real numbers and $D = p^3 + q^2$ is the discriminant, then:

(i) If $D > 0$, one root is real and two roots are complex conjugates.

(ii) If $D = 0$, all the roots are real and at least two are equal.

(iii) If $D < 0$, all the roots are real and unequal.

If $D < 0$, the roots can be written

x_1=2*(-p)^(1/2)*cos(θ/3)-a/3;
x_(2,3)=-2*(-p)^(1/2)*cos(θ/3\+π/3)-a/3

where cosθ=q/(-p^3)^(1/2)

Relations among the roots

$$x_1 + x_2 + x_3 = -a,$$

$$x_1x_2 + x_2x_3 + x_3x_1 = b,$$

$$x_1x_2x_3 = -c,$$

where x_1, x_2, x_3 are the three roots.

Quartic equation (fourth-degree equation) **The**

$$x^4 + ax^3 + bx^2 + cx + d = 0 \qquad \text{(i)}$$

Roots

If u is a root of the third degree equation

$$u^3 - bu^2 + (ac - 4d)u + 4bd - c^2 - a^2 d = 0 \qquad \text{(ii)}$$

then the four roots of (i) are roots of the two second degree equations

$$2x^2 + [a \pm (a^2 - 4b + 4u)^{1/2}]x + u \pm \varepsilon(u^2 - 4d)^{1/2} = 0 \qquad \text{(iii)}$$

Exa

The value of ε is 1 or -1 depending on the choice of the square roots so that $\varepsilon(a^2 - 4b + 4u)^{1/2}(u^2 - 4d)^{1/2} = au - 2c$. The calculation is simplified if we use a root of (ii) which gives real coefficients (if possible) for equations (iii).

Relations among the roots

$$x_1 + x_2 + x_3 + x_4 = -a$$

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = b$$

$$x_1x_2x_3 + x_2x_3x_4 + x_1x_2x_4 + x_1x_3x_4 = -c$$

$$x_1x_2x_3x_4 = d$$

where x_1, x_2, x_3, x_4 are the four roots.

Systems of linear equations

A system of two linear equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

with unknowns x_1, x_2 has solution (roots)

```
x_1=(b_1*a_(22)-b_2*a_(12))/(a_(11)*a_(22)-a_(12)*a_(21));
x_2=(b_2*a_(11)-b_1*a_(21))/(a_(11)*a_(22)-a_(12)*a_(21))
```

A system of three linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

with unknowns x_1, x_2, x_3 has solution (roots)

x_1=(1/D)*|...|, x_2=(1/D)*|...|, x_3=(1/D)*|...|

with $D = \det[a_{ik}] = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$.

A system of n linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

with unknowns $x_1, x_2, \ldots, x_n$ has solution (roots)

x_k=D_k/D $\quad k = 1, 2, \ldots, n$

where $D = \det[a_{ik}]$ and D_k is the determinant obtained from D after replacing the elements $a_{1k}, a_{2k}, \ldots, a_{nk}$ of the kth column by $b_1, b_2, \ldots, b_n$. The solution exists and is unique iff $D \neq 0$ (linearly independent equations).

Expansion in partial fractions

A *rational function* of x has the form

R(x)=P(x)/Q(x)

where $P(x)$ and $Q(x)$ are polynomials of x of degrees m amd n respectively, without common roots.

If $m \geq n$, we can divide $P(x)$ by $Q(x)$ and write $R(x)$ as a sum of a polynomial and another rational fraction with $m < n$. If $m < n$, $R(x)$ can be written as a sum of terms of the form $A(x - x_0)^{-r}$, where $x = x_0$ is a root of the denominator $Q(x)$. In this sum, every (real or complex) root $x = x_0$ of multiplicity k contributes terms of the form

A_1/(x-x_0)+A_2/(x-x_0)^2+...+A_k/(x-x_0)^k $\quad$ (i) $\quad$ Exa

where A_i are constants. Summing up the contributions of all roots, we have an expansion of $R(x)$ into a sum of *partial fractions*.

Let the coefficients of the polynomials $P(x)$ and $Q(x)$ be real. The complex roots of $Q(x)$ appear in pairs of complex conjugates, $a + ib$ and $a - ib$, each of multiplicity k. The terms that correspond to a pair of complex conjugate roots can be written in the form

$$\frac{A_1 x+B_1}{(x-a)^2+b^2}+\frac{A_2 x+B_2}{((x-a)^2+b^2)^2}+\ldots+\frac{A_k x+B_k}{((x-a)^2+b^2)^k} \qquad \text{(ii)}$$

where all coefficients are real. Thus, every rational function of polynomials with real coefficients can be expanded in a sum of partial fractions, i.e. a sum of terms of the forms (i) and (ii).

2.7 Hyperbolic Functions

Definitions

Hyperbolic sine $\sinh x=\frac{e^x-e^{-x}}{2}$

Hyperbolic cosine $\cosh x=\frac{e^x+e^{-x}}{2}$

Hyperbolic tangent $\tanh x=\frac{\sinh x}{\cosh x}$

Hyperbolic cotangent $\coth x=\frac{\cosh x}{\sinh x}$

The hyperbolic secant ($\operatorname{sech} x = 1/\cosh x$) and the hyperbolic cosecant ($\operatorname{csch} x = 1/\sinh x$) are not often used. **Ext**

The functions $\sinh x$ and $\cosh x$ have period $2\pi i$, while the functions $\tanh x$ and $\coth x$ have period πi. Thus, for k integer we have

$$\sinh(x + 2k\pi i) = \sinh x \qquad \cosh(x + 2k\pi i) = \cosh x$$

$$\tanh(x + k\pi i) = \tanh x \qquad \coth(x + k\pi i) = \coth x$$

Identities

$$\sinh x+\cosh x=\frac{1}{\cosh x-\sinh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh x+\cosh x=\frac{1}{\cosh x-\sinh x}$$

$$(\sinh x + \cosh x)^n = \sinh nx + \cosh nx$$

$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$

$\tanh(-x) = -\tanh x \qquad \coth(-x) = -\coth x$

sinh(x/2)=\+((coshx-1)/2)^(1/2)

cosh(x/2)=((coshx+1)/2)^(1/2)

tanh(x/2)=\+((coshx-1)/(coshx+1))^(1/2)

Pro

sinh(2*x)=2*sinhx*coshx=2*tanhx/(1-tanhx^2)

cosh(2*x)=coshx^2+sinhx^2=2*coshx^2-1=(1+tanhx^2)/(1-tanhx^2)

tanh(2*x)=2*tanhx/(1+tanhx^2)

coth(2*x)=(cothx^2+1)/(2*cothx)

$\sinh 3x = 3\sinh x + 4\sinh^3 x$

$\cosh 3x = 4\cosh^3 x - 3\cosh x$

tanh(3*x)=(3*tanhx+tanhx^3)/(1+3*tanhx^2)

coth(3*x)=(3*cothx+cothx^3)/(1+3*cothx^2)

$\sinh 4x = 8\sinh^3 x\cosh x + 4\sinh x\cosh x$

$\cosh 4x = 8\cosh^4 x - 8\cosh^2 x + 1$

tanh(4*x)=(4*tanhx+4*tanhx^3)/(1+6*tanhx^2+tanhx^4)

coth(4*x)=(1+6*cothx^2+cothx^4)/(4*cothx+4*cothx^3)

$$\sinh^2 x = \tfrac{1}{2}\cosh 2x - \tfrac{1}{2}$$

$$\cosh^2 x = \tfrac{1}{2}\cosh 2x + \tfrac{1}{2}$$

$$\sinh^3 x = \tfrac{1}{4}\sinh 3x - \tfrac{3}{4}\sinh x$$

$$\cosh^3 x = \tfrac{1}{4}\cosh 3x + \tfrac{3}{4}\cosh x$$

$$\sinh^4 x = \tfrac{1}{8}\cosh 4x - \tfrac{1}{2}\cosh 2x + \tfrac{3}{8}$$

$$\cosh^4 x = \tfrac{1}{8}\cosh 4x + \tfrac{1}{2}\cosh 2x + \tfrac{3}{8}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

tanh(x\+y)=(tanhx\+tanhy)/(1\+tanhx*tanhy)

coth(x\+y)=(cothx*cothy\+1)/(cothy\+cothx)

$$\sinh x + \sinh y = 2\sinh\tfrac{1}{2}(x+y)\cosh\tfrac{1}{2}(x-y)$$

$$\sinh x - \sinh y = 2\cosh\tfrac{1}{2}(x+y)\sinh\tfrac{1}{2}(x-y)$$

$$\cosh x + \cosh y = 2\cosh\tfrac{1}{2}(x+y)\cosh\tfrac{1}{2}(x-y)$$

$$\cosh x - \cosh y = 2\sinh\tfrac{1}{2}(x+y)\sinh\tfrac{1}{2}(x-y)$$

tanhx\+tanhy=sinh(x\+y)/(coshx*coshy)

cothx+cothy=sinh(x+y)/(sinhx*sinhy)

$$\sinh^2 x - \sinh^2 y = \cosh^2 x - \cosh^2 y = \sinh(x+y)\sinh(x-y)$$

$$\sinh^2 x + \cosh^2 y = \cosh^2 x + \sinh^2 y = \cosh(x+y)\cosh(x-y)$$

$$\sinh x \sinh y = \tfrac{1}{2}\{\cosh(x+y) - \cosh(x-y)\}$$

$$\cosh x \cosh y = \tfrac{1}{2}\{\cosh(x+y) + \cosh(x-y)\}$$

$$\sinh x \cosh y = \tfrac{1}{2}\{\sinh(x+y) + \sinh(x-y)\}$$

Graphs

Graph of

$y = \sinh x$

Fig. 2-2 $y = \sinh x$

Graph of

$y = \cosh x$

Fig. 2-3 $y = \cosh x$

Graph of

$y = \tan x$

Fig. 2-4 $y = \tanh x$

Graph of

$y = \coth x$

Fig. 2-5 $y = \coth x$

Relations with trigonometric functions

$\sin(ix) = i \sinh x \qquad \cos(ix) = \cosh x$

$\tan(ix) = i \tanh x \qquad \cot(ix) = -i \coth x$

$\sinh(ix) = i \sin x \qquad \cosh(ix) = \cos x$

$\tanh(ix) = i \tan x \qquad \coth(ix) = -i \cot x$

$\sinh(x \pm iy) = \sinh x \cos y \pm i \cosh x \sin y$

$\cosh(x \pm iy) = \cosh x \cos y \pm i \sinh x \sin y$

tanh(x\+i*y)=(tanhx\+i*tany)/(1\+i*tanhx*tany)

coth(x\+i*y)=(cothx*coty\+i)/(coty\+i*cothx)

Inverse hyperbolic functions

If $x = \sinh y$, then the *inverse* function is denoted by $y = \sinh^{-1} x$. This notation is similar for the other hyperbolic functions.

According to this notation, the *principal branches* of the inverse hyperbolic functions can be expressed in terms of logarithms as follows:

sinh^(-1)x=ln(x+(x^2+1)^(1/2)), $-\infty < x < \infty$

cosh^(-1)x=ln(x+(x^2-1)^(1/2)), $x \geq 1$

tanh^(-1)x=(1/2)*ln((1+x)/(1-x)) , $-1 < x < 1$

coth^(-1)x=(1/2)*ln((x+1)/(x-1)) , $x < -1$ or $x > 1$

Ext

Properties

$\sinh^{-1}(-x) = -\sinh^{-1} x$

$\cosh^{-1}(-x) = \cosh^{-1} x$

$\tanh^{-1}(-x) = -\tanh^{-1} x$

$\coth^{-1}(-x) = -\coth^{-1} x$

$\coth^{-1} x = \tanh^{-1}(1/x)$

sinh^(-1)x=tanh^(-1)(x/(x^2+1)^(1/2))

cosh^(-1)x=sinh^(-1)((x^2-1)^(1/2))=tanh^(-1)((x^2-1)^(1/2)/x)

sinh^(-1)x\+sinh^(-1)y=

cosh^(-1)x\+cosh^(-1)y=

tanh^(-1)x\+tanh^(-1)y=tanh^(-1)((x\+y)/(1\+x*y))

Graphs

Graph of $y = \sinh^{-1} x$

Fig. 2-6 $y = \sinh^{-1} x$

Graph of $y = \cosh^{-1} x$

Fig. 2-7 $y = \cosh^{-1} x$

Graph of $y = \tanh^{-1} x$

Fig. 2-8 $y = \tanh^{-1} x$

Graph of $y = \coth^{-1} x$

Fig. 2-9 $y = \coth^{-1} x$

Relations with inverse trigonometric functions

$$\left.\begin{array}{ll}
\sinh^{-1} x = -i \sin^{-1}(ix) & \sin^{-1} x = -i \sinh^{-1}(ix) \\
\cosh^{-1} x = \pm i \cos^{-1} x & \cos^{-1} x = \pm i \cosh^{-1} x \\
\tanh^{-1} x = -i \tan^{-1}(ix) & \tan^{-1} x = -i \tanh^{-1}(ix) \\
\coth^{-1} x = i \cot^{-1}(ix) & \cot^{-1} x = i \coth^{-1}(ix)
\end{array}\right\}$$

3 TRIGONOMETRY

3.1 Definitions

The triangle *ABC* has a 90° angle at *A*. The trigonometric numbers of the angle *B* are defined as follows:

Sine	$\sin B = b/a$	
Cosine	$\cos B = c/a$	
Tangent	$\tan B = b/c$	
Cotangent	$\cot B = c/b$	
Secant	$\sec B = a/c$	Ext
Cosecant	$\csc B = a/b$	Ext

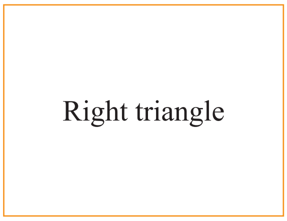

Fig. 3-1

Trigonometric circle

In a Cartesian system of coordinates (Fig. 3-2), the axes have positive directions from x' to x and from y' to y. The circle with centre O and radius $OA = 1$ is the *trigonometric circle*. The trigonometric numbers of the arc $\tau = \overset{\frown}{AP}$ are defined from the coordinates x and y of the point P as folows:

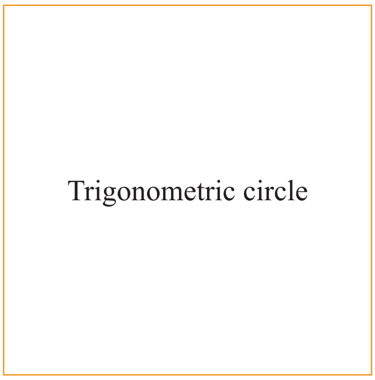

Fig. 3-2

Sine	$\sin\tau = OP' = y$	$-1 \le \sin\tau \le 1$
Cosine	$\cos\tau = OP'' = x$	$-1 \le \cos\tau \le 1$
Tangent	$\tan\tau = AQ' = y/x$	$-\infty < \tan\tau < +\infty$
Cotangent	$\cot\tau = BQ'' = x/y$	$-\infty < \cot\tau < +\infty$

The trigonometric numbers of the arc $\tau = \overset{\frown}{AP}$ are identical to those of the angle $\varphi = \widehat{AOP}$. The previous definitions are valid for arces (or angles) of any size, smaller or greater than 90°.

Trigonometric functions

For x in radians, the trigonometric functions $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \cot x$ have the following graphs:

Graph of $y = \sin x$

Fig. 3-3 $y = \sin x$

Graph of $y = \tan x$

Fig. 3-5 $y = \tan x$

Graph of $y = \cos x$

Fig. 3-4 $y = \cos x$

Graph of $y = \cot x$

Fig. 3-6 $y = \cot x$

Values of trigonometric functions

Ext

x in degrees	x in radians	$\sin x$	$\cos x$	$\tan x$	$\cot x$
0°	0	0	1	0	$\pm\infty$
15°	$\pi/12$	$(\sqrt{6} - \sqrt{2})/4$	$(\sqrt{6} + \sqrt{2})/4$	$2 - \sqrt{3}$	$2 + \sqrt{3}$
30°	$\pi/6$	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1
60°	$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$	$\sqrt{3}/3$
75°	$5\pi/12$	$(\sqrt{6} + \sqrt{2})/4$	$(\sqrt{6} - \sqrt{2})/4$	$2 + \sqrt{3}$	$2 - \sqrt{3}$
90°	$\pi/2$	1	0	$\pm\infty$	0

For angles in other quadrants, we can use the transformation formulas of Section 3.2.

3.2 Identities

Basic

$\sin^2 x + \cos^2 x = 1$

tanx=sinx/cosx

cotx=1/tanx=cosx/sinx

1/cosx^2-tanx^2=1

1/sinx^2-cotx^2=1

secx=1/cosx

cscx=1/sinx

Transformations

Pro

$\sin(-x) = -\sin x$ $\tan(-x) = -\tan x$

$\cos(-x) = \cos x$ $\cot(-x) = -\cot x$

sin(π/2\+x)=cosx

tan(π/2\+x)=\-cotx

cos(π/2\+x)=\-sinx

cot(π/2\+x)=\-tanx

$\sin(\pi \pm x) = \mp\sin x$ $\tan(\pi \pm x) = \pm\tan x$

$\cos(\pi \pm x) = -\cos x$ $\cot(\pi \pm x) = \pm\cot x$

$\sin(2\pi \pm x) = \pm\sin x$ $\tan(2\pi \pm x) = \pm\tan x$

$\cos(2\pi \pm x) = \cos x$ $\cot(2\pi \pm x) = \pm\cot x$

Sums

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

tan(x\+y)=(tanx\+tany)/(1\-tanx*tany)

cot(x\+y)=(cotx*coty\-1)/(coty\+cotx)

$$\sin x + \sin y = 2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)$$

$$\sin x - \sin y = 2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y)$$

$$\cos x + \cos y = 2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)$$

$$\cos x - \cos y = 2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(y-x)$$

tanx\+tany=sin(x\+y)/(cosx*cosy)

cotx\+coty=sin(y\+x)/(sinx*siny)

$$\sin x \sin y = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x \cos y = \tfrac{1}{2}[\cos(x-y) + \cos(x+y)]$$

$$\sin x \cos y = \tfrac{1}{2}[\sin(x-y) + \sin(x+y)]$$

$$\sin^2 x - \sin^2 y = \sin(x+y)\sin(x-y)$$

$$\cos^2 x - \cos^2 y = \sin(x+y)\sin(y-x)$$

$$\cos^2 x - \sin^2 y = \cos(x+y)\cos(x-y)$$

$$\sin(x \pm iy) = \sin x \cosh y \pm i\cos x \sinh y$$

$$\cos(x \pm iy) = \cos x \cosh y \mp i \sin x \sinh y$$

tan(x\+i*y)=(tanx\+i*tanhy)(1\-i*tanx*tanhy)

cot(x\+i*y)=(cotx*cothy\-i)(cothy\+i*cotx)

Multiple angles

sin(x/2)=\+((1-cosx)/2)^(1/2)

cos(x/2)=\+((1+cosx)/2)^(1/2)

Exa

tan(x/2)=(1-cosx)/sinx=sinx/(1+cosx)=\+((1-cosx)/(1+cosx))^(1/2)

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

sin(2*x)=2*tanx/(1+tanx^2)

cos(2*x)=(1-tanx^2)/(1+tanx^2)

tan(2*x)=2*tanx/(1-tanx^2)

$$\sin 3x = 3\sin x - 4\sin^3 x \qquad \cos 3x = 4\cos^3 x - 3\cos x$$

tan(3*x)=(3*tanx-tanx^3)/(1-3*tanx^2)

$$\sin 4x = 4\sin x\cos x - 8\sin^3 x\cos x \qquad \cos 4x = 8\cos^4 x - 8\cos^2 x + 1$$

tan(4*x)=(4*tanx-4*tanx^3)/(1-6*tanx^2+tanx^4)

$$\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x \qquad \cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$$

tan(5*x)=(5*tanx-10*tanx^3+tanx^5)/(1-10*tanx^2+5*tanx^4)

\S{sin(n*x)}

\S{cos(n*x)}

Pro

Powers

$$\sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x \qquad \cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x$$

$$\sin^3 x = \tfrac{3}{4}\sin x - \tfrac{1}{4}\sin 3x \qquad \cos^3 x = \tfrac{3}{4}\cos x + \tfrac{1}{4}\cos 3x$$

$$\sin^4 x = \tfrac{3}{8} - \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x \qquad \cos^4 x = \tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x$$

$$\sin^5 x = \frac{5}{8}\sin x - \frac{5}{16}\sin 3x + \frac{1}{16}\sin 5x$$

$$\cos^5 x = \frac{5}{8}\cos x + \frac{5}{16}\cos 3x + \frac{1}{16}\cos 5x$$

$$\sin^6 x = \frac{5}{16} - \frac{15}{32}\cos 2x + \frac{3}{16}\cos 4x - \frac{1}{32}\cos 6x$$

$$\cos^6 x = \frac{5}{16} + \frac{15}{32}\cos 2x + \frac{3}{16}\cos 4x + \frac{1}{32}\cos 6x$$

\S{sinx^(2*n+1)}

\S{cosx^(2*n+1)}

\S{sinx^(2*n)}

\S{cosx^(2*n)}

Pro

Inequalities

sinx\<x\<tanx:0\<x\<π/2

2/π<sinx/x:-π/2<x<π/2

cosx\<sinx/x\<1:0\<x\<π

π<sin(π*x)/(x*(1-x))\<4:0<x<1

3.3 Inverse Trigonometric Functions

If $x = \sin y$, then the *inverse* function is denoted as $y = \sin^{-1} x$. Similar notation is used for the other trigonometric functions. The *principal branches* of the inverse trigonometric functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$ have values in the following intervals:

$$-\pi/2 \le \sin^{-1} x \le \pi/2 \qquad 0 \le \cos^{-1} x \le \pi$$

$$-\pi/2 < \tan^{-1} x < \pi/2 \qquad 0 < \cot^{-1} x < \pi$$

Properties Pro

$$\sin^{-1}(-x) = -\sin^{-1} x \qquad \cos^{-1}(-x) = \pi - \cos^{-1} x$$

$$\tan^{-1}(-x) = -\tan^{-1} x \qquad \cot^{-1}(-x) = \pi - \cot^{-1} x$$

$$\sin^{-1} x + \cos^{-1} x = \pi/2 \qquad \tan^{-1} x + \cot^{-1} x = \pi/2$$

$$\tan^{-1} x + \tan^{-1}(1/x) = \pi/2 \quad \text{for } x > 0 \qquad \tan^{-1} x + \tan^{-1}(1/x) = -\pi/2 \quad \text{for } x < 0$$

$$\cot^{-1} x = \tan^{-1}(1/x) \quad \text{for } x > 0 \qquad \cot^{-1} x - \tan^{-1}(1/x) = \pi \quad \text{for } x < 0$$

Graphs

Graph of $y = \sin^{-1} x$

Fig. 3-7 $y = \sin^{-1} x$

Graph of $y = \cos^{-1} x$

Fig. 3-8 $y = \cos^{-1} x$

Graph of $y = \tan^{-1} x$

Fig. 3-9 $y = \tan^{-1} x$

Graph of $y = \cot^{-1} x$

Fig. 3-10 $y = \cot^{-1} x$

3.4 Plane Triangle

In a triangle with sides a, b, c and angles A, B, C the following relations hold:

$$|a - b| < c < a + b, \qquad A + B + C = 180°$$

Law of sines

a/sinA=b/sinB=c/sinC

Law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \text{etc.}$$

Law of tangents

(a-b)/(a+b)=tan((A-B)/2)/tan((A+B)/2) etc.

Projection formula

$$c = a\cos B + b\cos A \qquad \text{etc.}$$

Plane triangle

Fig. 3-11

Formulas with perimeter $2s = a + b + c$

sin(A/2)=((s-b)*(s-c)/(b*c))^(1/2)   cos(A/2)=(s*(s-a)/(b*c))^(1/2)

sinA=(2/(b*c))(s*(s-a)*(s-b)*(s-c))^(1/2)   etc.

Radius of circumscribed circle } r=a/(2*sinA)   *Radius of inscribed circle* } ρ=(s-a)*tan(A/2)

Area $E = \frac{1}{2} bc \sin A$

Ext

3.5 Spherical Triangle

In every spherical triangle, the sides a, b, c (arcs of maximum circles) and the angles A, B, C (dihedral angles) are connected by the following relations:

Law of sines

sina/sinA=sinb/sinB=sinc/sinC

Law of cosines

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \quad \text{etc.}$$

Law of tangents

tan((A+B)/2)/tan((A-B)/2)
=tan((a+b)/2)/tan((a-b)/2)   etc.

Spherical triangle

Fig. 3-12

Formulas with $2s = a + b + c$ and $2S = A + B + C$

sinA=2(sins*sin(s-a)*sin(s-b)*(s-c))^(1/2)/(sinb*sinc)

sina=2(-cosS*cos(S-A)*cos(S-B)*cos(S-C))^(1/2)/(sinB*sinC)

sin(A/2)=(sin(s-b)*sin(s-c)/(sinb*sinc))^(1/2)   cos(A/2)=(sins*sin(s-a)/(sinb*sinc))^(1/2)

sin(a/2)=(-cosS*cos(S-A)/(sinB*sinC))^(1/2)   cos(a/2)=(cos(S-B)*cos(S-C)/(sinB*sinC))^(1/2)   etc.

Ext

4 GEOMETRY

4.1 Plane Geometry Ext

Triangle

Plane triangle

Fig. 4-1

Side $b^2 = a^2 + c^2 \pm 2a(BD)$
$(+ \text{ if } \theta > 90°, - \text{ if } \theta < 90°)$

Perimeter $\Pi = 2s = a + b + c$

Height h_a=(2/a)(s*(s-a)*(s-b)*(s-c))^(1/2)

Median μ_a^2=(b^2+c^2)/2-a^2/4

Bisector (inner) d_a=2(b*c*s*(s-a))^(1/2)/(b+c)

Bisector (outer) D_a=2*(b*c*(s-b)*(s-c))^(1/2)/|b-c|

Area E=a*h_a/2=a*c*sinθ/2=(s*(s-a)*(s-b)*(s-c))^(1/2)

Radius of inscribed circle ρ=(1/s)*(s*(s-a)*(s-b)*(s-c))^(1/2)
=(1/h_a+1/h_b+1/h_c)^(-1)

Radius of circumscribed circle R=(1/4)*a*b*c/(s*(s-a)*(s-b)*(s-c))^(1/2)

Right triangle For $\theta = 90°$, $b^2 = a^2 + c^2$ (Pythagorean theorem)

Inscribed circle

Fig. 4-2

Circumscribed circle

Fig. 4-3

Similar triangles

Fig. 4-4

Similar triangles (Fig. 4-4)

$A = A'$ $B = B'$ $C = C'$ AB/A'B'=BC/B'C'=CA/C'A'

Rectangle

Fig. 4-5

Perimeter $\Pi = 2a + 2b$

Area $E = ab$

Parallelogram

Fig. 4-6

Perimeter $\Pi = 2a + 2b$

Area $E = ah = ab\sin\theta$

Trapezoid

Fig. 4-7

Perimeter $\Pi = a + b + h\left(\frac{1}{\sin\theta} + \frac{1}{\sin\varphi}\right)$

Area $E = \frac{1}{2}h(a + b)$

Quadrilateral **Pro**

We have $BD = p$, $AC = q$, $A + B + C + D = 360°$.

Perimeter $\Pi = 2s = a + b + c + d$

Area
$$E = \frac{1}{2}q(h_B + h_D) = \frac{1}{2}pq\sin\theta$$
$$= \frac{1}{4}(b^2 + d^2 - a^2 - c^2)\tan\theta$$
$$= \frac{1}{4}\sqrt{4p^2q^2 - (b^2 + d^2 - a^2 - c^2)^2},$$
$$(s-a)(s-b)(s-c)(s-d)$$

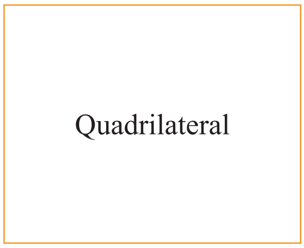

Fig. 4-8

Regular polygon **Ext**

Central angle $\theta = \frac{2\pi}{n}$

Perimeter $\Pi = na$

Apothem $\lambda = \frac{a}{2}\cot\frac{\pi}{n}$

Area $E = \frac{1}{4}na^2\cot\frac{\pi}{n}$

Fig. 4-9

where n is the number of the sides, a is the length of each side and λ is the apothem.

Regular polygon inscribed in a circle

Central angle θ=2*π/n

Side a=2*R*sin(π/n)

Apothem λ=(R^2-a^2/4)^(1/2)=R*cos(π/n)

Perimeter Π=2*n*R*sin(π/n)

Area E=(1/2)*n*R^2*sin(2*π/n)

Regular polygon inscribed in a circle

Fig. 4-10

where n is the number of the sides and R is the radius of the circumscribed circle.

Regular polygons inscribed in a circle of radius *R*

Polygon	Side	Apothem	Area
$n=3$	$\sqrt{3}\,R$	$\frac{1}{2}R$	$\frac{3}{4}\sqrt{3}\,R^2$
$n=4$	$\sqrt{2}\,R$	$\frac{1}{2}\sqrt{2}\,R$	$2R^2$
$n=5$	$\frac{1}{2}\sqrt{10-2\sqrt{5}}\,R$	$\frac{1}{4}\sqrt{6+2\sqrt{5}}\,R$	$\frac{5}{8}\sqrt{10+2\sqrt{5}}\,R^2$
$n=6$	R	$\frac{1}{2}\sqrt{3}\,R$	$\frac{3}{2}\sqrt{3}\,R^2$
$n=8$	$\sqrt{2-\sqrt{2}}\,R$	$\frac{1}{2}\sqrt{2+\sqrt{2}}\,R$	$2\sqrt{2}\,R^2$
$n=10$	$\frac{1}{2}(\sqrt{5}-1)R$	$\frac{1}{4}\sqrt{10+2\sqrt{5}}\,R$	$\frac{5}{4}\sqrt{10-2\sqrt{5}}\,R^2$
$n=12$	$\sqrt{2-\sqrt{3}}\,R$	$\frac{1}{2}\sqrt{2+\sqrt{3}}\,R$	$3R^2$

Regular polygon circumscribed on a circle

Central angle θ=2*π/n

Perimeter Π=2*n*r*tan(π/n)

Area E=n*r^2*tan(π/n)

Regular polygon circumscribed on a circle

Fig. 4-11

where n is the number of sides and r is the radius of the circle.

Circle

Diameter $d = 2r$

Perimeter $\Pi = 2\pi r$

Area $E = \pi r^2$

where r is the radius of the circle.

Circle

Fig. 4-12

Sector of a circle

Length of arc $s = r\theta$

Area $E = \frac{1}{2} r^2 \theta$ [θ in radians]

where r is the radius of the circle and θ is the angle of the sector.

Sector of a circle

Fig. 4-13

Segment of a circle

Length of chord AB $l = 2r\sin(\theta/2)$

Area of shaded segment $E = \frac{1}{2} r^2 (\theta - \sin\theta)$

where r is the radius of the circle.

Segment of a circle

Fig. 4-14

Ellipse

Perimeter

$$\Pi = 4a \int_0^{\pi/2} (1 - k^2 \sin^2\theta)^{1/2}\, d\theta \approx \pi \sqrt{2(a^2 + b^2)}$$

Area $E = \pi ab$

where $k = \frac{1}{a}\sqrt{a^2 - b^2}$.

Ellipse

Fig. 4-15

Segment of a parabola

With $AB = 2a$, $KM = b$, the length of the arc AMB (symmetric with respect to the axis KM) is

$$(AMB) = \sqrt{a^2 + 4b^2} + \frac{a^2}{2b} \ln\left(\frac{2b + \sqrt{a^2 + 4b^2}}{a}\right)$$

Area between AKB and AMB $E = \frac{2}{3} ab$

Segment of a parabola

Fig. 4-16

4.2 Solid Geometry

Rectangular parallelepiped

Area of the surface $E = 2(ab + ac + bc)$

Volume $V = abc$

where a, b, and c are the length, the width and the height of the parallelepiped, respectively.

Rectangular parallelepiped

Fig. 4-17

Parallelepiped

Height $h = c\sin\varphi$

Volume $V = Ah = abc\sin\theta\sin\varphi$

where A is the area of the horizontal cross-section of the parallelepiped.

Parallelepiped

Fig 4-18

Prism

Height $h = d\sin\theta$

Volume $V = Ah$

where A is the area of the base.

Prism

Fig. 4-19

Pyramid

Height $h = d\sin\theta$

Volume $V = \frac{1}{3}Ah$

where A is the area of the base.

If the base is a triangle, then

$$V = \tfrac{1}{3}Ah = \tfrac{1}{6}abd\sin\theta\sin\varphi$$

Pyramid

Fig. 4-20

Frustum from a pyramid or cone

Cutting a pyramid or a cone by a plane parallel to the base and throwing away the upper part gives a *frustum* (Fig. 4-21). Its volume is

V=(1/3)*h*(A_1+(A_1*A_2)^(1/2)+A_2)

where h is the height and A_1, A_2 are the areas of the bases.

For a cone frustum $A_1 = \pi r_1^2$, $A_2 = \pi r_2^2$ and

$$V = \tfrac{1}{3}\pi h(r_1^2 + r_1 r_2 + r_2^2)$$

Frustum from a pyramid or cone

Fig. 4-21

Regular polyhedra

There are five *regular convex polyhedra* (*Platonic solids*), given in Fig. 4-22: the *tetrahedron*, the *hexahedron* (or *cube*), the *octahedron*, the *dodecahedron* and the *icosahedron*. Each has identical *faces* (regular convex polygons) of side length a (the *edge* of the polyhedron).

Regular polyhedra

Fig. 4-22

Each of the above polyhedra can be constructed from the following plane figures, cut and bent appropriately.

Plane figures for construction of regular polyhedra

Fig. 4-23

The table that follows gives the radius R of the circumscribed sphere, the total surface area E and the volume V of each regular polyhedron in terms of its edge a. The radius of the inscribed sphere is always $\rho = 3V/E$.

Regular polyhedra inscribed in a sphere of radius *R*

Polyhedron	Radius R	Surface area E	Volume V
Tetrahedron (4 equilateral triangles)	$\frac{1}{4}\sqrt{6}\,a$	$\sqrt{3}\,a^2$	$\frac{1}{12}\sqrt{2}\,a^3$
Hexahedron or cube (6 equal squares)	$\frac{1}{2}\sqrt{3}\,a$	$6a^2$	a^3
Octahedron (8 equilateral triangles)	$\frac{1}{2}\sqrt{2}\,a$	$2\sqrt{3}\,a^2$	$\frac{1}{3}\sqrt{2}\,a^3$
Dodecahedron (12 regular pentagons)	$\frac{1}{4}(1+\sqrt{5})\sqrt{3}\,a$	$3\sqrt{5(5+2\sqrt{5})}\,a^2$	$\frac{1}{4}(15+7\sqrt{5})a^3$
Icosahedron (20 equilateral triangles)	$\frac{1}{4}\sqrt{2(5+\sqrt{5})}\,a$	$5\sqrt{3}\,a^2$	$\frac{5}{12}(3+\sqrt{5})a^3$

Right circular cone

Height h=l*sinθ=(l^2-r^2)^(1/2)

Area of conical surface E=π*r*l=π*r*(r^2+h^2)^(1/2)

Volume $V = \frac{1}{3}\pi r^2 h$

where r is the radius of the base and h is the height of the cone.

Right circular cone

Fig 4-24

Right circular cylinder

Area of cylindrical surface $E = 2\pi rh$

Volume $V = \pi r^2 h$

where r is the radius and h is the height of the cylinder.

Right circular cylinder

Fig. 4-25

Circular cylinder (oblique)

Height $h = l\sin\theta$

Area of cylindrical surface E=2*π*r*l=2*π*r*h/sinθ

Volume $V = \pi r^2 h = \pi r^2 l \sin\theta$

where r is the radius of the base, h is the height and l is the generator of the cylinder.

Circular cylinder (oblique)

Fig. 4-26

Frustum of a right cone

Height h=l*sinθ=(l^2-(a-b)^2)^(1/2)

Area of conical surface

E=π(a+b)*l=π*(a+b)*(h^2+(a-b)^2)^(1/2)

Volume $V = \frac{1}{3}\pi h(a^2 + ab + b^2)$

where a, b are the radii of the bases and h is the height.

Frustum of a right cone

Fig. 4-27

Sphere

Area of spherical surface $E = 4\pi r^2$

Volume $V = \frac{4}{3}\pi r^3$

Sphere

Fig. 4-28

Spherical segment or zone

Radii of bases $a^2 = r^2 - d^2, \quad b^2 = r^2 - (d + h)^2$

Area of the spherical zone $E = 2\pi rh$

Volume of spherical segment $V = \frac{1}{6}\pi h(3a^2 + 3b^2 + h^2)$

where r is the radius of the sphere, $h = AB$, and $d = AK$. Both bases are on the same side of the center.

Spherical segment or zone

Fig. 4-29

Spherical triangle

Area of the triangle ABC $E = (A + B + C - \pi)r^2$

where r is the radius of the sphere and A, B, C are the angles of the spherical triangle.

Spherical triangle

Fig. 4-30

Ellipsoid

Volume $V = \frac{4}{3}\pi abc$

where a, b, c are the semi-axes of the ellipsoid.

If $a = b > c$, then

Ellipticity ε=(1-c^2/a^2)^(1/2)

Area of surface E=2*π*a^2+(π*c^2/ε)*ln((1+ε)/(1-ε))

If $a = b < c$, then

Ellipticity ε=(1-a^2/c^2)^(1/2)

Area of surface E=2*π*a^2+2*π*(a*c*/ε)sin^(-1)ε

Ellipsoid

Fig. 4-31

Paraboloid of revolution

Area of curved surface E=(π*a^4/(6*b^2))*((4*b^2/a^2+1)^(3/2)-1)

Volume $V = \frac{1}{2}\pi a^2 b$

Paraboloid of revolution

Fig. 4-32

Torus

Area of surface $E = \pi^2(b^2 - a^2)$

Volume $V = \frac{1}{4}\pi^2(a + b)(b - a)^2$

where a and b are the internal and external radii, respectively.

Torus

Fig. 4-33

5 ANALYTIC GEOMETRY

5.1 In Two Dimensions

Systems of coordinates

To determine the position, the shape and the movement of the bodies in a Euclidean space (i.e. a flat space with positive distance between two points) we use *systems of coordinates*, that is, axes or curves appropriately numbered. In two dimensions, a *rectangular Cartesian* system of coordinates xOy is presented in Fig. 5-1. The position of the point P_1 is given by the *abscissa* x_1 and the *ordinate* y_1, which are defined by the perpendiculars to the axes.

Points

Distance between two points

The length of the straight line segment P_1P_2 is

$$d=((x_2-x_1)^2+(y_2-y_1)^2)^{(1/2)}$$

(x_1, y_1) = Cartesian coordinates of point P_1

(x_2, y_2) = Cartesian coordinates of point P_2

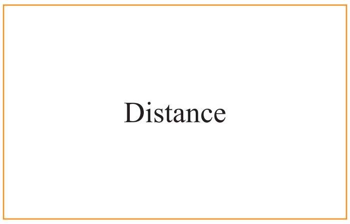

Fig. 5-1

Area of a triangle

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the *vertices* of a triangle, its area is equal to the absolute value of the determinant

$$(1/2)*\det(1,1,1;x_1,x_2,x_3;y_1,y_2,y_3)$$

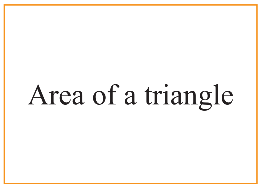

Fig. 5-2

Straight line

Line from two points Exa

If a straight line passes through two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, then every point $P(x, y)$ of the line satisfies the equation

$$(y-y_1)/(y_2-y_1)=(x-x_1)/(x_2-x_1),$$
$$\det(x,y,1;x_1,y_1,1;x_2,y_2,1)$$

The *slope* of the straight line is (Fig. 5-1)

$$m=\frac{y_2-y_1}{x_2-x_1}=\tan\theta$$

The line intersects the y axis at a point with ordinate (Fig. 5-3)

$$b=y_1-m x_1=\frac{x_2 y_1-x_1 y_2}{x_2-x_1}$$

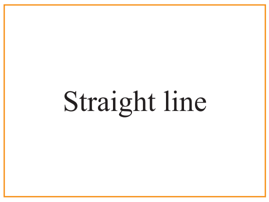

Fig. 5-3

The equation of the line is also written as

$$y-y_1=m(x-x_1)$$

or

$$y=mx+b$$

or

$$\frac{x}{a}+\frac{y}{b}=1$$

where a is the abscissa of the intersection with the x axis.

Normal form of the equation of a line (Fig. 5-3)

$$x\cos\varphi+y\sin\varphi=p$$

where p is the distance from the origin O to the line and φ is the angle of the perpendicular to the line with the positive semi-axis x.

General form of the equation of a line

$$Ax+By+C=0$$

Distance of a point from a line

$$\frac{|Ax_1+By_1+C|}{(A^2+B^2)^{1/2}}$$

(x_1, y_1) = Cartesian coordinates of the point

Angle between two lines

If m_1 and m_2 are the slopes of two lines, then the tangent of their angle is

$$\tan\psi=\frac{m_2-m_1}{m_1 m_2+1}$$

The lines are parallel or coincident, only if $m_1=m_2$.

The lines are perpendicular, if and only if $m_2=-1/m_1$.

Transformations of coordinates

The coordinates of one point in two systems of coordinates are interconnected by relations called *transformations*.

Translation

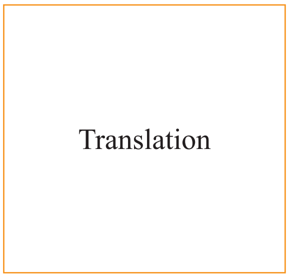

Fig. 5-4

$$x = x' + x_0; \quad y = y' + y_0$$

or

$$x' = x - x_0; \quad y' = y - y_0$$

(x, y) = coordinates with respect to the system xOy

(x', y') = coordinates with respect to the system $x'O'y'$

(x_0, y_0) = coordinates of O' with respect to xOy

Rotation

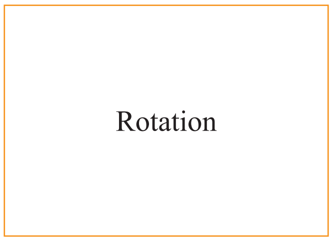

Fig. 5-5

$$x = x'\cos\varphi - y'\sin\varphi; \quad y = x'\sin\varphi + y'\cos\varphi$$

or

$$x' = x\cos\varphi + y\sin\varphi; \quad y' = -x\sin\varphi + y\cos\varphi$$

φ = angle of rotation

Translation and rotation

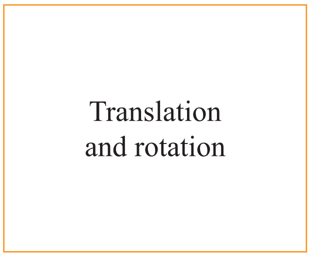

Fig. 5-6

$$x = x'\cos\varphi - y'\sin\varphi + x_0; \quad y = x'\sin\varphi + y'\cos\varphi + y_0$$

or

$$x' = (x - x_0)\cos\varphi - (y - y_0)\sin\varphi; \quad y' = -(x - x_0)\sin\varphi + (y - y_0)\cos\varphi$$

(x_0, y_0) = coordinates of O' with respect to xOy

Polar coordinates

Fig. 5-7

$$x = \rho\cos\varphi; \quad y = \rho\sin\varphi$$

or

$$\rho = (x^2 + y^2)^{1/2}; \quad \varphi = \tan^{-1}(x/y)$$

(x, y) = Cartesian coordinates

(ρ, φ) = polar coordinates

Plane curves

In general, an equation of the form $f(x, y) = 0$ represents a curve on the plane xy. A curve can be given in a *parametric* form with two equations $x = x(t)$, $y = y(t)$, where t is a parameter. A straight line is the simplest form of curve, since $f(x, y)$ is linear with respect to x and y. If $f(x, y)$ is a second-degree polynomial, we have a *conic section* (circle, ellipse, hyperbola or parabola).

Circle

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

(x_0, y_0) = Cartesian coordinates of the center

R = radius of the circle

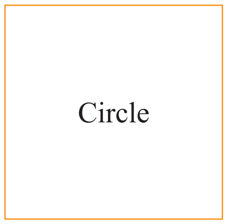

Fig. 5-8

Conic sections

Let ε=PO/PK be the ratio of the distances of a point P from a point O and a line AA', respectively. The geometric locus of the points P, for which ε = constant, is a *conic section* (like a section of a cone by a plane) with equation

r=ε*d/(1-ε*cosθ) **Exa**

There are three cases: (1) If $\varepsilon < 1$, we have an ellipse. (2) If $\varepsilon = 1$, we have a parabola. (3) If $\varepsilon > 1$, we have a hyperbola. The constant ε is called *eccentricity*, while AA' is called the *directrix*.

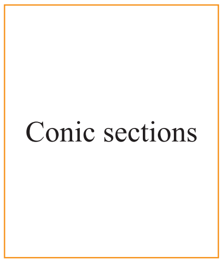

Fig. 5-9

In Cartesian coordinates the general quadratic equation for a conic section and some degenerate special cases is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$ **Ext**

Ellipse

The ellipse with a major axis $2a$ and a minor axis $2b$, parallel to the coordinate axes Ox and Oy respectively, has equation

(x-x_0)^2/a^2+(y-y_0)^2/b^2=1

where (x_0, y_0) are the coordinates of the center K of the ellipse. In Fig. 5-10 we have

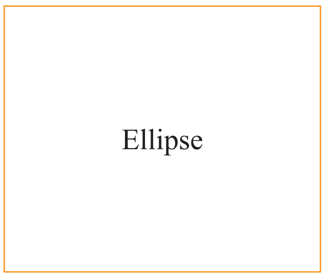

Fig. 5-10

$AB = 2a$

$CD = 2b$

KF=KF'=(a^2-b^2)^(1/2)

$PF + PF' = 2a$ (P is any point of the ellipse)

ε = eccentricity (a^2-b^2)^(1/2)/a

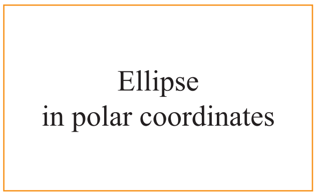

Fig. 5-11

In polar coordinates, if the origin O coincides with one of the foci, we have Fig. 5-11, and with $p = a(1 - \varepsilon^2) = b^2/a$, the equation for the ellipse becomes

r=p/(1-ε*cosθ), r'=p/(1+ε*cosθ') **Pro**

Hyperbola

The hyperbola with a major axis $2a$ and a minor axis $2b$, parallel to the coordinate axes Ox and Oy respectively, has equation

(x-x_0)^2/a^2-(y-y_0)^2/b^2=1

where (x_0, y_0) are the coordinates of the center K of the hyperbola. In Fig. 5-12 we have

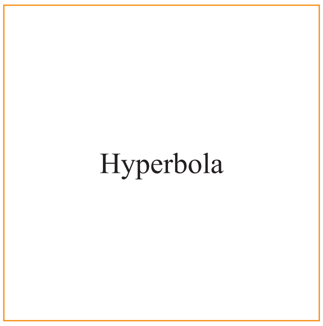

Fig. 5-12

$AB = 2a$

KF=KF'=(a^2+b^2)^(1/2)

$|PF - PF'| = 2a$
(P is any point of the hyperbola)

ε = eccentricity (a^2+b^2)^(1/2)/a

tanφ=b/a

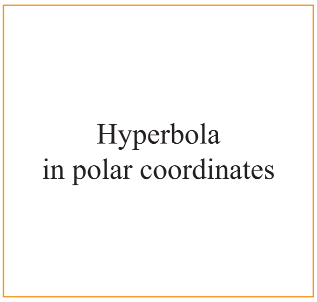

Fig. 5-13

In polar coordinates, if the origin O coincides with one of the foci, we have Fig. 5-13. With $p = a(\varepsilon^2 - 1) = b^2/a$, the equation for the right branch of the hyperbola becomes

r=p/(ε*cosθ-1), r'=p/(1-ε*cosθ') **Pro**

Parabola

The parabola with its axis of symmetry parallel to Ox (Fig. 5-14) has equation

$$(y - y_0)^2 = 4a(x - x_0),$$

where (x_0, y_0) are the coordinates of A and $AF = a$.

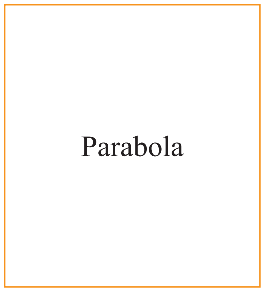

Fig. 5-14

If F coincides with O, then the parabola (Fig. 5-15) has equation

$$r = \frac{2a}{1-\cos\theta}$$

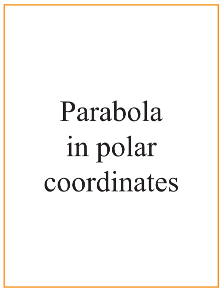

Fig. 5-15

The equation

$$y = ax^2 + bx + c, \qquad a > 0,$$

represents a parabola (Fig. 5-16) with coordinates of the lowest point

$$x_m = -\frac{b}{2a}, \qquad y_m = \frac{4ac - b^2}{4a}$$

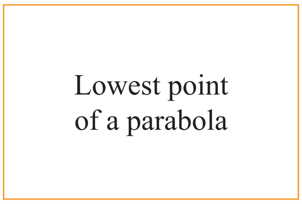

Fig. 5-16

Summary of conic sections (with $x_0 = y_0 = 0$)

	Ellipse	Hyperbola	Parabola
Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$y^2 = 4px$
Eccentricity	$\varepsilon = \left(1 - \frac{b^2}{a^2}\right)^{1/2}$	$\varepsilon = \left(1 + \frac{b^2}{a^2}\right)^{1/2}$	$\varepsilon = 1$
Foci	$(-a\varepsilon, 0)$ $(a\varepsilon, 0)$	$(-a\varepsilon, 0)$ $(a\varepsilon, 0)$	$(p, 0)$
Directrix	$x = -a/\varepsilon$ $x = a/\varepsilon$	$x = -a/\varepsilon$ $x = a/\varepsilon$	$x = -p$
Latus rectum (LM)	$2b^2/a$	$2b^2/a$	$4p$
Equation in polar coordinates	$r^2 = \frac{b^2}{1 - \varepsilon^2\cos^2\theta}$	$r^2 = \frac{-b^2}{1 - \varepsilon^2\cos^2\theta}$	$r = \frac{4p\cos\theta}{1 - \cos^2\theta}$

Cubic parabola (Fig. 5-17)

Equation

$$y = x^2(x + a)$$

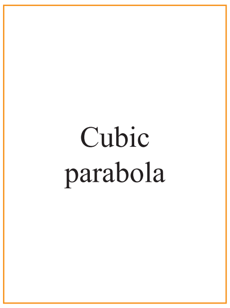

Fig. 5-17

Semi-cubic parabola (Fig. 5-18)

Equation

$$y = ax^{3/2}$$

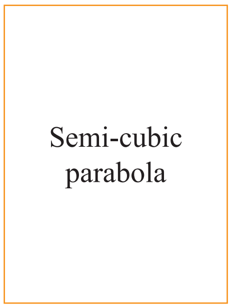

Fig. 5-18

Cissoid of Diocles (Fig. 5-19)

Point Q moves along QQ' (tangent of the circle and perpendicular to the axis Ox) and we have $OP = RQ$. Point P describes the curve.

Equations

$$y^2 = x^3/(d-x)$$

or $$r = d\sin\theta\tan\theta$$

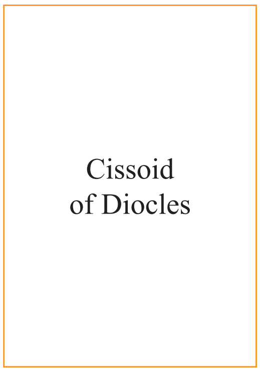

Fig. 5-19

Strophoid (Fig. 5-20)

Equation

$$x^3 + x(a^2 + y^2) = 2a(x^2 + y^2)$$

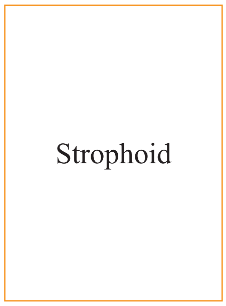

Fig. 5-20

Folium of Descartes (Fig. 5-21)

Equations

$$x^3 + y^3 = 3axy$$

or $$x = 3at/(1+t^3), \quad y = 3at^2/(1+t^3)$$

Asymptote $x + y + a = 0$

Area $E = (3/2)a^2$

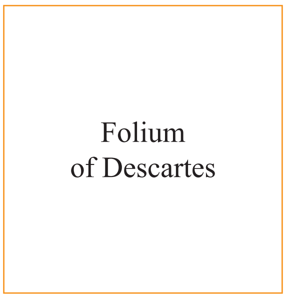

Fig. 5-21

Trisectrix (Fig. 5-22)

Equation

$$y^2 = x^2(3a+x)/(a-x)$$

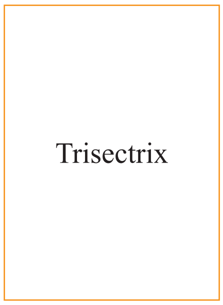

Fig. 5-22

Witch of Agnesi (Fig. 5-23)

Point A moves along the line $y = d$. B is the intersection of OA with the constant circle of center $(0, d/2)$ and diameter d. P is the intersection of the lines drawn from A and B and parallel to the axes.

Equations

y=d^3/(x^2+d^2) or x=d*cotφ; y=d*sinφ^2

where φ=AOx.

Witch of Agnesi

Fig. 5-23

Ovals of Cassini (Fig. 5-24)

The distances of P from two fixed points have a fixed product b^2.

Equation

$$(x^2 + y^2 + a^2)^2 - 4a^2x^2 = b^4$$

If $a < b$, we have Fig. 5-24a. If $a > b$, we have Fig. 5-24b.

Ovals of Cassini

Fig. 5-24a

Ovals of Cassini

Fig. 5-24b

Lemniscate of Bernoulli (Fig. 5-25)

Equation

$$(x^2 + y^2)^2 = a^2(x^2 - y^2) \quad \text{or} \quad r^2 = a^2 \cos 2\theta.$$

Area (total) $\quad E = a^2$

Lemniscate of Bernoulli

Fig. 5-25

Conhoid of Nicomide (Fig. 5-26)

Equation

$(x^2 + y^2)(x - a)^2 = b^2x^2$ or r=a/cosθ+b

Conhoid of Nicomide

Fig. 5-26

Limacons of Pascal (Fig. 5-27)

Point A moves on a constant circle of diameter d. On the line connecting A with the origin, we take two points P and P' with $PA = P'A = b < 2d$. These points describe the curve.

Equation $\quad r = b + d\cos\theta$

If $d < b$, we have Fig. 5-27a. If $b < d$, we have Fig. 5-27b.

Limacons of Pascal

Fig. 5-27a **Fig.5-27b**

Cardioid (Fig. 5-28)

The circle (B, a) rolls on the outside of the constant circle (A, a). The point P describes the curve.

Equation

$$(x^2 + y^2 - 2ax)^2 = 4a^2(x^2 + y^2)$$

or $$r = 2a(1 + \cos\theta)$$

Length (total) $= 16a$

Area $E = 6\pi a^2$

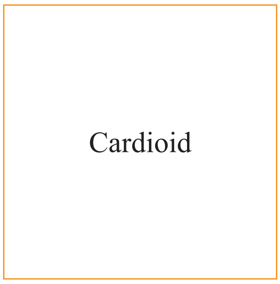

Fig. 5-28

Astroid (Fig. 5-29)

The small circle of radius $a/4$ rolls inside the large circle with radius a. The point P describes the curve.

Equations

$$x^{2/3} + y^{2/3} = a^{2/3}$$ or x=a*cosφ^3; y=a*sinφ^3

Lenght (total) $= 6a$

Area $E = \frac{3}{8}\pi a^2$

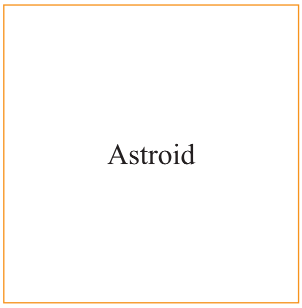

Fig. 5-29

Three-leaved rose (Fig. 5-30)

Equation

$$r = a\cos 3\theta$$

In general, if n is odd, then

$$r = a\cos n\theta$$

has n leaves.

Fig. 5-30

Four-leaved rose (Fig. 5-31)

Equation

$$r = a\cos 2\theta$$

In general, if n is even, then

$$r = a\cos n\theta$$

has $2n$ leaves.

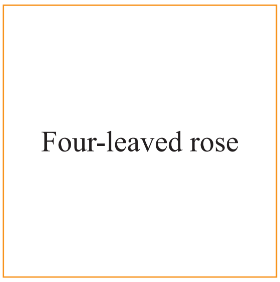

Fig. 5-31

Epicycloid (Fig. 5-32)

A small circle rolls on the outside of a large circle. Point P is fixed with respect to the small circle at a distance c from its center and describes the curve.

Equations (in Fig. 5-32, $b = c$)

```
x=(a+b)*cosφ-c*cos((a+b)*φ/b);
y=(a+b)*sinφ-c*sin((a+b)*φ/b)
```

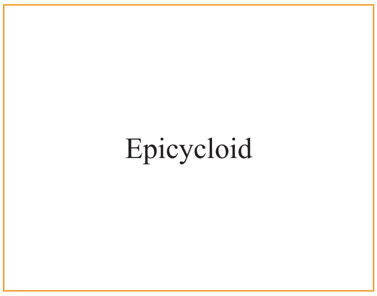

Fig. 5-32

Hypocycloid (Fig. 5-33)

A small circle rolls in the inside of a large circle. Point P is fixed with respect to the small circle at a distance c from its center and describes the curve.

Equations (in Fig. 5-33, $b = c$)

```
x=(a-b)*cosφ+c*cos((a-b)*φ/b);
y=(a-b)*sinφ-c*sin((a-b)*φ/b)
```

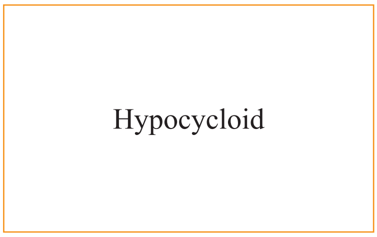

Fig. 5-33

Cycloid (Fig. 5-34)

The circle (K, a) rolls on the axis Ox. Point P is fixed on the circle and describes the curve.

Equations

$$x = a(\varphi - \sin\varphi), \qquad y = a(1 - \cos\varphi)$$

$OA = 2\pi a$, length of arc $OPA = 8a$,

area of a segment $E = 3\pi a^2$

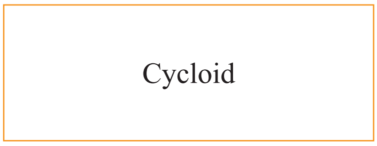

Fig. 5-34

Trochoid (Fig. 5-35)

A circle of radius a rolls on the axis Ox. Point P is fixed with respect to the circle at a distance b from its center and describes the curve.

Equations

$$x = a\varphi - b\sin\varphi, \qquad y = a - b\cos\varphi$$

In Fig. 5-35a, $b < a$. In Fig. 5-35b, $b > \alpha$. For $b = a$, we obtain the cycloid of Fig. 5-34.

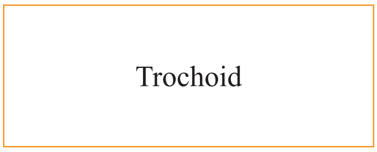

Fig. 5-35a

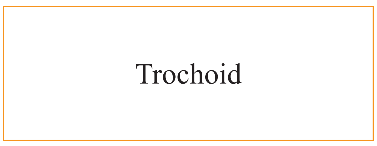

Fig. 5-35b

Spirals (Fig. 5-36)

Linear (of Archimedes)
Equation $r = a\theta$

Linear spiral

Fig. 5-36a

Parabolic
Equation $r^2 = 4p\theta$

Parabolic spiral

Fig. 5-36b

Logarithmic
Equation $r = a e^{b\theta}$

Logarithmic spiral

Fig. 5-36c

Involute of a circle (Fig. 5-37)

P is the end point of a string which is wound around a circle of a radius *a*. The string unwinds and *P* describes the curve.

Equations

$$x = a(\cos\varphi + \varphi\sin\varphi)$$

$$y = a(\sin\varphi - \varphi\cos\varphi)$$

Length of the arc $s = \frac{1}{2}a\varphi^2$

Involute of a circle

Fig. 5-37

Catenary (Fig. 5-38)

This is the shape that a chain of uniform density obtains, when suspended from two points *A* and *B*.

Equation

y=(a/2)*(e^(x/a)+e^(-x/a))

Length from $-x$ to x $s = 2a\sinh(x/a)$

Catenary

Fig. 5-38

Tractrix (Fig. 5-39)

The curve starts from the point $A(0, a)$. The tangent at any point *P* intersects the axis *Ox* at *B*. The length *PB* remains constant and equal to *a*.

Equations

x=a*(cosφ+ln(tan(φ/2)));
y=a*sinφ

Tractrix

Fig. 5-39

5.2 In Three Dimensions

Systems of coordinates

In the three dimensional Euclidean space, we use *systems of coordinates* with three coordinates.

The *Cartesian coordinates* x, y, z are measured from the origin O along the three rectangular axes Ox, Oy, Oz. A point P_0 is represented by three numbers (x_0, y_0, z_0), which define three *coordinate planes* $x = x_0$, $y = y_0$, $z = z_0$.

Cartesian coordinates

Fig. 5-40

In the directions of the axes Ox, Oy, Oz we define the *base unit vectors* **i**, **j**, **k**, respectively. Every point corresponds to a *position vector*

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

The *length* or *magnitude* of the vector **r** is denoted by $|\mathbf{r}|$ or r and is

$$r=|\mathbf{r}|=(x^2+y^2+z^2)^{1/2}$$

In general, the position of a point in space can be determined by three *curvilinear coordinates*, u, v, w, which are related to x, y, z and are measured along three *coordinate curves* [defined as intersections of the three *coordinate surfaces* $u = u_1$ (fixed), $v = v_2$ (fixed), $w = w_3$ (fixed)]. In each case, the system of coordinates is chosen so that the description of the physical system is simplified. The most frequently used coordinates (after the Cartesian) are the cylindric and the spherical coordinates. The Cartesian coordinates x, y, z, the cylindrical coordinates ρ, z and the spherical coordinate r measure distance and have dimensions of length (we assume that a unit of length has been adopted). The remaining coordinates θ and φ measure angles, are dimensionless and are expressed in degrees or radians.

Cylindrical coordinates (Fig. 5-41)

$$x=\rho\cos\varphi,\quad y=\rho\sin\varphi,\quad z=z$$

or

$$\rho=(x^2+y^2)^{1/2},\quad \varphi=\tan^{-1}(y/x),\quad z=z$$

(x, y, z) = Cartesian coordinates

(ρ, φ, z) = cylindrical coordinates

Cylindrical coordinates

Fig. 5-41

Spherical coordinates (Fig. 5-42)

$$x=r*\sin\theta*\cos\varphi$$
$$y=r*\sin\theta*\sin\varphi$$
$$z=r*\cos\theta$$

or

$$r=(x^2+y^2+z^2)^{(1/2)}$$
$$\theta=\cos^{(-1)}(z/(x^2+y^2+z^2)^{(1/2)})$$
$$\varphi=\tan^{(-1)}(y/x)$$

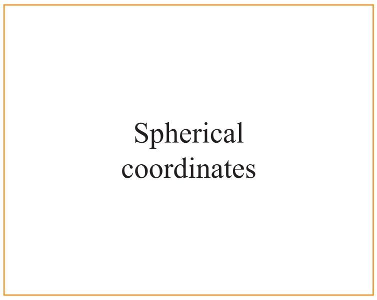

Fig. 5-42

(x, y, z) = Cartesian coordinates

(r, θ, φ) = spherical coordinates

Points

Distance between two points (Fig. 5-43)

$$d=|r_2-r_1|=((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2)^{(1/2)}$$

(x_i, y_i, z_i) = Cartesian coordinates of the point P_i $(i = 1, 2)$.

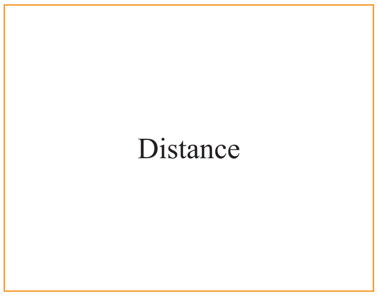

Fig. 5-43

Distance of a point from a plane

$$d=|A*x_0+B*y_0+C*z_0+D|/(A^2+B^2+C^2)^{(1/2)}$$

where $P(x_0, y_0, z_0)$ is the point and $Ax + By + Cz + D = 0$ is the plane.

Straight line

Equations of line that passes from two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

$$(x-x_1)/(x_2-x_1)=(y-y_1)/(y_2-y_1)=(z-z_1)/(z_2-z_1)$$

The *direction cosines* are

$$l=\cos\alpha=(x_2-x_1)/d,\ m=\cos\beta=(y_2-y_1)/d,\ n=\cos\gamma=(z_2-z_1)/d$$

where α, β, γ are the angles of the line P_1P_2 with the positive semi-axes x, y, z and d is the length of the line segment P_1P_2. We have

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad \text{or} \quad l^2 + m^2 + n^2 = 1$$

The equations of a line are also written as

$$(x-x_1)/l=(y-y_1)/m=(z-z_1)/n \quad \text{or} \quad x = x_1 + lt, \ \ y = y_1 + mt, \ \ z = z_1 + nt$$

where t is a parameter.

With vectors, the equations of a line are written as

$$\mathbf{r} = \mathbf{r}_1 + \kappa(\mathbf{r}_2 - \mathbf{r}_1) \quad \text{or} \quad \mathbf{r} = \mathbf{r}_1 + \lambda\mathbf{a} \quad \text{where} \quad a=(r_2-r_1)/|r_2-r_1|$$

and κ, λ are parameters.

Line from a point and parallel to a vector

$$(x-x_0)/X=(y-y_0)/Y=(z-z_0)/Z \quad \text{or} \quad \mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$$

where $\mathbf{r}_0 = (x_0, y_0, z_0)$ is the point and $\mathbf{v} = (X, Y, Z)$ is the parallel vector.

Line from a point and perpendicular to a plane

$$(x-x_0)/A=(y-y_0)/B=(z-z_0)/C \quad \text{or} \quad x = x_0 + At, \ \ y = y_0 + Bt, \ \ z = z_0 + Ct$$

where (x_0, y_0, z_0) is the point and $Ax + By + Cz + D = 0$ is the plane.

Angle between two lines

For the angle ψ between two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2, we have

$$\cos\psi = l_1 l_2 + m_1 m_2 + n_1 n_2$$

If the parametric equations of the lines are $\mathbf{r} = \mathbf{r}_1 + \lambda_1\mathbf{a}_1$, $\mathbf{r} = \mathbf{r}_2 + \lambda_2\mathbf{a}_2$, then ($\mathbf{a}\cdot\mathbf{b}$ is the internal product of two vectors)

$$\cos\psi=a_1 * a_2/(|a_1||a_2|)$$

Distance from a point to a line

$$d'_0=(|r_0-r_1|^2-((r_0-r_1) * v_0)^2)^{(1/2)}$$

where $\mathbf{r}_0$ is the given point, $\mathbf{r}_1$ is any point of the line and $\mathbf{v}_0$ is a unit vector along the line.

Plane

General equation

$$Ax + By + Cz + D = 0 \quad \text{or} \quad \mathbf{A}\cdot\mathbf{r} + D = 0,$$

The vector $\mathbf{A} = (A, B, C)$ is perpendicular to the plane.

Plane from three points

det[x-x_1,y-y_1,z-z_1;
x_2-x_1,y_2-y_1,z_2-z_1;
x_3-x_1,y_3-y_1,z_3-z_1]=0 or $[(\mathbf{r}-\mathbf{r}_1)(\mathbf{r}-\mathbf{r}_2)(\mathbf{r}-\mathbf{r}_3)] = 0$

where (x_i, y_i, z_i), $i = 1, 2, 3$, are the coordinates of the three points and **[abc]** the triple mixed product of three vectors.

Plane from the intersects with the axes (Fig. 5-44)

x/a+y/b+z/c=1

Plane from the intersects with the axes

Fig. 5-44

Regular form of the equation of a plane

$$x\cos\alpha + y\cos\beta + z\cos\gamma = d$$

where d is the distance of O from the plane and α, β, γ are the angles of the normal OP with the positive axes Ox, Oy, Oz.

Transformations of coordinates

The coordinate transformations that follow are all *linear*. Thus, any n degree polynomial of x, y, z remains an n degree polynomial of x', y', z' after the transformation.

Translation

x=x'+x_0
y=y'+y_0
z=z'+z_0

or

x'=x-x_0
y'=y-y_0
z'=z-z_0

Translation

Fig. 5-45

(x, y, z) = coordinates with respect to the system O

(x', y', z') = coordinates with respect to the system O'

(x_0, y_0, z_0) = coordinates of point O' with respect to the system O

Rotation

```
x=l_1*x'+l_2*y'+l_3*z'
y=m_1*x'+m_2*y'+m_3*z'
z=n_1*x'+n_2*y'+n_3*z'
```

or

```
x'=l_1*x+m_1*y+n_1*z
y'=l_2*x+m_2*y+n_2*z
z'=l_3*x+m_3*y+n_3*z
```

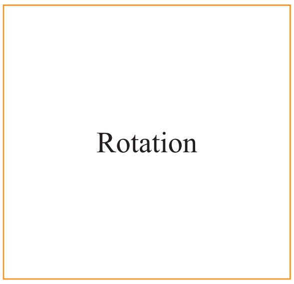

Fig. 5-46

where (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are the direction cosines of the axes Ox', Oy', Oz' with respect to the system O.

Translation and rotation

```
x=l_1*x'+l_2*y'+l_3*z'+x_0
y=m_1*x'+m_2*y'+m_3*z'+y_0
z=n_1*x'+n_2*y'+n_3*z'+z_0
```

or

```
x'=l_1*(x-x_0)+m_1*(y-y_0)+n_1*(z-z_0)
y'=l_2*(x-x_0)+m_2*(y-y_0)+n_2*(z-z_0)
z'=l_3*(x-x_0)+m_3*(y-y_0)+n_3*(z-z_0)
```

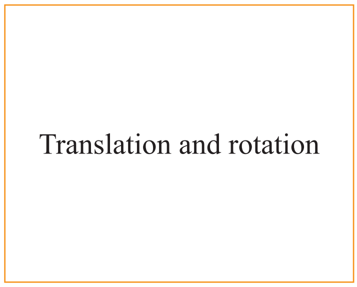

Fig. 5-47

(x_0, y_0, z_0) = coordinates of the point O' with respect to the system O.

Consecutive transformations

Two or more linear transformations (translations, rotations or combination of those) are equivalent to an appropriate linear transformation.

Surfaces

Sphere (Fig. 5-48) **Ext**

Equation of the surface in Cartesian co-ordinates

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

(x_0, y_0, z_0) = center of the sphere

R = radius of the sphere

Each section with plane whose distance from the center is smaller than R, is a circle.

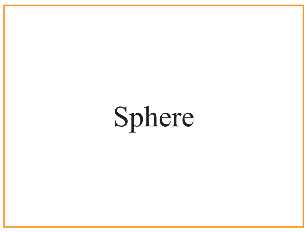

Fig. 5-48

Ellipsoid (Fig. 5-49)

Equation of the surface

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1$$

(x_0, y_0, z_0) = center
a, b, c = semi-axes

Each real section with a plane is a closed curve of second degree. Hence, it is an ellipse (the circle is a special case).

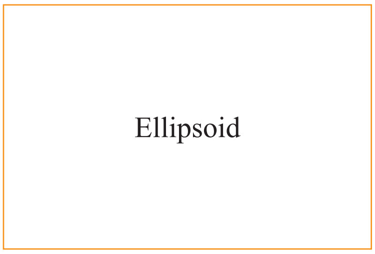

Fig. 5-49

Elliptic cylinder (Fig. 5-50)

For an elliptic cylinder parallel to the axis *Oz*, the equation of the surface is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where a, b are the semi-axes of the horizontal cross-section.

Every cross section with a plane non-parallel to the z axis is an ellipse (the circle is a special case).

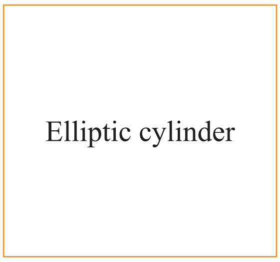

Fig. 5-50

Elliptic cone (Fig. 5-51)

For an elliptic cone with axis parallel to the axis *Oz,* the equation of the surface is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

A cross section with a plane is an ellipse (if it is a closed curve), a hyperbola (if it has two open parts) or a parabola (if it is has one open part).

Fig. 5-51

Hyperboloid of one sheet (Fig. 5-52)

Equation of surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

The horizontal cross sections are ellipses. The vertical cross sections are hyperbolas.

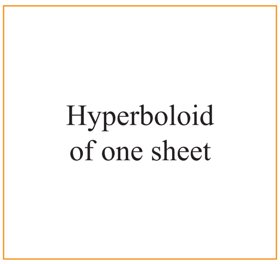

Fig. 5-52

Hyperboloid of two sheets (Fig. 5-53)

Equation of surface

$$z^2/c^2-x^2/a^2-y^2/b^2=1$$

The horizontal cross sections for $|z| > c$ are ellipses. The vertical cross sections (with a plane of the form $Ax + By + C = 0$) are hyperbolas.

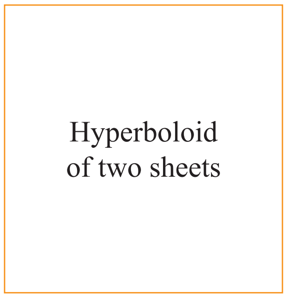

Fig. 5-53

Elliptic paraboloid (Fig. 5-54)

Equation of surface

$$x^2/a^2+y^2/b^2-z/c=0$$

The cross sections with a plane (horizontal with $z > 0$ or oblique) are ellipses. The vertical sections (with a plane of the form $Ax + By + C = 0$) are parabolas.

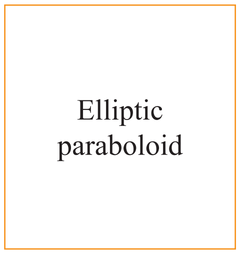

Fig. 5-54

Hyperbolic paraboloid (Fig. 5-55)

Equation of surface

$$x^2/a^2-y^2/b^2-z/c=0$$

The cross sections with a plane are hyperbolas, if they have two segments (e.g. horizontal cross sections with z = fixed) or parabolas, if they have one segment (e.g. vertical cross sections with x = const. or y = const.).

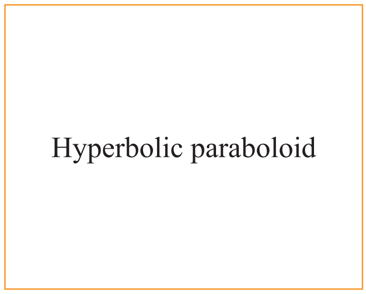

Fig. 5-55

6 DERIVATIVES

6.1 Definitions

Functions

In general, with the term *function* we signify a mapping (correspondence as specified by a rule) from a set D to a set R, so that each element of D corresponds to only one element of R. The sets D and R are called *domain of definition* and *range*, respectively, and constitute an integral part of the function's definition. D and R are usually sets of numbers (e.g. the set of positive integers, a straight line segment or a two dimensional domain). A function is symbolised as $f: X \to Y$ or simply as $y = f(x)$, where the *independent variable* x can represent one or more real or complex variables, with y being the corresponding value of the *dependent variable*. The value of x is mapped to a *unique value* of y. **The**

Limits

A function $y = f(x)$ of an independent variable x *has a limit* L (or *tends to* L) when x tends to x_0, if for any positive number ε there is a positive number δ such that $0 < |x - x_0| < \delta$ implies $|f(x) - L| < \varepsilon$. Note that this definition does not assume the existence of $f(x_0)$. In general, the limit is denoted by lim[x,x_0]{f(x)}=L .

The definition is also valid for $x_0 = \infty$ or $-\infty$, if for any positive number ε there is a number M such that $x > M$ implies $|f(x) - L| < \varepsilon$.

A function $y = f(x)$ *has a limit* or *tends to* ∞ (or $-\infty$) when x tends to x_0, if for any positive (or negative) number M there is a number δ such that $0 < |x - x_0| < \delta$ implies $f(x) > M$ (or $f(x) < M$).

Properties of limits

lim[x,x_0]{f(x)+g(x)}

lim[x,x_0]{a*f(x)}

lim[x,x_0]{f(x)*g(x)}

lim[x,x_0]{f(x)/g(x)}

Important limits

lim[n,\o]{(1+1/n)^n}=lim[x,0]{(1+x)^(1/x)}=e

lim[x,0]{(c^x-1)/x}=lnc, lim[x,0]{x^x}=1

lim[x,0]{x^a*lnx}=lim[x,\o]{x^(-a)*lnx}=lim[x,\o]{x^a*e^(-x)}=0

lim[x,x_0]{sinx/x}=lim[x,0]{tanx/x}=lim[{sinhx/x}=lim[x,0](tanhx/x}=1 **Pro**

Indeterminate forms

Sometimes, the calculation of limits leads to expressions without precise meaning, such as $0/0$, ∞/∞, $0\cdot\infty$, 0^0, ∞^0, 1^∞, $\infty - \infty$. Such an expression can be reduced to $0/0$ (by division or using logarithms) and calculated with the following *L'Hôpital rule*:

Let the functions $f(x)$ and $g(x)$ be differentiable on an open interval (a, b) that includes x_0 and $f(x_0) = g(x_0) = 0$, but $g'(x) \neq 0$ at every point of (a, b) except perhaps at x_0. Then

lim[x,x_0]{f(x)/g(x)}=lim[x,x_0]{f'(x)/g'(x)}

with the assumption that the derivatives and the limit at the right hand side exist. The same rule applies for unilateral limits, for $x_0 \to \pm\infty$ and for $f(x) \to \infty$, $g(x) \to \infty$.

Derivatives

If $y = f(x)$, the *derivative* of y or $f(x)$ with respect to x at a point (x, y) is defined by the relation

y'=lim[h,0]{(f(x+h)-f(x))/h}

=lim[Δx,0]{(f(x+Δx)-f(x))/Δx}

where $h = \Delta x$. The derivative is also denoted as $f'(x)$, dy/dx or df/dx.

The derivative of a function $y = f(x)$ at a point $x = x_0$ equals the tangent of the angle φ that is formed by the tangent line at this point and the x axis, i.e. $y' = \tan\varphi$. **Ext**

Tangent at a point of a curve

Fig. 6-1

From this property and the definition, it is clear that y' expresses essentially the rate of change of y at the point $x = x_0$.

6.2 General Rules of Differentiation

In the following formulas, (a) u, v, w are functions of x, (b) c, n are constants, (c) $e = 2.71828\ldots$ is the base of natural logarithms, (d) $\ln u$ is the natural logarithm of u (where $u > 0$) and all the angles are in radians.

- (d/dx){c}=0
- (d/dx){c*x}=c
- (d/dx){x^n}=n*x^(n-1) [n is integer or real] (d/dx){x^(1/2}, (d/dx){1/x}
- (d/dx){u\+v\+w\+...}

(d/dx){c*u}=c*(d/dx){u}

- (d/dx){u*v}=u*(d/dx){v}+v*(d/dx){u}

(d/dx){u*v*w}

- (d/dx){u/v}=(v*(d/dx){u}-u*(d/dx){v})/v^2
- (d/dx){u^n}=n*u^(n-1)*(d/dx){u}
- (d/dx){y}=(d/du){y}*(d/dx){u} [derivative of composite function]
- (d/dy){x}=1/(d/dx){y},
(d/dy)^2{x}=-(d/dx)^2{y}/((d/dx){y})^3 [derivative of inverse function]

- (d/dx){y}=(d/dt){y}/(d/dt){x} if the function is given in parametric form $x = x(t)$, $y = y(t)$
- (d/dx){y}=-(\d/\ dx){F}/(\d/\dy){F} if the function $y = f(x)$ is given in complex form $F(x, y) = 0$

(d/dx){u^v}=(d/dx){e^(v*lnu)}=e^(v*lnu)*(d/dx){v*lnu}

Second derivative (d/dx){(d/dx){y}}=(d/dx)^2{y}

Third derivative (d/dx){(d/dx)^2{y}}=(d/dx)^3{y}

Derivative of order *n* (d/dx){(d/dx)^(n-1){y}}=(d/dx)^n{y}

(d/dx)^n{x^a}=a*(a-1)*...*(a-n+1)*x^(a-n)

6.3 Derivatives of Elementary Functions

Trigonometric functions

- (d/dx){sinx}=cosx (d/dx)^n{sinx}=sin(x+n*π/2)
- (d/dx){cosx}=-sinx (d/dx)^n{cosx}=cos(x+n*π/2)
- (d/dx){tanx}=1/cosx^2
- (d/dx){cotx}=-1/sinx^2
- (d/dx){sin^(-1)x}=1/(1-x^2)^(1/2)

- $\frac{d}{dx}\{\cos^{-1}x\} = -\frac{1}{(1-x^2)^{1/2}}$

- $\frac{d}{dx}\{\tan^{-1}x\} = \frac{1}{1+x^2}$

- $\frac{d}{dx}\{\cot^{-1}x\} = -\frac{1}{1+x^2}$

[$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$ represent the principal branches.]

Exponential and logarithmic functions

- $\frac{d}{dx}\{e^x\} = e^x$

- $\frac{d}{dx}\{\log_c x\} = \frac{\log_c e}{x}$

- $\frac{d}{dx}\{c^x\} = c^x \cdot \ln c$

- $\frac{d}{dx}\{\ln x\} = \frac{1}{x}$

$$\left(\frac{d}{dx}\right)^n\{\ln x\} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

Hyperbolic functions

- $\frac{d}{dx}\{\sinh x\} = \cosh x$

- $\frac{d}{dx}\{\cosh x\} = \sinh x$

- $\frac{d}{dx}\{\tanh x\} = \frac{1}{\cosh^2 x}$

- $\frac{d}{dx}\{\coth x\} = -\frac{1}{\sinh^2 x}$

- $$\frac{d}{dx}\{\sinh^{-1}x\} = \frac{1}{(x^2+1)^{1/2}}$$

- $$\frac{d}{dx}\{\cosh^{-1}x\} = \frac{1}{(x^2-1)^{1/2}}$$

- $$\frac{d}{dx}\{\tanh^{-1}x\} = \frac{1}{1-x^2}$$

- $$\frac{d}{dx}\{\coth^{-1}x\} = \frac{1}{1-x^2}$$

[$\sinh^{-1}x$, $\cosh^{-1}x$, $\tanh^{-1}x$, $\coth^{-1}x$ represent the principal branches.]

6.4 Partial Derivatives

If $f(x, y)$ is a function of the independent variables x and y, the *partial derivative* of $f(x, y)$ with respect to x is defined by the relation

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\{f\} = \lim_{\Delta x \to 0}\left\{\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}\right\} \quad \text{with } y = \text{constant}$$

Similarly, the partial derivative of $f(x, y)$ with respect to y is defined by the relation

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\{f\} = \lim_{\Delta y \to 0}\left\{\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right\} \quad \text{with } x = \text{constant}$$

Partial derivatives of higher order can be defined as follows:

$$\left(\frac{\partial}{\partial x}\right)^2\{f\} = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\{f\}\right\}, \ldots.$$

The two previous relations $\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\{f\}\right\}$ and $\frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}\{f\}\right\}$ give the same result, if the function and the partial derivatives are continuous. In this case, the order of differentiation is of no importance. In general, the partial derivative with respect to an independent variable is calculated with a simple differentiation with respect to that variable, while regarding the other independent variables as constants.

Often we use the following abbreviations for the partial derivatives:

$$f_x = \frac{\partial}{\partial x}\{f\},\ f_y = \frac{\partial}{\partial y}\{f\}, \ldots$$

Rules of differentiation

In general, for the calculation of the partial derivatives, we apply the differentiation rules of the Section 6.2. Additionaly, we distinguish the following cases:

If $y = f(u, v, ..., w)$, where $u, v, ...$, are functions of only one independent variable x, then

(\d/\dx){f}=(\df/\du)*(du/dx)+(\df/\dv)*(dv/dx)+...+(\df/\dw)*(dw/dx)

If $y = f(u, v, ..., w)$, where $u, v, ...$, are functions of the independent variables x_1, x_2, ... , x_n, then

(\d/\dx_k){f}=(\df/\du)*(du/dx_k)+(\df/\dv)*(dv/dx_k)+...+(\df/\dw)*(dw/dx_k) for $k = 1, 2, ..., n$

6.5 Differentials

If $y = f(x)$ is a function, for an increase of the independent variable by Δx, the dependent variable increases by $\Delta y = f(x+\Delta x) - f(x)$. We define

Differential of x: $dx = \Delta x$

Differential of y: $dy = f'(x)\,dx$

We have Δy/Δx=(f(x+Δx)-f(x))/Δx=f'(x)+ε=(dy/dx)+ε, where $\varepsilon \to 0$, as $\Delta x \to 0$.

Thus $\Delta y = f'(x)\Delta x + \varepsilon \Delta x$.

$d(u \pm v \pm w \cdots) = du \pm dv \pm dw \cdots$

$d(uv) = udv + vdu$

d(u/v)=(v*du-u*dv)/v^2

$d(u^n) = nu^{n-1}du$

$d(\sin u) = \cos u\, du$

$d(\cos u) = -\sin u\, du$

For a function $f(x, y)$ of two independent variables, the *differential* is defined by the relation

▶ df=(\df/\dx)*dx+(\df/\dy)*dy

where $dx = \Delta x$ and $dy = \Delta y$. Extention to multivariable functions takes place easily.

6.6 Maxima and Minima

Definitions

Let a function $f(x)$ be defined on an interval (a, b) or $[a, b]$ and $x = x_0$ an interior point of (a, b). The function $f(x)$ has a *relative maximum* at $x = x_0$, if $f(x) \le f(x_0)$ for every x in some open interval (x_1, x_2) with $a < x_1 < x_0 < x_2 < b$.

Similarly, $f(x)$ has a *relative* (or *local*) *minimum* at $x = x_0$, if $f(x) \ge f(x_0)$ for every x in some open interval (x_1, x_2) with $a < x_1 < x_0 < x_2 < b$.

If $f(x) \le f(x_0)$ or $f(x) \ge f(x_0)$ for all x in (a, b) or $[a, b]$ then $f(x)$ has an *absolute maximum* or an *absolute minimum* at $x = x_0$.

Conditions for maxima and minima

Let $f(x)$ have $f'(x_0) = 0$ [$x = x_0$ is then called a *stationary point*] and $f''(x_0)$ be continuous on (x_1, x_2) with $a < x_1 < x_0 < x_2 < b$. If $f''(x_0) < 0$, then $f(x)$ has a relative maximum at $x = x_0$. If $f''(x_0) > 0$, then $f(x)$ has a relative minimum at $x = x_0$.

Generally, let $f(x)$ have continuous derivatives on (x_1, x_2) with

$$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0, \quad f^{(n)}(x_0) \neq 0, \qquad n > 1$$

Then the following hold:

(1) If n is even and $f^{(n)}(x_0) < 0$, $f(x)$ has a relative maximum at $x = x_0$.

(2) If n is even and $f^{(n)}(x_0) > 0$, $f(x)$ has a relative minimum at $x = x_0$.

(3) If n is odd, $f(x)$ has a *point of inflection* at $x = x_0$. Exa

Two independent variables

Let $f(x, y)$ be a function of two independent variables x and y, defined on a two dimensional region D, with continuous partial derivatives of first and second order $f_x, f_y, f_{xx}, f_{yy}, f_{xy}$. At an interior point (x_0, y_0), the function $f(x, y)$ has a *relative maximum*, if $f(x, y) \le f(x_0, y_0)$ for every point (x, y) in some open disk with a center at (x_0, y_0). Similarly, $f(x, y)$ has a *relative minimum*, if $f(x, y) \ge f(x_0, y_0)$.

For the point (x_0, y_0) that satisfies the equations $f_x = f_y = 0$ the following hold:

(1) If $f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} < 0$, $f(x, y)$ has a relative maximum at (x_0, y_0).

(2) If $f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$, $f(x, y)$ has a relative minimum at (x_0, y_0).

(3) If $f_{xx} f_{yy} - f_{xy}^2 < 0$, $f(x, y)$ has a *saddle point* at (x_0, y_0) [$f(x, y)$ increases or decreases as we move away from (x_0, y_0) depending on the direction].

(4) If $f_{xx} f_{yy} - f_{xy}^2 = 0$, then no conclusion is possible from these properties.

7 INDEFINITE INTEGRALS

7.1 Definitions

A function $f(x)$ has an *indefinite integral* (or antiderivative)

F(x)=\I[x]{f(x)}

in the interval (a, b) if and only if there is a function $F(x)$ such that the derivative $F'(x)$ exists and $F'(x) = f(x)$. The function $f(x)$ is the *integrand* and x is the *integration variable*. If $F(x)$ is an indefinite integral of $f(x)$, then each function of the form $F(x) + c$, where c is an arbitrary constant, is also an indefinite integral of $f(x)$.

Since $F'(x) = f(x)$, we can have the identity relation

F(x)=\I[x]{F'(x)}=\I[x]{dF/dx}=\I[F]{1}

7.2 General Rules

- \I[x]{f'(x)}=f(x)
- \I[x]{a*f(x)}=a*\I[x]{f(x)}
- \I[x]{u\+v}=\I[x]{u}\+\I[x]{v}
- \I[v]{u}=u*v-\I[u]{v}
- \I[x]{f'(x)*g(x)}=f(x)*g(x)-\I[x]{f(x)*g'(x)}

\I[x]{f^((n))*g}=f^((n-1))*g-f^((n-2))*g+...+(-1)^n\I[x]{fg^((n))}

} Integration by parts

\I[x]{(u'*v-u*v')/v^2}=u/v

\I[x]{(u'*v-u*v')/(u*v)}=ln(u/v)

\I[x]{f(a*x)}=(1/a)*\I[u]{f(u)}   where $u = ax$

▶ \I[x]{g(f(x))}=\I[u]{g(u)*(dx/du)}   where $u = f(x)$

7.3 Calculation Methods for Indefinite Integrals

In practice, an integral can be simplified by changing the integration variable or the integrand.

If the integration variable x appears only within a specific expression u in the integrand, then we try u as a new integration variable. Thus, we have the following relations:

\I[x]{f(a*x+b)}=(1/a)*\I[u]{f(u)}   with $u = ax + b$

\I[x]{f((a*x+b)^(1/2))}=(2/a)*\I[u]{u*f(u)}   with u=(a*x+b)^(1/2)

\I[x]{f((a*x+b)^(1/n))}=(n/a)*\I[u]{u^(n-1)*f(u)}   with u=(a*x+b)^(1/n)

\I[x]{f(((a*x+b)/(c*x+d))^(1/n))}=n*(a*d-b*c)*\
I[u]{f(u)*u^(n-1)/(c*u^n-a)^2}   with u=((a*x+b)/(c*x+d))^(1/n)

If the integrand depends only on an expression of the form $(a^2 \pm x^2)^{1/2}$, then we try a trigonometric function of x.

\I[x]{f((a^2-x^2)^(1/2))}   with $x = a \sin u$

\I[x]{f((x^2+a^2)^(1/2))}   with $x = a \tan u$

If the integrand depends only on an expression of the form $(x^2 \pm a^2)^{1/2}$, then we try a hyperbolic function of x.

\I[x]{f((x^2-a^2)^(1/2))}   with $x = a \cosh u$

\I[x]{f((x^2+a^2)^(1/2))}   with $x = a \sinh u$

If the integrand contains only e^{ax} or $\ln x$ or $\sin^{-1}x$, etc., we use this expression as a new integration variable. Thus

`\I[x]{f(e^(a*x))}=(1/a)\I[u]{f(u)/u}` with $u = e^{ax}$

`\I[x]{f(lnx)}=\I[u]{f(u)*e^u}` with $u = \ln x$

`\I[x]{f(sin^(-1)(x/a))}=a*\I[u]{f(u)*cosu}` with `u=sin^(-1)(x/a)`

(similarly for other inverse trigonometric functions)

If the integrand contains only $\sin x$ and $\cos x$, we use $\tan(x/2)$ as a new integration variable.

`\I[x]{f(sinx,cosx)}=2*\I[u]{f(2*u/(1+u^2),(1-u^2)/(1+u^2))/(1+u^2)}` with `u=tan(x/2)`

Similarly, if the integrand contains only $\sinh x$ and $\cosh x$, we use $\tanh(x/2)$ as a new integration variable.

`\I[x]{f(sinhx,coshx)}=2*\I[u]{f(2*u/(1-u^2),(1+u^2)/(1-u^2))/(1-u^2)}` with `u=tanh(x/2)`

If $f(x)$ is an rational function of x (i.e. the ratio of two polynomials), it can be expressed as a sum of a polynomial of x and terms of the form [ses See. 2.6, Expansion in partial fractions]

`c/(x-a)^k, (c*x+d)/(x^2+a*x+b)^m`

Then the indefinite integral can be calculated as the sum of the integrals of those terms.

If $f(x)$ contains x in an expression of the form $x^m(ax^n + b)^p$ we try the following replacements: If $(m + 1)/n$ = integer, we set $u = x^n$ and we continue with integration by parts or we set $au + b = w$. If $(m + 1)/n + p$ = integer, we set $u = x^{-n}$ and we continue with integration by parts or we set $a + bu = w$. If p = integer, we expand $(ax^n + b)^p$ and we obtain a sum of powers of x as integrand.

If $f(x)$ contains the expression $(ax^2 + bx + c)^{1/2}$ with $b^2 - 4ac \neq 0$, we try the replacement $u = (2ax + b)/|b^2 - 4ac|^{1/2}$.

7.4 Basic Integrals Inf

▶ Most important.
To calculate these integrals:
❶ Use derivatives of Sec. 6.3.
❷ Write integrand as sum.
❸ Integrate by parts.
❹ Use multiples of x.

▶ \I[x]{a} ❶

▶ \I[x]{x^n}, x^(n+1), ln|x| ❶ ❶

▶ \I[x]{sinx}, cosx ❶

▶ \I[x]{cosx}, sinx ❶

▶ \I[x]{1/sinx^2}, cotx ❶

▶ \I[x]{1/cosx^2}, tanx ❶

\I[x]{tanx}, ln|cosx| (set $u = \cos x$)

\I[x]{cotx}, ln|sinx| (set $u = \sin x$)

\I[x]{sinx^2}, sin(2*x), sinx*cosx ❹

\I[x]{cosx^2}, sin(2*x), sinx*cosx ❹

\I[x]{tanx^2}, tanx-x ❷

\I[x]{cotx^2}, -cotx-x ❷

▶ \I[x]{e^x} ❶

▶ \I[x]{a^x} ❶

▶ \I[x]{sinhx}, coshx ❶

▶ \I[x]{coshx}, sinhx ❶

▶ \I[x]{1/sinhx^2}, cothx ❶

▶ \I[x]{1/coshx^2}, tanhx ❶

\I[x]{tanhx}, ln(coshx) (set $u = \cosh x$)

\I[x]{cothx}, ln|sinhx| (set $u = \sinh x$)

\I[x]{sinhx^2}, sinh(2*x), sinhx*coshx ❹

\I[x]{coshx^2}, sinh(2*x), sinhx*coshx ❹

\I[x]{tanhx^2}, x-tanhx ❷

\I[x]{cothx^2}, x-cothx ❷

▶ \I[x]{1/(x^2+1)}, tanh^(-1)x ❶

▶ \I[x]{1/(x^2-1)}, coth^(-1)x, ln((x-1)/(x+1)) ❶ ❷

▶ \I[x]{1/(1-x^2)}, tanh^(-1)x, ln((1+x)/(1-x)) ❶ ❷

▶ \I[x]{1/(1-x^2)^(1/2)}, sinh^(-1)x ❶

▶ \I[x]{1/(x^2+1)^(1/2)}, sinh^(-1)x, ln(x+(x^2+1)^(1/2)) ❶

▶ \I[x]{1/(x^2-1)^(1/2)}, cosh^(-1)x, ln(x+(x^2-1)^(1/2)) ❶

\I[x]{1/(x*(x^2-1)^(1/2))}, cos^(-1)(1/|x|) (set $u = 1/\cos x$)

\I[x]{1/(x*(x^2+1)^(1/2))}, ln((1+(x^2+1)^(1/2))/|x|) (set $x = \tan u$)

\I[x]{1/(x*(1-x^2)^(1/2))}, ln((1+(1-x^2)^(1/2))/|x|) (set $u = \cos x$)

7.5 Various Integrals Inf

With $ax + b$

\I[x]{(a*x+b)^n}, (a*x+b)^(n+1), ln(a*x+b) ❶

\I[x]{x/(a*x+b)}, ln(a*x+b) ❶

\I[x]{x^2/(a*x+b)}, ln(a*x+b) ❶

\I[x]{x^3/(a*x+b)}, ln(a*x+b) ❶

\I[x]{1/(x*(a*x+b))}, ln(x/(a*x+b)) ❷

\I[x]{1/(x^2*(a*x+b))}, ln((a*x+b)/x) ❷

\I[x]{1/(x^3*(a*x+b))}, ln(x/(a*x+b)) ❷

\I[x]{x/(a*x+b)^2}, ln(a*x+b) ❶

\I[x]{x^2/(a*x+b)^2}, ln(a*x+b) ❶

\I[x]{x^3/(a*x+b)^2}, ln(a*x+b) ❶

\I[x]{1/(x*(a*x+b)^2)}, ln(x/(a*x+b)) ❷

To calculate these integrals:

❶ Set $u = ax + b$.

❷ Expand in partial fractions.

❸ Integrate by parts.

The argument of every logarithm is assumed to be positive, otherwise the absolute value of the argument should be taken.

\I[x]{1/(x^2*(a*x+b)^2)}, ln((a*x+b)/x) ❷

\I[x]{1/(x^3*(a*x+b)^2)}, ln(x/(a*x+b)) ❷

\I[x]{x/(a*x+b)^3}, 1/(a*x+b)^2 ❶

\I[x]{x^2/(a*x+b)^3}, 1/(a*x+b)^2, ln(a*x+b) ❶

\I[x]{x^3/(a*x+b)^3}, 1/(a*x+b)^2, ln(a*x+b) ❶

\I[x]{1/(x*(a*x+b)^3))}, 1/(a*x+b)^2, ln((a*x+b)/x) ❷

\I[x]{1/(x^2*(a*x+b)^3))}, 1/(a*x+b)^2, ln((a*x+b)/x) ❷

\I[x]{1/(x^3*(a*x+b)^3))}, 1/(a*x+b)^2, ln((a*x+b)/x) ❷

\I[x]{x*(a*x+b)^n}, (a*x+b)^(n+2), (a*x+b)^(n+1) ❶

\I[x]{x^n/(a*x+b)}, x^n, x^(n-1), ln(a*x+b) Pro

\I[x]{x^2*(a*x+b)^n}, (a*x+b)^(n+1), (a*x+b)^(n+2), (a*x+b)^(n+3) ❶

\I[x]{x^m*(a*x+b)^n},
x^(m+1)*(a*x+b)^n, \I[x]{x^m*(a*x+b)^(n-1)},
x^m*(a*x+b)^(n+1), \I[x]{x^(m-1)*(a*x+b)^n},
x^(m+1)*(a*x+b)^(n+1), \I[x]{x^m*(a*x+b)^(n+1)} ❸

With roots

❶ Set $u = ax + b$.
❷ Set $u = (ax + b)^{1/2}$.
❸ Integrate by parts.
❹ See **Ext**.

\I[x]{(a*x+b)^(1/2)}, (a*x+b)^(3/2) ❶ ❷

\I[x]{x*(a*x+b)^(1/2)}, (a*x+b)^(3/2) ❶ ❷

\I[x]{x^2*(a*x+b)^(1/2)}, (a*x+b)^(3/2) ❶ ❷

\I[x]{x^3*(a*x+b)^(1/2)}, (a*x+b)^(3/2) ❶ ❷

\I[x]{(a*x+b)^(1/2)/x}, \I[x]{1/(x*(a*x+b)^(1/2))} ❹

\I[x]{(a*x+b)^(1/2)/x^2}, \I[x]{1/(x*(a*x+b)^(1/2))} ❸ ❹

\I[x]{(a*x+b)^(1/2)/x^3}, \I[x]{1/(x*(a*x+b)^(1/2))} ❸ ❹

\I[x]{1/(a*x+b)^(1/2)} ❶ ❷

\I[x]{x/(a*x+b)^(1/2)} ❶ ❷

\I[x]{x^2/(a*x+b)^(1/2)} ❶ ❷

\I[x]{x^3/(a*x+b)^(1/2)} ❶ ❷

\I[x]{1/(x*(a*x+b)^(1/2))}, ln(((a*x+b)^(1/2)-b^(1/2))/((a*x+b)^(1/2)+b^(1/2))), tan^(-1)(((a*x+b)/(-b))^(1/2)) ❷

\I[x]{1/(x^2*(a*x+b)^(1/2))},
\I[x]{1/(x*(a*x+b)^(1/2)} ❹

\I[x]{1/(x^3*(a*x+b)^(1/2))},
\I[x]{1/(x*(a*x+b)^(1/2)} ❹

\I[x]{x^n*(a*x+b)^(1/2))}, (a*x+b)^(3/2) ❹
\I[x]{x^(n-1)*(a*x+b)^(1/2)}

\I[x]{(a*x+b)^(1/2)/x^n}, (a*x+b)^(3/2) ❸
\I[x]{1/(x^(n-1)*(a*x+b)^(1/2))},
\I[x]{(a*x+b)^(1/2)/x^(n-1)} ❹

\I[x]{x^n/(a*x+b)^(1/2)},
x^n*(a*x+b)^(1/2) ❹
\I[x]{x^(n-1)/(a*x+b)^(1/2)},

\I[x]{1/(x^n*(a*x+b)^(1/2))},
\I[x]{1/(x^(n-1)*(a*x+b)^(1/2))}, ❹

\I[x]{x/(a*x+b)^(3/2)},
1/(a*x+b)^(1/2) ❷

\I[x]{1/(x*(a*x+b)^(3/2))},
\I[x]{1/(x*(a*x+b)^(1/2))} ❷

\I[x]{(a*x+b)^(n/2)},
(a*x+b)^((n+2)/2) ❶ ❷

\I[x]{x*(a*x+b)^(n/2)},
(a*x+b)^((n+4)/2), (a*x+b)^((n+2)/2) ❶ ❷

\I[x]{x^2*(a*x+b)^(n/2)}, (a*x+b)^((n+6)/2),
(a*x+b)^((n+4)/2), (a*x+b)^((n+2)/2) ❶ ❷

\I[x]{(a*x+b)^(n/2)/x}, (a*x+b)^(n/2),
\I[x]{(a*x+b)^((n-2)/2)}

\I[x]{(a*x+b)^(n/2)/x^2}, (a*x+b)^((n+2)/2),
\I[x]{(a*x+b)^(n/2)/x}

\I[x]{1/(x*(a*x+b)^(n/2))},
\I[x]{1/(x*(a*x+b)^((n-2)/2))}

\I[x]{1/(x^2*(a*x+b)^(n/2))},
\I[x]{1/(x*(a*x+b)^(n/2))}

Cal

With $ax + b$ and $cx + d$

❶ Expand in partial fractions.
❷ Integrate by parts.
Restriction: $ad - bc \neq 0$

\I[x]{(a*x+b)/(c*x+d)},
ln(c*x+d) ❶

\I[x]{1/((a*x+b)*(c*x+d))},
ln((a*x+b)/(c*x+d)) ❶

\I[x]{x/((a*x+b)*(c*x+d))},
ln(a*x+b), ln(c*x+d) ❶

\I[x]{1/((a*x+b)^2*(c*x+d))},
ln((a*x+b)/(c*x+d)) ❶

\I[x]{x/((a*x+b)^2*(c*x+d))},
ln((a*x+b)/(c*x+d)) ❶

\I[x]{x^2/((a*x+b)^2*(c*x+d))},
ln(a*x+b), ln(c*x+d) ❶

\I[x]{(a*x+b)^m*(c*x+d)^n},
\I[x]{(a*x+b)^m*(c*x+d)^(n-1)} ❷

\I[x]{1/((a*x+b)^m*(c*x+d)^n)},
\I[x]{1/((a*x+b)^m*(c*x+d)^(n-1))} ❷

\I[x]{(a*x+b)^m/(c*x+d)^n)}, ❷
\I[x]{(a*x+b)^m/(c*x+d)^(n-1)},
\I[x]{(a*x+b)^(m-1)/(c*x+d)^n}, ❷
\I[x]{(a*x+b)^(m-1)/(c*x+d)^(n-1)} ❷

With roots

❶ Set $u = ax + b$.
❷ Set $u = (ax + b)^{1/2}$.
❸ Integrate by parts.

\I[x]{(c*x+d)/(a*x+b)^(1/2)} ❶ ❷

\I[x]{(c*x+d)*(a*x+b)^(1/2)},
(a*x+b)^(3/2) ❶ ❷

\I[x]{1/((c*x+d)*(a*x+b)^(1/2))}, ❷
tan^(-1)((c^2*(a*x+b)/K)^(1/2)) ❷

\I[x]{(a*x+b)^(1/2)/(c*x+d)}, ❷
tan^(-1)((c^2*(a*x+b)/K)^(1/2)) ❷

\I[x]{1/((a*x+b)*(c*x+d))^(1/2)}, ❷
ln((a*c*(a*x+b))^(1/2)+a*(c*x+d)^(1/2))
tan^(-1)(((-c*(a*x+b))/(a*(c*x+d)))^(1/2)) ❷

\I[x]{x/((a*x+b)*(c*x+d))^(1/2)},
\I[x]{1/((a*x+b)*(c*x+d))^(1/2)}

\I[x]{((a*x+b)*(c*x+d))^(1/2)},
\I[x]{1/((a*x+b)*(c*x+d))^(1/2)}

\I[x]{((c*x+d)/(a*x+b))^(1/2)},
((a*x+b)*(c*x+d))^(1/2), \I[x]{1/((a*x+b)*(c*x+d))^(1/2)}

\I[x]{1/((a*x+b)*(c*x+d)^3)^(1/2)},
(a*x+b)^(1/2)/(c*x+d)^(1/2)

Cal

\I[x]{(c*x+d)^n*(a*x+b)^(1/2)},
\I[x]{(c*x+d)^n/(a*x+b)^(1/2)}

\I[x]{1/((c*x+d)^n*(a*x+b)^(1/2))}, (a*x+b)^(1/2)/(c*x+d)^(n-1),
\I[x]{1/((c*x+d)^(n-1)*(a*x+b)^(1/2))}

\I[x]{(c*x+d)^n/(a*x+b)^(1/2)},
\I[x]{(c*x+d)^(n-1)/(a*x+b)^(1/2)}

\I[x]{(a*x+b)^(1/2)/(c*x+d)^n)}, (a*x+b)^(1/2)/(c*x+d)^(n-1),
\I[x]{1/((c*x+d)^(n-1)*(a*x+b)^(1/2))} ❸

Cal

With $x^2 + a^2$

\I[x]{1/(x^2+a^2)},
tan^(-1)(x/a) ❸

\I[x]{x/(x^2+a^2)},
ln(x^2+a^2) ❶

❶ Set $u = x^2 + a^2$.
❷ Expand in fractions. Cal
❸ Set $x = a \tan u$.

\I[x]{x^2/(x^2+a^2)},
tan^(-1)(x/a) ❷

\I[x]{x^3/(x^2+a^2)},
ln(x^2+a^2) ❶❷

\I[x]{1/(x*(x^2+a^2))},
ln(x^2/(x^2+a^2)) ❶❷

\I[x]{1/(x^2*(x^2+a^2))},
tan^(-1)(x/a) ❷

\I[x]{1/(x^3*(x^2+a^2))},
ln(x^2/(x^2+a^2)) ❶❷

\I[x]{1/((c*x+d)*(x^2+a^2))},
ln(c*x+d), ln(x^2+a^2), tan^(-1)(x/a) ❷

\I[x]{1/(x^2+a^2)^2},
x/(x^2+a^2), tan^(-1)(x/a) ❸

\I[x]{x/(x^2+a^2)^2},
1/(x^2+a^2) ❶

\I[x]{x^2/(x^2+a^2)^2},
x/(x^2+a^2), tan^(-1)(x/a) ❷❸

\I[x]{x^3/(x^2+a^2)^2},
1/(x^2+a^2), ln(x^2+a^2) ❶❷

\I[x]{1/(x*(x^2+a^2)^2)},
1/(x^2+a^2), ln(x^2/(x^2+a^2)) ❶❷

\I[x]{1/(x^2*(x^2+a^2)^2)},
x/(x^2+a^2), tan^(-1)(x/a) ❷❸

\I[x]{1/(x^3*(x^2+a^2)^2)},
1/(x^2+a^2), ln(x^2/(x^2+a^2)) ❶❷

\I[x]{1/(x^2+a^2)^3},
x/(x^2+a^2)^2, x/(x^2+a^2), tan(-1)(x/a) ❸

\I[x]{x/(x^2+a^2)^3},
1/(x^2+a^2)^2 ❶❷

\I[x]{x^2/(x^2+a^2)^3},
x/(x^2+a^2)^2, x/(x^2+a^2), tan^(-1)(x/a) ❷❸

\I[x]{x^3/(x^2+a^2)^3},
1/(x^2+a^2)^2, 1/(x^2+a^2) ❶❷

\I[x]{1/(x*(x^2+a^2)^3)},
1/(x^2+a^2)^2, 1/(x^2+a^2), ln(x^2/(x^2+a^2)) ❶❷

\I[x]{1/(x^2*(x^2+a^2)^3)},
x/(x^2+a^2)^2, x/(x^2+a^2), tan^(-1)(x/a) ❷❸

\I[x]{1/(x^3*(x^2+a^2)^3)},
1/(x^2+a^2)^2, 1/(x^2+a^2), ln(x^2/(x^2+a^2)) ❶❷

\I[x]{x^m/(x^2+a^2)},
\I[x]{x^(m-2)/(x^2+a^2)}

\I[x]{1/(x^2+a^2)^n},
x/(x^2+a^2)^(n-1), \I[x]{1/(x^2+a^2)^(n-1)}

\I[x]{x/(x^2+a^2)^n},
1/(x^2+a^2)^(n-1) ❶

\I[x]{1/(x*(x^2+a^2)^n)},
1/(x^2+a^2)^(n-1), \I[x]{1/(x*(x^2+a^2)^(n-1))}

\I[x]{x^m/(x^2+a^2)^n},
x^(m-1)/(x^2+a^2)^(n-1), \I[x]{x^(m-2)/(x^2+a^2)^n}

\I[x]{1/(x^m*(x^2+a^2)^n)},
1/(x^(m-1)*(x^2+a^2)^(n-1)), \I[x]{1/(x^(m-2)*(x^2+a^2)^n)}

Cal

With roots

❶ Set $u = x^2 + a^2$.
❷ Set $x = a\tan u$.
❸ Use reduction relation.

\I[x]{(x^2+a^2)^(1/2)},
ln(x+(x^2+a^2)^(1/2)) ❷

\I[x]{x*(x^2+a^2)^(1/2)},
(x^2+a^2)^(3/2)) ❶

\I[x]{x^2*(x^2+a^2)^(1/2)},
(x^2+a^2)^(3/2), ln(x+(x^2+a^2)^(1/2)) ❸

\I[x]{(x^2+a^2)^(1/2)/x},
ln((a+(x^2+a^2)^(1/2))/|x|) ❷

\I[x]{(x^2+a^2)^(1/2)/x^2},
ln(x+(x^2+a^2)^(1/2)) ❷❸

\I[x]{1/(x^2+a^2)^(1/2)},
ln(x+(x^2+a^2)^(1/2)), sinh^(-1)(x/a) ❷

\I[x]{x/(x^2+a^2)^(1/2)} ❶

\I[x]{x^2/(x^2+a^2)^(1/2)},
ln(x+(x^2+a^2)^(1/2)) ❸

\I[x]{1/(x*(x^2+a^2)^(1/2))},
ln((a+(x^2+a^2)^(1/2))/|x|) ❷

\I[x]{1/(x^2*(x^2+a^2)^(1/2))},
(x^2+a^2)^(1/2))/x ❸

\I[x]{(x^2+a^2)^(3/2)}, (x^2+a^2)^(1/2), ln(x+(x^2+a^2)^(1/2)) ❷

\I[x]{x*(x^2+a^2)^(3/2)},
(x^2+a^2)^(5/2) ❶

\I[x]{x^2*(x^2+a^2)^(3/2))}, (x^2+a^2)^(1/2)/x, ln(x+(x^2+a^2)^(1/2)) ❸

\I[x]{(x^2+a^2)^(3/2)/x}, (x^2+a^2)^(1/2), ln((a+(x^2+a^2)^(1/2))/x) ❷

\I[x]{(x^2+a^2)^(3/2)/x^2}, (x^2+a^2)^(1/2), ln(x+(x^2+a^2)^(1/2)) ❸

\I[x]{1/(x^2+a^2)^(3/2)}, (x^2+a^2)^(1/2) ❷

\I[x]{x/(x^2+a^2)^(3/2)}, 1/(x^2+a^2)^(1/2) ❶

\I[x]{x^2/(x^2+a^2)^(3/2)}, (x^2+a^2)^(1/2), ln(x+(x^2+a^2)^(1/2)) ❸

\I[x]{1/(x*(x^2+a^2)^(3/2))}, 1/(x^2+a^2)^(1/2), ln((a+(x^2+a^2)^(1/2))/x) ❷

\I[x]{1/(x^2*(x^2+a^2)^(3/2))}, x/(x^2+a^2)^(1/2) ❸

\I[x]{x^(n+1)*(x^2+a^2)^(1/2)}, (x^2+a^2)^(3/2), \I[x]{x^(n-1)*(x^2+a^2)^(1/2)}

\I[x]{(x^2+a^2)^(1/2)/x^(n+1)}, (x^2+a^2)^(3/2), \I[x]{(x^2+a^2)^(1/2)/x^(n-1)}

\I[x]{(x^(n+1)/(x^2+a^2)^(1/2)}, x^n*(x^2+a^2)^(1/2), \I[x]{x^(n-1)/(x^2+a^2)^(1/2)}

\I[x]{1/(x^(n+1)*(x^2+a^2)^(1/2))}, (x^2+a^2)^(1/2)/x^n, \I[x]{1/(x^(n-1)*(x^2+a^2)^(1/2))}

\I[x]{x^(n+1)*(x^2+a^2)^(3/2)}, x^n*(x^2+a^2)^(5/2), \I[x]{x^(n-1)*(x^2+a^2)^(3/2)}

\I[x]{(x^2+a^2)^(3/2)/x^(n+1)}, (x^2+a^2)^(5/2)/x^n, \I[x]{(x^2+a^2)^(3/2)/x^(n-1)}

Ext

\I[x]{x^(n+1)/(x^2+a^2)^(3/2)}, x^n/(x^2+a^2)^(1/2),
\I[x]{x^(n-1)/(x^2+a^2)^(3/2)}

\I[x]{1/(x^(n+1)*(x^2+a^2)^(3/2))}, 1/(x^n*(x^2+a^2)^(1/2),
\I[x]{1/(x^(n-1)*(x^2+a^2)^(3/2)}

Ext

\I[x]{x/(x^2+a^2)^((2*n+1)/2))},
(x^2+a^2)^((2*n-1)/2) ❶

\I[x]{x^3/(x^2+a^2)^((2*n+1)/2))}, (x^2+a^2)^((2*n-3)/2),
(x^2+a^2)^((2*n-1)/2) ❶

\I[x]{x*(x^2+a^2)^((2*n+1)/2))},
(x^2+a^2)^((2*n+3)/2) ❶

\I[x]{x^3*(x^2+a^2)^((2*n+1)/2))}, (x^2+a^2)^((2*n+3)/2),
(x^2+a^2)^((2*n+5)/2) ❶

With $x^2 - a^2$

\I[x]{1/(x^2-a^2)}, ❷❸
ln|(x-a)/(x+a)|,
coth^(-1)(x/a),
tanh^(-1)(x/a)

\I[x]{x/(x^2-a^2)},
ln|x^2-a^2)| ❶

\I[x]{x^2/(x^2-a^2)},
ln|(x-a)/(x+a)| ❷❸

\I[x]{1/(x*(x^2-a^2))},
ln|(x^2-a^2)/x^2|| ❷❸

\I[x]{1/(x^2*(x^2-a^2))},
ln|(x-a)/(x+a)|| ❷❸

❶ Set $u = x^2 - a^2$.
❷ Expand in fractions.
❸ Set $x = a \sin\theta$ or $x = a/\cos\theta$.
❹ Integrate by parts.

\I[x]{1/(x^2-a^2)^2},
ln|(x-a)/(x+a)|| ❷❸

\I[x]{x/(x^2-a^2)^2} ❶

\I[x]{x^2/(x^2-a^2)^2}, ln|(x-a)/(x+a)| ❷❸

\I[x]{1/(x*(x^2-a^2)^2)}, ln|x^2/(x^2-a^2)| ❷❸

\I[x]{1/(x^2*(x^2-a^2)^2)}, ln|(x-a)/(x+a)| ❷❸

\I[x]{1/(x^2-a^2)^n)}, x/(x^2-a^2)^(n-1),
\I[x]{1/(x^2-a^2)^(n-1)} ❹

\I[x]{x/(x^2-a^2)^n)},
1/(x^2-a^2)^(n-1) ❶

\I[x]{1/(x*(x^2-a^2)^(n+1))}, 1/(x^2-a^2)^n,
\I[x]{1/(x*(x^2-a^2)^n)} ❹

\I[x]{x^m*(x^2-a^2)^(n+1)}, x^(m+1)*(x^2-a^2)^(n+1),
\I[x]{x^m*(x^2-a^2)^n} ❹

\I[x]{x^m/(x^2-a^2)^(n+1)}, x^(m-1)/(x^2-a^2)^n,
\I[x]{x^(m-2)/(x^2-a^2)^n} ❹

\I[x]{(x^2-a^2)^(n+1)/x^m)}, (x^2-a^2)^(n+1)/x^(m-1),
\I[x]{(x^2-a^2)^n/x^m)} ❹

\I[x]{1/(x^m*(x^2-a^2)^(n+1))}, 1/(x^(m-1)*(x^2-a^2)^n,
\I[x]{1/(x^m*(x^2-a^2)^n)} ❹

\I[x]{1/(x^(m+1)*(x^2-a^2)^n)}, 1/(x^m*(x^2-a^2)^(n-1),
\I[x]{1/(x^(m-1)*(x^2-a^2)^n)} ❹

With roots of $x^2 - a^2$ $(x^2 > a^2)$

❶ Set $u = x^2 - a^2$.
❷ Set $x = a/\cos u$.
❸ Integrate by parts.

\I[x]{(x^2-a^2)^(1/2)}, x*(x^2-a^2)^(1/2), ln|x+(x^2-a^2)^(1/2)| ❷

\I[x]{x*(x^2-a^2)^(1/2)}, (x^2-a^2)^(3/2) ❶

\I[x]{x^2*(x^2-a^2)^(1/2)}, x*(x^2-a^2)^(3/2), x*(x^2-a^2)^(1/2), ln|x+(x^2-a^2)^(1/2)| ❷

\I[x]{(x^2-a^2)^(1/2)/x}, cos^(-1)|a/x| ❷

\I[x]{(x^2-a^2)^(1/2)/x^2}, ln|x+(x^2-a^2)^(1/2)| ❷

\I[x]{1/(x^2-a^2)^(1/2)}, ln|x+(x^2-a^2)^(1/2)| ❷

\I[x]{x/(x^2-a^2)^(1/2)} ❶

\I[x]{x^2/(x^2-a^2)^(1/2)}, x*(x^2-a^2)^(1/2), ln|x+(x^2-a^2)^(1/2)| ❷

\I[x]{1/(x*(x^2-a^2)^(1/2))}, cos^(-1)|a/x| ❷

\I[x]{1/(x^2*(x^2-a^2)^(1/2))}, (x^2-a^2)^(1/2)/x ❷

\I[x]{(x^2-a^2)^(3/2)}, x*(x^2-a^2)^(3/2), x*(x^2-a^2)^(1/2), ln|x+(x^2-a^2)^(1/2)| ❷

\I[x]{x*(x^2-a^2)^(3/2)}, (x^2-a^2)^(5/2) ❶

\I[x]{x^2*(x^2-a^2)^(3/2)}, (x^2-a^2)^(1/2), ln|x+(x^2-a^2)^(1/2)| ❷

\I[x]{(x^2-a^2)^(3/2)/x},
(x^2-a^2)^(1/2), cos^(-1)|a/x| ❷

\I[x]{(x^2-a^2)^(3/2)/x^2}, x*(x^2-a^2)^(1/2), ln|x+(a^2-a^2)^(1/2)| ❷

\I[x]{1/(x^2-a^2)^(3/2)},
x/(x^2-a^2)^(1/2) ❷

\I[x]{x/(x^2-a^2)^(3/2)},
1/(x^2-a^2)^(1/2) ❶

\I[x]{x^2/(x^2-a^2)^(3/2)}, x/(x^2-a^2)^(1/2),
ln|x+(x^2-a^2)^(1/2)| ❷

\I[x]{1/(x*(x^2-a^2)^(3/2))},
1/(x^2-a^2)^(1/2), cos(-1)|a/x| ❷

\I[x]{1/(x^2*(x^2-a^2)^(3/2))},
(x^2-a^2)^(1/2)/x, x/(x^2-a^2)^(1/2) ❷

\I[x]{x*(x^2-a^2)^((2*n+1)/2)},
(x^2-a^2)^((2*n+3)/2) ❶

\I[x]{x^3*(x^2-a^2)^((2*n+1)/2)},
(x^2-a^2)^((2*n+5)/2), (x^2-a^2)^((2*n+3)/2) ❶

\I[x]{x/(x^2-a^2)^((2*n+1)/2)},
(x^2-a^2)^((2*n-1)/2) ❶

\I[x]{x^3/(x^2-a^2)^((2*n+1)/2)},
1/(x^2-a^2)^((2*n-1)/2), 1/(x^2-a^2)^((2*n-3)/2) ❶

\I[x]{x^(n+1)*(x^2-a^2)^(1/2)},
x^n*(x^2-a^2)^(3/2), \I[x]{x^(n-1)*(x^2-a^2)^(1/2)} ❸

\I[x]{x^(n+1)/(x^2-a^2)^(1/2)},
x^n*(x^2-a^2)^(1/2), \I[x]{x^(n-1)/(x^2-a^2)^(1/2)} ❸

\I[x]{x^(n+1)*(x^2-a^2)^(3/2)},
x^n*(x^2-a^2)^(5/2), \I[x]{x^(n-1)*(x^2-a^2)^(3/2)} ❸

\I[x]{x^(n+1)/(x^2-a^2)^(3/2)},
x^n/(x^2-a^2)^(1/2), \I[x]{x^(n-1)*(x^2-a^2)^(3/2)} ❸

With roots of $a^2 - x^2$ $(a^2 > x^2)$

❶ Set $u = a^2 - x^2$.
❷ Set $x = a \sin u$.
❸ Integrate by parts.

\I[x]{(a^2-x^2)^(1/2)},
x*(a^2-x^2)^(1/2), sin^(-1)(x/a) ❷

\I[x]{x*(a^2-x^2)^(1/2)},
(a^2-x^2)^(3/2) ❶

\I[x]{x^2*(a^2-x^2)^(1/2)},
x*(a^2-x^2)^(3/2), x*(a^2-x^2)^(1/2), sin^(-1)(x/a) ❷

\I[x]{(a^2-x^2)^(1/2)/x},
(a^2-x^2)^(1/2), ln|(a+(a^2-x^2)^(1/2))/x| ❷

\I[x]{(a^2-x^2)^(1/2)/x^2},
(a^2-x^2)^(1/2)/x, sin^(-1)(x/a) ❷

\I[x]{1/(a^2-x^2)^(1/2)},
sin^(-1)(x/a) ❷

\I[x]{x/(a^2-x^2)^(1/2)} ❶

\I[x]{x^2/(a^2-x^2)^(1/2)},
x*(a^2-x^2)^(1/2), sin^(-1)(x/a) ❷

\I[x]{1/(x*(a^2-x^2)^(1/2))},
ln|(a+(a^2-x^2)^(1/2))/x| ❷

\I[x]{1/(x^2*(a^2-x^2)^(1/2))},
(a^2-x^2)^(1/2)/x ❷

\I[x]{(a^2-x^2)^(3/2)},
x*(a^2-x^2)^(3/2), x*(a^2-x^2)^(1/2), sin^(-1)(x/a) ❷

\I[x]{x*(a^2-x^2)^(3/2)},
(a^2-x^2)^(5/2) ❶

\I[x]{x^2*(a^2-x^2)^(3/2)}, x*(a^2-x^2)^(5/2), x*(a^2-x^2)^(1/2), sin^(-1)(x/a) ❷

\I[x]{(a^2-x^2)^(3/2)/x}, (a^2-x^2)^(1/2), ln|(a+(a^2-x^2)^(1/2))/x| ❷

\I[x]{(a^2-x^2)^(3/2)/x^2}, x*(a^2-x^2)^(1/2), sin^(-1)(x/a) ❷

\I[x]{1/(a^2-x^2)^(3/2)},
x/(a^2-x^2)^(1/2) ❷

\I[x]{x/(a^2-x^2)^(3/2)},
1/(a^2-x^2)^(1/2) ❶

\I[x]{x^2/(a^2-x^2)^(3/2)},
x/(a^2-x^2)^(1/2), sin^(-1)((x/a) ❷

\I[x]{1/(x*(a^2-x^2)^(3/2))}, 1/(a^2-x^2)^(1/2),
ln|(a+(a^2-x^2)^(1/2))/x| ❷

\I[x]{1/(x^2*(a^2-x^2)^(3/2))},
(a^2-x^2)^(1/2)/x, x/(a^2-x^2)^(1/2) ❷

\I[x]{x*(a^2-x^2)^((2*n+1)/2))},
(a^2-x^2)^((2*n+3)/2) ❶

\I[x]{x^3*(a^2-x^2)^((2*n+1)/2))},
(a^2-x^2)^((2*n+5)/2), (a^2-x^2)^((2*n+3)/2) ❶

\I[x]{x/(a^2-x^2)^((2*n+1)/2)},
1/(a^2-x^2)^((2*n-1)/2) ❶

\I[x]{x^3/(a^2-x^2)^((2*n+1)/2)},
1/(a^2-x^2)^((2*n-3)/2), 1/(a^2-x^2)^((2*n-1)/2) ❶

\I[x]{x^(n+1)*(a^2-x^2)^(1/2)},
x^n*(a^2-x^2)^(3/2), \I[x]{x^(n-1)*(a^2-x^2)^(1/2)} ❸

\I[x]{x^(n+1)/(a^2-x^2)^(1/2)},
x^n*(a^2-x^2)^(1/2), \I[x]{x^(n-1)/(a^2-x^2)^(1/2)} ❸

\I[x]{x^(n+1)*(a^2-x^2)^(3/2)},
x^n*(a^2-x^2)^(5/2), \I[x]{x^(n-1)*(a^2-x^2)^(3/2)} ❸

\I[x]{x^(n+1)/(a^2-x^2)^(3/2)},
x^n/(a^2-x^2)^(1/2), \I[x]{x^(n-1)/(a^2-x^2)^(3/2)} ❸

With $ax^k + b$

❶ Set $u = ax^3 + b$.
❷ Expand in fractions.
❸ Integrate by parts.

\I[x]{1/(a*x^2+b)}, tan^(-1)(x*(a/b)^(1/2)),
ln((b+x*(-a*b)^(1/2))/(b-x*(-a*b)^(1/2))) ❷

\I[x]{1/(a*x^3+b)}, ln((x+λ)^2/(x^2-λ*x+λ^2)),
tan^(-1)((2*x-λ)/(λ*3^(1/2))) ❷

\I[x]{x/(a*x^3+b)}, ln((x+λ)^2/(x^2-λ*x+λ^2)),
tan^(-1)((2*x-λ)/(λ*3^(1/2))) ❷

\I[x]{x^2/(a*x^3+b)},
ln(a*x^3+b) ❶

\I[x]{x^3/(a*x^3+b)},
\I[x]{1/(a*x^3+b)} ❷

\I[x]{1/(a*x^3+b)^2},
\I[x]{1/(a*x^3+b)} ❸

\I[x]{x/(a*x^3+b)^2},
\I[x]{x/(a*x^3+b)} ❸

\I[x]{x^2/(a*x^3+b)^2,
1/(a*x^3+b)} ❶

\I[x]{x^3/(a*x^3+b)^2},
\I[x]{1/(a*x^3+b)} ❸

\I[x]{1/(x*(a*x^3+b)^2)},
ln(x^3/(a*x^3+b)) ❷

\I[x]{1/(x^2*(a*x^3+b))}, \
I[x]{x/(a*x^3+b)} ❷

\I[x]{1/(x^3*(a*x^3+b))}, \I[x]{1/
(a*x^3+b)} ❷❸

\I[x]{1/(x*(a*x^3+b)^2)}, 1/(a*x^3+b),
ln(x^3/(a*x^3+b)) ❷❸

\I[x]{x^n/(a*x^3+b)^m)}, x^(n+1)/(a*x^3+b)^(m-1),
\I[x]{x^n/(a*x^3+b)^(m-1)}, x^(n-2)/(a*x^3+b)^(m-1),
\I[x]{x^(n-3)/(a*x^3+b)^m}

\I[x]{1/(x^n*(a*x^3+b)^m)}, 1/(x^(n-1)*(a*x^3+b)^(m-1)),
\I[x]{1/(x^n*(a*x^3+b)^(m-1))},
1/(x^(n-1)*(a*x^3+b)^(m-1)), \I[x]{1/(x^(n-3)*(a*x^3+b)^m)}

Cal

\I[x]{x^n*(a*x^k+b)^m)}, x^(n-k+1)*(a*x^k+b)^(m+1)),
\I[x]{x^(n-k)*(a*x^k+b)^m},
x^(n+1)*(a*x^k+b)^(m+1), \I[x]{x^(n+k)*(a*x^k+b)^m},
x^(n+1)*(a*x^k+b)^m, \I[x]{x^n*(a*x^k+b)^(m-1)} ❸

With $ax^2 + bx + c$

1. Set $u = x + b/(2a)$.
2. Expand in fractions.
3. Use reduction relations.

In the following we have set $R = ax^2 + bx + c$. If $b^2 = 4ac$, then $R = a(x + b/2a)^2$ and it may be preferable to reduce the integral to a simpler one, before the calculation. The same is true for other special cases, where one of a, b, c is zero.

\I[x]{1/R}, tan^(-1)((2*a*x+b)/(4*a*c-b^2)^(1/2)), ln|(2*a*x+b-(b^2-4*a*c)^(1/2))/(2*a*x+b+(b^2-4*a*c)^(1/2))|, ln|(x-x_1)/(x-x_2)| 1; 1 2; 2 Ext

\I[x]{x/R}, ln|R|, \I[x]{1/R} 2

\I[x]{x^2/R}, ln|R|, \I[x]{1/R} 2 Cal

\I[x]{1/(x*R)}, ln(x^2/|R|), \I[x]{1/R} 2

\I[x]{1/(x^2*R)}, ln(|R|/x^2), \I[x]{1/R} 2 3

\I[x]{1/R^2}, (2*a*x+b)/R, \I[x]{1/R} 3

\I[x]{x/R^2}, (b*x+2*c)/R, \I[x]{1/R} 2

\I[x]{x^2/R^2}, ((b^2-2*a*c)*x+b*c)/R, \I[x]{1/R} 2 3

\I[x]{1/(x*R^2)}, (2*a*c-b^2-a*b*x)/R, ln(x^2/|R|), \I[x]{1/R} 2

\I[x]{1/(x^2*R^2)}, (b*x+c)/(x*R), (2*a*x+b)/R, ln(x^2/|R|), \I[x]{1/R} ❷❸
\I[x]{x^n/R}, \I[x]{x^(n-2)/R}, \I[x]{x^(n-1)/R} ❷
\I[x]{1/(x^n*R)}, \I[x]{1/(x^(n-2)*R)}, \I[x]{1/(x^(n-1)*R)} ❷
\I[x]{1/R^m}, (2*a*x+b)/R^(m-1), \I[x]{1/R^(m-1)} Cal
\I[x]{x^n/R^m}, x^(n-1)/R^(m-1), \I[x]{x^(n-2)/R^m}, \I[x]{x^(n-1)/R^m} Cal
\I[x]{1/(x^n*R^m)},1/(x^(n-1)*R^(m-1)), \I[x]{1/(x^(n-2)*R^m)}, \I[x]{1/(x^(n-1)*R^m)} Cal

With roots of $ax^2 + bx + c$

We have set $R = ax^2 + bx + c$. If $b^2 = 4ac$ or one of a, b, c is equal to zero, it may be preferable to reduce the integral to a simpler one, before the calculation. We always assume that $R > 0$ under a square root.

❶ Set $u = x + b/(2a)$.
❷ Expand in fractions.
❸ Integrate by parts.
❸ See Ext.

\I[x]{1/R^(1/2)}, ln|2*a^(1/2)*R^(1/2)+2*a*x+b|, sinh(-1)((2*a*x+b)/(4*a*c-b^2)^(1/2)), sin(-1)((2*a*x+b)/(b^2-4*a*c)^(1/2)) ❶
\I[x]{x/R^(1/2)}, R^(1/2), \I[x]{1/R^(1/2)} ❷❹
\I[x]{x^2/R^(1/2)}, (2*a*x-3*b)*R^(1/2), \I[x]{1/R^(1/2)} ❷❹

\I[x]{1/(x*R^(1/2))}, ln|(2*c^(1/2)*R^(1/2)+b*x+2*c)/x|,
sinh^(-1)((b*x+2*c)/(|x|*(4*a*c-b^2)^(1/2))),
sin^(-1)((b*x+2*c)/(|x|*(b^2-4*a*c)^(1/2))) ❶

\I[x]{1/(x^2*R^(1/2))},
R^(1/2)/x, \I[x]{1/(x*R^(1/2))} ❸

\I[x]{R^(1/2)},
(2*a*x+b)*R^(1/2), \I[x]{1/R^(1/2)} ❸❹

\I[x]{x*R^(1/2)}, R^(3/2),
(2*a*x+b)*R^(1/2), \I[x]{1/R^(1/2)} ❷❹

\I[x]{x^2*R^(1/2)}, R^(1/2),
R^(3/2), R^(1/2), \I[x]{1/R^(1/2)} ❷❹

\I[x]{R^(1/2)/x}, R^(1/2),
\I[x]{1/R^(1/2)}, \I[x]{1/(x*R^(1/2))} ❷❹

\I[x]{R^(1/2)/x^2}, R^(1/2)/x,
\I[x]{1/R^(1/2)}, \I[x]{1/(x*R^(1/2))} ❸

\I[x]{1/R^(3/2)},
(4*a*x+2*b)/R^(1/2) ❸❹

\I[x]{x/R^(3/2)},
(2*b*x+4*c)/R^(1/2) ❷

\I[x]{x^2/R^(3/2)}, 1/R^(1/2),
\I[x]{1/R^(1/2)} ❷❹

\I[x]{1/(x*R^(3/2))}, 1/R^(1/2),
\I[x]{1/(x*R^(1/2))} ❷❹

\I[x]{1/(x^2*R^(3/2))}, 1/(x*R^(1/2)),
\I[x]{1/R^(3/2)}, \I[x]{1/(x*R^(1/2))} ❸

\I[x]{R^((2*n+1)/2)}, (2*a*x+b)*R^((2*n+1)/2),
\I[x]{R^((2*n-1)/2)} ❸

\I[x]{x*R^((2*n+1)/2)}, R^((2*n+3)/2),
\I[x]{R^((2*n+1)/2)} ❷

\I[x]{R^((2*n+1)/2)/x}, R^((2*n+1)/2),
\I[x]{R^(2*n-1)/2)}, \I[x]{R^((2*n-1)/2)/x} ❷

\I[x]{1/R^((2*n+1)/2)}, 1/R^((2*n-1)/2),
\I[x]{1/R^(2*n-1)/2)} ❸

\I[x]{x/R^((2*n+1)/2)}, R^((2*n-1)/2),
\I[x]{1/R^(2*n+1)/2)} ❷

\I[x]{1/(x*R^((2*n+1)/2)}, 1/R^((2*n-1)/2),
\I[x]{1/R^(2*n-1)/2)}, \I[x]{1/R^(2*n+1)/2)} ❷

\I[x]{R^(κ+1)}, (2*a*x+b)*R^(κ+1), \I[x]{R^κ} ❸ (κ real)

Ext

With $x^3 + a^3$ (see also $ax^k + b$)

❶ Set $u = x^3 + a^3$.
❷ Expand in fractions.
❸ Use reduction relation.

\I[x]{1/(x^3+a^3)}, ln((x+a)^2/(x^2-a*x+a^2)),
tan^(-1)((2*x-a)/a*3^(1/2)) ❷

\I[x]{x/(x^3+a^3)}, ln((x+a)^2/(x^2-a*x+a^2)),
tan^(-1)((2*x-a)/a*3^(1/2)) ❷

\I[x]{x^2/(x^3+a^3)},
ln|x^3+a^3| ❶

\I[x]{x^3/(x^3+a^3)}, ln((x+a)^2)/(x^2-a*x+a^2)),
tan^(-1)((2*x-a)/(a*3^(1/2))) ❷

Ext

\I[x]{1/(x*(x^3+a^3))},
ln|x^3/(x^3+a^3)| ❶

\I[x]{1/(x^2*(x^3+a^3))}, ln((x^2-a*x+a^2)/(x+a)^2),
tan^(-1)((2*x-a)/(a*3^(1/2))) ❷

\I[x]{1/(x^3*(x^3+a^3))}, ln((x^2-a*x+a^2)/(x+a)^2),
tan^(-1)((2*x-a)/(a*3^(1/2))) ❷

\I[x]{1/(x^3+a^3)^2}, x/(x^3+a^3), ln((x^2-a*x+a^2)/(x+a)^2),
tan^(-1)((2*x-a)/(a*3^(1/2))) ❸

\I[x]{x/(x^3+a^3)^2}, x^2/(x^3+a^3), ln((x^2-a*x+a^2)/(x+a)^2),
tan^(-1)((2*x-a)/(a*3^(1/2))) ❸

\I[x]{x^2/(x^3+a^3)^2},
1/(x^3+a^3) ❶

\I[x]{x^3/(x^3+a^3)^2}, x/(x^3+a^3), ln((x^2-a*x+a^2)/(x+a)^2),
tan^(-1)((2*x-a)/(a*3^(1/2))) ❷

\I[x]{1/(x*(x^3+a^3)^2)}, 1/(x^3+a^3),
ln|x^3/(x^3+a^3)| ❶

\I[x]{1/(x^2*(x^3+a^3)^2)}, x^2/(x^3+a^3),
ln((x^2-a*x+a^2)/(x+a)^2), tan^(-1)((2*x-a)/(a*3^(1/2))) ❷

\I[x]{1/(x^3*(x^3+a^3)^2)}, x/(x^3+a^3), ln((x^2-a*x+a^2)/(x+a)^2),
tan^(-1)((2*x-a)/(a*3^(1/2))) ❷

\I[x]{x^n/(x^3+a^3)^2},
\I[x]{x^(n-3)/(x^3+a^3)} ❷

\I[x]{1/(x^n*(x^3+a^3)^2)},
\I[x]{1/(x^(n-3)*(x^3+a^3))} ❷

\I[x]{x^n/(x^3+a^3)^m}, x^(n-2)/(x^3+a^3)^(m-1), ❸
\I[x]{x^(n-3)/(x^3+a^3)^m}, x^(n+1)/(x^3+a^3)^(m-1),
\I[x]{x^n/(x^3+a^3)^(m-1)} ❸

Ext

With $x^4 \pm a^4$ (see also $ax^k + b$)

❶ Set $u = x^2$ or $u = x^4$.
❷ Expand in fractions.

Ext

\I[x]{1/(x^4+a^4)}, tan^(-1)(a*x*2^(1/2)/(x^2-a^2)),
ln((x^2+a*x*2^(1/2)+a^2)/(x^2-a*x*2^(1/2)+a^2)) ❷

\I[x]{1/(x^4-a^4)}, ln|(x-a)/(x+a)|,
tan^(-1)(x/a) ❷

\I[x]{x/(x^4+a^4)},
tan^(-1)(x^2/a^2) ❶

\I[x]{x/(x^4-a^4)},
ln|(x^2-a^2)/(x^2+a^2)| ❶ ❷

\I[x]{x^2/(x^4+a^4)}, ln((x^2-a*x*2^(1/2)+a^2)/
(x^2+a*x*2^(1/2)+a^2)), tan^(-1)(a*x*2^(1/2)/(x^2-a^2)) ❷

\I[x]{x^2/(x^4-a^4)}, ln|(x-a)/(x+a)|,
tan^(-1)(x/a) ❷

\I[x]{x^3/(x^4\+a^4)}, ln|x^4\+a^4| ❶

\I[x]{1/(x*(x^4\+a^4))},
ln|x^4/(x^4\+a^4)| ❶

\I[x]{1/(x^2*(x^4+a^4))}, ln((x^2-a*x*2^(1/2)+a^2)/
(x^2+a*x*2^(1/2)+a^2)), tan^(-1)(a*x*2^(1/2)/(x^2-a^2)) ❷

\I[x]{1/(x^2*(x^4-a^4))},
ln|(x-a)/(x+a)|, tan^(-1)(x/a) ❷

\I[x]{1/(x^3*(x^4\+a^4))},
tan^(-1)(x^2/a^2) ❶

\I[x]{1/(x^3*(x^4-a^4))},
ln|(x^2-a^2)/(x^2+a^2)| ❶ ❷

\I[x]{x^n/(x^4\+a^4)^m)}, x^(n+1)/(x^4\+a^4)^(m-1),
\I[x]{x^n/(x^4\+a^4)^(m-1)} Ext

With $x^n \pm a^n$ (see also $ax^k + b$)

❶ Set $u = x^n \pm a^n$.
❷ Expand in fractions.

\I[x]{1/(x*(x^n\+a^n))}, ln|x^n/(x^n\+a^n)| ❶

\I[x]{x^(n-1)/(x^n\+a^n)}, ln|x^n\+a^n| ❶

\I[x]{x^m/(x^(2*n)-a^(2*n))}, \I[x]{x^m/(x^n-a^n)}, \I[x]{x^m/(x^n+a^n)} ❷

\I[x]{x^m/(x^n\+a^n)^s}, \I[x]{x^(m-n)/(x^n\+a^n)^(s-1)}, \I[x]{x^(m-n)/(x^n\+a^n)^s} ❷

\I[x]{1/(x^m*(x^n\+a^n)^s)}, \I[x]{1/(x^m*(x^n\+a^n)^(s-1))}, \I[x]{1/(x^(m-n)*(x^n\+a^n)^s)} ❷

\I[x]{1/(x*(x^n+a^n)^(1/2))}, ln|((x^n+a^n)^(1/2)-a^(n/2))/((x^n+a^n)^(1/2)+a^(n/2))|

\I[x]{1/(x*(x^n-a^n)^(1/2))}, cos^(-1)((a/x)^(n/2))

With sin ax

❶ Integrate by parts.
❷ Expand in powers of x.
❸ Set $u = \tan(ax/2)$.
❹ Use multiples of ax.
❺ Write integrand as sum.

\I[x]{sin(a*x+b)}, cos(a*x+b)

\I[x]{x*sin(a*x)}, x*cos(a*x) ❶

\I[x]{x^2*sin(a*x)}, x*sin(a*x), cos(a*x) ❶

\I[x]{x^3*sin(a*x)}, cos(a*x) ❶

\I[x]{sin(a*x)/x}, \S{Si(a*x)} ❷

\I[x]{sin(a*x)/x^2},
\I[x]{cos(a*x)/x} ❶

\I[x]{sin(a*x)/x^3},
sin(a*x)/x^2, cos(a*x)/x, \I[x]{sin(a*x)/x} ❶

\I[x]{1/sin(a*x)}, ln|tan(a*x/2)|, cos(a*x) ❸

\I[x]{x/sin(a*x)}, B_(2*k) ❷

($0 < |ax| < \pi$, B_k = Bernoulli numbers)

\I[x]{x^2/sin(a*x)}, B_(2*k) ❷

\I[x]{1/(x*sin(a*x))}, B_(2*k) ❷

Ext

\I[x]{sin(a*x)^2}, sin(2*a*x),
sin(a*x)*cos(a*x) ❶ ❹

\I[x]{x*sin(a*x)^2}, x*sin(2*a*x),
cos(2*a*x) ❶ ❹

\I[x]{x^2*sin(a*x)^2}, sin(2*a*x), x*cos(2*a*x) ❶ ❹

\I[x]{x^3*sin(a*x)^2}, sin(2*a*x), cos(2*a*x) ❶ ❹

\I[x]{sin(a*x)^2/x}, ln|a*x|, \I[x]{cos(2*a*x)/x} ❹

\I[x]{sin(a*x)^3}, cos(a*x)^3, cos(3*a*x) ❹

\I[x]{x*sin(a*x)^3}, x*cos(3*a*x), sin(3*a*x),
x*cos(a*x) ❶

\I[x]{sin(a*x)^4}, sin(2*a*x), sin(4*a*x) ❹

\I[x]{1/sin(a*x)^2}, cot(a*x) ❸

\I[x]{x/sin(a*x)^2}, x*cot(a*x), ln|sin(a*x)| ❶

\I[x]{1/sin(a*x)^3}, cos(a*x)/sin(a*x)^2, ln|tan(a*x/2)| ❸

\I[x]{x/sin(a*x)^3}, x*cos(a*x)/sin(a*x)^2, \I[x]{x/sin(a*x)} ❶

\I[x]{sin(a*x)*sin(b*x)}, sin((a-b)*x), sin((a+b)*x) ❺

\I[x]{sin(a*x+b)*sin(c*x+d)}, sin((a-c)*x+b-d), sin((a+c)*x+b+d) ❺

\I[x]{sin(a*x)*sin(b*x)*sin(c*x)}, cos((a+b+c)*x), cos((b+c-a)*x), cos(a-b+c)*x), cos(a+b-c)*x) ❺

\I[x]{1/(1\+sin(a*x))}, tan(a*x/2\-π/4) ❸

\I[x]{x/(1\+sin(a*x))}, tan(π/4\-a*x/2), ln|sin(π/4\+a*x/2)| ❶

\I[x]{1/(1\+sin(a*x)^2)}, tan(π/4\-a*x/2)^3 ❸

\I[x]{1/(b+c*sin(a*x))}, tan(a*x/2), tan^(-1)((b*tan(a*x/2)+c)/(b^2-c^2)^(1/2)), ln|(b*tan(a*x/2)+c-(c^2-b^2)^(1/2))/(b*tan(a*x/2)+c+(c^2-b^2)^(1/2))| ❸ ❸

\I[x]{sin(a*x)/(b+c*sin(a*x))}, \I[x]{1/(b+c*sin(a*x))} ❺

\I[x]{1/(b+c*sin(a*x))^2}, cos(a*x)/(b+c*sin(a*x)), \I[x]{1/(b+c*sin(a*x))} **Ext**

\I[x]{1/(b^2+c^2*sin(a*x)^2)}, tan^(-1)((b^2+c^2)^(1/2)*tan(a*x)/b) ❸❹

\I[x]{1/(b^2-c^2*sin(a*x)^2)}, tan^(-1)((b^2-c^2)^(1/2)*tan(a*x)/b), ln|((c^2-b^2)^(1/2)*tan(a*x)+b)/((c^2-b^2)^(1/2)*tan(a*x)-b)| ❸❹

\I[x]{x*sin(x^2)}, cos(x^2)

\I[x]{x^3*sin(x^2)}, x^2*cos(x^2)

\I[x]{sin(a*x)^n}, sin(a*x)^(n-1)*cos(a*x), \I[x]{sin(a*x)^(n-2)} ❶

\I[x]{sinx^(2*n)}, (2*n;k)

\I[x]{sinx^(2*n+1)}, (2*n+1;k)

\I[x]{sin(2*n*x)/sinx}

\I[x]{sin((2*n+1)*x)/sinx}

Cal

\I[x]{x^n*sin(a*x)}, x^n*cos(a*x), x^(n-1)*sin(a*x), \I[x]{x^(n-2)*sin(a*x)} ❶

\I[x]{sin(a*x)/x^n}, sin(a*x)/x^(n-1),
\I[x]{cos(a*x)/x^(n-1)}, ❶

\I[x]{1/sin(a*x)^n}, cos(a*x)/sin(a*x)^(n-1),
\I[x]{1/sin(a*x)^(n-2)} ❶

\I[x]{x/sin(a*x)^n}, x*cos(a*x)/sin(a*x)^(n-1),
1/sin(a*x)^(n-2), \I[x]{x/sin(a*x)^(n-2)} ❶

With roots (K = (1-k^2*sinx^2)^(1/2))

❶ Set $u = \cos ax$.

\I[x]{sinx/(1+k^2*sinx^2)^(1/2)},
sin^(-1)(k*cosx/(1+k^2)^(1/2)) ❶

\I[x]{1/(sinx*(1+k^2*sinx^2)^(1/2))},
ln(((1+k^2*sinx^2)^(1/2)-cosx)/((1+k^2*sinx^2)^(1/2)+cosx)) ❶

\I[x]{sinx*(1+k^2*sinx^2)^(1/2)},
cosx*(1+k^2*sinx^2)^(1/2), sin^(-1)(k*cosx/(1+k^2)^(1/2)) ❶

\I[x]{K*sinx},
K*cosx, ln(K+k*cosx) ❶

\I[x]{sinx/K},
ln(K+k*cosx) ❶

\I[x]{K/sinx},
ln((K-cosx)/(K+cosx)), ln(K+k*cosx)) ❶

\I[x]{1/(K*sinx)},
ln((K-cosx)/(K+cosx)) ❶

\I[x]{sinx/K^3},
cosx/K ❶

\I[x]{sinx/(a^2*sinx^2-1)^(1/2)},
sin^(-1)(a*cosx/(a^2-1)^(1/2)) ❶

\I[x]{1/(sinx*(a^2*sinx^2-1)^(1/2)},
tan^(-1)(cosx/(a^2*sinx^2-1)^(1/2)) ❶

With cos *ax*

❶ Integrate by parts.
❷ Expand in powers of x.
❸ Set $u = \tan(ax/2)$.
❹ Use multiples of ax.
❺ Write integrand as sum.

\I[x]{cos(a*x+b)}, sin(a*x+b)
\I[x]{x*cos(a*x)}, cos(a*x), x*(sin(a*x) ❶
\I[x]{x^2*cos(a*x)}, x*cos(a*x), sin(a*x) ❶
\I[x]{x^3*cos(a*x)}, cos(a*x), sin(a*x) ❶
\I[x]{cos(a*x)/x}, ln|a*x| ❷
\I[x]{cos(a*x)/x^2}, cos(a*x)/x, \I[x]{sin(a*x)/x} ❶
\I[x]{cos(a*x)/x^3}, cos(a*x)/x^2, sin(a*x)/x, \I[x]{cos(a*x)/x} ❶
\I[x]{1/cos(a*x)}, ln|tan(a*x/2+π/4)|, ln|(1+sin(a*x))/(1-sin(a*x))| ❸
\I[x]{x/cos(a*x)}, E_(2*k) ❷
$(\|x\| < \pi/2,\ E_k = \text{Euler numbers})$
\I[x]{x^2/cos(a*x)}, E_(2*k) ❷
\I[x]{1/(x*cos(a*x))}, ln|a*x|, E_(2*k) ❷
\I[x]{cos(a*x)^2}, sin(2*a*x), sin(a*x)*cos(a*x) ❶ ❹
\I[x]{x*cos(a*x)^2}, x*sin(2*a*x), cos(2*a*x) ❶ ❹

Ext

\I[x]{x^2*cos(a*x)^2}, sin(2*a*x), x*cos(2*a*x) ❶ ❹

\I[x]{x^3*cos(a*x)^2}, sin(2*a*x), cos(2*a*x) ❶ ❹

\I[x]{cos(a*x)^2/x}, ln|a*x|,
\I[x]{cos(2*a*x)/x} ❹

\I[x]{cos(a*x)^3}, sin(a*x)^3, sin(3*a*x) ❹

\I[x]{x*cos(a*x)^3}, x*sin(3*a*x), cos(3*a*x),
x*sin(a*x), cos(a*x) ❶

\I[x]{cos(a*x)^4}, sin(2*a*x),
sin(4*a*x) ❹

\I[x]{1/cos(a*x)^2},
tan(a*x) ❸

\I[x]{x/cos(a*x)^2}, x*tan(a*x),
ln|cos(a*x)| ❶

\I[x]{1/cos(a*x)^3}, sin(a*x)/cos(a*x)^2,
ln|tan(π/4+a*x/2)| ❸

\I[x]{x/cos(a*x)^3}, x*sin(a*x)/cos(a*x)^2,
\I[x]{x/cos(a*x)} ❶

\I[x]{cos(a*x)*cos(b*x)}, sin((a-b)*x),
sin((a+b)*x) ❺

\I[x]{cos(a*x+b)*cos(c*x+d)}, sin((a-c)*x+b-d), sin((a+c)*x+b+d) ❺

\I[x]{cos(a*x)*cos(b*x)*cos(c*x)},
sin((a+b+c)*x), sin((b+c-a)*x), sin((a+c-b)x), sin((a+b-c)*x) ❺

\I[x]{1/(1-cos(a*x))},
cot(a*x/2) ❸

\I[x]{x/(1-cos(a*x))}, x*cot(a*x/2),
ln|sin(a*x/2)| ❶

\I[x]{1/(1+cos(a*x))},
tan(a*x/2) ❸

\I[x]{x/(1+cos(a*x))}, x*tan(a*x/2),
ln|cos(a*x/2)| ❶

\I[x]{1/(1-cos(a*x))^2}, cot(a*x/2)^3 ❸

\I[x]{1/(1+cos(a*x))^2}, tan(a*x/2)^3 ❸

❸
\I[x]{1/(b+c*cos(a*x))},
tan^(-1)((b-c)*tan(a*x/2)/(b^2-c^2)^(1/2)),
ln|(tan(a*x/2)+(c^2-b^2)^(1/2)/(c-b))/(tan(a*x/2)-(c^2-b^2)^(1/2)/(c-b))|
❸

\I[x]{cos(a*x)/(b+c*cos(a*x))},
\I[x]{1/(b+c*cos(a*x))} ❺

\I[x]{1/(b+c*cos(a*x))^2}, sin(a*x)/(b+c*cos(a*x)),
\I[x]{1/(b+c*cos(a*x))} ❸ Ext

\I[x]{1/(b^2+c^2*cos(a*x)^2)},
tan^(-1)(b*tan(a*x)/(b^2+c^2)^(1/2)) ❸❹

\I[x]{1/(b^2-c^2*cos(a*x)^2)}, tan^(-1)(b*tan(a*x)/(b^2-c^2)^(1/2)), ❸❹
ln|(b*tan(a*x)-(c^2-b^2)^(1/2))/(b*tan(a*x)+(c^2-b^2)^(1/2))| ❸❹

\I[x]{1/(1-2*b*cos(a*x)+b^2)},
tan^(-1)((1+b)*tan(a*x/2)/(1-b)) ❸

\I[x]{cos(a*x)/(1-2*b*cos(a*x)+b^2)},
tan^(-1)((1+b)*tan(a*x/2)/(1-b)) ❸

\I[x]{x*cos(x^2)}, sin(x^2)

\I[x]{x^3*cos(x^2)}, x^*sin(x^2), cos(x^2)

\I[x]{cos(a*x)^n}, sin(a*x)*cos(a*x)^(n-1),
\I[x]{cos(a*x)^(n-2)} ❶

\I[x]{cosx^(2*n)}, (2*n;k), sin((2*n-2*k)*x)

\I[x]{cosx^(2*n+1)},
(2*n+1;k), sin((2*n-2*k+1)*x)

\I[x]{cos(2*n*x)/cosx)},
sin((2*k-1)*x), ln|tan(x/2+π/4)|

\I[x]{cos((2*n+1)*x)/cosx)}, sin(2*k*x)

Cal

\I[x]{x^n*cos(a*x)}, x^n*sin(a*x), x^(n-1)*cos(a*x),
\I[x]{x^(n-2)*cos(a*x)}, sin(a*x+k*π/2) ❶

\I[x]{cos(a*x)/x^n}, cos(a*x)/x^(n-1),
\I[x]{sin(a*x)/x^(n-1)} ❶

\I[x]{1/cos(a*x)^n}, sin(a*x)/cos(a*x)^(n-1),
\I[x]{1/cos(a*x)^(n-2)} ❶

\I[x]{x/cos(a*x)^n}, x*sin(a*x)/cos(a*x)^(n-1),
1/cos(a*x)^(n-2), \I[x]{x/cos(a*x)^(n-2)} ❶

With sin *ax* and cos *ax*

❶ Set $u = \sin ax$ or $u = \cos ax$.
❷ Set $u = \tan(ax/2)$.
❸ Integrate by parts.
❹ Write integrand as sum.
❺ Use multiples of ax.

\I[x]{sin(a*x)*cos(a*x)}, sin(a*x)^2, 1-cos(2*a*x) ❶ ❹

\I[x]{sin(a*x+b)*cos(a*x)}, x*sinb, cos(2*a*x+b) ❹

\I[x]{sin(a*x)*cos(b*x)}, cos((a-b)x), cos((a+b)*x) ❹

\I[x]{sin(a*x+b)*cos(c*x+d)}, cos((a+c)x+b+d), cos((a-c)*x+b-d) ❹

\I[x]{sin(a*x)*cos(b*x)*cos(c*x)}, cos((b+c-a)*x), cos((a+c-b)*x), cos((a+b-c)*x), cos((a+b+c)*x) ❹

\I[x]{cos(a*x)*sin(b*x)*sin(c*x)}, sin((a+b-c)*x), sin((a+c-b)*x), sin((b+c-a)*x), sin((a+b+c)*x) ❹

\I[x]{sin(a*x)^n*cos(a*x)}, sin(a*x)^(n+1) ❶

\I[x]{sin(a*x)*cos(a*x)^n}, cos(a*x)^(n+1) ❶

\I[x]{sin(a*x)^2*cos(a*x)^2}, sin(4*a*x) ❹ ❺

\I[x]{sin(a*x)/cos(a*x)}, ln|cos(a*x)| ❶

\I[x]{cos(a*x)/sin(a*x)}, ln|sin(a*x)| ❶

\I[x]{sin(a*x)^2/cos(a*x)}, sin(a*x), ln|tan(a*x/2+π/4)| ❷ ❹

\I[x]{sin(a*x)/cos(a*x)^2}, 1/cos(a*x) ❶

\I[x]{cos(a*x)^2/sin(a*x)},
cos(a*x), ln|tan(a*x/2)| ❷❹

\I[x]{cos(a*x)/sin(a*x)^2},
1/sin(a*x) ❶

\I[x]{1/(sin(a*x)*cos(a*x))},
ln|tan(a*x)| ❶❺

\I[x]{1/(sin(a*x)^2*cos(a*x))},
ln|tan(a*x/2+π/4)|, 1/sin(a*x) ❶❹

\I[x]{1/(sin(a*x)*cos(a*x)^2)},
ln|tan(a*x/2)|, 1/cos(a*x) ❶❹

\I[x]{1/(sin(a*x)^2*cos(a*x)^2)}, cot(2*a*x),
tan(a*x), cot(a*x) ❹❺

\I[x]{sin(a*x)/(b+c*cos(a*x))},
ln|b+c*cos(a*x)| ❶

\I[x]{cos(a*x)/(b+c*sin(a*x))},
ln|b+c*sin(a*x)| ❶

\I[x]{sin(a*x)/(b+c*cos(a*x))^n},
1/(b+c*cos(a*x))^(n-1) ❶

\I[x]{cos(a*x)/(b+c*sin(a*x))^n},
1/(b+c*sin(a*x))^(n-1) ❶

\I[x]{1/(sin(a*x)\+cos(a*x))},
ln|tan(a*x/2\+π/8)| ❷

\I[x]{sin(a*x)/(sin(a*x)\+cos(a*x))},
ln|sin(a*x)\+cos(a*x)| ❷❹

\I[x]{cos(a*x)/(sin(a*x)\+cos(a*x))},
ln|sin(a*x)\+cos(a*x)| ❷❹

\I[x]{1/(sin(a*x)\+cos(a*x))^2},
tan(a*x\-π/4) ❺

Cal

\I[x]{1/(b*sin(a*x)+c*cos(a*x))},
ln|tan((a*x+tan^(-1)(c/b))/2)| ❷

\I[x]{sin(a*x)/(b*sin(a*x)+c*cos(a*x))},
ln|b*sin(a*x)+c*cos(a*x)| ❷

\I[x]{cos(a*x)/(b*sin(a*x)+c*cos(a*x))},
ln|b*sin(a*x)+c*cos(a*x)| ❷

\I[x]{1/(b*sin(a*x)+c*cos(a*x))^2},
(c*sin(a*x)-b*cos(a*x))/(b*sin(a*x)+c*cos(a*x)) ❷

Cal

\I[x]{1/(b^2*sin(a*x)^2+c^2*cos(a*x)^2)},
tan^(-1)(b*tan(a*x)/c) ❺

\I[x]{1/(b^2*sin(a*x)^2-c^2*cos(a*x)^2)},
ln|(b*tan(a*x)-c)/(b*tan(a*x)+c)| ❺

\I[x]{1/(cos(a*x)*(1\+sin(a*x)))},
1/(1\+sin(a*x)), ln|tan(a*x/2+π/4)| ❶ ❷

\I[x]{1/(sin(a*x)*(1\+cos(a*x)))},
1/(1\+cos(a*x)), ln|tan(a*x/2)| ❶ ❷

\I[x]{1/(1+sin(a*x)+cos(a*x))},
ln|1+tan(a*x/2)| ❷

\I[x]{1/(1+sin(a*x)-cos(a*x))},
ln|tan(a*x/2)/(1+tan(a*x/2))| ❷

\I[x]{1/(b*sin(a*x)+c*cos(a*x)+d)},
tan(a*x/2) ❷

\I[x]{cos(2*n*x)/sinx},
cos((2*k-1)*x), ln|tan(x/2)|

\I[x]{cos((2*n+1)*x)/sinx},
cos(2*k*x), ln|sinx|

\I[x]{sin(2*n*x)/cosx}, cos((2*k-1)*x)

\I[x]{sin((2*n+1)*x)/cosx}, cos(2*k*x), ln|cosx|

Cal

\I[x]{sinx^(2*n)/cosx}, sinx^(2*k-1),
ln|tan(x/2+π/4)|

\I[x]{sinx^(2*n+1)/cosx},
sinx^(2*k), ln|cosx|

\I[x]{cosx^(2*n)/sinx},
cosx^(2*k-1), ln|tan(x/2)|

\I[x]{cosx^(2*n+1)/sinx},
cosx^(2*k), ln|sinx|

Cal

\I[x]{sin(a*x)^m*cos(a*x)^n}, sin(a*x)^(m-1)*cos(a*x)^(n+1),
\I[x]{sin(a*x)^(m-2)*cos(a*x)^n}, sin(a*x)^(m+1)*cos(a*x)^(n-1),
\I[x]{sin(a*x)^m*cos(a*x)^(n-2)}

❸

\I[x]{sin(a*x)^m/cos(a*x)^n},
sin(a*x)^(m-1)/cos(a*x)^(n-1), \I[x]{sin(a*x)^(m-2)/cos(a*x)^n},
sin(a*x)^(m+1)/cos(a*x)^(n-1), \I[x]{sin(a*x)^m/cos(a*x)^(n-2)},
sin(a*x)^(m-1)/cos(a*x)^(n-1), \I[x]{sin(a*x)^(m-2)/cos(a*x)^(n-2)}

❸

\I[x]{cos(a*x)^m/sin(a*x)^n},
cos(a*x)^(m-1)/sin(a*x)^(n-1), \I[x]{cos(a*x)^(m-2)/sin(a*x)^n},
cos(a*x)^(m+1)/sin(a*x)^(n-1), \I[x]{cos(a*x)^m/sin(a*x)^(n-2)},
cos(a*x)^(m-1)/sin(a*x)^(n-1), \I[x]{cos(a*x)^(m-2)/sin(a*x)^(n-2)} ❸

\I[x]{1/(sinx^(2*n)*cosx)},
sinx^(2*n-2*k+1), ln|tan(x/2+π/4)|

\I[x]{1/(sinx^(2*n+1)*cosx)},
sinx^(2*n-2*k+2), ln|tanx|

\I[x]{1/(sinx*cosx^(2*n))},
cosx^(2*n-2*k+1), ln|tan(x/2)|

\I[x]{1/(sinx*cosx^(2*n+1))},
cosx^(2*n-2*k+2), ln|tanx|

Cal

\I[x]{1/(sin(a*x)^m*cos(a*x)^n)},
1/(sin(a*x)^(m-1)*cos(a*x)^(n-1)), \I[x]{1/(sin(a*x)^m*cos(a*x)^(n-2)),
1/(sin(a*x)^(m-1)*cos(a*x)^(n-1)), \I[x]{1/(sin(a*x)^(m-2)*cos(a*x)^n) ❸

With roots ($K =$ (1-k^2*sinx^2)^(1/2))

❶ Set $u = \sin ax$.

\I[x]{K*cosx}, K*sinx, sin^(-1)(k*sinx) ❶

\I[x]{K/cosx}, ln((K+(1-k^2)^(1/2)*sinx)/(K-(1-k^2)^(1/2)*sinx)),
sin^(-1)(k*sinx) ❶

\I[x]{K*sinx*cosx}, K^3 ❶

\I[x]{K/(sinx*cosx)}, ln((1-K)/(1+K)),
ln((K+(1-k^2)^(1/2))/(K-(1-k^2)^(1/2))) ❶

\I[x]{K*sinx/cosx)},
ln((K+(1-k^2)^(1/2))/(K-(1-k^2)^(1/2))) ❶

\I[x]{K*cosx/sinx},
ln((1-K)/(1+K)) ❶

\I[x]{cosx/K}, tan^(-1)(k*sinx/K),
sin^(-1)(k*sinx) ❶

\I[x]{1/(K*cosx)},
ln((K-(1-k^2)^(1/2)*sinx)/(K+(1-k^2)^(1/2)*sinx)) ❶

\I[x]{sinx*cosx/K}, K ❶

\I[x]{1/(K*sinx*cosx)}, ln((1-K)/(1+K))
ln((K+(1-k^2)^(1/2))/(K-(1-k^2)^(1/2))) ❶

\I[x]{sinx/(K*cosx)},
ln((K+(1-k^2)^(1/2))/(K-(1-k^2)^(1/2))) ❶

\I[x]{cosx/(K*sinx)},
ln((1-K)/(1+K)) ❶

\I[x]{cosx/K^3},
sinx/K ❶

\I[x]{sinx*cosx/K^3},
1/K ❶

\I[x]{cosx*(1+k^2*sinx^2)^(1/2)},
sinx*(1+k^2*sinx^2)^(1/2), ln(k*sinx+(1+k^2*sinx^2)^(1/2)) ❶

\I[x]{cosx/(1+k^2*sinx^2)^(1/2)},
ln(k*sinx+(1+k^2*sinx^2)^(1/2)) ❶

\I[x]{1/(cosx*(1+k^2*sinx^2)^(1/2))},
ln(((1+k^2*sinx^2)^(1/2)+(1+k^2)^(1/2)*sinx)/((1+k^2*sinx^2)^(1/2)-(1+k^2)^(1/2)*sinx)) ❶

\I[x]{cosx/(a^2*sinx^2-1)^(1/2)},
ln(a*sinx+(a^2sinx^2-1)^(1/2)) ❶

\I[x]{1/(cosx*(a^2*sinx^2-1)^(1/2))},
ln(((a^2-1)^(1/2)*sinx+(a^2*sinx^2-1)^(1/2))/((a^2-1)^(1/2)*sinx-(a^2*sinx^2-1)^(1/2))) ❶

With tan *ax*

❶ Set u = sin ax or u = cos ax.
❷ Set u = tan ax.
❸ Write integrand as sum.

\I[x]{tan(a*x)}, ln|cos(a*x)| ❶

\I[x]{tan(a*x)^2}, tan(a*x)/a-x ❸

\I[x]{tan(a*x)^3}, tan(a*x)^2, ln|cos(a*x)| ❸

\I[x]{1/tan(a*x)}, ln|sin(a*x)| ❶

\I[x]{1/(tan(a*x)*cos(a*x)^2)}, ln|tan(a*x)| ❷

\I[x]{x*tan(a*x)}, B_(2*k)

\I[x]{tan(a*x)/x}, B_(2*k)

Cal

\I[x]{x*tan(a*x)^2}, x*tan(a*x), ln|cos(a*x)| ❸

\I[x]{1/(b*tan(a*x)+c)}, \I[x]{cos(a*x)/(b*sin(a*x)+c*cos(a*x))}, ln|b*sin(a*x)+c*cos(a*x)|

\I[x]{tan(a*x)/(b*tan(a*x)+c)}, \I[x]{sin(a*x)/(b*sin(a*x)+c*cos(a*x))}, ln|b*sin(a*x)+c*cos(a*x)|

Cal

\I[x]{1/(b*tan(a*x)^2+c)}, tan^(-1)((b/c)^(1/2)*tanx) ❷

\I[x]{tanx/(1+k^2*tanx^2)}, ln(cosx^2+k^2*sinx^2) ❶

\I[x]{tan(a*x)^n}, tan(a*x)^(n-1), \I[x]{tan(a*x)^(n-2)} ❸

\I[x]{tan(a*x)^n/cos(a*x)^2}, tan(a*x)^(n+1) ❷

\I[x]{tanx^(2*n)}, tanx^(2*n-2*k+1)

\I[x]{tanx^(2*n+1)}, ln|cosx|, tanx^(2*n-2*k+2)

Cal

\I[x]{tanx*(1+κ*sinx^2)^(1/2)}, ln|((1+κ*sinx^2)^(1/2)+(1+κ)^(1/2))/((1+κ*sinx^2)^(1/2)-(1+κ)^(1/2))| ❶

\I[x]{tanx/(1+κ*sinx^2)^(1/2)}, ln|((1+κ*sinx^2)^(1/2)+(1+κ)^(1/2))/((1+κ*sinx^2)^(1/2)-(1+κ)^(1/2))| ❶

\I[x]{tanx/(a^2*sinx^2-1)^(1/2)}, ln(((a^2-1)^(1/2)+(a^2*sinx^2-1)^(1/2))/((a^2-1)^(1/2)-(a^2*sinx^2-1)^(1/2))) ❶

\I[x]{tanx/(b*tanx^2+c)^(1/2)}, cos^(-1)(((b-c)/b)^(1/2)*cosx) ❶

With cot ax

\I[x]{cot(a*x)}, ln|sin(a*x)| ❶

\I[x]{cot(a*x)^2}, -cot(a*x)/a-x ❸

\I[x]{cot(a*x)^3}, cot(a*x)^2, ln|sin(a*x)| ❸

\I[x]{1/cot(a*x)}, ln|cos(a*x)| ❶

\I[x]{1/(sin(a*x)^2*cot(a*x))}, ln|cot(a*x)| ❷

\I[x]{x*cot(a*x)^2}, x*cot(a*x), ln|sin(a*x)| ❸

❶ Set $u = \sin ax$ or $u = \cos ax$.
❷ Set $u = \cot ax$.
❸ Write integrand as sum.

\I[x]{x*cot(a*x)}, x^(2*k+1), B_(2*k)

\I[x]{cot(a*x)/x}, x^(2*k-1), B_(2*k)

Cal

\I[x]{1/(b*cot(a*x)+c)}, \I[x]{sin(a*x)/(c*sin(a*x)+b*cos(a*x)),
ln|c*sin(a*x)+b*cos(a*x)|

\I[x]{cot(a*x)/(b*cot(a*x)+c)}, \I[x]{cos(a*x)/(c*sin(a*x)+b*cos(a*x)),
ln|c*sin(a*x)+b*cos(a*x)|

Cal

\I[x]{1/(b*cotx^2+c)},
tan^(-1)((b/c)^(1/2)*cotx) ❷

\I[x]{cotx/(1+k^2*cotx^2)},
ln(sinx^2+k^2*cosx^2) ❶

\I[x]{cot(a*x)^n}, cot(a*x)^(n-1),
\I[x]{cot(a*x)^(n-2)} ❸

\I[x]{cot(a*x)^n/sin(a*x)^2},
cot(a*x)^(n+1) ❷

\I[x]{cotx^(2*n)}, cotx^(2*n-2*k+1)

\I[x]{cotx^(2*n+1)}, ln|sinx|,
cotx^(2*n-2*k+2)

Cal

\I[x]{cotx*(1+κ*sinx^2)^(1/2)},
ln|(1-(1+κ*sinx^2)^(1/2))/(1+(1+κ*sinx^2)^(1/2))| ❶

\I[x]{cotx/(1+κ*sinx^2)^(1/2)},
ln|(1-(1+κ*sinx^2)^(1/2))/(1+(1+κ*sinx^2)^(1/2))| ❶

\I[x]{cotx/(a^2*sinx^2-1)^(1/2)},
tan^(-1)((a^2*sinx^2-1)^(1/2)) ❶

With inverse trigonometric functions Inf

❶ Set $u = \sin^{-1}(x/a)$.
❷ Set $u = \cos^{-1}(x/a)$.
❸ Set $u = \tan^{-1}(x/a)$.
❹ Set $u = \cot^{-1}(x/a)$.
❺ Integrate by parts.

\I[x]{sin^(-1)(x/a)},
x*sin^(-1)(x/a), (a^2-x^2)^(1/2) ❶❺

\I[x]{x*sin^(-1)(x/a)},
sin^(-1)(x/a), x*(a^2-x^2)^(1/2) ❶❺

\I[x]{x^2*sin^(-1)(x/a)},
x^3*sin^(-1)(x/a), (a^2-x^2)^(1/2) ❶❺

\I[x]{x^3*sin^(-1)(x/a)},
sin^(-1)(x/a), (a^2-x^2)^(1/2) ❶❺

\I[x]{sin^(-1)(x/a)/x}, x/a, (x/a)^3, (x/a)^5 Cal

\I[x]{sin^(-1)(x/a)/x^2}, sin^(-1)(x/a)/x,
ln|(a+(a^2-x^2)^(1/2))/x| ❶❺

\I[x]{sin^(-1)(x/a)/x^3},
sin^(-1)(x/a)/x^2, (a^2-x^2)^(1/2)/x ❶❺

\I[x]{sin^(-1)(x/a)^2},
sin^(-1)(x/a)^2, (a^2-x^2)^(1/2)*sin^(-1)(x/a) ❶❺

\I[x]{cos^(-1)(x/a)},
x*cos^(-1)(x/a), (a^2-x^2)^(1/2) ❷❺

\I[x]{x*cos^(-1)(x/a)},
cos^(-1)(x/a), x*(a^2-x^2)^(1/2)

\I[x]{x^2*cos^(-1)(x/a)},
x^3*cos^(-1)(x/a), (a^2-x^2)^(1/2) ❷❺

\I[x]{x^3*cos^(-1)(x/a)},
cos^(-1)(x/a), (a^2-x^2)^(1/2) ❷❺

\I[x]{cos^(-1)(x/a)/x},
ln|x/a|, \I[x]{sin^(-1)(x/a)/x}, (x/a)^3, (x/a)^5 Cal

\I[x]{cos^(-1)(x/a)/x^2},
cos^(-1)(x/a)/x, ln|(a+(a^2-x^2)^(1/2))/x| ❷❺

\I[x]{cos^(-1)(x/a)/x^3},
cos^(-1)(x/a)/x^2, (a^2-x^2)^(1/2)/x ❷❺

\I[x]{cos^(-1)(x/a)^2},
x*cos^(-1)(x/a)^2, (a^2-x^2)^(1/2)*cos^(-1)(x/a) ❷❺

\I[x]{tan^(-1)(x/a)},
x*tan^(-1)(x/a), ln(x^2+a^2) ❸❺

\I[x]{x*tan^(-1)(x/a)},
(x^2+a^2)*tan^(-1)(x/a) ❸❺

\I[x]{x^2*tan^(-1)(x/a)},
x^3*tan^(-1)(x/a), ln(x^2+a^2) ❸❺

\I[x]{x^3*tan^(-1)(x/a)},
(x^4-a^4)*tan^(-1)(x/a) ❸❺

\I[x]{tan^(-1)(x/a)/x},
(x/a)^3, (x/a)^5, ln|x/a|, (a/x)^3, (a/x)^5 Cal

\I[x]{tan^(-1)(x/a)/x^2},
tan^(-1)(x/a)/x, ln((x^2+a^2)/x^2) ❸❺

\I[x]{tan^(-1)(x/a)/x^3}, tan^(-1)(x/a) ❸❺

\I[x]{cot^(-1)(x/a)}, x*cot^(-1)(x/a),
ln(x^2+a^2) ❹❺

\I[x]{x*cot^(-1)(x/a)}, cot^(-1)(x/a) ❹❺

\I[x]{x^2*cot^(-1)(x/a)}, x^3*cot^(-1)(x/a),
ln(x^2+a^2) ❹❺

\I[x]{x^3*cot^(-1)(x/a)}, cot^(-1)(x/a) ❹❺

\I[x]{cot^(-1)(x/a)/x}, ln|x/a|,
\I[x]{tan^(-1)(x/a)/x} ❹❺

\I[x]{cot^(-1)(x/a)/x^2}, cot^(-1)(x/a)/x,
ln((x^2+a^2)/x^2) ❹❺

\I[x]{cot^(-1)(x/a)/x^3},
cot^(-1)(x/a)/x^2, tan^(-1)(x/a) ❹❺

\I[x]{x^n*sin^(-1)(x/a)}, x^(n+1)*sin^(-1)(x/a),
\I[x]{x^(n+1)/(a^2-x^2)^(1/2)} ❺

\I[x]{x^n*cos^(-1)(x/a)}, x^(n+1)*cos^(-1)(x/a),
\I[x]{x^(n+1)/(a^2-x^2)^(1/2)} ❺

\I[x]{x^n*tan^(-1)(x/a)}, x^(n+1)*tan^(-1)(x/a),
\I[x]{x^(n+1)/(x^2+a^2)} ❺

\I[x]{x^n*cot^(-1)(x/a)}, x^(n+1)*cot^(-1)(x/a), \
I[x]{x^(n+1)/(x^2+a^2)} ❺

Cal

\I[x]{x*sin^(-1)(x/a)/(a^2-x^2)^(1/2)},
(a^2-x^2)^(1/2)*sin^(-1)(x/a) ❶❺

\I[x]{x^2*sin^(-1)(x/a)/(a^2-x^2)^(1/2)},
x*(a^2-x^2)^(1/2)*sin^(-1)(x/a), sin^(-1)(x/a)^2 ❶❺

\I[x]{x^3*sin^(-1)(x/a)/(a^2-x^2)^(1/2)},
(a^2-x^2)^(1/2)*sin^(-1)(x/a) ❶❺

With e^{ax}

\I[x]{e^(a*x)} ❶

\I[x]{a^x}, \I{e^(x*lna)} ❶

\I[x]{x*e^(a*x)} ❷

❶ Set $u = e^{ax}$.
❷ Integrate by parts.
❸ Expand in powers of x.

\I[x]{x^2*e^(a*x)} ❷

\I[x]{x^3*e^(a*x)} ❷

\I[x]{e^(a*x)/x}, ln|a*x|, (a*x)^2, (a*x)^3 ❸

\I[x]{1/(b*e^(a*x)+c)},
ln|b*e^(a*x)+c| ❶

\I[x]{1/(b*e^(a*x)+c)^2}, ln|b*e^(a*x)+c| ❶

\I[x]{1/(b*e^(a*x)+c*e^(-a*x))},
tan^(-1)((b/c)^(1/2)*e^(a*x)), ❶
ln|(e^(a*x)-(-c/b)^(1/2))/(e^(a*x)+(-c/b)^(1/2))| ❶

\I[x]{e^(a*x)*sin(b*x)},
e^(a*x)*cos(b*x) ❷

\I[x]{e^(a*x)*sin(b*x)^2},
e^(a*x)*sin(b*x)^2, e^(a*x)*sin(b*x)*cos(b*x) ❷

\I[x]{e^(a*x)*cos(b*x)},
e^(a*x)*cos(b*x), e^(a*x)*sin(b*x) ❷

\I[x]{e^(a*x)*cos(b*x)^2},
e^(a*x)*cos(b*x)^2, e^(a*x)*sin(b*x)*cos(b*x) ❷

\I[x]{x*e^(a*x)*sin(b*x)}, e^(a*x)*cos(b*x), e^(a*x)*sin(b*x) ❷

\I[x]{x*e^(a*x)*cos(b*x)}, e^(a*x)*cos(b*x), e^(a*x)*sin(b*x) ❷

\I[x]{x^n*e^(a*x)}, \I[x]{x^(n-1)*e^(a*x)} ❷

\I[x]{e^(a*x)/x^n}, e^(a*x)/x^(n-1),
\I[x]{e^(a*x)/x^(n-1)} ❷

\I[x]{e^(a*x)*sin(b*x)^n}, e^(a*x)*sin(b*x)^(n-1),
\I[x]{e^(a*x)*sin(b*x)^(n-2)} ❷

\I[x]{e^(a*x)*cos(b*x)^n}, e^(a*x)*cos(b*x)^(n-1), \
I[x]{e^(a*x)*cos(b*x)^(n-2)} ❷

\I[x]{e^(-a*x^2-2*b*x-c}, erf(a^(1/2)*x+b/a^(1/2))

\I[x]{1/(b*e^(a*x)+c)^(1/2)},
ln|((b*e^(a*x)+c)^(1/2)-c^(1/2))/
((b*e^(a*x)+c)^(1/2)+c^(1/2))| ❶
tan^(-1)((b*e^(a*x)+c)^(1/2)/(-c)^(1/2)) ❶

With ln*x*

❶ Set $u = \ln x$.
❷ Integrate by parts.
Restriction: Arguments of logarithms are positive.

\I[x]{lnx}, x*lnx ❶❷

\I[x]{x*lnx}, x^2*lnx ❶❷

\I[x]{x^2*lnx}, x^3*lnx ❶❷

\I[x]{lnx/x},
lnx^2 ❶❷

\I[x]{lnx/x^2}, lnx/x ❶❷

\I[x]{lnx/(a*x+b)^2},
lnx/(a*x+b), ln|x/(a*x+b)| ❷

\I[x]{ln|a*x+b|/x^2},
lnx, ln|a*x+b| ❷

\I[x]{e^(a*x)*lnx}, \I[x]{e^(a*x)/x} Cal

\I[x]{lnx^2}, x*lnx^2 ❶❷

\I[x]{ln(x^2+a^2)},
x*ln(x^2+a^2), tan^(-1)(x/a) ❷

\I[x]{ln|x^2-a^2|},
x*ln|x^2-a^2|, ln|(x+a)/(x-a)| ❷

\I[x]{x*ln|x^2\+a^2|},
(x^2\+a^2)*ln|x^2\+a^2| ❷

\I[x]{1/lnx}, ln|lnx|, lnx^k ❶ Cal

\I[x]{1/(x*lnx)}, ln|lnx| ❶

\I[x]{sin(lnx)}, x*sin(lnx), x*cos(lnx) ❶❷

\I[x]{cos(lnx)}, x*sin(lnx), x*cos(lnx) ❶❷

\I[x]{x^n*lnx}, x^(n+1)*lnx ❷

\I[x]{x^n*ln|a*x+b|}, x^(n+1)*ln|a*x+b|,
\I[x]{x^(n+1)/(a*x+b)} ❷

\I[x]{x^n*lnx^2}, x^(n+1)*lnx^2, x^(n+1)*lnx ❶❷

\I[x]{x^n/lnx}, ln|lnx|, lnx^k ❶ Cal

\I[x]{lnx^n}, x*lnx^n,
\I[x]{lnx^(n-1)} ❶❷

\I[x]{lnx^n/x}, lnx^(n+1) ❶❷

\I[x]{x^m*lnx^n}, x^(m+1)*lnx^n, \I[x]{x^m*lnx^(n-1)}, x^(m+1)*lnx^(n+1), \I[x]{x^m*lnx^(n+1) ❷ ❷

\I[x]{x^n*ln|x^2\+a^2|}, x^(n+1)*ln|x^2\+a^2|, \I[x]{x^(n+2)/(x^2\+a^2)} ❷

\I[x]{ln|x+(x^2\+a^2)^(1/2)|}, x*ln|x+(x^2\+a^2)^(1/2)|, (x^2\+a^2)^(1/2) ❷

\I[x]{x*ln|x+(x^2\+a^2)^(1/2)|}, x^2*ln|x+(x^2\+a^2)^(1/2)|, x*(x^2\+a^2)^(1/2) ❷

\I[x]{x^n*ln|x+(x^2\+a^2)^(1/2)|}, x^(n+1)*ln|x+(x^2\+a^2)^(1/2)|, \I[x]{x^(n+1)/(x^2\+a^2)^(1/2)} ❷

With sinh ax

❶ Set $u = \sinh ax$ or $u = \cosh ax$.
❷ Set $u = \tanh ax$ or $u = \tan(ax/2)$.
❸ Integrate by parts.
❹ Write integrand as sum.
❺ Use multiples of ax.

\I[x]{sinh(a*x)}, cosh(a*x) ❶

\I[x]{x*sinh(a*x)}, x*cosh(a*x) ❸

\I[x]{x^2*sinh(a*x)}, x^2*cosh(a*x), x*sinh(a*x) ❸

\I[x]{sinh(a*x)/x}, (a*x)^(2*k+1) **Cal**

\I[x]{sinh(a*x)/x^2}, sinh(a*x), \I[x]{cosh(a*x)/x} ❸

\I[x]{1/sinh(a*x)}, ln|tanh(a*x/2)|, ln|(e^(a*x)-1)/(e^(a*x)+1)| ❷

\I[x]{x/sinh(a*x)}, B_(2*k), x^(2*k+1)

\I[x]{1/(x*sinh(a*x))}, B_(2*k), (a*x)^(2*k-1)

Cal

\I[x]{sinh(a*x)^2},
sinh(2*a*x) ❸ ❺

\I[x]{x*sinh(a*x)^2}, x*sinh(2*a*x),
cosh(2*a*x) ❸ ❺

\I[x]{1/sinh(a*x)^2},
coth(a*x) ❷

\I[x]{x/sinh(a*x)^2}, x*coth(a*x),
ln|sinh(a*x)| ❸

\I[x]{sinh(a*x)*sinh(b*x)}, sinh((a+b)*x), sinh((a-b)*x) ❹

\I[x]{sinh(a*x+b)*sinh(c*x+d)}, sinh((a+c)*x+b+d), sinh((a-c)*x+b-d) ❹

\I[x]{sinh(a*x)*sin(b*x)}, cosh(a*x)*sin(b*x),
sinh(a*x)*cos(b*x) ❸

\I[x]{sinh(a*x)*cos(b*x)}, cosh(a*x)*cos(b*x),
sinh(a*x)*sin(b*x) ❸

\I[x]{1/(b+c*sinh(a*x))}, **Cal**
ln|(c*e^(a*x)+b-(b^2+c^2)^(1/2))/(c*e^(a*x)+b+(b^2+c^2)^(1/2))| ❷

\I[x]{1/(b+c*sinh(a*x))^2}, cosh(a*x)/(b+c*sinh(a*x)), **Cal**
\I[x]{1/(b+c*sinh(a*x))}

\I[x]{1/(b^2+c^2*sinh(a*x)^2)},
tan^(-1)((c^2-b^2)^(1/2)*tanh(a*x)/b), ❷
ln((b+(b^2-c^2)^(1/2)*tanh(a*x))/(b-(b^2-c^2)^(1/2)*tanh(a*x))) ❷

\I[x]{1/(b^2-c^2*sinh(a*x)^2)},
ln|(b+(b^2+c^2)^(1/2)*tanh(a*x))/(b-(b^2+c^2)^(1/2)*tanh(a*x))| ❷

\I[x]{x^n*sinh(a*x)}, x^n*cosh(a*x), \I[x]{x^(n-1)*cosh(a*x)} ❸

\I[x]{sinh(a*x)^n}, sinh(a*x)^(n-1)*cosh(a*x), \I[x]{sinh(a*x)^(n-2)} ❸

\I[x]{sinh(a*x)/x^n}, sinh(a*x)/x^(n-1), \I[x]{cosh(a*x)/x^(n-1)} ❸

\I[x]{x^n/sinh(a*x)}, B_(2*k), x^(n+2*k) **Cal**

\I[x]{1/sinh(a*x)^n}, cosh(a*x)/sinh(a*x)^(n-1), \I[x]{1/sinh(a*x)^(n-2)} ❸

\I[x]{x/sinh(a*x)^n}, x*cosh(a*x)/sinh(a*x)^(n-1), 1/sinh(a*x)^(n-2), \I[x]{x/sinh(a*x)^(n-2)} ❸

With cosh ax

❶ Set $u = \sinh ax$ or $u = \cosh ax$.
❷ Set $u = \tanh ax$ or $u = \tan(ax/2)$.
❸ Integrate by parts.
❹ Write integrand as sum.
❺ Use multiples of ax.

\I[x]{cosh(a*x)}, sinh(a*x) ❶

\I[x]{x*cosh(a*x)}, x*sinh(a*x), cosh(a*x) ❸

\I[x]{x^2*cosh(a*x)}, sinh(a*x), x*cosh(a*x) ❸

\I[x]{cosh(a*x)/x}, ln|a*x|, (a*x)^(2*k) **Cal**

\I[x]{cosh(a*x)/x^2}, cosh(a*x)/x, \I[x]{sinh(a*x)/x} ❸

\I[x]{1/cosh(a*x)}, tan^(-1)(sinh(a*x)) ❶ or \I[x]{1/cosh(a*x)}, tan^(-1)(e^(a*x)) (set $u = e^{ax}$)

\I[x]{x/cosh(a*x)}, E_(2*k), x^(2*k+2)

\I[x]{1/(x*cosh(a*x))}, ln|a*x|, E_(2*k), (a*x)^(2*k+2)

Cal

\I[x]{cosh(a*x)^2}, sinh(2*a*x) ❸ ❺

\I[x]{x*cosh(a*x)^2}, x*sinh(2*a*x), cosh(2*a*x) ❸ ❺

\I[x]{1/cosh(a*x)^2}, tanh(a*x) ❷

\I[x]{x/cosh(a*x)^2}, x*tanh(a*x), ln(cosh(a*x)) ❸

\I[x]{cosh(a*x)*cosh(b*x)}, sinh((a+b)*x), sinh((a-b)*x) ❹

\I[x]{cosh(a*x+b)*cosh(c*x+d)}, sinh((a+c)*x+b+d), sinh((a-c)*x+b-d) ❹

\I[x]{cosh(a*x)*sin(b*x)}, sinh(a*x)*sin(b*x), cosh(a*x)*cos(b*x) ❸

\I[x]{cosh(a*x)*cos(b*x)}, sinh(a*x)*cos(b*x), cosh(a*x)*sin(b*x) ❸

\I[x]{1/(cosh(a*x)+1)}, tanh(a*x/2) ❷

\I[x]{1/(cosh(a*x)-1)}, coth(a*x/2) ❷

\I[x]{x/(cosh(a*x)+1)}, x*tanh(a*x/2), ln(cosh(a*x/2)) ❸

\I[x]{x/(cosh(a*x)-1)}, x*coth(a*x/2), ln|sinh(a*x/2)| ❸

\I[x]{1/(cosh(a*x)+1)^2}, tanh(a*x/2), tanh(a*x/2)^3 ❷

\I[x]{1/(cosh(a*x)-1)^2}, coth(a*x/2), coth(a*x/2)^3 ❷

\I[x]{1/(cosh(a*x)^2+1)},
ln((e^(2*x)+3-2*2^(1/2))/(e^(2*x)+3+2*2^(1/2))) [5]

\I[x]{1/(b+c*cosh(a*x))},
ln|(c*e^(a*x)+b-(b^2-c^2)^(1/2))/(c*e^(a*x)+b+(b^2-c^2)^(1/2))|, Cal
tan^(-1)((c*e^(a*x)+b)/(c^2-b^2)^(1/2)))

\I[x]{1/(b+c*coshx)^2}, sinhx/(b+c*coshx), Cal
\I[x]{1/(b+c*coshx)}

\I[x]{1/(b^2+c^2*cosh(a*x)^2)},
ln|(b*tanh(a*x)+(b^2+c^2)^(1/2))/(b*tanh(a*x)-(b^2+c^2)^(1/2))|, [2]
tanh^(-1)(b*tanh(a*x)/(b^2+c^2)^(1/2)) [2]

\I[x]{1/(b^2-c^2*cosh(a*x)^2)},
ln|(b*tanh(a*x)+(b^2-c^2)^(1/2))/(b*tanh(a*x)-(b^2-c^2)^(1/2))|, [2]
tanh^(-1)(b*tanh(a*x)/(c^2-b^2)^(1/2)) [2]

\I[x]{x^n*cosh(a*x)}, x^n*sinh(a*x),
\I[x]{x^(n-1)*sinh(a*x)} [3]

\I[x]{cosh(a*x)^n}, sinh(a*x)*cosh(a*x)^(n-1),
\I[x]{cosh(a*x)^(n-2)} [3]

\I[x]{cosh(a*x)/x^n}, cosh(a*x)/x^(n-1),
\I[x]{sinh(a*x)^(n-1)} [3]

\I[x]{x^n/cosh(a*x)}, E_(2*k), x^(n+2*k+1) Cal

\I[x]{1/cosh(a*x)^n}, sinh(a*x)/cosh(a*x)^(n-1),
\I[x]{1/cosh(a*x)^(n-2)} [3]

\I[x]{x/cosh(a*x)^n}, x*sinh(a*x)/cosh(a*x)^(n-1), 1/cosh(a*x)^(n-2),
\I[x]{x/cosh(a*x)^(n-2)} [3]

With sinh *ax* and cosh *ax*

❶ Set $u = \sinh ax$ or $u = \cosh ax$.
❷ Set $u = \tanh ax$ or $u = \tan(ax/2)$.
❸ Write integrand as sum.
❹ Use multiples of ax.

\I[x]{sinh(a*x)*cosh(a*x)}, sinh(a*x)^2 ❶❹

\I[x]{sinh(a*x)^2*cosh(a*x)^2}, sinh(4*a*x) ❹

\I[x]{sinh(a*x)*cosh(b*x)}, cosh((a+b)*x), cosh((a-b)*x) ❸

\I[x]{sinh(a*x+b)*cosh(c*x+d)}, cosh((a+c)*x+b+d), cosh((a-c)*x+b-d)

\I[x]{1/(sinh(a*x)*cosh(a*x))}, ln|tanh(a*x)| ❷❹

\I[x]{1/(sinh(a*x)^2*cosh(a*x))}, 1/sinh(a*x), tan^(-1)(sinh(a*x)) ❶

\I[x]{1/(sinh(a*x)*cosh(a*x)^2)}, 1/cosh(a*x), ln|tanh(a*x/2)| ❸

\I[x]{1/(sinh(a*x)^2*cosh(a*x)^2)}, cosh(2*a*x) ❹

\I[x]{sinh(a*x)^2/cosh(a*x)}, sinh(a*x), tan^(-1)(sinh(a*x)) ❶❸

\I[x]{cosh(a*x)^2/sinh(a*x)}, cosh(a*x), ln|tanh(a*x/2)| ❸

\I[x]{sinh(a*x)/cosh(a*x)^2}, 1/cosh(a*x) ❶

\I[x]{cosh(a*x)/sinh(a*x)^2}, 1/sinh(a*x) ❶

\I[x]{1/(sinh(a*x)*(cosh(a*x)+1))}, ln|tanh(a*x/2)|, tanh(a*x/2)^2 ❷

\I[x]{1/(sinh(a*x)*(cosh(a*x)-1))}, ln|tanh(a*x/2)|, coth(a*x/2)^2 ❷

\I[x]{1/(cosh(a*x)*(sinh(a*x)+1))},
ln|(sinh(a*x)+1)/cosh(a*x)|, tan^(-1)(sinh(a*x))| ❸

\I[x]{sinh(a*x)^n*cosh(a*x)},
sinh(a*x)^(n+1) ❶

\I[x]{sinh(a*x)*cosh(a*x)^n},
cosh(a*x)^(n+1) ❶

\I[x]{1/(sinh(a*x)*cosh(a*x)^n)}, 1/cosh(a*x)^(n-1),
\I[x]{1/(sinh(a*x)*cosh(a*x)^(n-2))} ❹

\I[x]{1/(sinh(a*x)^n*cosh(a*x))}, 1/sinh(a*x)^(n-1),
\I[x]{1/(sinh(a*x)^(n-2)*cosh(a*x))} ❹

With tanh *ax*

❶ Set $u = \sinh ax$ or $u = \cosh ax$.
❷ Set $u = \tanh ax$.
❸ Integrate by parts.
❹ Write integrand as sum.

\I[x]{tanh(a*x)},
ln(cosh(a*x)) ❶

\I[x]{tanh(a*x)^2},
tanh(a*x) ❹

\I[x]{tanh(a*x)^3}, tanh(a*x)^2,
ln(cosh(a*x)) ❹

\I[x]{1/(cosh(a*x)^2*tanh(a*x))},
ln|tanh(a*x)| ❷

\I[x]{1/tanh(a*x)},
ln|sinh(a*x)| ❶

\I[x]{x*tanh(a*x)}, B_(2*k), x^(2*k+1) Cal

\I[x]{x*tanh(a*x)^2}, x*tanh(a*x),
ln(cosh(a*x)) ❸

\I[x]{tanh(a*x)/x}, B_(2*k), (a*x)^(2*k-1) Cal

\I[x]{1/(b+c*tanh(a*x))}, ln|c*sinh(a*x)+b*cosh(a*x)| Cal

\I[x]{tanh(a*x)^n}, tanh(a*x)^(n-1),
\I[x]{tanh(a*x)^(n-2)} ❹

\I[x]{tanh(a*x)^n/cosh(a*)^2},
tanh(a*x)^(n+1) ❷

\I[x]{x^n*tanh(a*x)}, B_(2*k), x^(n+2*k) Cal

\I[x]{tanh(a*x)^(1/2)}, tanh^(-1)(tanh(a*x)^(1/2)),
tan^(-1)(tanh(a*x)^(1/2)) ❷

With coth ax

❶ Set u = sinh ax or u = cosh ax.
❷ Set u = coth ax.
❸ Integrate by parts.
❹ Write integrand as sum.

\I[x]{coth(a*x)},
ln|sinh(a*x)| ❶

\I[x]{coth(a*x)^2},
coth(a*x) ❹

\I[x]{coth(a*x)^3}, coth(a*x)^2,
ln|sinh(a*x)| ❹

\I[x]{1/(sinh(a*x)^2*coth(a*x))},
ln(coth(a*x)) ❷

\I[x]{1/coth(a*x)},
ln(cosh(a*x)) ❶

\I[x]{x*coth(a*x)}, B_(2*k), x^(2*k+1) Cal

\I[x]{x*coth(a*x)^2}, x*coth(a*x),
ln|sinh(a*x)| ❸

\I[x]{coth(a*x)/x}, B_(2*k), (a*x)^(2*k-1) Cal

\I[x]{1/(b+c*coth(a*x))}, ln|b*sinh(a*x)+c*cosh(a*x)| Cal

\I[x]{coth(a*x)^n}, coth(a*x)^(n-1),
\I[x]{coth(a*x)^(n-2)} ❹

\I[x]{coth(a*x)^n/sinh(a*x)^2}, coth(a*x)^(n+1) ❷

\I[x]{x^n*coth(a*x)}, B_(2*k), x^(n+2*k) Cal

\I[x]{coth(a*x)^(1/2)}, coth^(-1)(coth(a*x)^(1/2)), tan^(-1)(coth(a*x)^(1/2)) ❷

With inverse hyperbolic functions Inf

❶ Set $u = \sinh^{-1}(x/a)$.
❷ Set $u = \cosh^{-1}(x/a)$.
❸ Set $u = \tanh^{-1}(x/a)$.
❹ Set $u = \coth^{-1}(x/a)$.
❺ Integrate by parts.

\I[x]{sinh^(-1)(x/a)}, x*sinh^(-1)(x/a), (x^2+a^2)^(1/2) ❶❺

\I[x]{x*sinh^(-1)(x/a)}, sinh^(-1)(x/a), x*(x^2+a^2)^(1/2), $a > 0$ ❶❺

\I[x]{x^2*sinh^(-1)(x/a)}, x^3*sinh^(-1)(x/a), x^2*(x^2+a^2)^(1/2) ❶❺

\I[x]{sinh^(-1)(x/a)/x}, (x*a)^(2*k+1), (a/x)^(2*k), ln|x/a|^2 Cal

\I[x]{sinh^(-1)(x/a)/x^2}, sinh^(-1)(x/a)/x, ln|(a+(x^2+a^2)^(1/2))/x| ❶❺

\I[x]{cosh^(-1)(x/a)}, x*cosh^(-1)(x/a), (x^2-a^2)^(1/2) ❷❺

\I[x]{x*cosh^(-1)(x/a)}, x^2*cosh^(-1)(x/a), x*(x^2-a^2)^(1/2) ❷❺

\I[x]{x^2*cosh^(-1)(x/a)}, x^3*cosh^(-1)(x/a), x^2*(x^2-a^2)^(1/2) ❷❺

\I[x]{cosh^(-1)(x/a)/x}, ln(2*x/a)^2, (a/x)^(2*k) Cal

\I[x]{cosh^(-1)(x/a)/x^2}, cosh^(-1)(x/a)/x, cos^(-1)(a/x) ❷ ❺

\I[x]{tanh^(-1)(x/a)}, x*tanh^(-1)(x/a), ln(a^2-x^2) ❸ ❺

\I[x]{x*tanh^(-1)(x/a)}, (x^2-a^2)*tanh^(-1)(x/a) ❸ ❺

\I[x]{x^2*tanh^(-1)(x/a)}, x^3*tanh^(-1)(x/a), ln(a^2-x^2) ❸ ❺

\I[x]{tanh^(-1)(x/a)/x}, (x/a)^(2*k+1) Cal

\I[x]{tanh^(-1)(x/a)/x^2}, tanh^(-1)(x/a)/x, ln(x^2/(a^2-x^2)) ❸ ❺

\I[x]{coth^(-1)(x/a)}, x*coth^(-1)(x/a), ln(a^2-x^2) ❹ ❺

\I[x]{x*coth^(-1)(x/a)}, (x^2-a^2)*coth^(-1)(x/a) ❹ ❺

\I[x]{x^2*coth^(-1)(x/a)}, x^3*coth^(-1)(x/a), ln(x^2-a^2) ❹ ❺

\I[x]{coth^(-1)(x/a)/x}, (a/x)^(2*k+1) Cal

\I[x]{coth^(-1)(x/a)/x^2}, coth^(-1)(x/a)/x, ln(x^2/(x^2-a^2)) ❹ ❺

\I[x]{x^n*sinh^(-1)(x/a)}, x^(n+1)*sinh^(-1)(x/a), \I[x]{x^(n+1)/(x^2+a^2)^(1/2)} ❶ ❺

\I[x]{x^n*cosh^(-1)(x/a)}, x^(n+1)*cosh^(-1)(x/a), \I[x]{x^(n+1)/(x^2-a^2)^(1/2)} ❷ ❺

\I[x]{x^n*tanh^(-1)(x/a)}, x^(n+1)*tanh^(-1)(x/a), \I[x]{x^(n+1)/(a^2-x^2)} ❸ ❺

\I[x]{x^n*coth^(-1)(x/a)}, x^(n+1)*coth^(-1)(x/a), \I[x]{x^(n+1)/(a^2-x^2)} ❹ ❺

8 DEFINITE INTEGRALS

8.1 Definitions

Let $f(x)$ be defined on the interval $a \le x \le b$. If this interval is divided into n equal subintervals of length $\Delta x = (b - a)/n$, then the *definite interval* of $f(x)$ from $x = a$ to $x = b$ is defined by the relation

\I[x,a,b]{f(x)}=lim[n,\o]{...}

If $f(x)$ is *piecewise continuous* (i.e. continuous on a finite number of subintervals that constitute the interval of integration), then the limit exists.

If f(x)=(d/dx){g(x)}, then according to the *fundamental theorem of the integral calculus*, we have

\I[x,a,b]{f(x)}=\I[x,a,b]{(d/dx){g(x)}}=g(b}-g(a)

If the interval is infinite, or if $f(x)$ has a singularity at some point of the integration interval, the definite interval is called the *improper integral* and can be defined by the appropriate limits. Thus, we have

\I[x,a,\o]{f(x)}=lim[b,\o]{\I[x,a,b]{f(x)}}

\I[x,-\o,\o]{f(x)}=lim[...]{...} independently of the way that $a \to -\infty$ and $b \to \infty$

\I[x,a,b]{f(x)}=lim[ε,0+]{...} if b is a singular point

\I[x,a,b]{f(x)}=lim[ε,0+]{...} if a is a singular point

Cauchy's principal value of an improper integral \I[x,-\o,\o]{f(x)} is defined as the limit lim[N,\o]{\I[x,-N,N]{f(x)}}. Similarly, for a singular point inside the integration interval ($a < c < b$), Cauchy's principal value is defined as lim[ε,0+]{...}.

Note that Cauchy's principal value of an improper integral may exist even if the improper integral itself does not exist.

8.2 General Rules and Properties Inf

- \I[x,a,b]{f(x)\+g(x)\+...}=\I[x,a,b]{f(x)}\+\I[x,a,b]{g(x)}\+...

\I[x,a,b]{c*f(x)}=c*\I[x,a,b]{f(x)}

\I[x,a,a]{f(x)}=0

- \I[x,a,b]{f(x)}=-\I[x,b,a]{f(x)}

- \I[x,a,b]{f(x)}=\I[x,a,c]{f(x)}+\I[x,c,b]{f(x)}

- \I[x,a,b]{f(x)*g'(x)}=f(b)*g(b)-f(a)*g(a)-\I[x,a,b]{f'(x)*g(x)} Integration by parts

\I[x,a,b]{f(x)}=(b-a)*f(c), for appropriate c between a and b

This is the *mean value theorem* for definite integrals, and is valid if $f(x)$ is continuous in the interval $a \le x \le b$.

\I[x,a,b]{f(x)*g(x)}=f(c)\I[x,a,b]{g(x)}, for appropriate c between a and b

This relation is a generalization of the mean value theorem, and is valid if $f(x)$ and $g(x)$ are continuous on the interval $a \le x \le b$ and $g(x) \ge 0$.

- (d/db){\I[x,a,b]{f(x)}}=f(b)

- (d/dλ){\I[x,a(λ),b(λ)]{f(x,λ)}}=... [Leibniz's rule]

\I[x,0,\o]{(f(a*x)-f(b*x))/x}=f(0+)*ln(b/a) [Frullani's integral] Ext

|\I[x,a,b]{(f(x)}|\<\I[x,a,b]{|f(x)|}\<M*(b-a), for $|f(x)| \le M,\ a < x < b$

8.3 Various Integrals **Ext**

❶ Use indefinite integral.
❷ Use beta function.
❸ Use complex integral.
❹ Expand in powers of x.

With algebraic functions

\I[x,0,1]{x^λ/(1-x)^(λ+1)} ❸

\I[x,0,1]{x^n/(1+x)} ❹

\I[x,0,1]{x^m*(1-x)^n}, \I[x,0,1]{(x^m+x^n)/(1+x)^(m+n+2)},
Γ(m+1)*Γ(n+1)/Γ(m+n+2) ❷

\I[x,0,1]{x^n/(1-x)^(1/2)} ❷

\I[x,0,1]{x^n/(x*(1-x))^(1/2)} ❷

\I[x,0,1]{x^p/(1-x^2)^(1/2)},
\I[x,0,1]{x^p*(1-x^2)^(1/2)},
Γ((p+1)/2)/Γ(p/2+1) ❷

\I[x,0,a]{1/(a^2-x^2)^(1/2)} ❶

\I[x,0,a]{(a^2-x^2)^(1/2)} ❶

\I[x,0,a]{x^m*(a^n-x^n)^p}, Γ((m+1)/n)*Γ(p+1)/Γ((m+1)/n+p+1) ❷

\I[x,-a,a]{(a+x)^(m-1)*(a-x)^(n-1)}, Γ(m)*Γ(n)/Γ(m+n) ❷

Ext

\I[x,0,\o]{1/(x^2+a^2)} ❶

\I[x,0,\o]{1/(x^2+a^2)^n} ❶

\I[x,0,\o]{x^(p-1)/(x+a)} ❷❸

\I[x,0,\o]{x^m/(x^n+a^n)} ❷❸

\I[x,0,\o]{1/(a*x^2+b*x+c)} ❶

\I[x,0,\o]{x^(-p)/(x^2+2*x*cosφ+1)}, sin(pφ)/sinφ ❸

\I[x,0,\o]{x^(m-1)/(a*x+b)^(m+n)}, Γ(m)*Γ(n)/Γ(m+n) ❷

\I[x,0,\o]{x^m/(x^n+a^n)^p}, Γ((m+1)/n)/Γ((m+1)/n-p+1) ❷

With trigonometric functions

Ext

❶ Use indefinite integral.
❷ Use beta function.
❸ Use complex integral.

\I[x,0,π/2]{sinx^2}, \I[x,0,π]{sinx^2} ❶

\I[x,0,π/2]{cosx^2}, \I[x,0,π]{cosx^2} ❶

\I[x,0,π/2]{sin(n*x)^2}, \I[x,0,π]{sin(n*x)^2}, \I[x,0,2*π]{sin(n*x)^2} ❶

\I[x,0,π/2]{cos(n*x)^2}, \I[x,0,π]{cos(n*x)^2}, \I[x,0,2*π]{cos(n*x)^2 ❶

\I[x,0,π]{sin(m*x)*sin(n*x)}, \I[x,0,π]{cos(m*x)*cos(n*x)} ❶

\I[x,0,π]{sin(m*x)*cos(n*x)} ❶

\I[x,0,π/2]{sinx^p}, \I[x,0,π/2]{cosx^p}, Γ((p+1)/2)/Γ(p/2+1) ❶ ❶ ❷

\I[x,0,π]{x*sinx^n}, \I[x,0,π/2]{sinx^n} Cal

\I[x,0,π/2]{tanx^p}, \I[x,0,π/2]{cotx^p} ❸

\I[x,0,π/2]{1/(1+tanx^n)} Cal

\I[x,0,π/2]{x/sinx}, G

\I[x,0,π/2]{x^2/sinx^2} Cal

\I[x,0,π/2]{x/tanx}

\I[x,0,π]{sinx^p*sin(m*x)}, Γ(p+1)/(Γ((p+m)/2+1)*Γ((p-m)/2+1)) ❷

\I[x,0,π/2]{cosx^p*cos(m*x)}, Γ(p+1)/(Γ((p+m)/2+1)*Γ((p-m)/2+1)) ❷

\I[x,0,π]{sinx^p*cos(m*x)}, Γ(p+1)/(Γ((p+m)/2+1)*Γ((p-m)/2+1)) ❷

\I[x,0,π/2]{sinx^p*cosx^q},
Γ((p+1)/2)*Γ((q+1)/2)/Γ((p+q)/2+1)) ❷

\I[x,0,π/2]{1/(a+b*cosx)}, tan^(-1)((a-b)/(a^2-b^2)^(1/2)) ❶

\I[x,0,π]{1/(a+b*cosx)}, 1/(a^2-b^2)^(1/2) ❶

\I[x,0,2*π]{1/(a+b*sinx)}, \I[x,0,2*π]{1/(a+b*cosx)},
1/(a^2-b^2)^(1/2) ❶ ❸

Cal

\I[x,0,2*π]{1/(1-2*a*cosx+a^2)}, 1/(1-a^2) ❶ ❸

\I[x,0,π]{1/(a^2\+2*a*b*cosx+b^2)}, 1/|a^2-b^2| ❶ ❸

\I[x,0,π]{x*sinx/(1-2*a*cosx+a^2)},
ln(1+a), ln(1+1/a) ❸

\I[x,0,π]{cos(m*x)/(1-2*a*cosx+a^2)}, 1/(1-a^2) ❸

\I[x,0,2*π]{1/(a+b*sinx)^2}, \I[x,0,2*π]{1/(a+b*cosx)^2},
1/(a^2-b^2)^(3/2) ❶

Cal

\I[x,0,π/2]{1/(a*sinx+b*cosx)^2} ❶

\I[x,0,2*π]{1/(a+b*cosx+c*sinx)}, 1/(a^2-b^2-c^2)^(1/2) ❶ ❸

\I[x,0,π/2]{1/(1\+a^2*sinx^2)}, \I[x,0,π/2]{1/(1\+a^2*cosx^2)},
1/(1\+a^2)^(1/2) ❶

\I[x,0,π/2]{1/(1\+a^2*sinx^2)^2}, \I[x,0,π/2]{1/(1\+a^2*cosx^2)^2},
1/(1\+a^2)^(3/2) ❶

\I[x,0,π/2]{1/(b^2+a^2*tanx^2)},
\I[x,0,π/2]{cosx^2/(a^2*sinx^2+b^2*cosx^2)}, 1/(a+b) ❶

\I[x,0,π/2]{1/(a^2+b^2*cotx^2)},
\I[x,0,π/2]{sinx^2/(a^2*sinx^2+b^2*cosx^2)}, 1/(a+b) ❶

\I[x,0,π/2]{1/(a^2*sinx^2+b^2*cosx^2)}, a^2+b^2 ❶

\I[x,0,1]{sin^(-1)x/x} ❸

\I[x,0,1]{tan^(-1)x/x} ❶

\I[x,0,\o]{sin(p*x)/x}, sgn(p) ❸ **Cal**

\I[x,0,\o]{sin(p*x)*sin(q*x)/x}, ln((p+q)/(p-q)) ❸

\I[x,0,\o]{sin(p*x)*cos(q*x)/x} ❸

\I[x,0,\o]{sin(p*x)*sin(q*x)/x^2} ❸

\I[x,0,\o]{sin(p*x)^2/x^2} ❸

\I[x,0,\o]{(1-cos(p*x))/x^2} ❸ **Cal**

\I[x,0,\o]{(cos(p*x)-cos(q*x))/x} ❸ (also using Frullani's integral)

\I[x,0,\o]{(cos(p*x)-cos(q*x))/x^2} ❸

\I[x,0,\o]{sin(m*x)^(2*n+1)/x} Cal

\I[x,0,\o]{cos(m*x)/(x^2+a^2)} ❸

\I[x,0,\o]{x*sin(m*x)/(x^2+a^2)} ❸

\I[x,0,\o]{sin(m*x)/(x*(x^2+a^2))} ❸

\I[x,0,\o]{sin(p*x)/x^(1/2)}=\I[x,0,\o]{cos(p*x)/x^(1/2)} ❸

\I[x,0,\o]{sin(p*x)/x^(3/2)} ❸

\I[x,0,\o]{sinx^3/x^3} ❸

\I[x,0,\o]{sinx^4/x^4} ❸

\I[x,0,\o]{sinx/x^p}, 1/Γ(p), Γ(1-p) ❸

\I[x,0,\o]{cosx/x^p}, 1/Γ(p), Γ(1-p) ❸

\I[x,0,\o]{tan(m*x)/x}, sgn(m) ❸

\I[x,0,\o]{(tan^(-1)(p*x)-tan^(-1)(q*x))/x}, ln(p/q) ❶ Cal

\I[x,0,\o]{(1/(1+x^2)-cosx)/x}=γ ❸

\I[x,0,\o]{sin(a*x^2)}=\I[x,0,\o]{cos(a*x^2)} ❸

\I[x,0,\o]{sin(a*x^p)}, Γ(1/p) ❸

\I[x,0,\o]{cos(a*x^p)}, Γ(1/p) ❸

\I[x,0,\o]{sin(a*x^2)*cos(2*b*x)}, cos(b^2/a)-sin(b^2/a) ❸

\I[x,0,\o]{cos(a*x^2)*cos(2*b*x)}, cos(b^2/a)+sin(b^2/a) ❷❸

\I[x,0,\o]{sin(a*x^2+b^2/a)*cos(2*b*x)},
\I[x,0,\o]{cos(a*x^2+b^2/a)*cos(2*b*x)} ❸

With exponential functions

❶ Use indefinite integral.
❷ Use complex integral.
❸ Use gamma function.

\I[x,0,1]{1/x^x}=\S[k=1,\o]{1/k^k} Cal

\I[x,0,1]{x*e^x/(1+x)^2},
\I[x,1,e]{lnx/(1+lnx)^2} ❶

\I[x,0,\o]{e^(-a*x)} ❶

\I[x,0,\o]{x*e^(-a*x)} ❶

\I[x,0,\o]{x^(1/2)*e^(-a*x)} ❸

\I[x,0,\o]{e^(-a*x)/(x^(1/2)} ❸

\I[x,0,\o]{(e^(-a*x)-e^(-b*x))/x}, ln(b/a) (use Frullani's integral)

\I[x,0,\o]{x/(e^(a*x)-1)} ❷

\I[x,0,\o]{1/(e^(a*x)+1)} ❶

\I[x,0,\o]{x/(e^(a*x)+1)} ❷

\I[x,0,\o]{(1/(1+x)-e^(-x))/x}=\I[x,0,\o]{1/(e^x-1)-e^(-x)/x}=γ

\I[x,0,\o]{x^p*e^(-a*x)}, Γ(p+1) ❶ ❸

\I[x,0,\o]{x^(p-1)/(e^(a*x)-1)}, B_n, Γ(p), ζ(p) ❷

[$\Gamma(x)$ is the gamma function, $\zeta(x)$ is the Riemann zeta function]

\I[x,0,\o]{x^(p-1)/(e^(a*x)+1)}, B_n, Γ(p), ζ(p) ❷

\I[x,0,\o]{x^(n-1)*e^(-p*x)/(e^x+1))} Cal

\I[x,0,\o]{e^(-a*x)*sin(b*x)}=b/(a^2+b^2) ❶

\I[x,0,\o]{e^(-a*x)*cos(b*x)}=a/(a^2+b^2) ❶

\I[x,0,\o]{x*e^(-a*x)*sin(b*x)}=2*a*b/(a^2+b^2)^2 ❶

\I[x,0,\o]{x*e^(-a*x)*cos(b*x)}=
(a^2-b^2)/(a^2+b^2)^2 ❶

\I[x,0,\o]{e^(-a*x)*sin(b*x)/x}=
tan^(-1)(b/a) Cal

\I[x,0,\o]{x^(p-1)*e^(-a*x)*sin(b*x)}, Γ(p) ❷

\I[x,0,\o]{x^(p-1)*e^(-a*x)*cos(b*x)}, Γ(p)

\I[x,0,\o]{e^(-a*x)*sin(b*x+c)} ❶

\I[x,0,\o]{e^(-a*x)*cos(b*x+c)} ❶

\I[x,0,\o]{e^(-a*x)*sin(b*x)*sin(c*x)} ❶

\I[x,0,\o]{e^(-a*x)*sin(b*x)*cos(c*x) ❶

\I[x,0,\o]{e^(-a*x)*cos(b*x)*cos(c*x)} ❶

\I[x,0,\o]{sin(p*x)*/(e^(a*x)-1)} ❷

\I[x,0,\o]{sin(p*x)*/(e^(a*x)+1)} ❷

\I[x,0,\o]{(e^(-a*x)-e^(-b*x))*cos(p*x)/x} ❷

\I[x,0,\o]{(e^(-a*x)-e^(-b*x))*sin(p*x)/x} Cal

\I[x,0,\o]{e^(-a*x)*(1-cos(b*x))/x} ❷

\I[x,0,\o]{e^(-a*x)*(1-cosx)/x^2} ❷

\I[x,0,\o]{e^(-a*x^2)} ❸

\I[x,0,\o]{x*e^(-a*x^2)} ❶

Cal

\I[x,0,\o]{e^(-a*x^2)*cos(b*x)} Cal

\I[x,0,\o]{(1-e^(-a*x^2))/x^2} Cal

\I[x,0,\o]{(e^(-a*x^2-b*x-c)}, erfc(b/(2*a^(1/2)), erfc(z)=\I[u,z,\o]{e^(-u^2)} Cal

\I[x,-\o,\o]{(e^(-a*x^2-b*x-c)}

\I[x,0,\o]{(e^(-a*x^2-b/x^2)} Cal

\I[x,0,\o]{(e^(-x^2)-e^(-x))/x}=γ/2 [γ = Euler number]

\I[x,0,\o]{(e^(-a*x^2)-e^(-b*x^2))/x^2}=γ/2 ❷

\I[x,0,\o]{x^p*e^(-a*x^2)}, Γ((p+1)/2) ❸ ❶ ❸ Cal

\I[x,0,\o]{x^p*e^(-(a*x)^q)}, Γ((p+1)/q) ❸

\I[x,0,\o]{x/(e^x-1)^(1/2)} [set $z = (e^x - 1)^{1/2}$, then ❷]

With logarithmic functions

❶ Use indefinite integral.
❷ Use complex integral.
❸ Set $x = e^{-y}$.
❹ Set $x = \exp(-z^2)$, $z > 0$.
❺ Use gamma function.

\I[x,0,1]{lnx/(1\+x)} (❸, then ❷) Cal

\I[x,0,1]{lnx/(1+x)^2} ❶

\I[x,0,1]{lnx/(1+x^2)} Cal

\I[x,0,1]{lnx/(1-x^2)^(1/2)} (set $x = \cos y$, then ❷)

\I[x,0,1]{x*lnx/(1-x^2)^(1/2)} (set $x = \cos y$, then ❷)

\I[x,0,1]{lnx*(1-x^2)^(1/2)} (set $x = \cos y$, then ❷)

\I[x,0,1]{x*lnx*(1-x^2)^(1/2)} (set $x = \cos y$, then ❷)

Cal

\I[x,0,1]{|lnx|^(1/2)} ❹

\I[x,0,1]{1/|lnx|^(1/2)} ❹

\I[x,0,1]{x^(p-1)/|lnx|^(1/2)} ❹

\I[x,0,1]{lnx*ln(1+x)}

\I[x,0,1]{lnx*ln(1-x)}

Cal

\I[x,0,1]{x^(p-1)*ln|lnx|}=\I[x,0,\o]{e^(-p*x)*lnx/e^(p*x)}, γ ❸❺ [γ = Euler constant]

\I[x,0,1]{x^p*lnx} ❶

\I[x,0,1]{(-lnx)^p}, Γ(p+1) ❸

\I[x,0,1]{x^(p-1)*lnx/(1+x)}

\I[x,0,1]{x^n*lnx/(1+x)}, \S[k=1,n]{(-1)^k/k^2}

\I[x,0,1]{x^n*lnx/(1-x)}, \S[k=1,n]{1/k^2}

\I[x,0,1]{x^n*ln(1-x)}, \S[k=1,n+1]{1/k} Cal

\I[x,0,1]{x^n*ln(1+x)},
\S[k=1,n+1]{(-1)^k/k}, \S[k=1,n+1]{(-1)^(k-1)/k}

\I[x,0,1]{ln(1\+x^p)/x}

\I[x,0,1]{x^p*(-lnx)^q}, Γ(q+1) ❸

\I[x,0,1]{(x^p-x^q)/lnx}, ln((p+1)/(q+1)) ❸

\I[x,0,1]{(x^(p-1)*sin(q*lnx)}, q/(p^2+q^2) ❸

\I[x,0,1]{(x^(p-1)*cos(q*lnx)}, p/(p^2+q^2) ❸

\I[x,0,π/4]{ln(sinx)}, G

\I[x,0,π/4]{ln(cosx)}, G

\I[x,0,π/4]{ln(tanx)}, G

\I[x,0,π/4]{ln(1+tanx)}

\I[x,0,π/4]{ln(1-tanx)}, G

\I[x,0,π/2]{ln(sinx)}, \I[x,0,π/2]{ln(cosx)} ❷

Cal

\I[x,0,π/2]{ln(sinx)^2}, \I[x,0,π/2]{ln(cosx)^2}

\I[x,0,π/2]{ln(1\+cosx)}, G

\I[x,0,π/2]{ln(1+tanx)}, G

Cal

\I[x,0,π/2]{ln(1+a*sinx^2)}, \I[x,0,π/2]{ln(1+a*cosx^2)}

\I[x,0,π/2]{ln(a^2*sinx^2+b^2*cosx^2)}

\I[x,0,π/2]{ln(a^2+b^2*tanx^2)}, \I[x,0,π/2]{ln(a^2+b^2*cotx^2)}

Cal

\I[x,0,π/2]{sinx*ln(sinx)}, \I[x,0,π/2]{cosx(ln(cosx)} {set $y = \cos x$, $y = \sin x$, then ❶}

\I[x,0,π/2]{ln(1+p*cosx)/cosx}, \I[x,0,π/2]{ln(1+p*sinx)/sinx}

\I[x,0,π]{x*ln(sinx)} (set $x = \pi - y$)

\I[x,0,π]{ln(a+b*cosx)}, ln((a+(a^2-b^2)^(1/2))/2)

\I[x,0,π]{ln(a^2\+2*a*b*cosx+b^2)}

\I[x,0,2*π]{ln(a+b*sinx)}, \I[x,0,2*π]{ln(a+b*cosx)}, ln((a+(a^2-b^2)^(1/2))/2)

Cal

\I[x,0,\o]{e^(-a*x)*lnx}, γ

\I[x,0,\o]{x*e^(-a*x)*lnx}, γ

\I[x,0,\o]{x^2*e^(-a*x)*lnx}, γ

\I[x,0,\o]{e^(-a*x)*lnx/x^(1/2)}, γ

\I[x,0,\o]{e^(-a*x^2)*lnx}, γ

Cal

\I[x,0,\o]{lnx*sin(a*x)/x}, γ

\I[x,0,\o]{ln(1\+e^(-x))} 3

\I[x,0,\o]{lnx/(x^2+a^2)} 2

\I[x,0,\o]{lnx/(x^2+1)^2} 2

\I[x,0,\o]{x^(1/2)*lnx/(x^2+1)} 2

\I[x,0,\o]{lnx/(x^2-1)} ❷

\I[x,0,\o]{lnx^2/(x^2+1)} ❷

\I[x,0,\o]{ln(1+x)/(a*x+b)^2}

\I[x,0,\o]{ln(1+x)/(x^2+1)}, G (❷ or set $x = \tan y$)

\I[x,0,\o]{ln(x^2+1)/(x^2+1)} (❷ or set $x = \tan y$)

\I[x,0,\o]{ln((x^2+a^2)/(x^2+b^2))}

\I[x,0,\o]{x^(p-1)*lnx/(1\+x)}

\I[x,0,\o]{ln(1+x^p)/x^(q+1)}

With hyperbolic functions

\I[x,0,\o]{x/sinh(a*x)} ❷❸

\I[x,0,\o]{1/cosh(a*x)} ❶

\I[x,0,\o]{x/cosh(a*x)}, G ❸

\I[x,0,\o]{x^(p-1)/sinh(a*x)}, Γ(p)*ζ(p)

[$\Gamma(p)$ is the gamma function and $\zeta(p)$ is the Riemann zeta function]

\I[x,0,\o]{x^(p-1)/cosh(a*x)}, Γ(p)

Cal

❶ Use indefinite integral.
❷ Use complex integral.
❸ Set $x = -\ln y$.

\I[x,0,\o]{sin(b*x)/sinh(a*x)} 2

\I[x,0,\o]{cos(b*x)/cosh(a*x)} 2

\I[x,0,\o]{sinh(a*x)/sinh(b*x)} 2

\I[x,0,\o]{cosh(a*x)/cosh(b*x)} 2

\I[x,0,\o]{sin(a*x)*tanh(b*x)}

\I[x,0,\o]{sin(a*x)/tanh(b*x)}

\I[x,0,\o]{sinh(a*x)/(e^(b*x)+1)}

\I[x,0,\o]{sinh(a*x)/(e^(b*x)-1)}

\I[x,0,\o]{x*e^(-a*x^2)*sinh(b*x)} Cal

\I[x,0,\o]{sinh(p*x)/sinh(q*x)}

\I[x,0,\o]{cosh(p*x)*sin(r*x)/sinh(q*x)}

\I[x,0,\o]{sinh(p*x)*sin(r*x)/cosh(q*x)}

\I[x,0,\o]{cosh(p*x)*cos(r*x)/cosh(q*x)}

9 ORDINARY DIFFERENTIAL EQUATIONS

9.1 Definitions

An *ordinary differential equation* (ODE) is an equation of the form

$$f[y^{(n)}(x), y^{(n-1)}(x), \ldots, y''(x), y'(x), y(x), x] = 0$$ **Exa**

where $y'(x), y''(x), \ldots, y^{(n-1)}(x), y^{(n)}(x)$ are the derivatives of $y(x)$ with respect to x (x is the *independent variable* and y is the *dependent variable*). The *order* of the differential equation is the order n of the highest derivative that appears in the equation. A *solution* of an ODE is a function $y(x)$ that satisfies the ODE on an interval of x.

The *general solution* of an ODE is a solution of the form $y = y(x, c_1, c_2, \ldots, c_n)$, which depends on x and n arbitrary constants $c_1, c_2, \ldots, c_n$. If arbitrary but specific values are assigned to the constants, we have a *particular solution* of the ODE. A solution of the ODE that is not obtained from the general solution (for some values of the constants) is called *singular solution* (meaning that it is a special solution).

Initial conditions are n algebraic equations of the form

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad \ldots, \quad y^{(n-1)}(x_0) = y_0^{(n-1)}$$ **App**

These are conditions at a point $x = x_0$, from which the constants $c_1, c_2, \ldots, c_n$ can be determined. Thus, a particular solution of the ODE is obtained. Alternatively, the constants can be determined from *boundary conditions*, meaning conditions at more than one point in an interval of x (or the end points of the interval).

A *system of ODEs* is a set of ODEs that contains two or more dependent variables $y_1(x), y_2(x), \ldots$ and their derivatives with respect to x. In general, a system of ODEs can be reduced to a single ODE of higher order. Conversely, any ODE can be reduced to a first order system of ODEs with the substitution of the higher order derivatives by new dependent variables.

An *initial value problem* is a problem that requires us to find the particular solution of a given ODE, which satisfies a set of given initial conditions. Similarly, a *boundary value problem* is a problem that requires us to find the particular solution of a given ODE that satisfies a set of given boundary conditions.

A *linear* ODE is an ODE that is linear with respect to the dependent variable and its derivatives. Otherwise it is *nonlinear.*

In the complex z-plane, if $f(w, z)$ is an analytic function of w and z in a domain D with $z_0 \in D$, then the ODE $w' = f(w, z)$ with given $w(z_0)$ (or a set of such ODEs) has a unique analytic solution in a neighborhood of $z = z_0$.

9.2 Simple ODEs

Certain categories of simple ODEs can be easily solved with various methods.

Separable ODEs

DE: $f_1(x)g_1(y)\,dx + f_2(x)g_2(y)\,dy = 0$ Exa

Solution: \I[x]{f_1(x)/f_2(x)}+\I[x]{g_2(x)/g_1(x)}=0

Exact or complete ODEs

DE: $M(x, y)\,dx + N(x, y)\,dy = 0$ with \dM/\dy=\dN/\dx Exa

Solution: \I[x]{M}+\I[y]{N-\d(\I[x]{M})/\dy}=c

where integration with ∂x signifies that y is regarded as constant. Sometimes an ODE $M(x, y)\,dx + N(x, y)\,dy = 0$ that is not exact, can become exact if it is multiplied with a function $I(x, y)$, which is called *integrating factor*.

Homogeneous first-order ODEs

DE: dy/dx=F(y/x) Exa

Solution: ln|x|=\I[v]{1/(F(v)-v)} where v=y/x

Linear first-order ODEs

DE: dy/dx+p(x)*y=q(x) Exa

Solution: y=e^(-R)*(\I[x]{q*e^R}+c) where R=\I[x]{p}

Bernoulli's ODE

DE: dy/dx+p(x)*y=q(x)*y^n Exa

Solution: y^(1-n)=e^(-R)*((1-n)\I[x]{q*e^R}+c) where R=(1-n)*\I[x]{p}

Homogeneous linear second-order ODEs with constant coefficients

DE: y"+a*y'+b*y=0 (a, b real constants) Exa

Solution: Let ρ_1, ρ_2 be the roots of the *characteristic equation* $\rho^2 + a\rho + b = 0$.

If ρ_1, ρ_2 are real and different,

y=c_1*e^(ρ_1*x)+c_2*e(ρ_2*x)

If ρ_1, ρ_2 are real and equal,

y=c_1*e^(ρ_1*x)+c_2*x*e(ρ_1*x)

If ρ_1, ρ_2 are complex conjugates with $\rho_1 = \rho_2{}^* = \kappa + \lambda i$,

$$y = e^{\kappa x}(c_1 \cos\lambda x + c_2 \sin\lambda x)$$

Nonhomogeneous linear second-order ODEs with constant coefficients

DE: y"+a*y'+b*y=r(x) (a, b real constants) Exa

Solution: Let ρ_1, ρ_2 be roots of the characteristic equation $\rho^2 + a\rho + b = 0$.

If ρ_1, ρ_2 are real and different,

e^(ρ_1*x), e^(ρ_2*x), \I[x]{e^(-ρ_1*x)*r(x)}, \I[x]{e^(-ρ_2*x)*r(x)}

If ρ_1, ρ_2 are real and equal,

e^(ρ_1*x), x*e^(ρ_1*x), \I[x]{e^(-ρ_1*x)*r(x)}, \I[x]{x*e^(-ρ_1*x)*r(x)}

If ρ_1, ρ_2 are complex conjugates with $\rho_1 = \rho_2{}^* = \kappa + \lambda i$,

e^(κ*x), \I[x]{e^(-κ*x)*r(x)*cos(λ*x)}, \I[x]{e^(-κ*x)*r(x)*sin(λ*x)}

Euler's or Cauchy's ODEs

DE: x^2*y"+a*x*y'+b*y=r(x) Ext

Solution: Substituting $x = e^t$, we obtain (d/dt)^2{y}+(a-1)*(d/dt){y}+b*y=r(e^t), a linear nonhomogeneous ODE with constant coefficients.

Legendre's equation

Exa

DE: (1-x^2)*y"-2*x*y'+n*(n+1)*y=0

Solution: $y = c_1 P_n(x) + c_2 Q_n(x)$ for integer $n \geq 0$

$P_n(x)$ and $Q_n(x)$ are the Legendre polynomials and functions, respectively.

Bessel's equation

Exa

DE: x^2*y"+x*y'+(x^2-v^2)*y=0

Solution: $y = c_1 J_v(x) + c_2 Y_v(x)$

$J_v(x)$ and $Y_v(x)$ are the Bessel functions of the first and second kind, respectively.

Modified Bessel's equation

DE: x^2*y"+x*y'-(x^2+v^2)*y=0

Solution: $y = c_1 I_v(x) + c_2 K_v(x)$

$I_v(x)$ and $K_v(x)$ are the modified Bessel functions.

Linear ODEs of order *n*

A linear ODE of order n has the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

with $a_n(x) \neq 0$. If $g(x) = 0$, the ODE is *homogeneous*, otherwise it is *nonhomogeneous*.

The basic property of a homogeneous linear ODE is that if $y_1(x)$ and $y_2(x)$ are solutions, then $c_1 y_1(x) + c_2 y_2(x)$ is also solution.

A homogeneous linear ODE of order n has n *linearly independent* solutions y_1, y_2, ..., y_n. The general solution of the homogeneous ODE is $y_h = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$. The general solution of the nonhomogeneous ODE is $y = y_h + y_p$, where y_h is the general solution of the corresponding homogeneous ODE and y_p is a particular solution of the nonhomogeneous ODE.

The

Finding a particular solution

A particular solution $y_p(x)$ can be found with the *method of variation of constants*. According to this method, we replace the constants c_i $(i=1,\cdots,n)$ of y_h with unknown functions $\upsilon_i(x)$, i.e. we set $y_h=\upsilon_1 y_1+\upsilon_2 y_2+\ldots+\upsilon_n y_n$. Then, we solve the following system of n linear algebraic equations with respect to υ_i' [knowing the derivatives $y_i^{(j)}(x)$ up to order n]:

$$\upsilon_1' y_1^{(j)}+\upsilon_2' y_2^{(j)}+\cdots+\upsilon_n'^{(j)} y_n=0,\quad j=0,1,2,\ldots,n-2$$

$$\upsilon_1' y_1^{(n)}+\upsilon_2' y_2^{(n)}+\cdots+\upsilon_n'^{(n)} y_n=g(x)/a_n(x)$$

Finally, we integrate υ_i' with respect to x and find the functions $\upsilon_i(x)$. Exa

Nonhomogeneous linear ODEs of order *n* with constant coefficients

A linear ODE of order n with constant coefficients has the form

$$a_n y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_2 y''+a_1 y'+a_0 y=g(x)$$

where $a_0, a_1, \ldots, a_n$ $(a_n\neq 0)$ are real constants. If $g(x)=0$, it is a homogeneous linear ODE. The corresponding *characteristic equation* is

$$a_n\rho^n+a_{n-1}\rho^{n-1}+\cdots+a_2\rho^2+a_1\rho+a_0=0$$

If the roots $\rho_1, \rho_2, \ldots, \rho_n$ are all real and different, the general solution for $g(x)=0$ is

y_h, e^(ρ_1*x), e^(ρ_2*x), e^(ρ_n*x) Exa

If the roots (some or all) are complex conjugates, the same expression for the solution is valid, but in this case, the terms with complex conjugate exponents can be combined into sine and cosine terms (e.g., the roots $a\pm ib$ give the terms $c_1e^{ax}\cos bx + c_2e^{ax}\sin bx$). If a root μ has multiplicity m, then the corresponding terms in the previous expression of the solution should be replaced by $e^{\mu x}, xe^{\mu x}, x^2e^{\mu x}, \ldots, x^{m-1}e^{\mu x}$.

Finding a particular solution for $g(x)\neq 0$ Ext

A particular solution $y_p(x)$ can be found with the general method of variation of constants. A simpler method, the *method of the undetermined coefficients*, can be used if $g(x)$ is of the form $e^{ax}p_m(x)\sin bx+e^{ax}q_m(x)\cos bx$ with $p_m(x)$ and $q_m(x)$ polynomials of degree m. According to this method, we accept that a particular solution is of the form $e^{ax}P_m(x)\sin bx+e^{ax}Q_m(x)\cos bx$, where $P_m(x)$ and $Q_m(x)$ are polynomials of degree m, and we find their coefficients with substitution in the differential equation. If a term of $y_p(x)$ is included in the solution $y_h(x)$ of the homogeneous ODE, then in the candidate $y_p(x)$, this term is multiplied by x^k, where k is the lowest positive integer for which y_h and the new y_p do not have any common term. Exa

Solutions in power series

A function $f(x)$ is *analytic* at $x = x_0$ if its Taylor series expansion about x_0

f(x)=\S[k=0,\o]{(f^((k))(x_0)*(x-x_0)^k/(k!)}

converges for $0 \le |x - x_0| < d$ (*interval* or *circle of convergence* with $d > 0$). Such a series converges absolutely and uniformly and can be differentiated term by term.

The homogeneous linear ODE

$$y'' + P(x)y' + Q(x) = 0$$ Exa

has an *ordinary point* at $x = x_0$ if both $P(x)$ and $Q(x)$ are analytic functions at $x = x_0$. Otherwise, it has a *singular point* at $x = x_0$.

An (isolated) singular point is a *regular singular point* if $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2Q(x)$ are analytic at $x = x_0$. Otherwise, it is an *irregular singular point*.

At an ordinary or regular singular point, there is at least one solution of the ODE of the form

y_1=(x-x_0)^(ρ_1)*\S[k=0,\o]{a_k*(x-x_0)^k} Exa

where the series converges and represents an analytic function for $0 \le |x - x_0| < d$. The exponent ρ_1, which determines the power of $x - x_0$ outside the series, is the larger root (or the root with the larger real part, if the roots are complex) of the *indicial equation*

$$\rho^2 + [p(x_0) - 1]\rho + q(x_0) = 0$$

The coefficients a_k are connected by a *recurrence relation* which is obtained by substituting the expansion of y_1 in the ODE and requiring that the final coefficient of every power of $x - x_0$ vanishes (Frobenius's method).

From y_1, a second linearly independent solution of the ODE can be found using

y_2=y_1(x)*\I[x]{e^(-R)/(y_1(x))^2)}, R=\I[x]{P(x)}

In series form y_2=κ*y_1(x)*lnx+(x-x_0)^(ρ_2)*\S[k=0,\o]{b_k*(x-x_0)^k} Exa

where $\kappa = 0$ if $\rho_1 - \rho_2$ is not an integer, $\kappa = 1$ if $\rho_1 = \rho_2$, and $\kappa =$ constant if $\rho_1 - \rho_2$ is a positive integer.

10 SERIES AND PRODUCTS

10.1 Definitions

A *sequence* $f(k)$ or $\{s_k\}$ is a function defined on a set of non-negative integers k with $k \geq k_0$. A translation can give $k_0 = 0$ or $k_0 = 1$.

A *series* $a_0 + a_1 + a_2 + a_3 + \cdots =$ lim[n,\o]\ S[k=0,n]{a_k} represents the sum of a finite or infinite number of terms, denoted briefly by Σa_k, and it *converges* if the sequence $\{s_n\}$ of the partial sums $s_n =$ \S[k=0,n]{a_k} has a (finite) limit s as $n \to \infty$. Otherwise, the series *diverges*.

Exa

If the series Σa_k converges, then lim[k,\ o]{a_k}. If lim[k,\ o]{a_k} or the limit does not exist, then the series diverges.

A series Σa_k *converges absolutely* if the series $|a_0| + |a_1| + |a_2| + \cdots = \Sigma|a_k|$ converges. If $\Sigma \alpha_k$ converges but $\Sigma|a_k|$ diverges, then Σa_k *converges conditionally*.

Adding a finite number of terms to a series does not affect its convergence.

Uniform convergence

If a_k is a function $a_k(x)$ of x, then we have a *series of functions* $\Sigma a_k(x)$ with partial sums $s_n(x)$ that depend on x. A series $\Sigma a_k(x)$ *converges uniformly* if the sequence of $s_n(x)$ converges uniformly to $s(x)$, i.e. if there exists a number N, such that for any given positive real number ε, we have $|s_n(x) - s(x)| < \varepsilon$ for all $n > N$, and all x in some interval. [Uniform convergence means essentially that N may depend on ε, but not on x.]

If a series $\Sigma a_k(x)$ converges uniformly on (a, b), then with $a < c < d < b$

(1) lim[x,c]{\S[k=0,\o]{a_k(x)}}= \S[k=0,\o]{lim[x,c]{a_k(x)}},

(2) the continuity of $a_k(x)$ implies continuity of $s(x)$,

(3) \I[x,c,d]{\S[k=0,\o]{a_k(x)}=\ S[k=0,\o]{\I[x,c,d]{a_k(x)}}

If $\Sigma a_k(x)$ converges to $s(x)$, $a_k'(x) = da_k/dx$ exist for $k = 0, 1, 2, \ldots$ and if $\Sigma a_k'(x)$ converges uniformly, then $s'(x) = \Sigma a_k'(x)$.

If a power series converges on $[a, b]$, then it converges uniformly and

lim[x,b-]{\S[k=0,\o]{a_k*x^k}=\S[k=0,\o]{a_k*b^k}}

10.2 Tests for Convergence

Exa

Let Σa_k and Σb_k be two series of real positive terms ($a_k > 0$, $b_k > 0$).

(1) If $a_k \le b_k$ and Σb_k converges, then Σa_k also converges. If $a_k \ge b_k$ and Σb_k diverges, then Σa_k also diverges (comparison test).

(2) If $\frac{a_{k+1}}{a_k} < \frac{b_{k+1}}{b_k}$ for sufficiently large k, and Σb_k converges, then Σa_k also converges (comparison of ratio test).

(3) If $\frac{a_{k+1}}{a_k} < L < 1$ for sufficiently large k, then Σa_k converges. If $\frac{a_{k+1}}{a_k} > 1$, then Σa_k diverges (ratio test).

(4) If $a_k^{1/k} < L < 1$ for sufficiently large k, then Σa_k converges. If $a_k^{1/k} > 1$, then Σa_k diverges (root test).

(5) Let $M_k = k\left(\frac{a_k}{a_{k+1}} - 1\right)$. If $M_k \ge M > 1$ for sufficiently large k (M is independent of k), then Σa_k converges. If $M_k \le 1$, then Σa_k diverges (Raabe's test).

(6) Let a constant h and a bounded function $g(k)$ exist for sufficiently large k, such that $\frac{a_k}{a_{k+1}} = 1 + \frac{h}{k} + \frac{g(k)}{k^2}$. If $h > 1$, Σa_k converges, while if $h \le 1$, Σa_k diverges (Gauss's test).

(7) If $a_k \le f(k)$, where $f(x)$ is a real positive decreasing continuous function of x for which the integral $\int_N^\infty f(x)\,dx$ exists for some finite integer N, then the series Σa_k converges. If the integral diverges, so does the series (integral test).

A series $\Sigma(-1)^k a_k$ of alternating positive and negative terms ($a_k > 0$) converges

(1) if $a_{k+1} < a_k$ and $\lim_{k\to\infty} a_k = 0$,

(2) if the sequence of the partial sums $s_n = \sum_{k=0}^{n} (-1)^k a_k$ is bounded and monotonic.

A series of functions $\Sigma a_k(x)$ converges uniformly in $[a, b]$,

(1) if there is a convergent series of constants Σh_k with $|a_k(x)| \le h_k$ (Weierstrass test),

(2) if $a_k(x) = c_k g_k(x)$, where $g_k(x)$ is a monotonic sequence [i.e. either $g_{k+1}(x) \le g_k(x)$ or $g_{k+1}(x) \ge g_k(x)$] of bounded functions [i.e. $g_k(x) < M$ for all x] and Σc_k converges (Abel's test).

10.3 Series of Constants

Arithmetic progression

Sequence: $a_1 = a, \quad a_2 = a + d, \quad a_3 = a + 2d, \quad \ldots, \quad a_n = a + (n-1)d$

Sum: $s_n = a_1 + a_2 + \cdots + a_n = \frac{1}{2}n(a_1 + a_n)$

Geometric progression

Sequence: $a_1 = a, \quad a_2 = ar, \quad a_3 = ar^2, \quad \ldots, \quad a_n = ar^{n-1}$

Sum: s_n=a_1+a_2+...+a_n=a*(1-r^n)/(1-r)

With $-1 < r < 1$ and $n \to \infty$, we have the sum of infinite terms

a+a*r+a*r^2+...=a/(1-r)

Series of integers

1+2+3+...+n=\S[k=1,n]{k}=n*(n+1) Pro

1+3+5+...+(2*n-1)=\S[k=1,n]{2*k-1}=n^2

1+8+16+...+8*n=1+\S[k=1,n]{8*k}=(2*n+1)^2

1^2+2^2+3^2+...+n^2=\S[k=1,n]{k^2}=n*(n+1)*(2*n+1)/6

1^3+2^3+3^3+...+n^3=\S[k=1,n]{k^3}=(1+2+...+n)^2=n^2*(n+1)^2/4

1^4+2^4+3^4+...+n^4=\S[k=1,n]{k^4}=n*(n+1)*(2*n+1)*(3*n^2+3*n-1)/30

1^m+2^m+3^m+...+n^m=\S[k=1,n]{k^m}=(B_(m+1)(n+1)-B_(m+1))/(m+1) Pro

1*1!+2*2!+3*3!...n*n!=\S[k=1,n]{k*k!}=(n+1)!-1

Inverse powers of integers

1-1/2+1/3-1/4+...=\S[k=1,\o]{(-1)^(k+1)/k}=
\S[k=0,\o]{(-1)^k/(k+1)}=ln2

1-1/3+1/5-1/7+...=\S[k=0,\o]{(-1)^k/(2*k+1)}=π/4

1-1/4+1/7-1/10+...=\S[k=0,\o]{(-1)^k/(3*k+1)}=π*3^(1/2)/9+ln2/3

1-1/5+1/9-1/13+...=\S[k=0,\o]{(-1)^k/(4*k+1)}=
π*2^(1/2)/8+2^(1/2)*ln(1+2^(1/2))/4

1/2-1/5+1/8-1/11+...=\S[k=1,\o]{(-1)^k/(3*k+2)}=
π*3^(1/2)/9-ln2/3

Cal

1/1^2+1/2^2+1/3^2+...=\S[k=1,\o]{1/k^2}=π^2/6

1/1^3+1/2^3+1/3^3+...=\S[k=1,\o]{1/k^3}=ζ(3)

1/1^4+1/2^4+1/3^4+...=\S[k=1,\o]{1/k^4}=π^4/90

1/1^5+1/2^5+1/3^5+...=\S[k=1,\o]{1/k^5}=ζ(5)

1/1^6+1/2^6+1/3^6+...=\S[k=1,\o]{1/k^6}=π^6/945

1/1^x+1/2^x+1/3^x+...=\S[k=1,\o]{1/k^x}=ζ(x)

Pro

1/1^2-1/2^2+1/3^2-...=\S[k=1,\o]{(-1)^(k+1)/k^2}=π^2/12

1/1^3-1/2^3+1/3^3-...=\S[k=1,\o]{(-1)^(k+1)/k^3}=3*ζ(3)/4

1/1^4-1/2^4+1/3^4-...=\S[k=1,\o]{(-1)^(k+1)/k^4}=7*π^4/720

1/1^6-1/2^6+1/3^6-...=\S[k=1,\o]{(-1)^(k+1)/k^6}=31*π^6/30240

1/1^x-1/2^x+1/3^x-...=\S[k=1,\o]{(-1)^(k+1)/k^x}=(1-2^(1-x))*ζ(x) Cal

1/1^2+1/3^2+1/5^2+...=\S[k=0,\o]{1/(2*k+1)^2}=π^2/8

1/1^3+1/3^3+1/5^3+...=\S[k=0,\o]{1/(2*k+1)^3}=7*ζ(3)/8

1/1^4+1/3^4+1/5^4+...=\S[k=0,\o]{1/(2*k+1)^4}=π^4/96

1/1^6+1/3^6+1/5^6+...=\S[k=0,\o]{1/(2*k+1)^6}=π^6/960

1/1^x+1/3^x+1/5^x+...=\S[k=0,\o]{1/(2*k+1)^x}=(1-1/2^x)*ζ(x) Cal

1/1^2-1/3^2+1/5^2-...=\S[k=0,\o]{(-1)^k/(2*k+1)^2}=G

[Catalan constant] Cal

1/1^3-1/3^3+1/5^3-...=\S[k=0,\o]{(-1)^k/(2*k+1)^3}=π^3/32

1/1^4-1/3^4+1/5^4-...=\S[k=0,\o]{(-1)^k/(2*k+1)^4}

1/1^3+1/3^3-1/5^3-1/7^3+...=
\S[k=0,\o]{(-1)^k/(1/(4*k+1)^3+1/(4*k+3)^3)}=3π^2*2^(1/2)/128

1/(1*2)+1/(2*3)+1/(3*4)+1/(4*5)+...=
\S[k=1,\o]{1/(k*(k+1))}=1

Cal

1/(1*3)+1/(3*5)+1/(5*7)+1/(7*9)+...=
\S[k=1,\o]{(1/((2*k-1)*(2*k+1))=1/2

```
1/(1*3)+1/(2*4)+1/(3*5)+1/(4*6)+...=
\S[k=1,\o]{(1/(k*(k+2))}=3/4

1/(1*3)+1/(2*5)+1/(3*7)+1/(4*9)+...=
\S[k=1,\o]{(1/(k*(2*k+1))=2-2*ln2

1/((a+b)*(2*a+b))+ 1/((2*a+b)*(3*a+b))+...=
\S[k=1,\o]{(1/((a*k+b)*(a*k+b+a))}=1/(a*(a+b))
```

Cal

```
1/(1^2*3^2)+1/(3^2*5^2)+1/(5^2*7^2)+...=
\S[k=1,\o]{(1/((2*k-1)*(2*k+1))^2}=(π^2-8)/16

1/(1^2*2^2*3^2)+1/(2^2*3^2*4^2)+1/(3^2*4^2*5^2)+...=
\S[k=1,\o]{(1/(k*(k+1)*(k+2))^2}=(4*π^2-39)/16
```

Cal

```
1/a-1/(a+b)+1/(a+2*b)-1/(a+3*b)+...=\I[0,1]{x^(a-1)/(1+x^b)}

1/1^(2*n)+1/2^(2*n)+1/3^(2*n)+...=\S[1,\o]{1/k^(2*n)}, B_(2*n)

1/1^(2*n)+1/3^(2*n)+1/5^(2*n)+...=\S[0,\o]{1/(2*k+1)^(2*n)}, B_(2*n)

1/1^(2*n)-1/2^(2*n)+1/3^(2*n)-...=\S[1,\o]{(-1)^(k+1)/k^(2*n)}, B_(2*n)

1/1^(2*n+1)-1/3^(2*n+1)+1/5^(2*n+1)-...=
\S[0,\o]{(-1)^k/(2*k+1)^(2*n+1)}, E_(2*n)
```

Cal

```
1/(2^1*1)+1/(2^2*2)+1/(2^3*3)+1/(2^4*4)+...=
\S[k=1,\o]{(1/(2^k*k)}=ln2

1/(2^1*1^2)+1/(2^2*2^2)+1/(2^3*3^2)+1/(2^4*4^2)+...=
\S[k=1,\o]{(1/(2^k*k^2)}=π^2/12-ln2^2/2

1/0!+1/1!+1/2!+1/3!+...=\S[k=0,\o]{(1/k!}=e

1/0!-1/1!+1/2!-1/3!+...=\S[k=0,\o]{(-1)^k/k!}=1/e
```

1/0!+1/2!+1/4!+1/6!+...=\S[k=0,\o]{(1/(2*k)!}=(e+e^(-1))/2

1/1!+1/3!+1/5!+1/7!+...=\S[k=0,\o]{(1/(2*k+1)!}=(e-e^(-1))/2

10.4 Series of Functions

Polynomials

$(x + a)^2 = x^2 + 2ax + x^2$

$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$

$(x + a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$

$(x + a)^5 = x^5 + 5ax^4 + 10a^2x^3 + 10a^3x^2 + 5a^4x + + a^5$

$(x + a)^6 = x^6 + 6ax^5 + 15a^2x^4 + 20a^3x^3 + 15a^4x^2 + 6a^5x + a^6$

$(x + a)^7 = x^7 + 7ax^6 + 21a^2x^5 + 35a^3x^4 + 35a^4x^3 + 21a^5x^2 + 7a^6x + a^7$

Sines and cosines

sinx+sin(2*x)+sin(3*x)+...+sin(n*x)=\S[k=1,n]{sin(k*x)}

cosx+cos(2*x)+cos(3*x)+...+cos(n*x)=\S[k=1,n]{cos(k*x)}

sinx+sin(3*x)+sin(5*x)+...+sin((2*n-1)*x)=
\S[k=1,n]{sin((2*k-1)*x)}=sin(n*x)^2/sinx

cosx+cos(3*x)+cos(5*x)+...+cos((2*n-1)*x)=
\S[k=1,n]{cos((2*k-1)*x)}=sin(2*n*x)/(2*sinx)

sinx-sin(3*x)+sin(5*x)-...+(-1)^(n+1)*sin((2*n-1)*x)=
\S[k=1,n]{(-1)^(k+1)*sin((2*k-1)*x)}=(-1)^(n+1)*sin(2*n*x)/(2*cosx)

cosx-cos(2*x)+cos(3*x)-...+(-1)^(n+1)*cos(n*x)=
\S[k=1,n]{(-1)^(k+1)*cos(k*x)}

Pro

sinx+2*sin(2*x)+3*sin(3*x)+...+n*sin(n*x)=\S[k=1,n]{k*sin(k*x)}

cosx+2*cos(2*x)+3*cos(3*x)+...+n*cos(n*x)=\S[k=1,n]{k*cos(k*x)}

r*sinx+r^2*sin(2*x)+r^3*sin(3*x)+...+r^n*sin(n*x)=\S[k=1,n]{r^k*sin(k*x)}

1+r*cosx+r^2*cos(2*x)+r^3*cos(3*x)+...+r^n*cos(n*x)=\S[k=0,n]{r^k*cos(k*x)}

r*sinx+r^2*sin(2*x)+r^3*sin(3*x)+...=\S[k=1,\o]{r^k*sin(k*x)}

1+r*cosx+r^2*cos(2*x)+r^3*cos(3*x)+...=\S[k=0,\o]{r^k*cos(k*x)}

Pro

Taylor and Maclaurin series

If $f(x)$ has continuous derivatives up to order n, then

f(x)=f(a)+f'(a)*(x-a)+f"(a)*(x-a)^2/2!+...+f^((n-1))(a)*(x-a)^(n-1)/(n-1)!+R_n

where R_n is the *balance* or *remainder* after n terms and, with an appropriate ξ between a and x, it is written as

R_n=f^((n))(ξ)*(x-a)^n/n! (Lagrange's formula) or R_n=f^((n))(ξ)*(x-ξ)^(n-1)*(x-a)/(n-1)! (Cauchy's formula)

If $\lim_{x\to\infty} R_n = 0$, we obtain the *Taylor series* or *Taylor expansion* of $f(x)$ for $x = a$. If $a = 0$, the series is often called *a Maclaurin series*. A Taylor (or Maclaurin) series generally converges for any x in an interval, which is called the *interval of convergence*, and it diverges outside this interval.

For functions of two variables, we have

$$f(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$$
$$+ \frac{1}{2!}[f_{xx}(a, b)(x-a)^2 + 2f_{xy}(a, b)(x-a)(y-b) + f_{yy}(a, b)(y-b)^2] + \cdots$$

where $f_x(a, b)$, $f_y(a, b)$, … are the partial derivatives with respect to $x, y, \ldots$ evaluated at the point $x = a$, $y = b$.

Binomial expansions

\S{1/(1+x)}

\S{1/(1+x)^2}

Cal

\S{1/(1+x)^3}

$(1+x)^{-4} = 1 - 4x + 10x^2 - 20x^3 + 35x^4 - \cdots, \qquad -1 < x < 1$

$(1+x)^{-5} = 1 - 5x + 15x^2 - 35x^3 + 70x^4 - \cdots, \qquad -1 < x < 1$

$(1+x)^{-6} = 1 - 6x + 21x^2 - 56x^3 + 126x^4 - \cdots, \qquad -1 < x < 1$

\S{1/(1+x)^(1/2)}

\S{(1+x)^(1/2)}

\S{1/(1+x)^(1/3)}

\S{(1+x)^(1/3)}

\S{1/(1+x)^(1/4)}

\S{(1+x)^(1/4)}

\S{1/(1+x)^(3/2)}

\S{(1+x)^(3/2)}

\S{1/(1+x)^(5/2)}

\S{(1+x)^(5/2)}

For $n = 1, 2, 3, \ldots$ and any x

(x+a)^n=x^n+n*a*x^(n-1)+n*(n-1)*a^2*x^(n-2)/2!+...=
x^n+(n;1)*a*x^(n-1)+(n;2)*a^2*x^(n-2)+...(n;n)*a^n=
\S[k=0,n]{(n;k)*a^k*x^(n-k)

For any real r and $-1 < x < 1$

(1+x)^r=1+r*x+r*(r-1)*x^2/2!+...=\S[k=0,\o]{(r;k)*x^k}

Ext

Rational functions

\S{(1+x)/(1-x)^2}, \S[k=0,\o]{(2*k+1)*x^k}

\S{(1+x)/(1-x)^3}, \S[k=0,\o]{(k+1)^2*x^k}

\S{(1+6*x+x^2)/(1-x)^3}, \S[k=0,\o]{(2*k+1)^2*x^k}

\S{(a+(b-a)*x)/(1-x)^2}, \S[k=0,\o]{(a+k*b)*x^k}

Trigonometric functions

\S{sinx}

\S{1/sinx}, B_(2*k)

Cal

\S{1/sinx}

\S{1/sinx^2}

\S{cosx}

\S{1/cosx}, E_(2*k) Cal

\S{1/cosx}

\S{1/cosx^2}

\S{tanx}, B_(2*k) Cal

\S{tanx}

\S{cotx}, B_(2*k)

\S{cotx}

\S{sin^(-1)x}

\S{cos^(-1)x}=\S{π/2-sin^(-1)x}

\S{tan^(-1)x} Cal

\S{cot^(-1)x}=\S{π/2-tan^(-1)x}

\S{sin^(-1)x^2}

Exponential and logarithmic functions

\S{e^x}=\S[k=0,\o]{x^k/k!}

\S{a^x}=\S{e^(x*lna)}

\S{ln(1+x)}

\S{ln|(1+x)/(1-x)|}

\S{lnx}

\S{lnx}

Cal

\S{ln(1+(1+x^2)^(1/2))}

\S{ln(1+x)/(1+x)}

\S{e^sinx}

\S{e^cosx}

\S{e^tanx}

\S{e^x*sinx}

\S{e^x*cosx}

Cal

\S{ln|sinx|}, ln|x|, B_(2*k)

\S{ln|cosx|}, B_(2*k)

Cal

\S{ln|tanx|}, ln|x|, B_(2*k)

Hyperbolic functions

\S{sinhx}, \S[k=0,\o]{x^(2*k+1)/(2*k+1)!}

\S{1/sinhx}, B_(2*k) Cal

\S{1/sinhx}

\S{coshx}, \S[k=0,\o]{x^(2*k)/(2*k)!}

\S{1/coshx}, E_(2*k) Cal

\S{1/coshx}

\S{tanhx}, B_(2*k) Cal

\S{tanhx}

\S{cothx}, B_(2*k) Cal

\S{cothx}

\S{sinh^(-1)x}, ln|x|

\S{cosh(-1)x}, ln|x|

\S{tanh^(-1)x}, \S[k=0,\o]{x^(2*k+1)/(2*k+1)}

\S{coth^(-1)x}, \S[k=0,\o]{1/((2*k+1)*x^(2*k+1))}

Cal

Multiplication of series

If f(x)=\S[k=0,\o]{a_k*x^k} and g(x)=\S[k=0,\o]{b_k*x^k}, then

f(x)*g(x), \S[k=0,\o]{\S[i=0,k]{a_(k-i)*b_i}x^k}

Inversion of series

If $$y = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + gx^7 + \cdots \qquad (a \neq 0)$$

then $$x = Ay + By^2 + Cy^3 + Dy^4 + Ey^5 + Fy^6 + Gy^7 + \cdots$$

where $aA = 1,\qquad a^3B = -b,$

$$a^5C = 2b^2 - ac, \qquad a^7D = 5abc - 5b^3 - a^2d,$$

$$a^9E = 6a^2bd + 3a^2c^2 - a^3e + 14b^4 - 21ab^2c,$$

$$a^{11}F = 7a^3be + 84ab^3c + 7a^3cd - 28a^2bc^2 - a^4f - 28a^2b^2d - 42b^5,$$

$$a^{13}G = 8a^4bf + 8a^4ce + 4a^4d^2 + 120a^2b^3d + 180a^2b^2c^2 + 132b^6 - a^5g - 36a^3b^2e - 72a^3bcd - 12a^3c^3 - 330ab^4c.$$

10.5 Infinite Products

Definitions

Let $\{g_k\}$ with $g_k > 0$ be a sequence of constants or functions defined for $k \geq 1$ (or $k \geq 0$). We form the sequence of *partial products*

p_n=...=\P[k=1,n]{g_k}

and define the *infinite product*

P=\P[k=1,\o]{g_k}=lim[n,\o]{p_n}

denoted briefly by Πg_k. If the limit exists, is finite and different than zero, then the infinite product *converges*. If the limit does not exist, is infinite, or is equal to zero, the infinite product *diverges*. Often we use the notation $g_k = 1 + a_k$ and write

P=lim[n,\o]{\P[k=1,n]{1+a_k}}

or briefly, $\Pi(1 + a_k)$. If $\Pi(1 + |a_k|)$ converges, $\Pi(1 + a_k)$ is said to converge *absolutely*.

Since $\ln(\Pi g_k) = \Sigma \ln g_k$, the convergence of an infinite product can be tested from the convergence of an infinite series. Some useful theorems include the following:

1. Πg_k converges if and only if $\Sigma \ln g_k$ converges.
2. If the infinite product $\Pi(1 + a_k)$ converges, then $a_k \to 0$ as $k \to \infty$.
3. If $0 \leq a_k < 1$ and Σa_k converges, then $\Pi(1 + a_k)$ and $\Pi(1 - a_k)$ converge. Alternatively, if $0 \leq a_k < 1$ and Σa_k diverges, then $\Pi(1 + a_k)$ and $\Pi(1 - a_k)$ diverge.

Numerical products

(1+1/1)*(1-1/2)*(1+1/3)...=\P[k=1,\o]{1-(-1)^k/k}=1

(1-1/2^2)*(1-1/3^2)*(1-1/4^2)...=\P[k=2,\o]{1-1/k^2}=1/2

(1-2/(2^3+1))*(1-2/(3^3+1)*(1-2/(4^3+1))...=\P[k=2,\o]{1-2/(k^3+1)}=2/3

(1+1/1)*(1-1/3)*(1+1/5)...=\P[k=1,\o]{1-(-1)^k/(2*k-1)}=2^(1/2)

(1-1/2^2)*(1-1/4^2)*(1-1/6^2)...=\P[k=1,\o]{1-1/(2*k)^2}=2/π

(1-1/3^2)*(1-1/5^2)*(1-1/7^2)...=\P[k=1,\o]{1-1/(2*k+1)^2}=π/4

(1+1/2^2)*(1+1/3^2)*(1+1/4^2)...=\P[k=2,\o]{1+1/k^2}=sinhπ/π

(2*2/(1*3))*(4*4/(3*5))*(6*6/(5*7))...=\P[k=1,\o]{1+1/((2*k-1)*(2*k+1))}=π/2

[Wallis's formula]

Products of functions

For expansions of $x^{2n}-y^{2n}$, $x^{2n}+y^{2n}$, $x^{2n+1}-y^{2n+1}$, $x^{2n+1}+y^{2n+1}$ see Sec. 2.1.

\P{1/(1-x)}

\P{sinx}

\P{cosx}

\P{sinhx}

\P{coshx}

\P{sinx/x}=...=\P[k=1,\o]{cos(x/2^k)}

\P{cos(π*x/4)-sin(π*x/4)}

\P{x!}

11 FOURIER SERIES

11.1 Definitions

Let $f(x)$ be a function defined on the interval $I = (c, c + 2L)$, where c and L are constants. The corresponding *Fourier series* $F(x)$ is a function defined in $(-\infty, \infty)$ by

F(x)=a_0/2+\S[k=1,\o]{a_k*cos(k*π*x/L)+b_k*sin(k*π*x/L)} **Ext**

where ($k = 0, 1, 2, \ldots$)

a_k=(1/L)*\I[x,c,c+2*L]{f(x)*cos(k*π*x/L)}, b_k=(1/L)*\I[x,c,c+2*L]{f(x)*sin(k*π*x/L)}

Obviously, $F(x)$ is defined in $(-\infty, \infty)$ but depends only on the values of $f(x)$ in I.

Convergence of *F*(*x*)

If $f(x)$ is defined on the interval I (except perhaps at a finite number of points), and $f(x)$ and $f'(x)$ are piecewise continuous, then at any point of I the Fourier series $F(x)$ converges. If $f(x)$ is continuous at x, then $F(x) = f(x)$. If $f(x)$ is discontinuous at x, then $F(x) = \frac{1}{2}\{f(x+0) + f(x-0)\}$. Usually, $I = (0, 2\pi)$ or $I = (-\pi, \pi)$. **Inf**

If $f(x)$ is defined outside I by its periodic extension, i.e. $f(x + 2L) = f(x)$, then the Fourier series $F(x)$ converges as above to $f(x)$ or to $\frac{1}{2}\{f(x+0) + f(x-0)\}$ at any point of $(-\infty, \infty)$. At a point $x = c \pm 2nL$, where $n = 0, 1, \ldots$, the Fourier series $F(x)$ converges to $\frac{1}{2}\{f(c+0) + f(c+2L-0)\}$. Thus, in all cases the values of $F(x)$ in $(-\infty, \infty)$ are determined by the values of $f(x)$ in $I = (c, c + 2L)$.

The previous are *sufficient* but *not neccessary* conditions. They are known as *Dirichlet conditions* and can be found in various formulations. **The**

Complex form of Fourier series

Using complex numbers, we can write the Fourier series of $f(x)$ in its complex form

F(x)=\S[k,-\o,\o]{c_k*e^(i*k*π*x/L)}

where c_k=(1/(2*L))*\I[x,c,c+2*L]{f(x)*e^(-i*k*π/L)} **Ext**

At the points of discontinuity, $F(x)$ converges always to $\frac{1}{2}\{f(x+0) + f(x-0)\}$.

11.2 Properties

Even and odd functions

If $f(x)$ is odd, i.e. if $f(-x) = -f(x)$, then the Fourier series $F(x)$ includes only sines. If $f(x)$ is even, i.e. if $f(-x) = f(x)$, then the Fourier series $F(x)$ includes only cosines and (possibly) a constant term.

Parseval's identities

Let the coefficients of the Fourier series of $f(x)$ and $g(x)$ be a_k, b_k and α_k, β_k, respectively. These coefficients satisfy *Parseval's identities*:

(1/L)*\I[x,c,c+2*L]{f(x)^2}=
a_0^2/2+\S[k=1,\o]{a_k^2+b_k^2}

(1/L)*\I[x,c,c+2*L]{f(x)*g(x)}=
a_0*α_0/2+\S[k=1,\o]{a_k*α_k+b_k*β_k}

Pro

Convergence

Let $f^{(i)}(x)$, $i = 0, \ldots, n$, satisfy the Dirichlet conditions, be bounded in $(c, c + 2L)$, and have $f^{(i)}(c + 0) = f^{(i)}(c + 2L - 0)$ for $i = 0, 1, \ldots, n - 1$. Then the Fourier coefficients f_k (a_k or b_k) of $f(x)$ satisfy $|f_k| < A/k^{n+1}$, where A is some positive constant. Near discontinuities, the *Gibbs phenomenon* dominates the behavior of $F(x)$. **Ext**

If a Fourier series $\Sigma u_k(x)$ converges uniformly in an interval (a, b), then it converges uniformly to a continuous function $f(x)$.

Differentiation of Fourier series

Let the Fourier series of $f(x)$ be $\Sigma u_k(x)$. If the series $\Sigma u_k'(x)$ of the derivatives $u_k'(x) = du_k(x)/dx$ converges uniformly, then

(d/dx){\S[,]{u_k(x)}=\S[,]{u_k'(x)}

i.e. the interchange of the order of differentiation and summation is allowed.

Integration of Fourier series

If the Fourier series $\Sigma u_k(x)$ of $f(x)$ converges, then \S[,]{\I[x]{u_k(x)}} also converges and we have

\I[x]{\S[,]{u_k(x)}}=
\S[,]{\I[x]{u_k(x)}}

and

\I[x,a,b]{\S[,]{u_k(x)}}=
\S[,]{\I[x,a,b]{u_k(x)}}

i.e. the interchange of the order of integration and summation is allowed.

11.3 Applications

Boundary value problems

Natural phenomena that evolve in time or extend in two or more dimensions are described simply by linear partial differential equations with w as the dependent and x, y as the independent variables. In many cases, separation of variables gives solutions of the form $w = \Sigma g_k(x)h_k(y)$, where $g_k(x)$ [or $h_k(y)$] satisfies the ordinary differential equation $g_k''(x) + \omega^2(k)g_k = 0$ with solutions $\cos\omega(k)x$ and $\sin\omega(k)x$. Hence, w is essentially a Fourier series with respect to x with coefficients that depend on y.

The displacement $y(x, t)$ at a point x of a string of length L satisfies the wave equation (\d/\dt)^2y= a^2*(\d/\dx)^2y. If at $t = 0$ the shape of the string is given by $y(x, 0) = f(x)$, we expand $y(x, t)$ and $f(x)$ in Fourier series so that the initial condition is fulfilled. Exa

The temperature $u(x, t)$ along a thin bar of length L is described by (\d/\dt)u= κ*(\d/\dx)^2u. If at $t = 0$ the distribution of the temperature is given by $f(x)$, we expand $f(x)$ in a Fourier series so that $u(x, 0) = \Sigma g_k(x)h_k(0) = f(x)$. Exa

The electric potential in a rectangle satisfies the Laplace equation (\d/\dx)^2V+ (\d/\dy)^2V=0. Each side of the rectangle is kept at a prescribed potential. We want $V(x, y)$. Exa

Calculation of series of constants

Since $F(x) = f(x)$ at a point of continuity, or $F(x) = \frac{1}{2}\{f(x + 0) + f(x - 0)\}$ at a point of discontinuity of $f(x)$, we can obtain equations of the form $F(x_0)$ = const., where $F(x_0)$ is a series of constants. Thus, we can evaluate series of constants using the Fourier series of known functions. The following are examples: Exa

$f(x) = |x|$, $-\pi \le x < \pi$, gives $F(-\pi) = f(-\pi) = \pi$ from which \S[k=1,\o]{1/(2*k-1)^2}=π^2/8.

$f(x) = |\sin x|$, $-\pi \le x < \pi$, gives $F(0) = f(0) = 0$ from which \S[k=1,\o]{1/((2*k-1)*(2*k+1))}=1/2.

$f(x) = \ln[\cos(x/2)]$, $-\pi < x < \pi$, gives $F(0) = f(0) = 0$ from which \S[k=1,\o]{(-1)^(k+1)/k}=ln2.

f(x)=π^4/90-π^2*x^2/12+π*x^3/12-x^4/48, $0 \le x \le 2\pi$, gives F(0)=f(0)=π^4/90, \S[k=1,\o]{1/k^4}=π^4/90.

$f(x) = \cosh ax$, $-\pi < x < \pi$, gives $F(0) = f(0) = 1$ and \S[k=1,\o]{(-1)^k/(k^2+a^2)}=(a*π-sinh(a*π))/(2*a^2*sinh(a*π))}.

11.4 Tables of Fourier series

In each case below, the following are given: the function $f(x)$ with the interval $I = (c, c+2L)$, the Fourier series $F(x)$, the discontinuities x_d of $f(x)$, and the values $F(x_d)$, the graphs of $f(x)$ in red and F_n(x)=a_0/2+\S[k=1,n]{a_k*cos(k*π*x/L)+b_k*sin(k*π*x/L)} in green.

Odd functions

f(x)=-1:-π<x<0;1:0<x<π

\F{-1:-π<x<0;1:0<x<π}

$x_d = n\pi,\ \ F(x_d) = 0$

Graphs of $f(x)$ and $F_{10}(x)$

Fig. 11-1: $f(x)$ ——, $F_{10}(x)$ ——

$f(x) = x, \quad -\pi < x < \pi$

\F{x:-π<x<π}

Cal

$x_d =$ (2n\F{π, $F(x_d) = 0$

Graphs of $f(x)$ and $F_{10}(x)$

Fig. 11-2: $f(x)$ ——, $F_{10}(x)$ ——

f(x)=-(π+x)/2:-π\<x<0;
(π-x)/2:0<x\<π}

\F(x)={-(π+x)/2:-π\<x<0;(π-
x)/2:0<x\<π}

Cal

$x_d = 2n\pi,\ \ F(x_d) = 0$

Graphs of $f(x)$ and $F_{10}(x)$

Fig. 11-3: $f(x)$ ——, $F_{10}(x)$ ——

f(x)=-x^2:-π<x<0;x^2:0\<x<π

\F{-x^2:-π<x<0;x^2:0\<x<π}

$x_d = (2n+1)\pi,\ \ F(x_d) = 0$

Graphs of $f(x)$ and $F_{10}(x)$

Fig. 11-4: $f(x)$ ——, $F_{10}(x)$ ——

f(x)=-cosx:-π<x<0;cosx:0<x<π

\F{-cosx:-π<x<0;cosx:0<x<π}

$x_d = n\pi,\ \ F(x_d) = 0$

Graphs of $f(x)$ and $F_{30}(x)$

Fig. 11-5: $f(x)$ —, $F_{30}(x)$ —

$f(x) = x(\pi - x)(\pi + x), \quad -\pi \leq x < \pi$

\F{x*(π-x)*(π+x):-π\<x<π}

x_d : None.

Graphs of $f(x)$ and $F_3(x)$

Fig. 11-6: $f(x)$ —, $F_3(x)$ —

f(x)=x*(π+x):-π<x<0;x*(π-x):0<x<π

\F{x*(π+x):-π<x<0;x*(π-x):0<x<π}

x_d : None.

Graphs of $f(x)$ and $F_2(x)$

Fig 11-7: $f(x)$ —, $F_2(x)$ —

$f(x) = \sin ax, \quad -\pi < x < \pi, \quad a \neq$ integer

\F{sin(a*x):-π<x<π}

$x_d = (2n+1)\pi,\ \ F(x_d) = 0$

Graphs of $f(x)$ and $F_{10}(x)$

Fig. 11-8: $a = 1.5$, $f(x)$ —, $F_{10}(x)$ —

$f(x) = \sinh ax, \quad -\pi < x < \pi$ **Cal**

\F{sinh(a*x):-π<x<π}

$x_d = (2n+1)\pi,\ \ F(x_d) = 0$

Graphs of $f(x)$ and $F_{20}(x)$

Fig. 11-9: $a = 2$, $f(x)$ —, $F_{20}(x)$ —

Even functions

f(x)=0:-π\<x<-a;
b:-a<x<a;0:a<x<π

\F{0:-π\<x<-a;
b:-a<x<π;0:a<x<π}

Cal

$x_d = 2n\pi \pm a,\ F(x_d) = b/2$

Graphs of $f(x)$ and $F_{10}(x)$

Fig. 11-10: $2ab = 1$, $f(x)$ ——, $F_{10}(x)$ ——

f(x)=|x|:-π\<x<π

\F{|x|:-π\<x<π}

x_d : None.

Graphs of $f(x)$ and $F_5(x)$

Fig. 11-11: $f(x)$ ——, $F_5(x)$ ——

$f(x) = x^2, \quad -\pi < x \le \pi$

\F{x^2:-π<x\<π}

x_d : None

Graphs of $f(x)$ and $F_5(x)$

Fig. 11-12: $f(x)$ ——, $F_5(x)$ ——

f(x)=-x*(π+x):-π\<x\<0;
x*(π-x):0\<x<π

\F{-x*(π+x):-π\<x\<0;
x*(π-x):0\<x<π}

Cal

x_d : None.

Graphs of $f(x)$ and $F_{10}(x)$

Fig. 11-13: $f(x)$ ——, $F_{10}(x)$ ——

$f(x) = |\sin x|, \quad -\pi \le x < \pi$

\F{|sinx|:-π\<x<π}

Cal

x_d : None.

Graphs of $f(x)$ and $F_5(x)$

Fig. 11-14: $f(x)$ ——, $F_5(x)$ ——

```
f(x)=1:-π<x<-π+a;
0:-π+a<x<π-a;1:π-a<x<π
```

```
\F{1:-π<x<-π+a;
0:-π+a<x<π-a;1:π-a<x<π}
```

$x_d = (2n+1)\pi \pm a,\ \ F(x_d) = 0.5$

Graphs of $f(x)$ and $F_{10}(x)$

Fig. 11-15: $a = 1$, $f(x)$ ——, $F_{10}(x)$ ——

$f(x) = \cos ax, \quad -\pi \le x < \pi, \ \ a \ne \text{integer}$

```
\F{cos(a*x):-π\<x<π}
```

x_d : None. **Cal**

Graphs of $f(x)$ and $F_5(x)$

Fig. 11-16: $a = 1.5$, $f(x)$ ——, $F_5(x)$ ——

$f(x) = \cosh ax, \quad -\pi \le x < \pi$

```
\F{cosh(a*x):-π\<x<π}
```

x_d : None.

Graphs of $f(x)$ and $F_{10}(x)$

Fig. 11-17: $a = 1$, $f(x)$ ——, $F_{10}(x)$ ——

$f(x) = \ln|\sin \frac{1}{2}x|, \quad -\pi \le x \le \pi, \ \ x \ne 0$

```
\F{ln|sin(x/2)|:-π\<x\<π}
```

Cal

$x_d = 2n\pi,\ \ F(x_d)$ diverges.

Graphs of $f(x)$ and $F_{10}(x)$

Fig. 11-18: $f(x)$ ——, $F_{10}(x)$ ——

$f(x) = \ln(\cos \frac{1}{2}x), \quad -\pi < x < \pi$

```
\F{ln(cos(x/2)):-π<x<π}
```

Cal

$x_d = (2n+1)\pi,\ \ F(x_d)$ diverges.

Graphs of $f(x)$ and $F_{10}(x)$

Fig. 11-19: $f(x)$ ——, $F_{10}(x)$ ——

Other functions **Inf**

$f(x) = x, \qquad 0 < x < 2\pi$

```
\F{x:0<x<2*π}
```

Cal

$x_d = 2n\pi,\ F(x_d) = \pi$

Graphs of $f(x)$ and $F_{10}(x)$

Fig. 11-20: $f(x)$ ——, $F_{10}(x)$ ——

$f(x) = \frac{1}{6}\pi^2 - \frac{1}{2}\pi x + \frac{1}{4}x^2, \qquad 0 \le x \le 2\pi$

```
\F{π^2/6-π*x/2+x^2/4:0\<x\<2*π}
```

x_d : None.

Graphs of $f(x)$ and $F_5(x)$

Fig. 11-21: $f(x)$ ——, $F_5(x)$ ——

```
f(x)=π^2*x/6-π*x^2/4+x^3/12
```

, $0 \le x \le 2\pi$

```
\F{π^2*x/6-π*x^2/4+x^3/12:0\<x\<2*π}
```

x_d : None.

Graphs of $f(x)$ and $F_3(x)$

Fig. 11-22: $f(x)$ ——, $F_3(x)$ ——

```
f(x)=π^4/90-π^2*x^2/12+π*x^3/12-x^4/48
```

$0 \le x \le 2\pi$

```
\F{π^4/90-π^2*x^2/12+π*x^3/12-x^4/48:0\<x\<2*π}
```

x_d : None.

Graphs of $f(x)$ and $F_3(x)$

Fig. 11-23: $f(x)$ ——, $F_3(x)$ ——

```
f(x)=π^4*x/90-π^2*x^3/36+π*x^4/48-x^5/240
```

$0 \le x \le 2\pi$

```
\F{π^4*x/90-π^2*x^3/36+π*x^4/48-x^5/240:0\<x\<2*π}
```

x_d : None.

Graphs of $f(x)$ and $F_3(x)$

Fig. 11-24: $f(x)$ ——, $F_3(x)$ ——

f(x)=0:-π<x<0;sinx:0\<x\<π

\F{0:-π<x<0;sinx:0\<x\<π}

x_d : None.

Graphs of $f(x)$ and $F_5(x)$

Fig. 11-25: $f(x)$ ——, $F_5(x)$ ——

$f(x) = e^{ax}, \quad -\pi < x < \pi$ App

\F{e^(a*x):-π<x<π}

$x_d = (2n+1)\pi,\ F(x_d) = \cosh a\pi$

Graphs of $f(x)$ and $F_{10}(x)$

Fig. 11-26: $a = 1$, $f(x)$ ——, $F_{10}(x)$ ——

f(x)=1/(1-2*a*cosx+a^2):-π\<x<π

\F{1/(1-2*a*cosx+a^2):-π\<x<π} Cal

x_d : None.

Graphs of $f(x)$ and $F_5(x)$

Fig. 11-27: $a = \frac{1}{2}$, $f(x)$ ——, $F_5(x)$ ——

f(x)=a*sinx/(1-2*a*cosx+a^2):-π\<x<π

\F{a*sinx/(1-2*a*cosx+a^2):-π\<x<π} Cal

x_d : None.

Graphs of $f(x)$ and $F_5(x)$

Fig. 11-28: $a = \frac{1}{2}$, $f(x)$ ——, $F_5(x)$ ——

f(x)=(1-a*cosx)/(1-2*a*cosx+a^2):-π\<x<π

\F{(1-a*cosx)/(1-2*a*cosx+a^2):-π\<x<π} Cal

x_d : None.

Graphs of $f(x)$ and $F_5(x)$

Fig. 11-29: $a = \frac{1}{2}$, $f(x)$ ——, $F_5(x)$ ——

$f(x) = \ln(1 - 2a\cos x + a^2),$
$-\pi \le x < \pi, \; |a| < 1$

\F{ln(1-2*a*cosx+a^2):-π\<x<π}

Cal

x_d : None.

Graphs of $f(x)$ and $F_3(x)$

Fig. 11-30: $a = \frac{1}{2}$, $f(x)$ ——, $F_3(x)$ ——

f(x)={ln((1+2*a*cosx+a^2)/(1-2*a*cosx+a^2):-π\<x<π

\F{ln((1+2*a*cosx+a^2)/(1-2*a*cosx+a^2):-π\<x<π}

Cal

x_d : None.

Graphs of $f(x)$ and $F_3(x)$

Fig. 11-31: $a = \frac{1}{2}$, $f(x)$ ——, $F_3(x)$ ——

f(x)=tan^(-1)(a*sinx/(1-a*cosx)):-π\<x<π

\F{tan^(-1)(a*sinx/(1-a*cosx)):-π\<x<π}

Cal

x_d : None.

Graphs of $f(x)$ and $F_3(x)$

Fig. 11-32: $a = \frac{1}{2}$, $f(x)$ ——, $F_3(x)$ ——

f(x)=(1/2)*tan^(-1)(2*a*sinx/(1-a^2)):-π\<x<π

\F{(1/2)*tan^(-1)(2*a*sinx/(1-a^2)):-π\<x<π}

Cal

x_d : None.

Graphs of $f(x)$ and $F_3(x)$

Fig. 11-33: $a = \frac{1}{2}$, $f(x)$ ——, $F_3(x)$ ——

f(x)=(1/2)*tan^(-1)(2*a*cosx/(1-a^2)):-π\<x<π

\F{(1/2)*tan^(-1)(2*a*cosx/(1-a^2)):-π\<x<π}

Cal

x_d : None.

Graphs of $f(x)$ and $F_3(x)$

Fig. 11-34: $a = \frac{1}{2}$, $f(x)$ ——, $F_3(x)$ ——

12 VECTOR ANALYSIS

12.1 Definitions

Scalars and vectors

A quantity that is expressed by only one (usually real) number is a *scalar*. Examples are the density and the temperature at a point.

A quantity that is expressed by an ordered set of (usually real) numbers is a *vector* (more precisely, a vector follows some specific transformation rules from one system of coordinates to another). Examples are the velocity of a moving fluid at a point and the gravitational field at a point in space (in Newtonian Mechanics). Ext

A vector is represented by an arrow. In a Euclidean space the *magnitude* $A = |\mathbf{A}|$ of a vector is its length and the *direction* is the arrow's direction. A *unit vector* is a vector that has a magnitude of 1. The

Components of a vector

In a Cartesian system of coordinates, the unit vectors in the directions of the axes x, y, and z are denoted $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively. The three components of the vector $\mathbf{A}$ are $A_1\mathbf{i}$, $A_2\mathbf{j}$, $A_3\mathbf{k}$. We have $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and

A=|A|=(A_1^2+A_2^2+A_3^2)^(1/2)

Components of a vector

Fig. 12-1

12.2 Summation, Subtraction and Multiplication

If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are vectors and r, s scalars, then

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	Commutative law for addition
$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$	Associative law for addition
$r\mathbf{A} = \mathbf{A}r, \quad r(s\mathbf{A}) = (rs)\mathbf{A} = s(r\mathbf{A})$	Rules for multiplication with scalar
$(r + s)\mathbf{A} = r\mathbf{A} + s\mathbf{A}$	Distributive law
$r(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B}$	Distributive law

These properties are also presented in Fig.12-2.

Summation, subtraction and multiplication

Fig. 12-2

12.3 Products of Vectors

Dot or scalar product

Ext

$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta \qquad 0 \le \theta \le \pi$

where $A = |\mathbf{A}|$, $B = |\mathbf{B}|$ and θ is the angle between **A** and **B**.

Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$. Then

$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$ Commutative law

$\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$ Distributive law

$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$

$r(\mathbf{A}\cdot\mathbf{B}) = (r\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(r\mathbf{B}) = (\mathbf{A}\cdot\mathbf{B})r$

$A_1 = \mathbf{A}\cdot\mathbf{i}, \ \ A_2 = \mathbf{A}\cdot\mathbf{j}, \ \ A_3 = \mathbf{A}\cdot\mathbf{k}$

$\mathbf{A}\cdot\mathbf{A} = \mathbf{A}^2 = A^2 = A_1^{\,2} + A_2^{\,2} + A_3^{\,2} \ge 0$

$\mathbf{A}\cdot\mathbf{B} \le |\mathbf{A}||\mathbf{B}| = AB$ Schwarz inequality

$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \quad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$ Ext

The *direction cosines* of **A** are

cosα=A*i/A=A_1/A, cosβ=A*j/A=A_2/A, cosγ=A*k/A=A_3/A

where α, β, and γ are the angles of **A** with the positive semiaxes Ox, Oy, and Oz.

Two vectors, **A** and **B**, are *orthogonal* to each other if $\theta = 90°$ or $\mathbf{A}\cdot\mathbf{B} = 0$ (the two statements are equivalent since the zero vector can have any orientation).

Cross or vector product

$$\mathbf{A} \times \mathbf{B} = AB \sin\theta \, \mathbf{n}, \qquad 0 \le \theta \le \pi$$

where $A = |\mathbf{A}|$, $B = |\mathbf{B}|$, θ is the angle between **A** and **B** and **n is** the unit vector perpendicular to the plane of **A** and **B**, so that **A**, **B**, **n** constitute a right-handed system.

Fig. 12-3

A\xB=det(i,j,k;A_1,A_2,A_3;B_1,B_2,B_3)
=(A_2*B_3-A_3*B_2)*i+(A_3*B_1-A_1*B_3)*j+(A_1*B_2-A_2*B_1)*k

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}, \quad \mathbf{A} \times \mathbf{A} = \mathbf{0}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

$$(r\mathbf{A}) \times (s\mathbf{B}) = rs(\mathbf{A} \times \mathbf{B})$$

$$r(\mathbf{A} \times \mathbf{B}) = (r\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (r\mathbf{B}) = (\mathbf{A} \times \mathbf{B})r$$

$|\mathbf{A} \times \mathbf{B}|$ = area of parallelogram with sides **A** and **B**

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}, \quad \mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

Other products

Scalar triple product

$[\mathbf{ABC}] = \mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) = \mathbf{B}\cdot(\mathbf{C} \times \mathbf{A}) = \mathbf{C}\cdot(\mathbf{A} \times \mathbf{B}) =$ det(A_1,A_2,A_3; B_1,B_2,B_3; C_1,C_2,C_3)

$$= A_1B_2C_3 + A_2B_3C_1 + A_3B_1C_2 - A_3B_2C_1 - A_2B_1C_3 - A_1B_3C_2$$

$|\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C})|$ = volume of parallelepiped with sides **A**, **B**, **C**

Vector triple product

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$$

$$(\mathbf{A} \times \mathbf{B})\cdot(\mathbf{C} \times \mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = \{\mathbf{A}\cdot(\mathbf{B} \times \mathbf{D})\}\mathbf{C} - \{\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C})\}\mathbf{D}$$
$$= \{\mathbf{A}\cdot(\mathbf{C} \times \mathbf{D})\}\mathbf{B} - \{\mathbf{B}\cdot(\mathbf{C} \times \mathbf{D})\}\mathbf{A}$$

12.4 Derivatives of a Vector

If $\mathbf{A}(t) = A_1(t)\mathbf{i} + A_2(t)\mathbf{j} + A_3(t)\mathbf{k}$ is a vector function of t, the derivative of **A** with respect to the (scalar) variable t is

dA/dt=...=(dA_1/dt)*i+(dA_2/dt)*j+(dA_3/dt)*k

d(A+B)/dt=...=dA/dt+dB/dt, d(c*A)=c*(dA/dt)

d(f(t)*A)/dt=(df(t)/dt)*A+f(t)*(dA/dt), d(A(u(t))/dt=(dA/du)*(du/dt)

d(A*B)/dt=A*(dB/dt)+(dA/dt)*B

d(A\xB)/dt=A\x(dB/dt)+(dA/dt)\xB

d([ABC])/dt=[(dA/dt)BC]+[A(dB/dt)C]+[AB(dC/dt)]

A*(dA/dt)=A*(dA/dt)

A*(dA/dt)=0 , if $|\mathbf{A}|$ or **A** is independent of t.

If $\mathbf{A} = \mathbf{A}(x, y, z)$ is a *vector field*, i.e. a vector function of the position, the partial derivatives \dA/\dx, \dA/\dy, \dA/\dz are defined in a similar manner.

12.5 Gradient, Divergence, Curl

Ext

In Cartesian coordinates, the *del operator* is defined by the relation

\D=i*(\d/\dx)+j*(\d/\dy)+k*(\d/\dz)

If Φ is a scalar field (i.e. a scalar function of the position), **A is** a vector field (i.e. a vector function of the position) of the independent variables x, y, z, and the partial derivatives exist, then we define the following:

Gradient of $\Phi = \nabla\Phi$

$$= \left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\Phi = \frac{\partial \Phi}{\partial x}\mathbf{i}+\frac{\partial \Phi}{\partial y}\mathbf{j}+\frac{\partial \Phi}{\partial z}\mathbf{k}$$

Divergence of $\mathbf{A} = \text{div}\mathbf{A} = \nabla\cdot\mathbf{A}$ Ext

$$= \left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}) = \frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}$$

A vector field $\mathbf{A}$ is *solenoidal* if $\nabla\cdot\mathbf{A} = 0$.

Curl of $\mathbf{A} = \text{curl}\mathbf{A}$

$$= \nabla\times\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times(A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\mathbf{k}$$

A vector field $\mathbf{A}$ is *irrotational* if $\nabla\times\mathbf{A} = \mathbf{0}$.

The *directional derivative* (the derivative of Φ in the direction of the unit vector $\mathbf{a} = \mathbf{A}/|\mathbf{A}|$) is defined by the relation

$$(\mathbf{a}\cdot\nabla)\Phi = \frac{(\mathbf{A}\cdot\nabla)\Phi}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\left(A_1\frac{\partial \Phi}{\partial x}+A_2\frac{\partial \Phi}{\partial y}+A_3\frac{\partial \Phi}{\partial z}\right)$$

The same formula is valid if Φ is replaced by a vector field $\mathbf{A}$.

The *Laplacian operator* ∇^2 is defined by the relation

$$\nabla^2\Phi = \nabla\cdot\nabla\Phi = \frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}+\frac{\partial^2 \Phi}{\partial z^2}$$

$$\nabla^2\mathbf{A} = \frac{\partial^2 \mathbf{A}}{\partial x^2}+\frac{\partial^2 \mathbf{A}}{\partial y^2}+\frac{\partial^2 \mathbf{A}}{\partial z^2}$$

The *biharmonic operator* ∇^4 is defined by the relation

$$\nabla^4\Phi = \nabla^2(\nabla^2\Phi) = \frac{\partial^4 \Phi}{\partial x^4}+\frac{\partial^4 \Phi}{\partial y^4}+\frac{\partial^4 \Phi}{\partial z^4}+2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \Phi}{\partial y^2}\right)+2\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 \Phi}{\partial z^2}\right)+2\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 \Phi}{\partial x^2}\right)$$

For expressions of the $\nabla\Phi$, $\nabla\cdot\mathbf{A}$, $\nabla\times\mathbf{A}$, and $\nabla^2\Phi$ in other systems of coordinates see Sec. 13.2-13.3.

Calculations with the del operator

Exa

If Φ, Ψ are scalar functions and **A**, **B** are vector functions of x, y, z (i.e. scalar and vector fields in the three dimensional Euclidean space) and their partial derivatives exist, we have the following formulas:

$\nabla(\Phi + \Psi) = \nabla\Phi + \nabla\Psi$

$\nabla(a\Phi) = a\nabla\Phi \quad (a = \text{constant})$

$\nabla\cdot(\mathbf{A} + \mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B}$

$\nabla\cdot(a\mathbf{A}) = a\nabla\cdot\mathbf{A}$

$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$

$\nabla \times (a\mathbf{A}) = a\nabla \times \mathbf{A}$

$\nabla(\Phi\Psi) = \Psi(\nabla\Phi) + \Phi(\nabla\Psi)$

$\nabla\cdot(\Phi\mathbf{A}) = (\nabla\Phi)\cdot\mathbf{A} + \Phi(\nabla\cdot\mathbf{A})$

$\nabla \times (\Phi\mathbf{A}) = (\nabla\Phi) \times \mathbf{A} + \Phi(\nabla \times \mathbf{A})$

$\nabla\cdot(\mathbf{A} \times \mathbf{B}) = \mathbf{B}\cdot(\nabla \times \mathbf{A}) - \mathbf{A}\cdot(\nabla \times \mathbf{B})$

$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$

$(\mathbf{A}\cdot\nabla)\mathbf{B} = \frac{1}{2}[\nabla \times (\mathbf{B} \times \mathbf{A}) + \nabla(\mathbf{A}\cdot\mathbf{B}) + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) - \mathbf{A} \times (\nabla \times \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{A})]$

$(\mathbf{A}\cdot\nabla)(\Phi\mathbf{B}) = \mathbf{B}(\mathbf{A}\cdot\nabla\Phi) + \Phi(\mathbf{A}\cdot\nabla)\mathbf{B}$

$\nabla \times (\nabla\Phi) = \mathbf{0}$, i.e. the curl of the gradient of Φ is zero.

$\nabla\cdot(\nabla \times \mathbf{A}) = 0$, i.e. the divergence of the curl of **A** is zero.

$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

$\nabla^2(\Phi\Psi) = \Phi\nabla^2\Psi + 2(\nabla\Phi)\cdot(\nabla\Psi) + \Psi\nabla^2\Phi$

Differential equations of mathematical physics

Laplace's equation	$\nabla^2\Phi = 0$
Poisson's equation	$\nabla^2\Phi = \rho$
Wave equation	(\D)^2{Φ}-c^(-2)*(\d/\dt)^2{Φ}=0
Heat equation	(\D)^2{Φ}-γ^(-2)*(\dΦ/\dt)=0
Helmholtz's equation	$\nabla^2\Phi + k^2\Phi = 0$
Maxwell's equations	\D*E=ρ/ε, \D\xE+\dB/\dt=0, \D*B=0, \D\xB-μ*ε*\dE/\dt=μ*J

12.6 Integrals with Vectors

Indefinite and definite integrals

If dB(t)/dt=A(t) the *indefinite integral* of **A**(t) is

\I[t]{A(t)}=B(t)+c (**c** = constant vector)

The *definite integral* of **A**(t) from a to b is

\I[t,a,b]{A(t)}=B(b)-B(a)

Line integrals

In a connected open domain D of a two or three dimensional Euclidean space, we have a piecewise smooth curve C, a scalar field Φ, and a vector field **A** with continuous first derivatives. The curve C connects two points $P_1(a_1, b_1, c_1)$ and $P_2(a_2, b_2, c_2)$ and is divided into n segments by $n - 1$ interior points (x_1, y_1, z_1), (x_2, y_2, z_2), …, $(x_{n-1}, y_{n-1}, z_{n-1})$. We also identify point P_2 as $P_2(x_n, y_n, z_n)$. We define several integrals.

Line integral

Fig. 12-4

The *line integral* of Φ on C is

\I_C{Φ*ds}=\I[s,P_1, P_2]{Φ}

where $\Delta s_k = [(\Delta x_k)^2 + (\Delta y_k)^2 + (\Delta z_k)^2]^{1/2}$, $\Delta x_k = x_{k+1} - x_k$, $\Delta y_k = y_{k+1} - y_k$, $\Delta z_k = z_{k+1} - z_k$ with all $\Delta s_k \to 0$ when $n \to \infty$.

The *line integral* of **A** on C is

\I_C{A*dr}=\I[r,P_1, P_2]{A*dr}

Exa

where $\Delta \mathbf{r}_k = \Delta x_k \mathbf{i} + \Delta y_k \mathbf{j} + \Delta z_k \mathbf{k}$, $\Delta x_k = x_{k+1} - x_k$, $\Delta y_k = y_{k+1} - y_k$, $\Delta z_k = z_{k+1} - z_k$ with $|\Delta \mathbf{r}_k| \to 0$ when $n \to \infty$.

The *line integrals* \I_C{Φ*dr}, \I_C{A*ds}, \I_C{A\xdr} are defined in a similar way.

The line integrals can be written in several forms. If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and the curve C has a parametric representation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ with tangent unit vector $\mathbf{T} = d\mathbf{r}/ds$ and normal unit vector $\mathbf{N} = (d\mathbf{T}/dt)/|d\mathbf{T}/dt|$, then the scalar field Φ may depend on $\mathbf{A}$, $\mathbf{T}$ and $\mathbf{N}$, e.g. $\Phi = \mathbf{A}\cdot\mathbf{N}$.

For the line integral of **A** we can write

\I_C{A*dr}=\I_C{A*T*ds}=...

where a prime means differentiation with respect to t. Thus the line integral is reduced to a definite integral with respect to t. Two useful properties are the following

\I_C[r,P_1, P_2]{A*dr}=-\I_C[r,P_2, P_1]{A*dr}

\I_C[r,P_1, P_2]{A*dr}=\I_C[r,P_1, P_3]{A*dr}+\I_C[r,P_3, P_2]{A*dr}

In general, the line integral depends on the integration curve. The necessary and sufficient condition for the line integral to be independent of the integration curve (i.e. depends only on the end points P_1 and P_2) is that in D

$$\nabla \times \mathbf{A} = 0$$

This means that there has to be a scalar function Φ such that $\mathbf{A} = \nabla\Phi$. In this case, the field **A** is called *conservative* or *irrotational* with *potential function* Φ and we have

\I_C[r,P_1, P_2]{A*dr}=Φ(P_2)-Φ(P_1) **Exa**

Surface integrals

In a connected open domain D of the three dimensional Euclidean space (Fig. 12-5), we have a piecewise smooth surface S, a scalar field Φ, and a vector field **A** with continuous partial derivatives. The surface S is divided into n small segments ΔS_k of area ΔE_k, $k = 1, 2, \ldots, n$, and $\mathbf{N}_k$ is the unit normal at some point (x_k, y_k, z_k) of ΔS_k. We define the *surface integral* of Φ on S

\I_S{Φ*dS}=...

where we suppose that ΔS_k shrinks to a point when $n \to \infty$.

Surface integral

Fig. 12-5

We define the *surface integral* of **A** on S (S' is the projection of S on xOy)

\I_S{A*dS}=\I_S'{(A*N/|N*k|)*dx*dy}=...

Also, \I_S{A\xdS}=... is defined in a similar way.

Theorems of Gauss, Stokes and Green

In a three dimensional Euclidean space, let S be a piecewise smooth oriented closed surface (Fig. 12-6), which encloses a bounded, simply connected region V. Let **N** be the unit vector normal to S toward the outside and $d\mathbf{S} = \mathbf{N}dS$. Then, according to *Gauss's theorem*, for a vector field **A** with continuous partial derivatives, we have

\I_V{\D*A*dV}=\I_S{A*dS}

Gauss's theorem

Fig. 12-6

Let S be a piecewise smooth oriented open surface whose boundary is a piecewise smooth simple closed curve C (Fig. 12-7), and $d\mathbf{S} = \mathbf{N}dS$ (**N** the unit vector normal to S). Then according to *Stokes's theorem*, for a vector field **A** with continuous partial derivatives we have

\I_C{A*dr}=\I{\D\xA*dS}

where the line integral on the closed curve C has been obtained with the appropriate direction (someone walking on S, on the side of **N** and close to C, has the inside of S at his left).

Stokes's theorem

Fig. 12-7

If D is a domain of the xy plane containing a piecewise smooth and simple closed curve C and its interior R, then according to *Green's theorem in the plane*, we have

\I_C{A*T*ds}=\I_C{P*dx+Q*dy}=\I_R{(\dQ/\dx-\dP/\dy)*dx*dy} **Ext**

This can be obtained from Stokes's theorem with $\mathbf{A} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ and C a closed curve in the xy plane.

Another form of *Green's theorem in the plane* is obtained if we replace P with $-Q$ and Q with P. In the xy plane the outward unit normal to C is $\mathbf{N} = (dy\,\mathbf{i} - dx\,\mathbf{j})/ds$, so we have

\I_C{A*N*ds}=\I_C{P*dy-Q*dx}=\I_R{(\dP/\dx+\dQ/\dy)*dx*dy} Exa

In both expressions of Green's theorem in the plane, the integral on C is taken in the positive (counterclockwise) sense.

If Φ and Ψ are scalar fields, then $\mathbf{A} = \Phi\nabla\Psi$ is a vector field and Gauss's theorem implies *Green's first theorem*

\I_V{(Φ*(\D)^2{Ψ}+\DΦ*\DΨ)*dV}=\I_S{(Φ*\DΨ)*dS}

Similarly, setting $\mathbf{A} = \Phi\nabla\Psi - \Psi\nabla\Phi$ in Gauss's theorem we obtain *Green's second theorem*

\I_V{(Φ*(\D)^2Ψ-Ψ(\D)^2Φ)*dV}=\I_S{(Φ*\DΨ-Ψ*\DΦ)*dS}

Two more integral formulas in vector form can be obtained from Gauss's theorem. Setting $\mathbf{A} = \Phi\mathbf{B}$, where Φ is a scalar field and $\mathbf{B}$ an arbitrary constant vector, we obtain

\I_V{\DΦ*dV}=\I_S{(Φ*dS}

Also from Gauss's theorem, replacing $\mathbf{A}$ with $\mathbf{A} \times \mathbf{B}$, where $\mathbf{B}$ is again an arbitrary constant vector, we find

\I_V{\D\xA*dV}=\I_S{dS\xA}

Two more integral formulas in vector form can be obtained from Stokes's theorem. Setting $\mathbf{A} = \Phi\mathbf{B}$, where Φ is a scalar field and $\mathbf{B}$ an arbitrary constant vector, we obtain

\I_C{Φ*dr}=\I_S{dS\x\DΦ}

Also from Stokes's theorem, replacing $\mathbf{A}$ with $\mathbf{A} \times \mathbf{B}$, where $\mathbf{B}$ is again an arbitrary constant vector, we find

\I_C{dr\xA}=\I_S{(dS\x\D)\xA}

13 CURVILINEAR COORDINATES

13.1 General Definitions

The position of a point P in the three dimensional Euclidean space can be determined by the *rectangular* (or *orthogonal*) *Cartesian coordinates* (x, y, z), which are measured along three straight lines, normal to each other. Thus $Oxyz$ in Fig. 13-1 is a right handed rectangular Cartesian system of coordinates. If

Fig. 13-1

$$u_1 = u_1(x, y, z)$$
$$u_2 = u_2(x, y, z)$$
$$u_3 = u_3(x, y, z)$$

are three functions of x, y, z, the equations $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$ (c_1, c_2, c_3 are three constants) define three families of two dimensional surfaces. We assume that each pair of these surfaces intersects at a curve along which u_1, u_2, or u_3 varies. Thus, a point can also be defined by the *curvilinear coordinates* (u_1, u_2, u_3). The two systems of coordinates are connected by the above three (continuously differentiable) *transformation equations* or their inverses

$$x = x(u_1, u_2, u_3), \quad y = y(u_1, u_2, u_3), \quad z = z(u_1, u_2, u_3)$$

The *Jacobian of the transformation* (which is a 3×3 determinant)

\d(x,y,z)/\d(u_1,u_2,u_3)=
det(\dx/\du_1,\dx/\du_2,\dx/\du_3;
\dy/\du_1,\dy/\du_2,\dy/\du_3;\dz/\du_1,\dz/\du_2,\dz/\du_3)

is assumed to be different from zero, so that the transformation equations are inversible. The system of coordinates u_1, u_2, u_3, remains Cartesian (but generally not rectangular), if and only if the transformation equations are linear.

If u_2 and u_3 remain constant and u_1 is the only one that changes, the point P that is defined by the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, describes a curve that is called the *coordinate curve* u_1 passing through the point P. The coordinate curves u_2 and u_3 that pass through the point P are defined in a similar manner.

If u_1 is the only one that remains constant and u_2, u_3 change, the point P describes a two dimensional surface, which is called the *coordinate surface* u_1. The coordinate surfaces u_2 and u_3 are defined in a similar manner.

The vectors $\partial\mathbf{r}/\partial u_1$, $\partial\mathbf{r}/\partial u_2$, $\partial\mathbf{r}/\partial u_3$ are tangent on the coordinate curves u_1, u_2, u_3. If $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the unit vectors that are tangent on these curves, then

\dr/\du_1=h_1*e_1, \dr/\du_2=h_2*e_2, \dr/\du_3=h_3*e_3

The vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ form a *local base* of vectors, which we assume to be right handed. Any vector can be written as $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$. The quantities

h_1=|\dr/\du_1|, h_2=|\dr/\du_2|, h_3=|\dr/\du_3|

are called *scale factors*. If any two of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are perpendicular (i.e. if $\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$), the curvilinear system of coordinates is *orthogonal*.

Differentials

In curvilinear coordinates, the *elementary vector displacement* is

dr=(\dr/\du_1)*du_1+(\dr/\du_2)*du_2+(\dr/\du_3)*du_3
=h_1*du_1*e_1+h_2*du_2*e_2+h_3*du_3*e_3

If the coordinate surfaces are perpendicular to each other, as they are assumed to be from here on, then the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are also perpendicular to each other. We thus have an orthogonal coordinate system and the *line element* ds (> 0) is given by

$$ds^2 = d\mathbf{r}\cdot d\mathbf{r} = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$ Ext

Although u_1, u_2, u_3 may not be lengths, $ds_i = h_i du_i$ ($i = 1, 2, 3$) must be lengths.

The *area element* on the surface u_1 = const. is $dS_{23} = h_2 h_3 du_2 du_3$. Similarly, on u_2 = const. it is $dS_{31} = h_3 h_1 du_3 du_1$ and on u_3 = const. it is $dS_{12} = h_1 h_2 du_1 du_2$. The *vector area element* is

$$d\mathbf{S} = h_3 h_2 du_3 du_2 \mathbf{e}_1 + h_1 h_3 du_1 du_3 \mathbf{e}_2 + h_2 h_1 du_2 du_1 \mathbf{e}_3$$

The *volume element* is

dV=h_1*h_2*h_3*du_1*du_2*du_3=...
=|\d(x,y,z)/\d(u_1,u_2,u_3)|*du_1*du_2*du_3

Line, surface and volume integrals

In the following, Φ is a scalar function and $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ is a vector function of the orthogonal curvilinear coordinates u_1, u_2, u_3 (both A_1, A_2, A_3 and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ may depend on u_1, u_2, u_3).

Since $d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz = ds_1\mathbf{e}_1 + ds_2\mathbf{e}_2 + ds_3\mathbf{e}_3 = h_1du_1\mathbf{e}_1 + h_2du_2\mathbf{e}_2 + h_3du_3\mathbf{e}_3$, a line integral along a curve C described by $u_i = u_i(t)$ $(i = 1, 2, 3)$ takes the form

\I_C{A*dr}=\I_C[t]{(A_1*h_1*(du_1/dt)+A_2*h_2*(du_2/dt)+A_3*h_3*(du_3/dt))*dt}

Similarly, a surface integral takes the form

\I_S{A*dS}=\I_S{A_1*h_2*h_3*du_2*du_3+A_2*h_3*h_1*du_3*du_1+A_3*h_1*h_2*du_1*du_2}

For a volume integral we have $dV = ds_1ds_2ds_3 = h_1h_2h_3du_1du_2du_3$ and thus

\I_R{Φ*dV}=\I_R{Φ(x,y,z)*dx*dy*dz}=...

where the transformation $(x, y, z) \to (u_1, u_2, u_3)$ maps the region $\mathcal{R}$ onto $\mathcal{R}'$ and gives $\Psi(u_1, u_2, u_3)$ from $\Phi(x, y, z)$ after replacing x, y, z.

13.2 Gradient, Divergence, Curl

Let Φ be a scalar function and let $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ be a vector function of the orthogonal curvilinear coordinates u_1, u_2, u_3. The del operator is defined as

\D=(1/h_1)*e_1*(\d/\du_1)+(1/h_2)*e_2*(\d/\du_2)+(1/h_3)*e_3*(\d/\du_3)

Its actions on scalar or vector fields can be expressed in general as follows:

Gradient of Φ=gradΦ=\DΦ=(1/h_1)*(\dΦ/\du_1)*e_1+(1/h_2)*(\dΦ/\du_2)*e_2+(1/h_3)*(\dΦ/\du_3)*e_3

Divergence of A=divA=\D*A=(1/(h_1*h_2*h_3))*((\d/\du_1){h_2*h_3*A_1}+(\d/\du_2){h_3*h_1*A_2}+(\d/\du_3){h_1*h_2*A_3})

Curl of A=curlA=\D\xA=
(1/(h_1*h_2*h_3))*det(h_1*e_1,h_2*e_2,h_3*e_3;
\d/\du_1,\d/\du_2,\d/\du_3;
h_1*A_1,h_2*A_2,h_3*A_3)=...

Directional derivative = (A*\D){Φ}=(A_1/h_1)*\dΦ/\du_1+(A_2/h_2)*\dΦ/\du_2+(A_3/h_3)*\dΦ/\du_3

Laplacian of Φ=(\D)^2{Φ}=(1/(h_1*h_2*h_3))((\d/\du_1){(h_2*h_3/h_1)*\dΦ/\du_1}+(\d/\u_2){(h_3*h_1/h_2)*\dΦ/\du_2}+(\d/\u_3){(h_1*h_2/h_3)*\dΦ/\du_3})

From the same relation the biharmonic operator $\nabla^4\Phi = \nabla^2(\nabla^2\Phi)$ is obtained.

13.3 Various Coordinate Systems

Cylindrical coordinates (ρ, φ, z)

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = z$$

$$\rho = (x^2 + y^2)^{1/2}, \ \varphi = \cos^{-1}(x/\rho) = \sin^{-1}(y/\rho)$$

with $0 \le \rho < \infty, \ 0 \le \varphi < 2\pi, \ -\infty < z < \infty$.

$$h_1 = h_\rho = 1, \ h_2 = h_\varphi = \rho, \ h_3 = h_z = 1$$

$$\mathbf{e}_1 = \mathbf{e}_\rho, \ \mathbf{e}_2 = \mathbf{e}_\varphi, \ \mathbf{e}_3 = \mathbf{e}_z, \ \mathbf{A} = A_\rho\mathbf{e}_\rho + A_\varphi\mathbf{e}_\varphi + A_z\mathbf{e}_z$$

$$d\mathbf{r} = d\rho\,\mathbf{e}_\rho + \rho d\varphi\,\mathbf{e}_\varphi + dz\,\mathbf{e}_z$$

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$

$$dV = \rho\, d\rho\, d\varphi\, dz$$

$$d\mathbf{S} = \rho\, d\varphi dz\,\mathbf{e}_\rho + d\rho dz\,\mathbf{e}_\varphi + \rho\, d\rho d\varphi\,\mathbf{e}_z$$

Cylindrical coordinates

Fig. 13-2

\DΦ=(\dΦ/\dρ)*e_ρ+(1/ρ)*(\dΦ/\dφ)*e_φ+(\dΦ/\dz)*e_z

\D*A=(1/ρ)*(\d(ρ*A_ρ)/\dρ)+(1/ρ)*(\dA_φ/\dφ)+\dA_z/\dz

\D\xA=(1/ρ)*(\dA_z/\dφ-\d(ρ*A_φ)/\dz)*e_ρ+(\dA_ρ/\dz-\dA_z/\dρ)*e_θ+(1/ρ)*(\d(ρ*A_φ)/\dρ-\dA_ρ/\dφ)*e_φ

(A*\D){Φ}=A_ρ*\dΦ/\dρ+(A_φ/ρ)*\dΦ/\dφ+A_z*\dΦ/\dz

(\D)^2{Φ}=(\d/\dρ)^2{Φ}+(1/ρ)*\dΦ/\dρ+(1/ρ^2)*(\d/\dφ)^2{Φ}+(\d/\dz)^2{Φ}

Ext

Spherical coordinates (r, θ, φ)

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta$$

$$r = (x^2 + y^2 + z^2)^{1/2}, \quad \theta = \cos^{-1}(z/r),$$

φ=cos^(-1)(x/(x^2+y^2)^(1/2))=sin^(-1)(y/(x^2+y^2)^(1/2))

with $0 \le r < \infty, \ 0 \le \theta \le \pi, \ 0 \le \varphi < 2\pi.$

$$h_1 = h_r = 1, \quad h_2 = h_\theta = r, \quad h_3 = h_\varphi = r\sin\theta$$

$$\mathbf{e}_1 = \mathbf{e}_r, \ \mathbf{e}_2 = \mathbf{e}_\theta, \ \mathbf{e}_3 = \mathbf{e}_\varphi, \ \mathbf{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_\varphi\mathbf{e}_\varphi$$

$$d\mathbf{r} = dr\,\mathbf{e}_r + r\,d\theta\,\mathbf{e}_\theta + r\sin\theta\,d\varphi\,\mathbf{e}_\varphi$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\varphi^2$$

$$d\mathbf{S} = r^2\sin\theta\,d\theta\,d\varphi\,\mathbf{e}_r + r\sin\theta\,dr\,d\varphi\,\mathbf{e}_\theta + r\,dr\,d\theta\,\mathbf{e}_\varphi$$

$$dV = r^2\sin\theta\,dr\,d\theta\,d\varphi$$

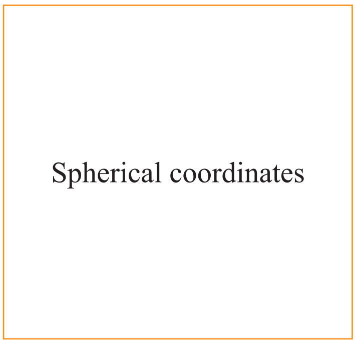

Fig. 13-3

\DΦ=(\dΦ/\dr)*e_r+(1/r)*(\dΦ/\dθ)*e_θ+(1/(r*sinθ))*(\dΦ/\dφ)*e_φ

\D*A=(1/(r^2*sinθ))((\d/\dr){A_r*r^2*sinθ}+(\d/\dθ){A_θ*r*sinθ}+(\d/\dφ){A_φ*r})

\D\xA=(1/(r^2*sinθ))((\d/\dθ){A_φ*r*sinθ}-(\d/\dφ){A_θ*r})*e_r+
(1/(r*sinθ))*((\d/\dφ){A_r}-(\d/\dr){A_φ*r*sinθ}*e_θ)+
(1/r)*((\d/\dr){A_θ*r}-(\d/\dθ){A_r})*e_φ

(A*\D){Φ}=A_r*\dΦ/\dr+(A_θ/r)*\dΦ/\dθ+(A_φ/(r*sinθ))*\dΦ/\dφ

(\D)^2{Φ}=(1/r^2)(\d/\dr){r^2*\dΦ/\dr}+(1/(r^2*sinθ))*\d/\dθ{sinθ*\dΦ/\dθ}+(1/(r^2*sinθ^2))*(\d/\dφ)^2{Φ}

Ext

Parabolic cylindrical coordinates (u, v, z)

$$x = \tfrac{1}{2}(u^2 - v^2), \quad y = uv, \quad z = z$$

with $-\infty < u < \infty, \quad 0 \le v < \infty, \quad -\infty < z < \infty$

$h_1 = h_u = h_2 = h_v =$ (u^2+v^2)^(1/2), $\quad h_3 = h_z = 1$

(\D)^2{Φ}=(1/(u^2+v^2))((\d/\du)^2{Φ}+(\d/\dv)^2{Φ})+(\d/\dz)^2{Φ}

Parabolic cylindrical coordinates

Fig. 13-4

The coordinate surfaces z = const. are planes. On each such plane, the traces of the u = const. and v = const. coordinate surfaces form two families of confocal parabolas with equations $x = \tfrac{1}{2}(y^2/v^2 - v^2)$ and $x = \tfrac{1}{2}(u^2 - y^2/u^2)$, plotted in Fig. 13-4.

Parabolic coordinates (u, v, φ)

$$x = uv\cos\varphi, \quad y = uv\sin\varphi, \quad z = \tfrac{1}{2}(u^2 - v^2)$$

with $0 \le u < \infty, \; 0 \le v < \infty, \; 0 \le \varphi < 2\pi$

$h_1 = h_u = h_2 = h_v =$ (u^2+v^2)^(1/2), $h_3 = h_z = uv$

(\D)^2{Φ}=(1/(u^2+v^2))((1/u)*(\d/\du){u*\dΦ/\du}+(1/v)*(\d/\dv){v*\dΦ/\dv})+(1/(u^2*v^2))*(\d/\dφ)^2{Φ}

Parabolic coordinates

Fig. 13-5

The coordinate surfaces result from the rotation of the parabolas in Fig. 13-4 around the x axis, which then becomes z. The traces of coordinate surfaces on planes that pass through the new z axis are given in Fig. 13-5.

Elliptic cylindrical coordinates (u, v, z)

$$x = a\cosh u\cos v, \quad y = a\sinh u\sin v, \quad z = z$$

with $0 \le u < \infty, \quad 0 \le v < 2\pi, \quad -\infty < z < \infty$

$h_1 = h_u = h_2 = h_v = a$ (sinhu^2+sinv^2)^(1/2), $\; h_3 = h_z = 1$

(\D)^2{Φ}=(1/(a^2(sinhu^2+sinv^2)))*((\d/\du)^2{Φ}+(\d/\dv)^2{Φ})+(\d/\dz)^2{Φ}

Elliptic cylindrical coordinates

Fig. 13-6

The traces of the coordinate surfaces on a z = const. plane are confocal ellipses and hyperbolas. The equations of those families are

x^2/coshu^2+y^2/sinhu^2=a^2 and x^2/cosv^2-y^2/sinv^2=a^2

from which we obtain the curves in Fig. 13-6 for various values of u and v.

Oblate spheroidal coordinates (ξ, η, φ)

$$x = a\cosh\xi\cos\eta\cos\varphi, \quad y = a\cosh\xi\cos\eta\sin\varphi,$$
$$z = a\sinh\xi\sin\eta$$

with $0 \le \xi < \infty, \; -\pi/2 \le \eta \le \pi/2, \; 0 \le \varphi < 2\pi$

$$h_1 = h_\xi = h_2 = h_\eta = a(\sinh^2\xi + \sin^2\eta)^{1/2},$$
$$h_3 = a\cosh\xi\cos\eta$$

Setting $w^2 = a^2(\sinh^2\xi + \sin^2\eta)$ we have

$$\nabla^2\Phi = \frac{1}{w^2\cosh\xi}\frac{\partial}{\partial\xi}\left(\cosh\xi\frac{\partial\Phi}{\partial\xi}\right) + \frac{1}{w^2\cos\eta}\frac{\partial}{\partial\eta}\left(\cos\eta\frac{\partial\Phi}{\partial\eta}\right) + \frac{1}{a^2\cosh^2\xi\cos^2\eta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

Oblate spheroidal coordinates

Fig. 13-7

Two families of coordinate surfaces result from the rotation of Fig. 13-6 around its y axis, which then becomes z. The third family of coordinate surfaces consists of planes that include this axis. In a plane that includes the new z axis, the coordinate curves (Fig. 13-7) are given for various values of ξ and η by the equations

$$\frac{\rho^2}{\cosh^2\xi} + \frac{z^2}{\sinh^2\xi} = a^2 \quad \text{and} \quad \frac{\rho^2}{\cos^2\eta} - \frac{z^2}{\sin^2\eta} = a^2 \quad \text{where } \rho = (x^2 + y^2)^{1/2}$$

Prolate spheroidal coordinates (ξ, η, φ)

$$x = a\sinh\xi\sin\eta\cos\varphi, \quad y = a\sinh\xi\sin\eta\sin\varphi,$$
$$z = a\cosh\xi\cos\eta$$

with $0 \le \xi < \infty, \quad 0 \le \eta \le \pi, \quad 0 \le \varphi < 2\pi$

$$h_1 = h_\xi = h_2 = h_\eta = a(\sinh^2\xi + \sin^2\eta)^{1/2},$$
$$h_3 = h_\varphi = a\sinh\xi\sin\eta$$

Setting $w^2 = a^2(\sinh^2\xi + \sin^2\eta)$ we have

$$\nabla^2\Phi = \frac{1}{w^2}\frac{\partial}{\partial\xi}\left(\sinh\xi\frac{\partial\Phi}{\partial\xi}\right) + \frac{1}{w^2\sin\eta}\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial\Phi}{\partial\eta}\right) + \frac{1}{a^2\sinh^2\xi\sin^2\eta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

Prolate spheroidal coordinates

Fig. 13-8

Two families of coordinate surfaces result from the rotation of Fig. 13-6 around its x axis, which then becomes z. The third family of coordinate surfaces consists of planes that include this axis. In a plane that includes the new z axis, the coordinate curves (Fig. 13-8) are given for various values of ξ and η by the equations

$$\frac{\rho^2}{\sinh^2\xi} + \frac{z^2}{\cosh^2\xi} = a^2 \quad \text{and} \quad \frac{z^2}{\cos^2\eta} - \frac{\rho^2}{\sin^2\eta} = a^2 \quad \text{where } \rho = (x^2 + y^2)^{1/2}$$

Bipolar cylindrical coordinates (u, υ, z)

x=a*sinhv/(coshv-cosu), y=a*sinu/(coshv-cosu), z=z

with $0 \le u < 2\pi$, $-\infty < \upsilon < \infty$, $-\infty < z < \infty$

h_1=h_u=h_2=h_v=a/(coshv-cosu), h_3=h_z=1

(\D)^2{Φ}=((coshv-cosu)^2/a^2)*((\d/\du)^2{Φ}+(\d/\dv)^2{Φ})+(\d/\dz)^2Φ

Since x^2+(y-a*cotu)^2=a^2/sinu^2 and (x-a*cothv)^2+y^2=a^2/sinhv^2, the traces of the cylindrical coordinate surfaces u = const. and υ = const. on the planes z = const. are circles as in Fig. 13-9.

Bipolar cylindrical coordinates

Fig. 13-9

Toroidal coordinates (u, υ, φ)

x=(a/w)*sinhv*cosφ, y=(a/w)*sinhv*sinφ, z=(a/w)*sinu

with $0 \le u < 2\pi$, $0 \le \upsilon < \infty$, $0 \le \varphi < 2\pi$

and $w = \cosh \upsilon - \cos u$.

h_1=h_u=h_2=h_v=a/w, h_3=h_φ=a*sinhv/w

(\D)^2{Φ}=(w^3/a^2)*(\d/\du){(1/w)*\dΦ/\du}+(w^3/(a^2*sinhv))*(\d/\dv){sinhv/w)*\dΦ/\dv}+(w^2/(a^2*sinhv^2))*(\d/\dφ)^2Φ

Toroidal coordinates

Fig. 13-10

The coordinate surfaces result from the rotation of Fig. 13-9 around its y axis, which then becomes z. With $\rho = (x^2 + y^2)^{1/2}$, the traces of the coordinate surfaces u and υ on a plane φ = const. are given in Fig. 13-10 by the equations

ρ^2+(z-a*cotu)^2=a^2/sinu^2, (ρ-a*cothv)^2+z^2=a^2/sinhv^2

Other orthogonal coordinate systems

There are other orthogonal coordinate systems, such as conical coordinates **Ext**, confocal ellipsoidal coordinates **Ext**, confocal paraboloidal coordinates **Ext**, and others that are used rather rarely.

14 BESSEL FUNCTIONS

14.1 Definitions

The functions that satisfy *Bessel's differential equation*

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$ Ext

are the *Bessel functions of order n*.

The general solution of Bessel's differential equation is

$y = c_1J_n(x) + c_2J_{-n}(x),$ $\quad n \neq 0, 1, 2, \ldots$

$y = c_1J_n(x) + c_2Y_n(x),$ $\quad$ for any n

$y = c_1H_n^{(1)}(x) + c_2H_n^{(2)}(x),$ $\quad$ for any n

y=c_1*J_n(x)+c_2*J_n(x)*\
I[x]{1/(x*J_n(x)^2)} $\quad$ for any n

where c_1 and c_2 are arbitrary constants and $J_n(x)$, $Y_n(x)$ are the Bessel functions of the first and second kind, respectively.

Graphs of Bessel functions $J_n(x)$, $Y_n(x)$

Fig. 14-1: $J_n(x)$, $Y_n(x)$, $n = 0$ ——, $n = 1$ ——, $n = 2$ ——, $n = 3$ ——

14.2 Bessel Functions of the First Kind

The Bessel functions of the first kind and order n are defined by the relations

\S{J_n(x)}, Γ(n+1), Γ(k+n+1), x^(2*k+n)

\S{J_(-n)(x)}, Γ(1-n), Γ(k-n+1), x^(2*k-n)

If $n \neq 0, 1, 2, \ldots, J_n(x)$ and $J_{-n}(x)$ are linearly independent.

If $n = 0, 1, 2, \ldots, J_{-n}(x) = (-1)^n J_n(x)$.

As $x \to 0$, $J_n(x)$ behaves as $x^n/[2^n \Gamma(n+1)]$.

The Wronskian (Wronski's determinant) is

W[J_n(x),J_(-n)(x)]=det(J_n(x),J_(-n)(x);J_n'(x),J_(-n)'(x))=-(2/(π*x))*sin(n*π)

Generating function

The *generating function* is

\S{e^((x/2)(t-1/t)}=\S[n,-\o,\o]{J_n(x)*t^n}

Properties

For $n = 0, 1$ we have

\S{J_0(x)}, x^(2*k)

\S{J_1(x)}, x^(2*k+1)

\P{J_n(x)}, Γ(n+1) [λ_k the positive roots of $J_n(x) = 0$]

J_n(x)=(-1)^n*x^n*((1/x)*d/dx)^n{J_0(x)}

J_n(x+y)=\S[k,-\o,\o]{J_k(x)*J_(n-k)(y)} [*Addition formula*]

|J_n(x)|\<1:n\>0, |J_n(x)|\<1/2^(1/2):n\>1

In general, for any n we have

J_(n+1)(x)=(2*n/x)*J_n(x)-J_(n-1)(x)

$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

$xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$

$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

(d/dx){x^n*J_n(x)}=x^n*J_(n-1)(x)

(d/dx){x^(-n)*J_n(x)}=-x^(-n)*J_(n+1)(x)

Recurrence relations [the same for $Y_n(x)$] **Pro**

In the case of order $n = (2k + 1)/2$ for $k = 0, 1, 2, \ldots$, the Bessel functions are expressed in terms of sines, cosines and R=(2/(π*x))^(1/2).

J_(1/2)(x)=R*sinx

J_(-1/2)(x)=R*cosx

J_(3/2)(x)=R*(sinx/x-cosx)

J_(-3/2)(x)=-R*(cosx/x+sinx)

J_(5/2)(x)=R*((3/x^2-1)*sinx-3*cosx/x)

J_(-5/2)(x)=R*(3*sinx/x+(3/x^2-1)*cosx)

J_(n+1/2)(x)=R*x^(n+1)*(-(1/x)*d/dx)^n{sinx/x}

Zeros of Bessel functions

For any real n we have the following:

1. $J_n(x) = 0$ has an infinite number of real roots and a finite number of complex roots. All these roots are *simple* except possibly $x = 0$.
2. If $n \geq -1$, all the roots of $J_n(x) = 0$ are real.
3. If $n = 1, 2, \ldots$, the positive or negative roots of $J_n(x) = 0$ and $J_{n+1}(x) = 0$ *interchange*, i.e. between any two consecutive roots of $J_n(x) = 0$, only one root of $J_{n+1}(x) = 0$ is included.
4. If λ_k, λ_{k+1} are two consecutive roots of $J_n(x) = 0$ with $\lambda_k < \lambda_{k+1}$, the difference $\lambda_{k+1} - \lambda_k$ tends to π, when $k \to \infty$. **Ext**

14.3 Bessel Functions of the Second Kind

The Bessel functions of the second kind and order n (also called *Neumann functions*) are defined by the relations

Y_n(x), J_n(x)*cos(n*π), J_(-n)(x)

The Bessel functions $Y_n(x)$ satisfy the same recurrence relations as $J_n(x)$.

For $n = 0, 1, 2, \ldots$, L' Hôpital rule gives

\S{Y_n(x)}, J_n(x), lnx, γ

where $\gamma = 0.5772156\ldots$ is the Euler constant and

P_0, P_k=1+1/2+1/3+...+1/k

For $n = 0$, we have

\S{Y_0(x)}, J_0(x), lnx, γ

For $n = 0, 1, 2, \ldots$ we have $Y_{-n}(x) = (-1)^n Y_n(x)$.

As $x \to 0$, $Y_n(x)$ behaves as $-2^n \Gamma(n) \pi^{-1} x^{-n}$.

The Wronskian (Wronski's determinant) is

W[J_n(x),Y_n(x)]=det(J_n(x),Y_n(x);J_n'(x),Y_n'(x))=2/(π*x)

The functions $Y_{1/2}(x)$, $Y_{3/2}(x)$, etc. can be obtained from the definition of $Y_n(x)$ for $n \neq 1, 2, \ldots$ and $J_{1/2}(x)$, $J_{3/2}(x)$, ….

Hankel functions of the first and second kind

Ext

$$H_n^{(1)}(x) = J_n(x) + iY_n(x) = -i2^n \Gamma(n) \pi^{-1} x^{-n} + \cdots, \quad n > 0$$

$$H_n^{(2)}(x) = J_n(x) - iY_n(x) = i2^n \Gamma(n) \pi^{-1} x^{-n} + \cdots, \quad n > 0$$

14.4 Modified Bessel Functions

Definitions

The functions that satisfy the *modified Bessel's differential equation*

$$x^2y'' + xy' - (x^2 + n^2)y = 0$$

Ext

are known as the *modified Bessel functions of order n*.

The general solution of the modified Bessel's differential equation is

$y = c_1I_n(x) + c_2I_{-n}(x),$ $\quad n \neq 0, 1, 2, \ldots$

$y = c_1I_n(x) + c_2K_n(x),$ $\quad$ for any n

y=c_1*I_n(x)+c_2*I_n(x)*\I[x]{1/(x*I_n(x)^2)}, for any n

where c_1 and c_2 are arbitrary constants and $I_n(x)$ and $K_n(x)$ are the modified Bessel functions of the first and second kind, respectively.

Graphs of modified Bessel functions $I_n(x)$, $K_n(x)$

Fig. 14-2: $I_n(x)$, $K_n(x)$, $n = 0$ ——, $n = 1$ ——, $n = 2$ ——, $n = 3$ ——

Modified Bessel functions of the first kind

I_n(x)=i^(-n)*J_n(i*x)=e^(-n*π*i/2)*J_n(i*x),
\S{I_n(x)}, Γ(n+1), Γ(n+k+1), x^(n+2*k)

As $x \to 0$, $I_n(x)$ behaves as $x^n/[2^n\Gamma(n + 1)]$ for $n \neq -1, -2, \ldots$.

I_(-n)(x)=i^n*J_(-n)(i*x)=e^(n*π*i/2)*J_(-n)(i*x),
\S{I_(-n)(x)}, Γ(1-n), Γ(k+1-n), x^(2*k-n)

For $n \neq 0, 1, 2, \ldots$, $I_n(x)$ and $I_{-n}(x)$ are linearly independent with Wronskian

$$W[I_n(x), I_{-n}(x)] = I_n(x)I_{-n-1}(x) - I_{n+1}(x)I_{-n}(x) = -2\sin n\pi/(\pi x)$$

For $n = 0, 1, 2, \ldots$ we have $I_{-n}(x) = I_n(x)$.

Generating function

The *generating function* is

\S{e^((x/2)(t+1/t)}=\S[n,-\o,\o]{I_n(x)*t^n}

Properties

For $n = 0, 1$ we have

\S{I_0(x)}

\S{I_1(x)}

$I_0'(x) = I_1(x)$

I_(n+1)'(x)=I_(n-1)(x)-(2*n/x)*I_n(x)

$I_n'(x) = \frac{1}{2}[I_{n-1}(x) + I_{n+1}(x)]$

$xI_n'(x) = xI_{n-1}(x) - nI_n(x)$

$xI_n'(x) = xI_{n+1}(x) + nI_n(x)$

(d/dx){x^n*I_n(x)}=x^n*I_(n-1)(x)

(d/dx){x^(-n)*I_n(x)}=x^(-n)*I_(n+1)(x)

Recurrence relations

In the case of order $n = (2k + 1)/2$ for $k = 0, 1, 2, \ldots$, the functions are expressed in terms of hyperbolic sines, cosines and R=((2/(π*x))^(1/2).

I_(1/2)(x)=R*sinhx

I_(-1/2)(x)=R*coshx

I_(3/2)(x)=R*(coshx-sinhx/x)

I_(-3/2)(x)=R*(sinhx-coshx/x)

I_(5/2)(x)=R*((3/x^2+1)*sinhx-(3/x)*coshx)

I_(-5/2)(x)=R*((3/x^2+1)*coshx-(3/x)*sinhx)

Other expressions can be obtained using the recurrence relation.

Modified Bessel functions of the second kind

K_n(x), I_(-n)(x), I_n(x)

$K_n(x) = (\pi i^{n+1}/2)H_n^{(1)}(ix)$

For $n = 0, 1, 2, \ldots$, we have $K_{-n}(x) = K_n(x)$ and L'Hôpital's rule gives

K_n(x), I_n(x), lnx, γ, x^(2*k-n), x^(2*k+n)

where $\gamma = 0.5772156\ldots$ is the Euler constant and

P_0, P_k=1+1/2+1/3+...+1/k

For $n = 0$, we have

\S{K_0(x)}, I_0(x), lnx, γ

As $x \to 0$, $K_0(x)$ behaves as $-\ln x$ and $K_n(x)$ as $2^{n-1}\Gamma(n)x^{-n}$ for $n > 0$.

The Wronskian is

$W[K_n(x), I_n(x)] = I_n(x)K_{n+1}(x) + I_{n+1}(x)K_n(x) = x^{-1}$

For any n we have

K_(n+1)(x)=K_(n-1)(x)+(2*n/x)*K_n(x)

$K_n'(x) = -\frac{1}{2}[K_{n-1}(x) + K_{n+1}(x)]$

$xK_n'(x) = -xK_{n-1}(x) - nK_n(x)$

$xK_n'(x) = nK_n(x) - xK_{n+1}(x)$

(d/dx){x^n*K_n(x)}=-x^n*K_(n-1)(x)

(d/dx){x^(-n)*K_n(x)}=-x^(-n)*K_(n+1)(x)

} Recurrence relations

$K_{n+1/2}(x)$ can be obtained from the definition of $K_n(x)$ and $I_{1/2}(x)$, $I_{3/2}(x)$, ….

14.5 The Functions Ber and Bei, Ker and Kei

The functions Ber and Bei

The general solution of the differential equation

$$x^2y'' + xy' - (ix^2 + n^2)y = 0, \qquad \text{i=(-1)^(1/2)}$$

is $y = c_1\{\mathrm{Ber}_n(x) + i\mathrm{Bei}_n(x)\} + c_2\{\mathrm{Ker}_n(x) + i\mathrm{Kei}_n(x)\}$.

$\mathrm{Ber}_n(x)$ and $\mathrm{Bei}_n(x)$ are the real and the imaginary part of $J_n(xe^{3\pi i/4})$ respectively and are often referred to as *Kelvin functions*.

$\mathrm{Ber}_n(x) + i\,\mathrm{Bei}_n(x) = J_n(xe^{3\pi i/4})$

\S{Ber_n(x)}, Γ(n+k+1), x^(2*k+n)

\S{Bei_n(x)}, Γ(n+k+1), x^(2*k+n)

For $n = 0$ we have

\S{Ber(x)}=\S{Ber_0(x)}, x^(4*k)

\S{Bei(x)}=\S{Bei_0(x)}, x^(4*k+2)

Graphs of $\mathrm{Ber}_n(x)$, $\mathrm{Bei}_n(x)$

Fig. 16-3: $\mathrm{Ber}_n(x)$, $\mathrm{Bei}_n(x)$, $n = 0$ ——, $n = 1$ ——, $n = 2$ ——, $n = 3$ ——

The functions Ker and Kei

$\mathrm{Ker}_n(x)$ and $\mathrm{Kei}_n(x)$ are the real and the imaginary part of $e^{-n\pi i/2}K_n(xe^{\pi i/4})$.

$\mathrm{Ker}_n(x) + i\,\mathrm{Kei}_n(x) = e^{-n\pi i/2}K_n(xe^{\pi i/4})$

\S{Ker_n(x)}, Ber_n(x), Bei_n(x), lnx, γ, P_k

\S{Kei_n(x)}, Ber_n(x), Bei_n(x), lnx, γ, P_k

where $\gamma = 0.5772156\ldots$ is the Euler constant and

P_0, P_k=1+1/2+1/3+...+1/k

For $n = 0$, we have

\S{Ker(x)}=\S{Ker_0(x)}, Ber(x), Bei(x), lnx, γ, P_k

\S{Kei(x)}=\S{Kei_0(x)}, Ber(x), Bei(x), lnx, γ, P_k

Graphs of $\mathrm{Ker}_n(x)$, $\mathrm{Kei}_n(x)$

Fig. 14-4: $\mathrm{Ker}_n(x)$, $\mathrm{Kei}_n(x)$, $n = 0$ ——, $n = 1$ ——, $n = 2$ ——, $n = 3$ ——

For $n = 0$ we have

\S{Ker(x)}=\S{Ker_0(x)}, Ber(x), Bei(x), lnx, γ, P_k

\S{Kei(x)}=\S{Kei_0(x)}, Ber(x), Bei(x), lnx, γ, P_k

14.6 Spherical Bessel Functions

Ext

The *spherical Bessel functions* of the first, second, third and fourth kind are, respectively, defined by the relations

j_n(x)=(π/(2*x))^(1/2)*J_(n+1/2)(x)

n_n(x)=(π/(2*x))^(1/2)*Y_(n+1/2)(x)

h_n^((1))(x)=j_n(x)+i*n_n(x)

h_n^((2))(x)=j_n(x)-i*n_n(x)

and satisfy the differential equation

$$x^2y'' + 2xy' + [x^2 - n(n + 1)]y = 0$$

For any integer n, the spherical Bessel functions are elementary functions:

j_0(x)=n_(-1)(x)=sinx/x

n_0(x)=-j_(-1)(x)=-cosx/x

h_0^((1))(x)=-i*e^(i*x)/x

h_0^((2))(x)=i*e^(-i*x)/x

j_1(x)=sinx/x^2-cosx/x

n_1(x)=-cosx/x^2-sinx/x

h_1^((1))(x)=-e^(i*x)*(1+i/x)/x

h_1^((2))(x)=-e^(-i*x)*(1-i/x)/x

j_n(x)=(-x)^n*((1/x)*(d/dx))^n{sinx/x}

n_n(x)=-(-x)^n*((1/x)*(d/dx))^n{cosx/x}

14.7 Various Expressions of Bessel Functions

Asymptotic expressions

Ext

For large x we have the asymptotic expressions

J_n(x)~(2/(π*x))^(1/2)*cos(x-n*π/2-π/4)

Y_n(x)~(2/(π*x))^(1/2)*sin(x-n*π/2-π/4)

I_n(x)~(1/(2*π*x))^(1/2)*e^x

K_n(x)~(π/(2*x))^(1/2)*e^(-x)

H_n^((1))(x)~(2/(π*x))^(1/2)*e^(i*(x-n*π/2-π/4))

H_n^((2))(x)~(2/(π*x))^(1/2)*e^(-i*(x-n*π/2-π/4))

J_(n+1/2)(x)~(2/(π*x))^(1/2)*sin(x-n*π/2)

For large n we have the asymptotic expressions

J_n(x)~(1/(2*π*n))^(1/2)*(e*x/(2*n))^n

Y_n(x)~-(2/(π*n))^(1/2)*(e*x/(2*n))^(-n)

The term "for large x" means that the ratio of the left to the right part tends to 1, when $x \to \infty$. The same for the expression "for large n", when $n \to \infty$.

Expressions with integrals

J_0(x)=(1/π)*\I[θ,0,π]{cos(x*sinθ)}, Y_0(x)=(-2/π)*\I[u,0,\o]{cos(x*coshu)}

I_0(x)=(1/π)*\I[θ,0,π]{cosh(x*sinθ)}=(1/π)*\I[θ,0,π]{e^(x*cosθ)}

J_n(x), \I[θ,0,π]{cos(x*sinθ-n*θ)}, \I[t,0,\o]{e^(-n*t-x*sinht)}, any n

Y_n(x), \I[θ,0,π]{sin(x*sinθ-n*θ)}, \I[t,0,\o]{(e^(n*t)+e^(-n*t)*cos(n*π))*e^(-x*sinht)}, any $n, x > 0$

J_n(x), \I[θ,0,π/2]{cos(x*cosθ)*sinθ^(2*n)}, Γ(n+1/2)

Pro

J_(2*n)(x), \I[θ,0,π/2]{cos(x*sinθ)*cos(2*n*θ)}

J_(2*n+1)(x), \I[θ,0,π/2]{sin(x*sinθ)*sin((2*n+1)*θ)}

I_n(x), \I[θ,0,π]{e^(x*cosθ)*cos(n*θ)}, \I[t,0,\o]{e^(-x*cosht-n*t)}

K_n(x)=\I[t,0,\o]{e^(-x*cosht)*cosh(n*t)}

14.8 Integrals with Bessel Functions

Indefinite integrals

The following relations also hold if $J_n(x)$ is replaced by $Y_n(x)$ or generally by $c_1 J_n(x) + c_2 Y_n(x)$, where c_1 and c_2 are constants.

\I[x]{x*J_0(x)}=x*J_1(x)

\I[x]{x^2*J_0(x)}=x^2*J_1(x)+x*J_0(x)-\I[x]{J_0(x)}

\I[x]{J_0(x)/x^2}, \I[x]{J_0(x)}, J_1(x)

\I[x]{x^m*J_0(x)}, \I[x]{x^(m-2)*J_0(x)}, x^m*J_1(x)

\I[x]{J_0(x)/x^m}, \I[x]{J_0(x)/x^(m-2)}, J_1(x)/x^(m-2)

\I[x]{J_1(x)}=-J_0(x)

\I[x]{x*J_1(x)}=-x*J_0(x)+\I[x]{J_0(x)}

\I[x]{J_1(x)/x}=-J_1(x)+\I[x]{J_0(x)}

\I[x]{x^m*J_1(x)}=-x^m*J_0(x)+m*\I[x]{x^(m-1)*J_0(x)}

\I[x]{J_1(x)/x^m}, \I[x]{J_0(x)/x^(m-1)}, J_1(x)/x^(m-1)

\S{\I[x]{J_n(x)}}, J_(n+2*k-1)(x)

\I[x]{x^n*J_(n-1)(x)}}=x^n*J_n(x)

\I[x]{x^(-n)*J_(n+1)(x)}}=-x^(-n)*J_n(x)

\I[x]{x^n*Y_(n-1)(x)}=x^n*Y_n(x)

\I[x]{x^n*I_(n-1)(x)}=x^n*I_n(x)

\I[x]{x^(-n)*I_(n+1)(x)}=x^(-n)*I_n(x)

\I[x]{x^n*K_(n-1)(x)}=-x^n*K_n(x)

\I[x]{x^(-n)*K_(n+1)(x)}=-x^(-n)*K_n(x)

\I[x]{x^m*J_n(x)}, \I[x]{x^(m-1)*J_(n-1)(x)}

\I[x]{x*J_n(α*x)*J_n(β*x)}

\I[x]{x*(J_n(α*x))^2}

Definite integrals

\I[x,0,\o]{e^(-a*x)*J_0(b*x)}

Ext

\I[x,0,\o]{e^(-a*x)*J_0(b*x^(1/2))}

\I[x,0,\o]{J_0(a*x)*sin(b*x)}

\I[x,0,\o]{J_0(a*x)*cos(b*x)}

\I[x,0,\o]{e^(-a*x)*J_n(b*x)}

\I[x,0,\o]{J_n(a*x)}=1/a

\I[x,0,\o]{J_n(a*x)/x}=1/n

\I[x,0,\o]{Y_n(x)}=-tan(n*π/2)

\I[x,0,\o]{x^(m-n)*J_n(α*x)}, Γ((m+1)/2), Γ(n-(m-1)/2)

\I[x,0,1]{x^(n+1)*J_n(a*x)}=(1/a)*J_(n+1)(a)

\I[x,0,1]{x^(n+1)*I_n(a*x)}=(1/a)*I_(n+1)(a)

\I[x,0,1]{x*J_n(α*x)*J_n(β*x)}

\I[x,0,1]{x*(J_n(a*x))^2)}

\I[x,0,1]{x*J_0(α*x)*I_0(β*x)}

\I[x,0,1]{J_n(a*x)*J_n(b*x)}=(1/a)*δ(a-b)

14.9 Expansions in Series of Bessel Functions

Orthogonality

If λ_k, λ_m are two roots of the equation $J_n(x) = 0$, then

\I[x,0,1]{x*J_n(λ_k*x)*J_n(λ_m*x)}

Thus, the Bessel functions are orthogonal in the interval $[0, 1]$ with weight function x.

In general, if λ_k, λ_m are two roots of the equation $aJ_n(x) + bxJ_n'(x) = 0$, where a and b are constants (at least one of which is different from zero), then

\I[x,0,1]{x*J_k(λ_k*x)*J_m(λ_m*x)}

Expansion in series

The

We assume that $f(x)$ and $f'(x)$ are piecewise continuous functions and $f(x)$ is defined as equal to $\frac{1}{2}[f(x+0)+f(x-0)]$ at the points of discontinuity.

With roots of $J_n(x) = 0$

If $0 < \lambda_1 < \lambda_2 < \lambda_3, \ldots$ are the positive roots of $J_n(x) = 0$ with $n > -1$, then

$$f(x) = a_1 J_n(\lambda_1 x) + a_2 J_n(\lambda_2 x) + a_3 J_n(\lambda_3 x) + \cdots$$

where

a_k=(2/(J_(n+1)(λ_k))^2*\I[x,0,1]{x*f(x)*J_n(λ_k*x)}

Exa

With roots of $cJ_n(x) + xJ_n'(x) = 0$

If $0 < \lambda_1 < \lambda_2 < \lambda_3 \ldots$ are the positive roots of $cJ_n(x) + xJ_n'(x) = 0$ with $n > -1$ and c is an arbitrary constant, then the expansion of $f(x)$ has the general form

$$f(x) = f_0(x) + a_1 J_n(\lambda_1 x) + a_2 J_n(\lambda_2 x) + a_3 J_n(\lambda_3 x) + \cdots$$

where

a_k=(2/(d_(nk))*\I[x,0,1]{x*f(x)*J_n(λ_k*x)}

d_(nk)=..., J_n(λ_k), J_(n+1)(λ_k)

and for $c > -n$, $f_0(x) = 0$,

for $c = -n$, $f_0(x) = a_0 x^n$, a_0=2*(n+1)*\I[x,0,1]{x^(n+1)*f(x)}.

for $c < -n$, $f_0(x) = a_0 I_n(\lambda_0 x)$, a_0=..., I_n(λ_0), I_(n-1)(λ_0), I_(n+1)(λ_0),

where $\pm i\lambda_0$ are the two additional imaginary roots that exist in this case.

The previous results can be extended to series of Bessel functions of the second kind.

Various expansions

\S{cos(x*sinθ)}=J_0(x)+2*\S[n,1,\o]{J_(2*n)(x)*cos(2*n*θ)}

\S{sin(x*sinθ)}=2*\S[n,1,\o]{J_(2*n-1)(x)*sin((2*n-1)*θ)}

\S{1}=\S[n,-\o,\o]{J_n(x)}

\S{1}=\S[n,-\o,\o]{(J_n(x))^2}

\S{x}=2*\S[n,1,\o]{(2*n-1)*J_(2*n-1)(x)}

\S{x^2}=2*\S[n,1,\o]{(2*n)^2*J_(2*n)(x)}

\S{x^n}, Γ(n+k), J_(n+2*k)(x)

\S{x^(1/2)}, J_(2*n+1/2)(x)

\S{sinx}=2*\S[n,1,\o]{(-1)^(n+1)*J_(2*n-1)(x)}

\S{cosx}=J_0(x)+2*\S[n,1,\o]{(-1)^n*J_(2*n)(x)}

\S{x*J_1(x)}=4*\S[n,1,\o]{(-1)^(n+1)*n*J_(2*n)(x)}

\S{1}=I_0(x)+2*\S[n,1,\o]{(-1)^n*I_(2*n)(x)}

\S{e^x}=I_0(x)+2*\S[n,1,\o]{I_n(x)}

\S{e^(x*cosθ)}=I_0(x)+2*\S[n,1,\o]{I_n(x)*cos(n*θ)}

\S{sinhx}=2*\S[n,1,\o]{I_(2*n-1)(x)}

\S{coshx}=I_0(x)+2*\S[n,1,\o]{I_(2*n)(x)}

15 LEGENDRE FUNCTIONS

15.1 Definitions

The functions that satisfy *Legendre's differential equation*

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

are called *Legendre functions of order n*.

The general solution of Legendre's differential equation is

$$y = c_1 u_n(x) + c_2 v_n(x)$$

where

```
u_n(x)=1-(n*(n+1)/2!)*x^2+(n*(n-2)*(n+1)*(n+3)/4!)*x^4-...
```

```
v_n(x)=x-((n-1)*(n+2)/3!)*x^3+((n-1)*(n-3)*(n+2)*(n+4)/5!)*x^5-...
```

and c_1, c_2 are arbitrary constants. Each series converges for $-1 < x < 1$ and any n. Below, we are limited to a non-negative integer n.

If $n = 0, 2, 4, \ldots$, the series of $u_n(x)$ terminates (i.e. the terms vanish after some term) and becomes a polynomial. If $n = 1, 3, 5, \ldots$, the series of $v_n(x)$ terminates and again becomes a polynomial. Thus, if n is a non-negative integer, Legendre's differential equation has the polynomial solution

```
\S{P_n(x)}, x^(n-2*k)
```

Pro

that converges for any x. This solution is normalized so that it equals 1 at $x = 1$ and it is called a *Legendre polynomial*.

The other series has an infinite number of terms and, after being multiplied by an appropriate constant, gives the *Legendre function of the second kind and order n*

```
\S{Q_n(x)}, u_n(x), v_n(x)
```

These series converge for $-1 < x < 1$ and diverge for $|x| \geq 1$. $Q_n(x)$ has been normalized so that it satisfies the same recurrence relations as $P_n(x)$.

15.2 Legendre Polynomials

The Legendre polynomials are given by *Rodrigues's formula*

P_n(x)=(1/(2^n*n!))*(d/dx)^n{(x^2-1)^n}

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$

$P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$

$P_8(x) = \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$

Graphs of Legendre polynomials $P_n(x)$

Fig. 15-1

Ext

With $x = \cos\theta$ we have

$P_0(\cos\theta) = 1$

$P_1(\cos\theta) = \cos\theta$

$P_2(\cos\theta) = \frac{1}{4}(1 + 3\cos 2\theta)$

$P_3(\cos\theta) = \frac{1}{8}(3\cos\theta + 5\cos 3\theta)$

$P_4(\cos\theta) = \frac{1}{64}(9 + 20\cos 2\theta + 35\cos 4\theta)$

$P_5(\cos\theta) = \frac{1}{128}(30\cos\theta + 35\cos 3\theta + 63\cos 5\theta)$

$P_6(\cos\theta) = \frac{1}{512}(50 + 105\cos 2\theta + 126\cos 4\theta + 231\cos 6\theta)$

$P_7(\cos\theta) = \frac{1}{1024}(175\cos\theta + 189\cos 3\theta + 231\cos 5\theta + 429\cos 7\theta)$

$P_8(\cos\theta) = \frac{1}{16384}(1225 + 2520\cos 2\theta + 2772\cos 4\theta + 3432\cos 6\theta + 6435\cos 8\theta)$

Note that all the numerical coefficients are positive. Hence at $\theta = 0$, $P_n(\cos\theta)$ receives its maximum value, which is $P_n(1) = 1$.

Generating function

1/(1-2*x*t+t^2)^(1/2),
\S[n,0,\o]{P_n(x)*t^n},
\S[n,0,\o]{P_n(x)*t^(-n-1)}

Recurrence relations

$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

$P'_{n+1}(x) - xP_n'(x) = (n+1)P_n(x)$

$xP_n'(x) - P'_{n-1}(x) = nP_n(x)$ **Pro**

$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$

$(x^2-1)P_n'(x) = nxP_n(x) - nP_{n-1}(x)$

The same recurrence relations are satisfied by $Q_n(x)$.

Orthogonality

\I[x,-1,1]{P_m(x)*P_n(x)}=(2/(2*n+1))*δ_(mn)

Since $P_m(x)$ and $P_n(x)$ satisfy the previous relation for $m \neq n$, they are *orthogonal* in $-1 \leq x \leq 1$.

Expansions in series of Legendre polynomials

The Legendre polynomials constitute a *complete set of functions*, i.e. any piecewise smooth function $f(x)$ in the interval $-1 < x < 1$ can be expanded into a series of Legendre polynomials of the form

$$f(x) = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + \cdots$$

a_n=((2*n+1)/2)*\I[x,-1,1]{f(x)*P_n(x)} **Ext**

At the points of discontinuity, the series gives the sum $\frac{1}{2}[f(x+0)+f(x-0)]$.

Properties

$P_n(1) = 1,\qquad P_n(-1) = (-1)^n,\qquad P_n(-x) = (-1)^n P_n(x)$

P_n(0)

P_n(x)=(1/π)*\I[φ,0,π]{(x+(x^2-1)^(1/2)*cosφ)^n}

\I[x]{P_n(x)}=(P_(n+1)(x)-P_(n-1)(x))/(2*n+1)

$|P_n(x)| \le 1 \quad \text{for } -1 \le x \le 1$

P_n(x)=(1/(2^(n+1)*π*i))*\I_C[z]{(z^2-1)^n/(z-x)^(n+1)} [Schläfli's integral]

where C is a simple closed curve on the complex z plane and x is an interior point.

P_n(x)=F(-n,n+1;1;(1-x)/2),
x^n*F(-n/2,(1-n)/2;1/2-n;1/x^2)

where F is the hypergeometric function. This relation is generally also valid for a non-integer n and consequently gives the *Legendre function of the first kind*. In this case $P_n(x)$ has singularities at $x = -1$ and $x = \infty$.

The equation $P_n(x) = 0$ has exactly n real roots, all in the interval $(-1, 1)$.

15.3 Legendre Functions of the Second Kind

Ext

The Legendre functions of the second kind $Q_n(x)$ are the solutions of Legendre's differential equation given in Sec. 15.1. These functions diverge at the end points of the interval $-1 < x < 1$ and satisfy the symmetry condition

$$Q_n(-x) = (-1)^{n+1} Q_n(x)$$

The divergence is logarithmic, as is apparent from the first functions, which are

Q_0(x)=(1/2)*ln|(1+x)/(1-x)|

Q_1(x)=(x/2)*ln|(1+x)/(1-x)|-1

Q_2(x)=((3*x^2-1)/4)*ln|(1+x)/(1-x)|-3*x/2

Graphs of Legendre functions of the second kind $Q_n(x)$

Fig. 15-2

Q_3(x), ln|(1+x)/(1-x)|

Q_4(x), ln|(1+x)/(1-x)|

Q_5(x), ln|(1+x)/(1-x)|

Generally we have

Q_n(x), P_n(x)*ln|(1+x)/(1-x)|, Q_0(x)*P_n(x),
\S[k,1,n]{(P_(k-1)(x)*P_(n-k)(x))/k}

Q_n(x), (1/x^(n+1))*F((n+2)/2,(n+1)/2;(2*n+3)/2;1/x^2), Γ(n+1), Γ(n+3/2)

where F is the hypergeometric function. This relation holds for a non-integer n and, consequently, $Q_n(x)$ has irregular points at $x = \pm 1$ and $x = \infty$.

Properties

The functions $Q_n(x)$ have exactly $n + 1$ zeros in $(-1, 1)$ and satisfy the same recurrence relations as the Legendre polynomials (Section 15.2). They diverge at $x = \pm 1$, while at $x = 0$

Q_n(0)

A simple integral expression is

Q_n(x)=(1/2^(n+1))*\I[z,-1,1]{(1-z^2)^n/(x-z)^(n+1)}

The general solution of Legendre's differential equation can be written as

$$y = c_3 P_n(x) + c_4 Q_n(x)$$

The Wronskian (Wronski's determinant) is

W[P_n(x), Q_n(x)]=(n/(1-x^2))*(P_n(x)*Q_(n-1)(x)-P_(n-1)(x)Q_n(x))=1/(1-x^2)

15.4 Associated Legendre Functions

The functions that satisfy the *associated Legendre's differential equation*

(1-x^2)*y"-2*x*y'+(n*(n+1)-m^2/(1-x^2))*y=0

Ext

are the *associated Legendre functions*. In the following, we are limited to the case where m and n are integers with $n \geq 0$, $-n \leq m \leq n$ and $-1 < x < 1$. Then the associated Legendre's ODE is obtained from Legendre's ODE after m differentiations.

The general solution of the associated Legendre's differential equation is

$$y = c_1 P_n^m(x) + c_2 Q_n^m(x)$$

where $P_n^m(x)$ and $Q_n^m(x)$ are the *associated Legendre functions of the first and second kind*, respectively.

Associated Legendre functions of the first kind

The associated Legendre functions of the first kind are defined from the Legendre polynomials with the relation

P_n^m(x)=(1-x^2)^(m/2)*(d/dx)^m{P_n(x)}

where $P_n(x)$ are the Legendre polynomials. Note that the right part of the previous relation is also meaningful for a negative m with $-n \leq m \leq n$. In general we have

P_n^(-m)(x)=(-1)^m*((n-m)!/(n+m)!)*P_n^m(x)}

We also define $P_n^m(x) = 0$ for $m < -n$ or $m > n$. For the values of $P_n^m(x)$ we have

$$P_n^m(-x) = (-1)^{m+n} P_n^m(x) \quad \text{and} \quad P_n^m(\pm 1) = 0 \text{ for } m \neq 0$$

The first functions are (with $x = \cos\theta$, $0 \leq \theta \leq \pi$)

$$P_n^0(x) = P_n(x) \quad \text{for any } n$$

$n = 1$ $\qquad P_1^1(x) = (1 - x^2)^{1/2} = \sin\theta$

$n = 2$ $\qquad P_2^1(x) = 3x(1 - x^2)^{1/2} = \frac{3}{2}\sin 2\theta$

$$P_2^2(x) = 3(1 - x^2) = \frac{3}{2}(1 - \cos 2\theta)$$

$n = 3$ $\quad P_3^1(x) = \frac{3}{2}(5x^2 - 1)(1 - x^2)^{1/2} = \frac{3}{8}(\sin\theta + 5\sin 3\theta)$

$$P_3^2(x) = 15x(1 - x^2) = \frac{15}{4}(\cos\theta - \cos 3\theta)$$

$$P_3^3(x) = 15(1 - x^2)^{3/2} = \frac{15}{4}(3\sin\theta - \sin 3\theta)$$

$n = 4$ $\quad P_4^1(x) = \frac{5}{2}(7x^3 - 3x)(1 - x^2)^{1/2} = \frac{5}{16}(2\sin 2\theta + 7\sin 4\theta)$

$$P_4^2(x) = \frac{15}{2}(7x^2 - 1)(1 - x^2) = \frac{15}{16}(3 + 4\cos 2\theta - 7\cos 4\theta)$$

$$P_4^3(x) = 105x(1 - x^2)^{3/2} = \frac{105}{8}(2\sin 2\theta - \sin 4\theta)$$

$$P_4^4(x) = 105(1 - x^2)^2 = \frac{105}{8}(3 - 4\cos 2\theta + \cos 4\theta)$$

Generating function

t^m/(1-2*x*t+t^2)^(m+1/2), \S[n,m,\o]{P_n^m(x)*t^n}

Recurrence relations

$$(n - m + 1)P_{n+1}^m(x) - (2n+1)xP_n^m(x) + (n + m)P_{n-1}^m(x) = 0$$

$$(1 - x^2)^{1/2}P_n^{m+1}(x) - 2mxP_n^m(x) + [n(n+1) - m(m - 1)](1 - x^2)^{1/2}P_n^{m-1}(x) = 0$$

$$(1 - x^2)P_n^{m\prime}(x) + nxP_n^m(x) - (m + n)P_{n-1}^m(x) = 0 \qquad ({}' = d/dx)$$

$$2(1 - x^2)^{1/2}P_n^{m\prime}(x) - P_n^{m+1}(x) + (m + n)(n - m + 1)P_n^{m-1}(x) = 0 \qquad ({}' = d/dx)$$

Orthogonality

The functions $P_l^m(x)$ and $P_n^m(x)$ (the same upper index) are orthogonal (with weight function equal to 1) in $-1 \le x \le 1$, i.e.

\I[x,-1,1,]{P_l^m(x)*P_n^m(x)}=(2/(2*n+1))*((n+m)!/(n-m)!)*δ_(ln)

The functions $P_n^k(x)$ and $P_n^m(x)$ (the same lower index) are orthogonal with weight function equal to $(1 - x^2)^{-1}$ in $-1 \le x \le 1$, i.e.

\I[x,-1,1,]{P_n^k(x)*P_n^m(x)/(1-x^2)}=(1/m)*((n+m)!/(n-m)!)*δ_(km)

Expansions in series

$$f(x) = a_m P_m^m(x) + a_{m+1} P_{m+1}^m(x) + a_{m+2} P_{m+2}^m(x) + \cdots$$

```
a_k=((2*k+1)/2)*((k-m)!/(k+m)!)*\I[x,-1,1]{f(x)*P_k^m(x)}
```

Relation with hypergeometric functions

```
P_n^m(x), (1-x^2)^(m/2)*F(m-n,m+n+1;m+1;(1-x)/2)
```

Associated Legendre functions of the second kind

The *associated Legendre functions of the second kind* are defined from the Legendre functions of the second kind with the relations

```
Q_n^m(x)=(1-x^2)^(m/2)*(d/dx)^m{Q_n(x)}, Q_n^(-m)(x)
```

These functions tend to infinity for $x \to \pm 1$, whereas $P_n^m(x)$ are finite for $x = \pm 1$.

The functions $Q_n^m(x)$ satisfy the same recurrence relations as $P_n^m(x)$.

An explicit expression in terms of hypergeometric functions is

```
Q_n^m(x), (x^2-1)^(m/2)*F((n+m+2)/2,(n+m+1)/2;(2*n+3)/2;1/x^2),
Γ(n+m+1), Γ(n+3/2)
```

The Wronskian of $P_n^m(x)$, $Q_n^m(x)$ for $|x| < 1$ is

```
W[P_n^m(x),Q_n^m(x)]=((n+m)!/(n-m)!)/(1-x^2)
```

The first associated Legendre functions of the second kind are

$Q_n^0(x) = Q_n(x)$ for any n

$n = 1$

```
Q_1^1(x)=(1-x^2)^(1/2)*((1/2)ln|(1+x)/(1-x)|+x/(1-x^2))
```

$n = 2$

```
Q_2^1(x)=(1-x^2)^(1/2)*((3*x/2)ln|(1+x)/(1-x)|+(3*x^2-2)/(1-x^2))
```

```
Q_2^2(x)=(3*(1-x^2)/2)*ln|(1+x)/(1-x)|-(3*x^3-5*x)/(1-x^2)
```

15.5 Spherical Harmonics

Various physics problems in the three dimensional Euclidean space lead to the mathematical problem of solving a differential equation with partial derivatives, such as the Laplace equation, the Helmholtz equation and the Schrödinger equation. Following the method of separation of variables in spherical coordinates r, θ, φ, we look for solutions of the form $R(r)Y(\theta, \varphi)$. Thus, we obtain for the function $Y(\theta, \varphi)$ the differential equation

```
(1/sinθ)*(\d/\dθ){sinθ*(\dY/\dθ)}+(1/sinθ^2)*(\d/\dφ)^2{Y}+n*(n+1)*Y=0
```
Ext

A further separation of the variables with the substitution $Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi)$, gives for the $\Phi(\varphi)$, the ODE

```
(d/dφ)^2{Φ}+m^2*Φ=0
```

with solutions $\Phi = e^{\pm im\varphi}$. For $\Theta(\theta)$ we obtain another ODE. Setting $x = \cos\theta$ and $P(x) = \Theta(\theta)$ leads to the differential equation for the associated Legendre functions. The boundary and other physical conditions of regularity and uniqueness require n and m to be integers and also $n \geq 0$, $-n \leq m \leq n$. Thus, the dependence of $Y(\theta, \varphi)$ on the angles θ and φ is expressed by the functions

```
Y_n^m(θ,φ), P_n^m(cosθ)*e^(i*m*φ)
```

These are the *spherical harmonics*. The coefficients have been chosen so that we have the condition for *orthonormality* (the asterisk implies the complex conjugate)

```
\I[φ,0,2*π]{\I[θ,0,π]{#(Y_n^m(θ,φ))*Y_n'^m'(θ,φ)*sinθ}}=δ_(nn')*δ_(mm')
```

and the condition for *completeness*

```
\S[n,0,\o]{\S[m,-n,n]{#(Y_n^m(θ',φ'))*Y_n^m(θ,φ)}}=δ(φ-φ')*δ(cosθ-cosθ')
```

The first spherical harmonics $Y_n^m(\theta, \varphi)$ are the following:

$n = 0$

```
Y_0^0=1/(4*π)^(1/2)
```

$n = 1$

```
Y_0^1=(3/(4*π))^(1/2)*cosθ
```

```
Y_1^1=-(3/(8*π))^(1/2)*sinθ*e^(i*φ)
```

$n = 2$ Y_2^0, (3/2)*cosθ-1/2 Y_2^1, sinθ*cosθ*e^(i*φ)

Y_2^2, sinθ^2*e^(2*i*φ)

$n = 3$ Y_3^0, (5/2*cosθ^3-(3/2)*cosθ Y_3^1, sinθ*(5*cosθ^2-1)*e^(i*φ)

Y_3^2, sinθ^2*cosθ*e^(2*i*φ) Y_3^3, sinθ^3*e^(3*i*φ)

For $m < 0$ the spherical harmonics are obtained from the relation

Y_n^(-m)(θ,φ)=(-1)^m*#(Y_n^m)(θ,φ))

The theorem of addition of spherical harmonics

In spherical coordinates (r, θ, φ) a pair of values for θ and φ defines a direction in space. The angle γ between two directions (θ_1, φ_1) and (θ_2, φ_2) satisfies the relation

$$\cos\gamma = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2)$$

In general, the *theorem of addition of spherical harmonics* is expressed by the relation

P_n(cosγ)=(4*π/(2*n+1))*\S[m,-n,n]{Y_n^m(θ_1,φ_1)*#(Y_n^m)(θ_2,φ_2)

Expansion of functions

A function $f(\theta, \varphi)$, defined on the surface of a sphere, differentiable and continuous, can be expanded in a series of spherical harmonics according to the formula

f(θ,φ)=\S[n,0,\o]{\S[m,-n,n]{c_(nm)*Y_n^m(θ,φ)}}

where the coefficients are $(d\Omega = \sin\theta\, d\theta\, d\varphi)$

c_(nm)=\I{f(θ,φ)*#(Y_n^m(θ,φ))*dΩ}

16 ORTHOGONAL POLYNOMIALS

16.1 Orthogonal Functions

Orthonormal set of functions

We consider two real functions $f(x)$ and $g(x)$, defined and (quadratically) integrable in an interval (a, b), apart perhaps, from a finite number of the points. If

\I[x,a,b]{f(x)*g(x)}=0

we then say that the functions $f(x)$ and $g(x)$ are *orthogonal* in the interval (a, b).

If $\{u_i(x)\}$, $i = 1, 2, \ldots$, is a set of real functions, such that

\I[x,a,b]{u_i(x)*u_j(x)}=δ_(ij) Exa

then the set $\{u_i(x)\}$ is *orthogonal* and *normal* or simply *orthonormal* in the interval (a, b). The Kronecker symbol δ_{ij} always equals 0 for $i \neq j$ and 1 for $i = j$. If the functions $u_i(x)$ are complex, all the above hold, but with $u_j^*(x)$ instead of $u_j(x)$ in the previous integral [$u_j^*(x)$ is the complex conjugate of $u_j(x)$].

If the integral has the general form

\I[x,a,b]{u_i(x)*u_j(x)*w(x)}=δ_(ij)

where $w(x)$ is a nonnegative function in (a, b), then we say that the set $\{u_i(x)\}$ is *orthogonal* and *normal* or simply *orthonormal* in the interval (a, b) with a *weight function* $w(x)$. In this case, the set $\{[w(x)]^{1/2}u_i\}$ is orthonormal.

Complete orthonormal set of functions

If $\{u_i(x)\}$ is an orthonormal set of functions and $f(x)$ a (quadratically) *integrable function* (a function for which \I[x,a,b]{|f(x)|^2}=δ_(ij) exists), we define the *generalized Fourier coefficients*

c_i=\I[x,a,b]{f(x)*u_i(x)}

and form the sum

S_M(x)=\S[i,1,M]{c_i*u_i(x)}

where M is a positive integer. The sum S_M approximates the function $f(x)$ with a *mean square error*

$$E_{rms}=\left(\frac{1}{b-a}\int_a^b (f(x)-S_M(x))^2\,dx\right)^{1/2}$$

which (with given u_i and M) is the smallest possible error of any other choice of c_i.

If $E_{rms} \to 0$ when $M \to \infty$, then we have *average convergence* or *convergence in the mean* of $S_M(x)$ to $f(x)$ and we write $\text{l.i.m.}\, S_M = f(x)$ (l.i.m. are the initials of *limit in the mean*). If this is true for any function $f(x)$ [*piecewise smooth* and *quadratically integrable* in the interval (a, b)], then the set $\{u_i(x)\}$ is called *complete*.

In any case, we have

$$\int_a^b f(x)^2\,dx \ge \sum_{i=1}^{M} c_i^2 \qquad \text{[Bessel's inequality]}$$

for any M. In the case of a complete set $\{u_i(x)\}$, we have $E_{rms} \to 0$ for $M \to \infty$ and

$$\int_a^b f(x)^2\,dx = \sum_{i=1}^{\infty} c_i^2 \qquad \text{[Parseval's identity]}$$

Expansion in series of orthonormal functions

Often, we simplify the notation and write the expansion

$$f(x)=\sum_{i=1}^{\infty} c_i u_i(x)$$

instead of explicitly writing an average convergence. Also, we restrict ourselves to piecewise continuous and differentiable functions $f(x)$. These conditions are sufficient (but not necessary) for average convergence and are usually satisfied by the functions that appear in the applications. Attention should be paid to the points of discontinuity where the expansion gives $\frac{1}{2}[f(x+0)+f(x-0)]$ instead of $f(x)$.

Orthonormalization method of Gram-Schmidt

Given a countable (finite or infinite) set of linearly independent functions $\{\varphi_i(x)\}$, $i = 1, 2, \ldots$, we can form a set of orthonormal functions $\{u_i(x)\}$ using the formulas

$$\chi_1(x)=\varphi_1(x), \quad u_i(x)=\frac{\chi_i(x)}{(\chi_i,\chi_i)^{1/2}},$$
$$\chi_{i+1}(x)=\varphi_{i+1}(x)-\sum_{k=1}^{i}(u_k,\varphi_{i+1})\,u_k(x)$$

where $(f,g)=\int_a^b f(x)g(x)\,dx$ is the *scalar product* of $f(x)$ and $g(x)$.

16.2 Orthogonal Functions from ODEs

Often, a physical problem leads to a differential equation with partial derivatives (PDE) and then, after separating the variables, to ordinary differential equations (ODE) of the form

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0$$

over an interval (a, b), where the constant λ results from the separation of the PDE. This ODE, together with the *boundary conditions* that $y(x)$ and $y'(x)$ have to satisfy, usually at the end points of $[a, b]$, form a *Sturm-Liouville system*. Because of the boundary and other conditions, acceptable solutions $y(x)$ exist only for particular values of λ, which are called *eigenvalues*. The corresponding solutions $y(x)$ are called *eigenfunctions*. Usually, for each value of λ there is only one corresponding eigenfunction, but sometimes there are more.

If $p(x)$, $q(x)$, $r(x)$ and the values of $y(x)$, $y'(x)$ at the end points $x = a$, $x = b$ are real and if $r(x) \geq 0$ in (a, b), then (a) the eigenvalues are real and (b) the eigenfunctions that correspond to different eigenvalues constitute an orthogonal set of functions with weight function $r(x)$.

Orthogonal polynomials from ODEs

Some orthogonal sets of functions that result from ODEs include simple and easy to use functions. Some of these are sines and cosines (that lead to Fourier series) and polynomials (probably in combination with other elementary functions). The following table gives the coefficients $p(x)$, $q(x)$, $r(x)$ of some ODEs, the weight function $w(x)$, the constant λ and the interval (a, b).

Polynomials *	$p(x)$	$q(x)$	$r(x)$	$w(x)$	λ	(a, b)
Legendre, $P_n(x)$	$1 - x^2$	0	1	1	$n(n+1)$	$[-1, 1]$
Associated Legendre, $P_n^m(x)$	$1 - x^2$	-m^2/(1-x^2)	1	1	$n(n+1)$	$[-1, 1]$
Hermite, $H_n(x)$	$\exp(-x^2)$	0	$\exp(-x^2)$	$\exp(-x^2)$	$2n$	$(-\infty, \infty)$
Laguerre, $L_n(x)$	xe^{-x}	0	e^{-x}	e^{-x}	n	$[0, \infty)$
Associated Laguerre, $L_n^m(x)$	$x^{m+1}e^{-x}$	0	$x^m e^{-x}$	$x^m e^{-x}$	n	$[0, \infty)$
Chebyshev I, $T_n(x)$	$(1 - x^2)^{1/2}$	0	$(1 - x^2)^{-1/2}$	$(1 - x^2)^{-1/2}$	n^2	$[-1, 1]$
Chebyshev II, $U_n(x)$	$(1 - x^2)^{3/2}$	0	$(1 - x^2)^{1/2}$	$(1 - x^2)^{1/2}$	$n(n+2)$	$[-1, 1]$

* These are all polynomials apart from $P_n^m(x)$ for odd m.

16.3 Hermite Polynomials

Basic relations

For $n = 0, 1, 2, \ldots$ *Hermite's differential equation*

$$y'' - 2xy' + 2ny = 0$$

is satisfied by the *Hermite polynomials*

H_n(x)=(-1)^n*e^(x^2)*(d/dx)^n{e^(-x^2)} (Rodrigues's formula)

Generating function

e^(2*x*t-t^2)=
\S[n,0,\o]{H_n(x)*t^n/n!}

Graphs of Hermite polynomials $H_n(x)$

Fig. 16-1: n 0 1 2 3 4 5

First polynomials

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$

$H_4(x) = 16x^4 - 48x^2 + 12$

$H_5(x) = 32x^5 - 160x^3 + 120x$

$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$

$H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x$

$H_8(x) = 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680$

\S{H_n(x)}, x^(n-2*k)

Properties

Recurrence relations

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$$

$$H_n'(x) - 2nH_{n-1}(x) = 0$$

Orthogonality

\I[x,-\o,\o]{H_m(x)*H_n(x)*e^(-x^2)}=2^*n!*π^(1/2)*δ_(mn)

Other properties

$H_n(-x) = (-1)^n H_n(x)$

|H_n(x)|<(2^n*n!*e^(x^2))^(1/2)

H_(2*n)(0), H_(2*n+1)(0)

\I[x,-\o,\o]{e^(-(x-y)^2)*H_n(x)}=2^n*π^(1/2)*y^n

\I[x,-\o,\o]{e^(-t^2)*H_(2*n)(x*t)}, (x^2-1)^n

\I[x,-\o,\o]{t*e^(-t^2)*H_(2*n+1)(x*t)}, x*(x^2-1)^n

\I[x,-\o,\o]{t^n*e^(-t^2)*H_n(x*t)}=π^(1/2)*n!*P_n(x)

\S{H_n(x+y)}=(1/2^(n/2))*\S[k,0,n]{(n;k)*H_
k(x*2^(1/2))*H_(n-k)(y*2^(1/2))

[*Addition formula* for Hermite polynomials]

\S[k,0,n]{H_k(x)*H_k(y)/(2^k*k!)}, H_n(x), H_(n+1)(x)

H_n(x)=(2^n/π^(1/2))*\I[t,-\o,\o]{(x+i*t)^n*e^(-t^2)}

\I[x,-\o,\o]{x^n*e^(-(x-a)^2)}, H_n(i*a)

Expansion in series

$$f(x) = a_0H_0(x) + a_1H_1(x) + a_2H_2(x) + \cdots$$ Exa

where

a_k=(1/(2^k*k!*π^(1/2)))*\I[x,-\o,\o]{e^(-x^2)*f(x)*H_k(x)}

16.4 Laguerre Polynomials

Basic relations

For $n = 0, 1, 2, \ldots$ *Laguerre's differential equation*

$$xy'' + (1 - x)y' + ny = 0$$

is satisfied by the *Laguerre polynomials*

L_n(x)=(e^x/n!)*(d/dx)^n{x^n*e^(-x)}   (Rodrigues's formula)

Generating function

e^(-x*t/(1-t))/(1-t)=
\S[n,0,\o]{L_n(x)*t^n}

where $|t| < 1$.

First polynomials

$L_0(x) = 1$

$L_1(x) = -x + 1$

$L_2(x) = \frac{1}{2}(x^2 - 4x + 2)$

$L_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$

$L_4(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)$

$L_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)$

$L_6(x) = \frac{1}{720}(x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720)$

$L_7(x) = \frac{1}{5040}(-x^7 + 49x^6 - 882x^5 + 7350x^4 - 29400x^3 + 52920x^2 - 35280x + 5040)$

$L_8(x) = \frac{1}{40320}(x^8 - 64x^7 + 1568x^6 - 18816x^5 + 117600x^4 - 376320x^3 + 564480x^2 - 322560x + 40320)$

Graphs of Laguerre polynomials $L_n(x)$

Fig. 16-2

\S{L_n(x)}, x^k

Properties

Recurrence relations

$$(n+1)L_{n+1}(x) - (2n+1-x)L_n(x) + nL_{n-1}(x) = 0$$

$$xL_n'(x) - nL_n(x) + nL_{n-1}(x) = 0$$

$$L_{n+1}'(x) - L_n'(x) + L_n(x) = 0$$

Orthogonality

\I[x,0,\o]{e^(-x)*L_m(x)*L_n(x)}=δ_(mn)

Other properties

$$L_n(0) = 1$$

$$|L_n(x)| \le e^{x/2} \quad \text{for} \quad x \ge 0$$

$$L_n(x) = M(-n, 1, x) \qquad [M(a, b, x) = \text{confluent hypergeometric function}]$$

\I[x,0,\o]{x^p*e^(-x)*L_n(x)}

\S[k,0,n]{L_k(x)*L_k(y)}, L_n(x), L_n(y)

\S[k,0,\o]{t^k*L_k(x)/k!}=e^t*J_0(2*(x*t)^(1/2))

L_n(x)=(e^x/n!)*\I[u,0,\o]{u^n*e^(-u)*J_0(2*(x*u)^(1/2))}

\I[t,0,x]{L_m(t)*L_n(x-t)}=L_(m+n)(x)-L_(m+n+1)(x)

Expansion in series

$$f(x) = a_0L_0(x) + a_1L_1(x) + a_2L_2(x) + \cdots$$ Exa

a_k=\I[x,0,\o]{e^(-x)*f(x)*L_k(x)}

16.5 Associated Laguerre Polynomials

Ext

Basic relations

The *associated Laguerre's differential equation* (m and n nonnegative integers)

$$xy'' + (m + 1 - x)y' + ny = 0$$

is satisfied by the *associated Laguerre polynomials*

L_n^m(x)=(e^x/(x^m*n!))*(d/dx)^n{x^(n+m)*e^(-x)} (Rodrigues's formula)

Using the Laguerre polynomials we have $L_n^0(x) = L_n(x)$ and

L_n^m(x)=(-1)^m*(d/dx)^m{L_(n+m)(x)}

Generating function

e^(-x*t/(1-t))/(1-t)^(m+1)= \S[n,0,\o]{L_n^m(x)*t^n}

Graphs of associated Laguerre polynomials $L_n^{10}(x)$

Fig. 16-3

First polynomials

$L_0^m(x) = 1$

$L_1^m(x) = -x + m + 1$

$L_2^m(x) = \frac{1}{2}[x^2 - 2(m + 2)x + (m + 1)(m + 2)]$

$L_3^m(x) = -\frac{1}{6}[x^3 - 3(m + 3)x^2 + 3(m + 2)(m + 3)x - (m + 1)(m + 2)(m + 3)]$

$$L_4^m(x) = \frac{1}{24}[x^4 - 4(m + 4)x^3 + 6(m + 3)(m + 4)x^2 - 4(m + 2)(m + 3)(m + 4)x + (m + 1)(m + 2)(m + 3)(m + 4)]$$

\S{L_n^m(x)}, M(-n,m+1,x), x^k

Properties

Recurrence relations

L_n^m(x), L_(n+1)^m(x), L_(n-1)^m(x)

L_n^m(x), L_n^(m-1)(x), L_(n-1)^m(x)

L_n^m(x), L_n^(m+1)(x), L_(n-1)^m(x)

(d/dx){L_n^m(x)}+L_(n-1)^(m+1)(x)=0,
(d/dx){L_(n+1)^m(x)}-(d/dx){L_n^m(x)}+L_n^m(x)=0

x*(d/dx){L_n^m(x)}-n*L_n^m(x)+(n+m)*L_(n-1)^m(x)=0

Orthogonality

\I[x,0,\o]{x^m*e^(-x)*L_n^m(x)*L_p^m(x)}=
((n+m)!/n!)*δ_(np)

Other properties

\I[x,0,\o]{x^(m+1)*e^(-x)*L_n^m(x)^2}=
(2*n+m+1)*(n+m)!/n!

Expansion in series

$$f(x) = A_0^{(m)} L_0^m(x) + A_1^{(m)} L_1^m(x) + A_2^{(m)} L_2^m(x) + \cdots$$ **Exa**

A_k^((m))=(k!/(k+m)!)*\I[x,0,\o]{e^(-x)*x^m*L_k^m(x)*f(x)}

16.6 Chebyshev Polynomials

Basic relations

For $n = 0, 1, 2, \ldots$ *Chebyshev's differential equation*

$$(1 - x^2)y'' - xy' + n^2 y = 0$$ **Ext**

is satisfied by the *Chebyshev polynomials of the first kind*

$$T_n(x) = \cos(n\theta), \quad x = \cos\theta, \quad |x| \le 1$$

and the functions $(1 - x^2)^{1/2} U_{n-1}(x)$, where

U_n(x)=sin((n+1)*θ)/sinθ

are the *Chebyshev polynomials of the second kind*. All roots of $T_n(x)$ and $U_n(x)$ are real and in the interval $[-1, 1]$.

The general solution of Chebyshev's differential equation is

y=c_1+c_2*sin^(-1)x,
y=c_1*T_n(x)+c_2*(1-x^2)^(1/2)*U_(n-1)(x)

16.7 Chebyshev Polynomials of the First Kind

Generating function

(1-x*t)/(1-2*x*t+t^2)=
\S[n,0,\o]{T_n(x)*t^n}

Graphs of Chebyshev polynomials of the first kind $T_n(x)$

Fig. 16-4: n 0 1 2 3 4 5

First polynomials

$T_0(x) = 1$

$T_1(x) = x = \cos\theta$

$T_2(x) = 2x^2 - 1 = \cos 2\theta$

$T_3(x) = 4x^3 - 3x = \cos 3\theta$

$T_4(x) = 8x^4 - 8x^2 + 1 = \cos 4\theta$

$T_5(x) = 16x^5 - 20x^3 + 5x = \cos 5\theta$

$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1 = \cos 6\theta$

$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x = \cos 7\theta$

$T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 = \cos 8\theta$

\S{T_n(x)}, x^(n-2*k)

T_n(x)=((-1)^n*2^n*n!/(2*n)!)*(1-x^2)^(1/2)*(d/dx)^n{(1-x^2)^n/(1-x^2)^(1/2)} (Rodrigues's formula)

Properties

Recurrence relations Ext

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0$$

$$(1 - x^2)T_n'(x) + nxT_n(x) - nT_{n-1}(x) = 0$$

Orthogonality

\I[x,-1,1]{T_m(x)*T_n(x)/(1-x^2)^(1/2)}

Values

$T_n(-x) = (-1)^n T_n(x)$

$T_{2n}(0) = (-1)^n$ $\quad T_{2n+1}(0) = 0$ $\quad T_n(1) = 1$ $\quad T_n(-1) = (-1)^n$

Expansion in series

$$f(x) = \tfrac{1}{2}a_0 T_0(x) + a_1 T_1(x) + a_2 T_2(x) + \cdots$$

Exa

a_k=(2/π)*\I[x,-1,1]{f(x)*T_k(x)/(1-x^2)^(1/2)}

16.8 Chebyshev Polynomials of the Second Kind

Differential equation

The polynomials $U_n(x)$ satisfy the differential equation

$$(1 - x^2)y'' - 3xy' + n(n + 2)y = 0$$

Generating function

1/(1-2*x*t+t^2)=\S[n,0,\o]{U_n(x)*t^n}

First polynomials

$U_0(x) = 1$

$U_1(x) = 2x$

$U_2(x) = 4x^2 - 1$

$U_3(x) = 8x^3 - 4x$

$U_4(x) = 16x^4 - 12x^2 + 1$

$U_5(x) = 32x^5 - 32x^3 + 6x$

$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$

$U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$

$U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$

Graphs of the Chebyshev polynomials of the decond kind $U_n(x)$

Fig. 16-5

\S{U_n(x)}, x^(n-2*k)

U_n(x)=((-1)^n*2^n*(n+1)!/((2*n+1)!*(1-x^2)^(1/2)))*(d/dx)^n{(1-x^2)^(1/2)*(1-x^2)^n} (Rodrigues's formula)

Properties

Recurrence relations

$$U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0$$

$$(1 - x^2)U_n'(x) + nxU_n(x) - (n+1)U_{n-1}(x) = 0$$

Orthogonality

\I[x,-1,1]{U_m(x)*U_n(x)*(1-x^2)^(1/2)}=(π/2)*δ_(mn)

Values

$$U_n(-x) = (-1)^n U_n(x)$$

$$U_{2n}(0) = (-1)^n \qquad U_{2n+1}(0) = 0$$

$$U_n(1) = n + 1 \qquad U_n(-1) = (-1)^n(n+1)$$

Expansion in series

$$f(x) = a_0U_0(x) + a_1U_1(x) + a_2U_2(x) + \cdots$$

a_k=(2/π)*[x,-1,1]{(1-x^2)^(1/2)*f(x)*U_k(x)}

Relations among Chebyshev polynomials and other functions

$$T_n(x) = U_n(x) - xU_{n-1}(x) = xU_{n-1}(x) - U_{n-2}(x) = \tfrac{1}{2}[U_n(x) - U_{n-2}(x)]$$

$$(1 - x^2)U_n(x) = xT_{n+1}(x) - T_{n+2}(x)$$

$$T_n{}'(x) = nU_{n-1}(x)$$

T_n(x)=(1/π)\I[v,-1,1]{(1-v^2)^(1/2)*U_(n-1)(v)/(x-v)}

U_n(x)=(1/π)*\I[v,-1,1]{T_(n+1)(v)/((v-x)*(1-v^2)^(1/2))}

T_n(x)=F(n,-n,1/2;(1-x)/2) [$F(a, b, c; x)$ = hypergeometric function]

U_n(x)=(n+1)*F(n+2,-n,3/2;(1-x)/2)

17 VARIOUS FUNCTIONS

17.1 The Gamma Function

Definitions

If $x > 0$, Γ(x)=\I[t,0,\o]{t^(x-1)*e^(-t)}=\I[t,0,1]{(-lnt)^(x-1)}

If $x \neq 0, -1, -2, \ldots$, Γ(x)=Γ(x+1)/x (recurrently)

The definition is also valid for complex x with $\mathrm{Re}(x) > 0$ or $\mathrm{Re}(x) \neq 0, -1, -2, \ldots$.

For any complex $z \neq 0, -1, -2, \ldots$ Γ(z)=\S[k,0,\o]{(-1)^k/(k!*(z+k))}+ \I[t,1,\o]{e^(-t)*t^(z-1)}

$\Gamma(z)$ is a meromorphic function of z with simple poles at $z = 0, -1, -2, \ldots$.

Other definitions Γ(x)=lim[k,\o]{...}, k^x

1/Γ(x)=x*e^(γ*x)*\P[k,1,\o]{(1+x/k)*e^(-x/k)} [γ is the Euler constant]

Graphs of the gamma function
$\Gamma(x)$ and $1/\Gamma(x)$

Fig. 17-1

Properties

Ext

$\Gamma(x+1) = x\Gamma(x)$

$\Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots$ with $0! = 1$.

(2*n)!!, (2*n+1)!!

Γ(x)*Γ(1-x)=π/sin(π*x)   Γ(x)*Γ(-x)=-π/(x*sin(π*x))

Γ(1/2+x)*Γ(1/2-x)=π/cos(π*x)

Γ(2*x)=(2^(2*x-1)/π^(1/2))*Γ(x)*Γ(x+1/2)   [duplication formula]

Γ(n*x), Γ(x)*Γ(x+1/n)*Γ(x+2/n)...Γ(x+(n-1)/n)

Asymptotic series of Stirling for large x

Γ(x)=(2*π*x)^(1/2)*x^(x-1)*e^(-x)*(1+1/(12*x)+1/(288*x^2)-...)

For $x = n$ (large positive integer) we have Stirling's formula

Γ(n+1)=n!\~(2*π*n)^(1/2)*n^n*e^(-n)   (The ratio of the two parts tends to 1 as $n \to \infty$.)

Values

$\Gamma(1) = \Gamma(2) = 1$,   Γ(1/2)=2*\I[x,0,\o]{e^(-x^2)}=π^(1/2), Γ(3/2)=π^(1/2)/2

Γ(n+1/2)

Γ(-n+1/2)

$\Gamma(1/4) = 3.62560\ 99082\ldots$, $\Gamma(1/3) = 2.67893\ 85347\ldots$

$\Gamma(2/3) = 1.35411\ 79394\ldots$, $\Gamma(3/4) = 1.22541\ 67024\ldots$

Γ'(1)=\I[x,0,\o]{e^(-x)*lnx}=-γ

The incomplete gamma function

γ(a,x)=\I[t,0,x]{t^(a-1)*e^(-t)} — Lower incomplete gamma function

Γ(a,x)=\I[t,x,\o]{t^(a-1)*e^(-t)} — Upper incomplete gamma function

γ(a,x)=Γ(a)-Γ(a,x)=\S[k,0,\o]{(-1)^k*x^(a+k)/(k!*(a+k))}

The psi function

ψ(x)=Γ'(x)/Γ(x)=(d/dx){ln(Γ(x)}=-γ+\S[k,0,\o]{1/(k+1)-1/(x+k)} **Pro**

ψ(x)=\I[t,0,\o]{e^(-t)/t-e^(-x*t)/(1-e^(-t))}=...=\I[t,0,1]{(1-t^(x-1))/(1-t)}-γ

$\psi(x+1) = \psi(x) + x^{-1}$

$\psi(1-x) = \psi(x) + \pi\cot\pi x$

ψ(2*x)=(1/2)*ψ(x)+(1/2)*ψ(x+1/2)+ln2

$\psi(1) = -\gamma, \qquad \psi(1/2) = -\gamma - 2\ln 2, \qquad \psi(1/4) = -\gamma - \pi/2 - 3\ln 2$ **App**

ψ(n+1)=-γ+\S[k,1,n]{1/k}

ψ(n+1/2)=-γ-2*ln2+2*\S[k,1,n]{1/(2*k-1)}

$\psi'(1) = \pi^2/6, \qquad \psi'(1/2) = \pi^2/2$

ψ'(n)=π^2/6-\S[k,1,n-1]{1/k^2}, ψ'(1/2\+n)=π^2/2\-4*\S[k,1,n]{1/(2*k-1)^2}

Polygamma functions

ψ_n(x)=(d/dx)^n{ψ(x)}=(d/dx)^n{ln(Γ(x))}

ψ_n(x)=(-1)^(n+1)*\I[t,0,\o]{t^n*e^(-x*t)/(1-e^(-t))}

$\psi_0(x) = \psi(x)$ is the digamma function, $\psi_1(x) = \psi'(x)$ is the trigamma function, etc.

$$\psi_n(x+1) = \psi_n(x) + (-1)^n n!\, x^{-n-1}$$

$$\psi_n(1) = (-1)^{(n+1)} n!\, \zeta(n+1), \qquad n = 1, 2, \ldots$$

$$\psi_n(1/2) = (-1)^{(n+1)} n!\, (2^{n+1} - 1)\zeta(n+1), \qquad n = 1, 2, \ldots$$

[$\zeta(n)$ is the Riemann zeta function]

17.2 The Beta Function

Definitions

B(x,y)=\I[t,0,1]{t^(x-1)*(1-t)^(y-1)}

The definition is also valid for complex x and y with Re(x) > 0, Re(y) > 0.

B(x,y)=Γ(x)*Γ(y)/Γ(x+y) (it can be used for extention to $x < 0$, $y < 0$) **Pro**

Properties

$B(x, y) = B(y, x)$

B(x,y+1)=(y/(x+y))*B(x,y)

B(x+1,y)=(x/(x+y))*B(x,y)

B(x,y)=2*\I[θ,0,π/2]{sinθ^(2*x-1)*cosθ^(2*y-1)}

B(x,y)=\I[t,0,\o]{t^(x-1)/(1+t)^(x+y)}

B(x,y)=r^y*(r+1)^x*\I[t,0,1]{t^(x-1)*(1-t)^(y-1)/(t+r)^(x+y)}

B(x,y)=\I[t,0,\o]{e^(-x*t)*(1-e^(-t))^(y-1)}

1/B(m,n)=m*((m+n-1);(n-1))=n*((m+n-1);(m-1))

The incomplete beta function

B_x(a,b)=\I[t,0,x]{t^(a-1)*(1-t)^(b-1)}

17.3 Bernoulli and Euler Polynomials and Numbers

Definitions

The *Bernoulli polynomials* $B_k(x)$ are defined by the relation

t*e^(x*t)/(e^t-1)=\S[k,0,\o]{B_k(x)*t^k/k!}

The *Euler polynomials* $E_k(x)$ are defined by the relation

2*e^(x*t)/(e^t+1)=\S[k,0,\o]{E_k(x)*t^k/k!}

The first Bernoulli and Euler polynomials are

B_0(x)	E_0(x)
B_1(x)	E_1(x)
B_2(x)	E_2(x)
B_3(x)	E_3(x)
B_4(x)	E_4(x)
B_5(x)	E_5(x)

From the polynomials we have the *Bernoulli numbers* B_k and the *Euler numbers* E_k

$B_k = B_k(0), \quad k \geq 0$

$B_0 = 1$	$B_1 = -\frac{1}{2}$
$B_2 = \frac{1}{6}$	$B_4 = -1/30$
$B_6 = 1/42$	$B_8 = -1/30$
$B_{10} = 5/66$	$B_{12} = -691/2730$

$B_3 = B_5 = \cdots = B_{2k+1} = 0, \quad k \geq 1$

$B_k(1) = B_k, \quad k \geq 2$

B_n(x)=\S[k,0,n]{(n;k)*B_k*x^(n-k)}

$E_k = 2^k E_k(\frac{1}{2}), \quad k \geq 0$

$E_0 = 1$	
$E_2 = -1$	$E_4 = 5$
$E_6 = -61$	$E_8 = 1385$
$E_{10} = -50521$	$E_{12} = 2702765$

$E_1 = E_3 = \cdots = E_{2k+1} = 0, \quad k \geq 0$

$E_k(1) = -E_k(0), \quad k \geq 0$

E_n(x)=\I[k,0,n]{(n;k)*E_k*(x-1/2)^(n-k)/2^k}

Properties **Ext**

(k+1)*x^k= S[i,0,k]{((k+1);i)*B_i(x)} $\quad k = 0, 1, \ldots$

$B_k'(x) = kB_{k-1}(x), \quad k = 1, 2, \ldots$

$B_k(x + 1) - B_k(x) = kx^{k-1}, \quad k = 0, 1, \ldots$

$E_k'(x) = kE_{k-1}(x), \quad k = 1, 2, \ldots$

$E_k(x + 1) + E_k(x) = 2x^k, \quad k = 0, 1, \ldots$

$B_k(1 - x) = (-1)^k B_k(x), \quad k = 0, 1, \ldots$

$(-1)^k B_k(-x) = B_k(x) + kx^{k-1}, \quad k = 0, 1, \ldots$

$E_k(1 - x) = (-1)^k E_k(x), \quad k = 0, 1, \ldots$

$(-1)^{k+1} E_k(-x) = E_k(x) - 2x^k, \quad k = 0, 1, \ldots$

B_(m+1)(n+1)-B_(m+1)=(m+1)*\S[k,1,n]{k^m}

E_m(n+1)+(-1)^n*E_m(0)=2*\S[k,1,n]{(-1)^(n-k)*k^m}

\S[i,0,k]{((k+1);i)*B_i}=0

B_(2*k), ζ(2*k)

B_(2*k), \I[x,0,\o]{x^(2*k-1)/(e^(2*π*x)-1)}

B_(2*k), \I[x,0,\o]{x^(2*k)/sinhx^2}

\I[t,a,x]{B_k(t)}=(B_(k+1)(x)-B_(k+1)(a))/(k+1)}

\I[t,a,x]{E_k(t)}=(E_(k+1)(x)-E_(k+1)(a))/(k+1)}

17.4 The Riemann Zeta Function

Definitions

The Riemann zeta function is defined by

ζ(x)=\S[k,1,\o]{1/k^x}

This definition holds for $x > 1$ and also for a complex argument z with $\mathrm{Re}(z) > 1$. It can be extended by analytic continuation to other values. The only singular point of $\zeta(z)$ is $z = 1$ and has residue 1.

Usually, for $x = n$ (positive integer) we also use the following functions:

β(n)=\S[k,0,\o]{(-1)^k/(2*k+1)^n}, η(n)=\S[k,1,\o]{(-1)^(k+1)/k^n},
λ(n)=\S[k,0,\o]{1/(2*k+1)^n}

Properties

ζ(x)=(1/Γ(x))*\I[t,0,\o]{t^(x-1)/(e^t-1)} (alternative definition)

ζ(x)=(1/((1-2^(1-x))*Γ(x)))*\I[t,0,\o]{t^(x-1)/(e^t+1)}

ζ(1-x)=2^(1-x)*π^(-x)*Γ(x)*cos(π*x/2)*ζ(x)

η(n)=(1-2^(1-n))*ζ(n)

λ(n)=(1-2^(-n))*ζ(n)

1/ζ(x)=\P[p,2,\o]{(1-1/p^x)}

(The infinite product contains all the prime numbers p.)

ζ(2*n+1), \I[x,0,1]{B_(2*n+1)(x)*cot(π*x)}

β(2*n), \I[x,0,1]{E_(2*n-1)(x)/cos(π*x)}

Values

ζ(0)=-1/2, ζ'(0)=-ln(2*π)/2

ζ(2)	β(1)	η(2)	λ(2)
ζ(4)	β(3)	η(4)	λ(4)
ζ(6)	β(5)	η(6)	λ(6)
ζ(8)	β(7)	η(8)	λ(8)
ζ(10)	β(9)	η(10)	λ(10)

$\beta(2) = G = 0.915\,965\,594\,177\ldots$ (G is the Catalan constant)

ζ(2*n), B_(2*n)

ζ(1-2*n)=-B_(2*n)/(2*n), ζ(-2*n)=0

β(2*n+1), |E_(2*n)|

17.5 Hypergeometric Functions

Differential equations

The *hypergeometric differential equation* (Gauss's ODE) is

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$ **Ext**

If c, $a-b$, and $c-a-b$ are not integers, the general solution for $|x| < 1$ is

$$y = c_1 F(a, b, c; x) + c_2 x^{1-c} F(a-c+1, b-c+1, 2-c; x)$$

where $F(a, b, c; x)$ is the *hypergeometric function* **Inf**

\S{F(a,b,c;x}, x^k

If the function has infinite terms and a, b, and c are real constants, it converges (uniformly and absolutely) in the interval $-1 < x < 1$ (in general, in the complex plane for $|z| < 1$, whereas it diverges for $|z| > 1$).

For $|x| = 1$ (or $|z| = 1$), the convergence depends on the quantity $s = c - (a + b)$: If $s \leq -1$, the series diverges. If $-1 < s \leq 0$, the series converges conditionally, except at $x = 1$, where it diverges. If $s > 0$, the series converges absolutely.

The *confluent hypergeometric differential equation* (Kummer's ODE)

$$xy'' + (b - x)y' - ay = 0$$

is obtained from the hypergeometric differential equation with the confluence of two irregular points. The general solution is

$$y = c_1 M(a, b; x) + c_2 x^{1-b} M(a - b + 1, 2 - b; x)$$

where $M(a, b; x)$ is the *confluent hypergeometric function*

\S{M(a,b;x)}, x^k

that converges for every finite x. Often, we use *Whittaker's function*

$$M_{\kappa\mu}(x) = e^{-x/2} x^{\mu+1/2} M(\mu - \kappa + \tfrac{1}{2}, 2\mu + 1; x)$$

which satisfies the ODE

$$x^2 y'' + (-\tfrac{1}{4}x^2 + \kappa x + \tfrac{1}{4} - \mu^2)y = 0$$

Relations with other functions

Ext

For particular values of the parameters, the hypergeometric functions reduce to elementary or simpler functions.

$F(-p, b, b; -x) = (1 + x)^p$

F(1,p,p;x)=1/(1-x)

F(1,1,2;-x)=ln(1+x)/x

F(1/2,1,3/2;x^2)=(1/(2*x)*ln((1+x)/(1-x))

F(n/2,-n/2,1/2;sinx^2)=cos(n*x)

F((n+1)/2,(1-n)/2,1/2;sinx^2)=cos(n*x)/cosx

F(1/2,1/2,3/2;x^2)=sin^(-1)x/x

F(1/2,1,3/2;-x^2)=tan^(-1)x/x

F(a,a+1/2,1/2;x^2)=(1/2)*(1/(1+x)^(2*a)+1/(1-x)^(2*a))

lim[n,\o]{F(1,n,1;x/n)=e^x

F(1/2,1/2,3/2;-x^2)=ln(x+(1+x^2)^(1/2))/x

F(n+1,-n,1;(1-x)/2)=P_n(x)

F(n,-n,1/2;(1-x)/2)=T_n(x)

F(n+2,-n,3/2;(1-x)/2)=U_n(x)/(n+1)

F(-n,n+2*α,α+1/2;(1-x)/2), C_n^((α))(x)

[$C_n^{(\alpha)}(x)$ are the Gegenbauer polynomials]

F(-n,α+β+n+1,α+1;(1-x)/2), P_n^((α,β))(x)

[$P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials]

$M(a, a; x) = e^x$ $M(1, 2; -2ix) = e^{-ix}x^{-1}\sin x$ $M(1, 2; 2x) = e^x x^{-1}\sinh x$

M(-n,1/2;x^2), H_(2*n)(x)

M(-n,3/2;x^2), H_(2*n+1)(x)/x

M(n+1/2,2*n+1;2*i*x)=2^n*n!*e^(i*x)*J_n(x)/x^n

M(n+1/2,2*n+1;2*x)=2^n*n!*e^x*I_n(x)/x^n

M(1/2,3/2;-x^2)=erf(x)/x, (1/x)*\I[t,0,x]{e^(-t^2)}

M(-n,m+1;x)=(n!*m!/(n+m)!)*L_n^m(x)

Properties

Ext

F(a,b,c;1)=Γ(c)*Γ(c-a-b)/(Γ(c-a)*Γ(c-b))

$F(a, b, c; x) = (1 - x)^{c-a-b}F(c - a, c - b; c; x)$

F(a,b,c;x)=F(a,c-b,c;-x/(1-x))/(1-x)^a

(d/dx){F(a,b,c;x)}=(a*b/c)*F(a+1,b+1,c+1;x)

F(a,b,c;x)}=(Γ(c)/(Γ(b)*Γ(c-b)))*\I[u,0,1]{u^(b-1)*(1-u)^(c-b-1)/(1-u*x)^a}

$M(a, b, x) = e^x M(b - a, b; -x)$

M(a,b;x)}=(Γ(b)/(Γ(a)*Γ(b-a)))*\I[u,0,1]{e^(x*u)*u^(a-1)*(1-u)^(b-a-1)}

17.6 Elliptic Functions

Elliptic integrals

The (*incomplete*) *elliptic integral of the first kind* is

F(φ,k)=\I[θ,0,φ]{(1/(1-k^2*sinθ^2)^(1/2)}=
\I[v,0,x]{1/((1-v^2)*(1-k^2*v^2))^(1/2)}

The *complete elliptic integral of the first kind* is

K(k)=F(π/2,k)=\I[θ,0,π/2]{1/(1-k^2*sinθ^2)^(1/2)=
\I[v,0,1]{1/((1-v^2)*(1-k^2*v^2))^(1/2)}=
(π/2)*(1+k^2/4+...)

The (*incomplete*) *elliptic integral of the second kind* is

E(φ,k)=\I[θ,0,φ]{(1-k^2*sinθ^2)^(1/2)}=
\I[v,0,x]{(1-k^2*v^2)^(1/2)/(1-v^2)^(1/2)}

The *complete elliptic integral of the second kind* is

E(k)=E(π/2,k)=\I[θ,0,π/2]{(1-k^2*sinθ^2)^(1/2)}=
\I[v,0,1]{(1-k^2*v^2)^(1/2)/(1-v^2)^(1/2)}=
(π/2)*(1-k^2/4-...)

The (*incomplete*) *elliptic integral of the third kind* is

Π(n,φ,k)=\I[θ,0,φ]{1/((1+n*sinθ^2)*(1-k^2*sinθ^2)^(1/2))}=
\I[v,0,x]{1/((1+n*v^2)*((1-v^2)*(1-k^2*v^2))^(1/2))}

The *complete elliptic integral of the third kind* is ($x = \sin\varphi$, $k^2 < 1$)

Π(n,π/2,k)=\I[θ,0,π/2]{1/((1+n*sinθ^2)*(1-k^2*sinθ^2)^(1/2))}=
\I[v,0,1]{1/((1+n*v^2)*((1-v^2)*(1-k^2*v^2))^(1/2))}

Catalan constant

G=(1/2)*\I[k,0,1]{K(k)}=(1/2)*\I[k,0,1]{\I[θ,0,π/2]{1/(1-k^2*sinθ^2)^(1/2)}}=
1/1^2-1/3^2+1/5^2-...

Elliptic functions

Ext

The inverse function of $u = F(\varphi, k)$ is $\varphi = \operatorname{am} u$ (it is called *amplitude* of u and its dependence on k is suppressed). The following are *Jacobi's elliptic functions*:

$\operatorname{sn} u = \sin\varphi = x = \sin(\operatorname{am} u)$, cnu=cosφ=(1-x^2)^(1/2)=cos(amu)=(1-snu^2)^(1/2),

dnu=(1-k^2*sinφ^2)^(1/2)=(1-k^2*x^2)^(1/2)=(1-k^2*snu^2)^(1/2)

Let K(k)=\I[θ,0,π/2]{1/(1-k^2*sinθ^2)^(1/2), K'(k)=\I[θ,0,π/2]{1/(1-(1-k^2)*sinθ^2)^(1/2) . Each of the three elliptic functions has two periods: $\operatorname{sn} u$ has periods $4K$ and $2iK'$, $\operatorname{cn} u$ has periods $4K$ and $2K + 2iK'$, and $\operatorname{dn} u$ has periods $2K$ and $4iK'$.

Identities

$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1, \qquad \operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u = 1, \qquad \operatorname{dn}^2 u - k^2 \operatorname{cn}^2 u = 1 - k^2$

Values

$\operatorname{am} 0 = 0 \qquad \operatorname{sn} 0 = 0 \qquad \operatorname{cn} 0 = 1 \qquad \operatorname{dn} 0 = 1$

Addition formulas

sn(u+v)=(snu*cnv*dnv+cnu*snv*dnu)/(1-k^2*snu^2*snv^2)

cn(u+v)=(cnu*cnv-snu*snv*dnu*dnv)/(1-k^2*snu^2*snv^2)

dn(u+v)=(dnu*dnv-k^2*snu*snv*cnu*cnv)/(1-k^2*snu^2*snv^2)

Derivatives

(d/du){snu}=cnu*dnu (d/du){cnu}=-snu*dnu (d/du){dnu}=-k^2*snu*cnu

Series expansions

\S{snu}, u^(2*k+1)

\S{cnu}, u^(2*k)

\S{dnu}, u^(2*k)

Integrals

\I[u]{snu}	\I[u]{cnu}
\I[u]{dnu}	\I[u]{1/snu}
\I[u]{1/cnu}	\I[u]{1/dnu}
\I[u]{snu/cnu}	\I[u]{snu/dnu}
\I[u]{cnu/snu}	\I[u]{cnu/dnu}
\I[u]{dnu/snu}	\I[u]{dnu/cnu}

17.7 Other Functions

Error function

erf(x)=(2/π^(1/2))*\I[u,0,x]{e^(-u^2)}=
(1/π^(1/2))*\I[v,0,x^2]{e^(-v)/v^(1/2)}=
(2*x*e^(-x^2)/π^(1/2))*M(1,3/2;x^2),
\S[erf(x)}, x^(2*k+1)

erf(x)~1-(e^(-x^2))/(π^(1/2)*x)*(1-1/(2*x^2)+...) — asymptotic expansion as $x \longrightarrow \infty$

$\mathrm{erf}(-x) = -\mathrm{erf}(x), \quad \mathrm{erf}(0) = 0, \quad \mathrm{erf}(-\infty) = -1, \quad \mathrm{erf}(\infty) = 1$

Complementary error function

Ext

erfc(x)=1-erf(x)=(2/π^(1/2))*\I[u,x,\o]{e^(-u^2)}=
1-(2/π^(1/2))*(x-x^3/(3*1!)+x^5/(5*2!)+...)

erfc(x)~(e^(-x^2)/(π^(1/2)*x)*(1-1/(2*x^2)+... — asymptotic expansion as $x \longrightarrow \infty$

$\mathrm{erfc}(-x)=2 - \mathrm{erfc}(x), \quad \mathrm{erfc}(0) = 1, \quad \mathrm{erfc}(-\infty) = 2, \quad \mathrm{erfc}(\infty) = 0$

Exponential integrals

Ext

E_n(x)=\I[u,1,\o]{e^(-x*u)/u^n}

Ei(x)=-E_1(-x)=-\I[v,-x,\o]{e^(-v)/v}=\I[v,-\o,x]{e^v/v}

Sine integral

Si(x)=\I[u,0,x]{sinu/u}=π/2-\I[u,x,\o]{sinu/u},
\S{Si(x)}, x^(2*k+1)

Si(x)~π/2-(sinx/x)*(1/x-3!/x^3+...)-(cosx/x)*(1-2!/x^2+...)

$\mathrm{Si}(-x) = -\mathrm{Si}(x), \quad \mathrm{Si}(0) = 0, \quad \mathrm{Si}(\infty) = \pi/2$

Cosine integral

Ci(x)=-\I[u,x,\o]{cosu/u}=γ+lnx+\I[u,0,x]{(cosu-1)/u}, \S{Ci(x)}

Ci(x)~(sinx/x)*(1-2!/x^2+...)-(cosx/x)*(1/x-3!/x^3+...)

$\mathrm{Ci}(\infty) = 0$

Fresnel's sine and cosine integrals

S(x)=(2/π)^(1/2)*\I[u,0,x]{sin(u^2)}, \S{S(x)}, x^(4*k+3)

C(x)=(2/π)^(1/2)*\I[u,0,x]{cos(u^2)}, \S{C(x)}, x^(4*k+1)

S(x)~1/2-(1/(2*π)^(1/2))*(cos(x^2)(1/x-...)+sin(x^2)*(1/(2*x^3)-...))

C(x)~1/2+(1/(2*π)^(1/2))*(sin(x^2)(1/x-...)-cos(x^2)*(1/(2*x^3)-...))

$S(-x) = -S(x), \quad S(0) = 0, \quad S(\infty) = \frac{1}{2} \qquad C(-x) = -C(x), \quad C(0) = 0, \quad C(\infty) = \frac{1}{2}$

18 FOURIER TRANSFORMS

18.1 Fourier's Integral Theorem

If (i) the functions $f(t)$ and $f'(t)$ are piecewise continuous in every finite open interval, (ii) \I[t,-\o,\o]{|f(t)|} converges, and (iii) $f(t)$ equals $\frac{1}{2}\{f(t+0)+f(t-0)\}$ at any point of discontinuity, then

Inf

f(t)=\I[ω,0,\o]{A(ω)*cos(ω*t)+B(ω)*sin(ω*t)}

where

A(ω)=(1/π)*\I[u,-\o,\o]{f(u)*cos(ω*u)},
B(ω)=(1/π)*\I[u,-\o,\o]{f(u)*sin(ω*u)}

Two other forms of *Fourier's theorem* are

f(t)=(1/(2*π))*\I[ω,-\o,\o]{\I[u,-\o,\o]{f(u)*cos(ω*(u-t))}}

and

f(t)=(1/(2*π))*\I[ω,-\o,\o]{e^(-i*ω*t)*\I[u,-\o,\o]{f(u)*e^(i*ω*u)}}
=(1/(2*π))*\I[ω,-\o,\o]{\I[u,-\o,\o]{f(u)*e^(i*ω*(u-t))}}

If $f(t)$ is an odd function [meaning $f(-t)=-f(t)$], then

f(t)=(2/π)*\I[ω,0,\o]{\I[u,0,\o]{f(u)*sin(ω*u)}*sin(ωt)}

If $f(t)$ is an even function [meaning $f(-t)=f(t)$], then

f(t)=(2/π)*\I[ω,0,\o]{\I[u,0,\o]{f(u)*cos(ω*u)}*cos(ωt)}

All of the above are summarized as follows: All the information that exists in the t space [meaning in $f(t)$] can be transferred to the ω space after an integration (with respect to t) and then, after a second integration (with respect to ω), it can be transferred back to the t space.

18.2 Fourier Transforms

The *Fourier transform* of $f(t)$ is defined by the relation

$$F(\omega) = \mathcal{F}\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

The *inverse Fourier transform* of $F(\omega)$ is defined by the relation

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega t}\, d\omega$$

The functions $f(t)$ and $F(\omega)$ are a *pair of Fourier transforms*. The function $f(t)$ represents the information in the *time space* and $F(\omega)$ in the *frequency space* (often t represents time and ω angular frequency).

Properties

If $F(\omega) = \mathcal{F}\{f(t)\}$ and $G(\omega) = \mathcal{F}\{g(t)\}$, then for constants a and b we have the following properties:

$\mathcal{F}\{af(t) + bg(t)\} = aF(\omega) + bG(\omega)$	Linearity
$\mathcal{F}\{f^*(t)\} = F^*(-\omega)$, (* for complex conj.)	Complex conjugation
$\mathcal{F}\{F(t)\} = f(-\omega)$	Duality
$\mathcal{F}\{f(at)\} = a^{-1}F(\omega/a), \quad a > 0$	Scale change
$\mathcal{F}\{f(t+a)\} = e^{-ia\omega}F(\omega)$	Translation
$\mathcal{F}\{t^n f(t)\} = (-i)^n \frac{d^n}{d\omega^n} F(\omega)$	Multiplication by power
$\mathcal{F}\{f(t)e^{iat}\} = F(\omega + a)$	Multiplication by e^{iat}
$\mathcal{F}\{f(t)\cos at\} = \frac{1}{2}F(\omega + a) + \frac{1}{2}F(\omega - a)$	Modulation

If $\lim_{a \to 0} \mathcal{F}\{f(t,a)\} = \mathcal{F}\{f(t)\}$, then $\lim_{a \to 0} f(t,a) = f(t)$ where $f(t)$ is continuous.

If (a) the derivatives $f^{(r)}(t)$ exist for $r = 0, 1, 2, \ldots, n$ for every t and are absolutely integrable functions and (b) $f^{(r)}(t) \to 0$ as $|t| \to \infty$ for any $r = 0, 1, 2, \ldots, n-1$, then

$$\mathcal{F}\{f^{(n)}(t)\} = (-i\omega)^n F(\omega)$$

If $f(t)$ and $\int_{-\infty}^{t} f(u)\, du$ are absolutely integrable functions of t, then

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(u)\, du\right\} = \frac{i}{\omega} F(\omega)$$

The convolution theorems

If $f \star g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du$ is the *convolution* (as defined for the Fourier transformation) of two absolutely integrable functions $f(t)$ and $g(t)$ with Fourier transforms $F(\omega)$ and $G(\omega)$, then

$$\mathcal{F}\{f \star g\} = FG \qquad \mathcal{F}\{fg\} = F \star G$$ Pro

i.e. the transform of the convolution is the product of the transforms and the transform of a product is the convolution of the transforms. The inverse of these relations are

$$f \star g = \mathcal{F}^{-1}\{FG\} \qquad fg = \mathcal{F}^{-1}\{F \star G\}$$

Parseval's identity

If $F(\omega) = \mathcal{F}\{f(t)\}$ and $G(\omega) = \mathcal{F}\{g(t)\}$, then

$$\int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt = \int_{-\infty}^{\infty} F(\omega)\, G^*(\omega)\, d\omega$$ Pro

where the asterisk indicates the complex conjugate. More specifically,

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega$$

Correlation

The *cross-correlation* $c(t)$ of two (real or complex) functions $f(t)$ and $g(t)$ is

$$c(t) = f^*(-t) \star g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^*(-u)\, g(t-u)\, du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^*(u)\, g(t+u)\, du$$

The *autocorrelation* $a(t)$ of a function is

$$a(t) = f^*(-t) \star f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^*(-u)\, f(t-u)\, du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^*(u)\, f(t+u)\, du$$

The corresponding Fourier transforms are

$$\mathcal{F}\{c(t)\} = F^*(\omega)G(\omega) \qquad \text{and} \qquad \mathcal{F}\{a(t)\} = |F(\omega)|^2$$

Moments

The *moment of order n* (or *n*th moment) of a function $f(t)$ is

$$P_{nf} = P_n\{f(t)\} = \int_{-\infty}^{\infty} t^n f(t)\, dt = \ldots = \sqrt{2\pi}\,(-i)^n F^{(n)}(0)$$

where $F^{(n)}(0) = [d^nF/d\omega^n]_{\omega=0}$. P_{nf} is linear in $f(t)$.

Area under $f(t)$ $\quad P_{0f} = \sqrt{2\pi}\,F(0)$

First moment $\quad P_{1f} = -i\sqrt{2\pi}\,F'(0)$

Center of $f(t)$ at $\quad x = P_{1f}/P_{0f} = -iF'(0)/F(0) \quad (\text{for } P_{0f} \neq 0)$

Second moment or moment of inertia $\quad P_{2f} = -\sqrt{2\pi}\,F''(0)$

Equivalent width $\quad W_f = P_{0f}/f(0) = 2\pi F(0)/P_{0F}$

Autocorrelation width $\quad W_a = P_{0a}/a(0)$

Pro

Uncertainty relation

If $f(t)$ is differentiable and $\lim[tf^2(t)] = 0$ as $|t| \to \infty$, then

(\I[t,-\o,\o]{t^2*|f(t)|^2})^(1/2)*(\I[ω,-\o,\o]{ω^2*|F(ω)|^2})^(1/2)\>(1/2)*\I[t,-\o,\o]{|f(t)|^2}

More generally, the *time duration* Δt and the *bandwidth* $\Delta\omega$ defined by

(Δt)^2=\I[t,-\o,\o]{(t-t_0)^2*|f(t)|^2}/\I[t,-\o,\o]{|f(t)|^2},
(Δω)^2=\I[ω,-\o,\o]{(ω-ω_0)^2*|F(ω)|^2}/\I[ω,-\o,\o]{|F(ω)|^2}

where t_0 and ω_0 are arbitrary, satisfy the *uncertainty relation*

$$\Delta t \cdot \Delta\omega \geq 1/2$$

Pro

18.3 Fourier Sine and Cosine Transforms

The *Fourier sine transform* of $f(t)$ (defined for $t > 0$) is

F_s(ω)=(2/π)^(1/2)*\I[t,0,\o]{f(t)*sin(ω*t)}

The *inverse Fourier sine transform* of $F_s(\omega)$ for $\omega > 0$ is

f(t)=(2/π)^(1/2)*\I[ω,0,\o]{F_s(ω)*sin(ω*t)}

The *Fourier cosine transform* of $f(t)$ (defined for $t > 0$) is

F_c(ω)=(2/π)^(1/2)*\I[t,0,\o]{f(t)*cos(ω*t)}

The *inverse Fourier cosine transform* of $F_c(\omega)$ for $\omega > 0$ is

f(t)=(2/π)^(1/2)*\I[ω,0,\o]{F_c(ω)*cos(ω*t)}

Ext

18.4 Tables of Fourier Transforms

In each case, we give (a) the function $f(t)$, (b) the corresponding Fourier transform $F(\omega)$ [or $F_s(\omega)$ or $F_c(\omega)$], (c) the graph of $f(t)$ in green, (d) the graph of $\mathrm{Re}\{F(\omega)\}$ in red, and (e) the graph of $\mathrm{Im}\{F(\omega)\}$ in purple. On the horizontal axis, the values of t and ω are given and on the vertical axis the values of $f(t)$ and $F(\omega)$ are given. For some $f(t)$ the integral \I[t,-\o,\o]{|f(t)|} does not exist, but the function $F(\omega)$ can be used in formal (not rigorous) calculations.

Methods to prove the formulas: ❶ Use delta function. ❷ Use definite integral. ❸ Use Fourier cosine integral. ❹ Use complex integral. ❺ Prove inverse.

Fourier transforms (— $f(t)$, — $\mathrm{Re}\{F(\omega)\}$, — $\mathrm{Im}\{F(\omega)\}$, $-\infty < \omega < \infty$) **Inf**

$f(t) = 1$ ❶

\F{1}, δ(ω)

Fig. 18-1

Graphs of $f(t)$ and $F(\omega)$

$f(t) = \delta(t - a)$ ❶

\F{δ(t-a)}, e^(i*a*ω)

Fig. 18-2

Graphs of $f(t)$ and $F(\omega)$

f(t)=1:|t|<a;0:|t|>a

Cal

\F{1:|t|<a;0:|t|>a}, sin(a*ω)/ω

Fig. 18-3

Graphs of $f(t)$ and $F(\omega)$

f(t)=1-|t|/a:|t|<a;0:|t|>a

Cal

\F{1-|t|/a:|t|<a;0:|t|>a}, (1-cos(a*ω))/ω^2

Fig. 18-4

Graphs of $f(t)$ and $F(\omega)$

$f(t) = \frac{1}{2}[1 + \text{sgn}(t)]$

Cal

\F{$\frac{1}{2}$[1 + sgn(t)]}, δ(ω), 1/ω

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-5

f(t)=1/t

Cal

\F{1/t}, sgn(ω)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-6

f(t)=1/|t|^(1/2)

2

\F{1/|t|^(1/2)}, 1/|ω|^(1/2)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-7

f(t)=1/|t|^a

Cal

\F{1/|t|^a}, |ω|^(a-1)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-8

f(t)=1/(t^2+a^2)

\F{1/(t^2+a^2)}, e^(-a*|ω|)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-9

Cal

f(t)=t/(t^2+a^2)

\F{t/(t^2+a^2)}, sgn(ω)*e^(-a*|ω|)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-10

$f(t) = e^{iat}$ ❶

\F{e^(i*a*t}, δ(ω+a)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-11

f(t)=e^(i*a*t):|t|<b;0:|t|>b

Cal

\F{e^(i*a*t):|t|<b;0:|t|>b},
sin(b*(ω+a))/(ω+a)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-12

$f(t) = e^{-a|t|}, \quad a > 0$

Cal

\F{e^(-a*|t|)}, 1/(ω^2+a^2)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-13

$f(t) = te^{-a|t|}, \quad a > 0$ ❷

\F{t*e^(-a*|t|)}, i*ω/(ω^2+a^2)^2

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-14

$f(t) = |t|e^{-a|t|}, \quad a > 0$ ❷

\F{|t|*e^(-a*|t|)}, (a^2-ω^2)/(ω^2+a^2)^2

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-15

$f(t) = \exp(-at^2), \quad a > 0$

Cal

\F{e^(-a*t^2)}, e^(-ω^2/(4*a))

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-16

$f(t) = \sin(at^2), \quad a > 0$ ❸

\F{sin(a*t^2)}, cos(ω^2/(4*a)+π/4)

Fig. 18-17

Graphs of $f(t)$ and $F(\omega)$

$f(t) = \cos(at^2)$ ❸

\F{cos(a*t^2)}, cos(ω^2/(4*|a|)-π/4)

Fig. 18-18

Graphs of $f(t)$ and $F(\omega)$

$f(t) = e^{-a|t|}\cos bt, \quad a > 0$ ❸

\F{e^(-a*|t|)*cos(b*t)},
1/((ω+b)^2+a^2), 1/((ω-b)^2+a^2)

Fig. 18-19

Graphs of $f(t)$ and $F(\omega)$

f(t)=e^(-a*t)*ln|1-e^(-t)| ❹

\F{e^(-a*t)*ln|1-e^(-t)|},
cot(π*a-i*π*ω)/(a-i*ω)

Fig. 18-20

Graphs of $f(t)$ and $F(\omega)$

f(t)=e^(-a*t)*ln|1+e^(-t)| ❹

\F{e^(-a*t)*ln|1+e^(-t)|},
1/((a-i*ω)*sin(π*a-i*π*ω))

Fig. 18-21

Graphs of $f(t)$ and $F(\omega)$

$f(t) = t^{-1}\sin at, \quad a > 0$ Cal

\F{sin(a*t)/t}, 1+sgn(a-|ω|)

Fig. 18-22

Graphs of $f(t)$ and $F(\omega)$

f(t)=t/sinht

❹

\F{t/sinht}, e^(π*ω)/(1+e^(π*ω))^2

Fig. 18-23

Graphs of $f(t)$ and $F(\omega)$

f(t)=cosh(a*t)/sinh(b*t)

❹

\F{cosh(a*t)/sinh(b*t)}, sinh(π*ω/b)/(cosh(π*ω/b)+cos(π*a/b))

Fig. 18-24

Graphs of $f(t)$ and $F(\omega)$

Fourier sine transforms (— $f(t)$, — $F_s(\omega)$, $0 < \omega < \infty$)

f(t)=1:0<t<a;0:0<a<t

Cal

\F_s{1:0<t<a;0:0<a<t}, (1-cos(a*ω))/ω

Fig. 18-25

Graphs of $f(t)$ and $F(\omega)$

f(t)=sint:0<t<a;0:0<a<t

Cal

\F_s{sint:0<t<a;0:0<a<t}, (cosa*sin(a*ω)-ω*sina*cos(a*ω))/(ω^2-1)

Fig. 18-26

Graphs of $f(t)$ and $F(\omega)$

$f(t) = t^{-1}$

Cal

\F_s{1/t}

Fig. 18-27

Graphs of $f(t)$ and $F(\omega)$

f(t)=1/t^(1/2)}

❷

\F_s{1/t^(1/2)}, 1/ω^(1/2)

Fig. 18-28

Graphs of $f(t)$ and $F(\omega)$

$f(t) = t^{-3/2}$ ❷

\F_s{1/t^(3/2)}, ω^(1/2)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-29

$f(t) = t^{-a}, \quad 0 < a < 2$ ❷

\F_s{1/t^a}, ω^(a-1)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-30

f(t)=t/(t^2+a^2)

Cal

\F_s{t/(t^2+a^2)}, e^(-a*ω)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-31

f(t)=t/(t^2+a^2)^2

❺

\F_s{t/(t^2+a^2)^2}, ω*e^(-a*ω)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-32

f(t)=1/(t*(t^2+a^2))

❷

\F_s{1/(t*(t^2+a^2))}, 1-e^(-a*ω)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-33

$f(t) = e^{-at}, \quad a > 0$

Cal

\F_s{e^(-a*t)}, ω/(ω^2+a^2)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-34

$f(t) = te^{-at}, \quad a > 0$ ❷

\F_s{t*e^(-a*t)}, ω/(ω^2+a^2)^2

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-35

$f(t) = t^n e^{-at}, \quad a > 0, \ n > -2$ ❷

\F_s{t^n*e^(-a*t)},
tan^(-1)(ω/a), 1/(ω^2+a^2)^((n+1)/2)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-36

f(t)=e^(-a*t)/t

Cal

\F_s{e^(-a*t)/t}, tan^(-1)(ω/a)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-37

$f(t) = t\exp(-at^2), \quad a > 0$

Cal

\F_s{t*e^(-a*t^2)}, ω*e^(-ω^2/(4*a))

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-38

$f(t) = \exp(-at^2)\sin bt, \quad a > 0$ ❷

\F_s{e^(-a*t^2)*sin(b*t)},
sinh(b*ω/(2*a)*e^(-(ω^2+b^2)/(4*a))

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-39

f(t)=sin(a*t)/t

Cal

\F_s{sin(a*t)/t},
ln|(ω+a)/(ω-a)|

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-40

f(t)=sin(a*t)/t^2

\F_s{sin(a*t)/t^2}, ω

Cal

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-41

f(t)=cos(a*t)/t

\F_s{cos(a*t)/t}, U_a(ω)

Cal

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-42

f(t)=sin(π*t)/(1-t^2)

\F_s{sin(π*t)/(1-t^2)}, U_a(ω)*sinω

Cal

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-43

f(t)=tan^(-1)(t/a)

\F_s{tan^(-1)(t/a)}, e^(-a*ω)/ω

5

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-44

f(t)=lnt/t

\F_s{lnt/t}, γ+lnω

4

(γ = Euler number)

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-45

f(t)=1/sinh(a*t)

\F_s{1/sinh(a*t)}, tanh(π*ω/(2*a))

4

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-46

f(t)=sinh(a*t)/cosh(a*t) ❹

\F_s{sinh(a*t)/cosh(a*t)},
sinh(π*ω/(2*b))/(cosh(π*ω/b)+cos(π*a/b))

Fig. 18-47

Graphs of $f(t)$ and $F(\omega)$

f(t)=1/(e^(2*t)-1) ❹

\F_s{1/(e^(2*t)-1)},
coth(π*ω/2), 1/ω

Fig. 18-48

Graphs of $f(t)$ and $F(\omega)$

Fourier cosine transforms (— $f(t)$, — $F_c(\omega)$, $0 < \omega < \infty$)

f(t)=1:0<t<a;0:0<a<t

Cal

\F_c{1:0<t<a;0:0<a<t}, sin(a*ω)/ω

Fig. 18-49

Graphs of $f(t)$ and $F(\omega)$

f(t)=1/t^(1/2) ❷

\F_c{1/t^(1/2)}, 1/ω^(1/2)

Fig. 18-50

Graphs of $f(t)$ and $F(\omega)$

$f(t) = t^{-a}, \quad 0 < a < 1$ ❷

\F_c{1/t^a}, ω^(a-1)

Fig. 18-51

Graphs of $f(t)$ and $F(\omega)$

f(t)=1/(t^2+a^2) ❷

\F_c{1/(t^2+a^2)}, e^(-a*ω)

Fig. 18-52

Graphs of $f(t)$ and $F(\omega)$

f(t)=1/(t^2+a^2)^2

5

\F_c{1/(t^2+a^2)^2},
(1+a*ω)*e^(-a*ω)

Fig. 18-53

Graphs of $f(t)$ and $F(\omega)$

$f(t) = e^{-at}, \quad a > 0$

\F_c{e^(-a*t)}, 1/(ω^2+a^2)

Fig. 18-54

Graphs of $f(t)$ and $F(\omega)$

Cal

$f(t) = te^{-at}, \quad a > 0$

\F_c{t*e^(-a*t)},
(a^2-ω^2)/(ω^2+a^2)^2

Fig. 18-55

Graphs of $f(t)$ and $F(\omega)$

f(t)=e^(-a*t)/t^(1/2)

2

\F_c{e^(-a*t)/t^(1/2)}, ω^2+a^2

Fig. 18-56

Graphs of $f(t)$ and $F(\omega)$

$f(t) = t^{n-1}e^{-at}, \quad a > 0, \; n > 0$ 2

\F_c{t^(n-1)*e^(-a*t)},
tan^(-1)(ω/a), 1/(ω^2+a^2)^(n/2)

Fig. 18-57

Graphs of $f(t)$ and $F(\omega)$

f(t)=e^(-a*t^2)

Cal

\F_c{e^(-a*t^2)}, e^(-ω^2/(4*a))

Fig. 18-58

Graphs of $f(t)$ and $F(\omega)$

f(t)=sin(a*t)/t

5

\F_c{sin(a*t)/t}, sgn(a-ω)

Fig. 18-59

Graphs of $f(t)$ and $F(\omega)$

$f(t) = t^{-1}e^{-t}\sin t$ 4

\F_c{(1/t)*e^(-t)*sint}, tan^(-1)(2/ω^2)

Fig. 18-60

Graphs of $f(t)$ and $F(\omega)$

$f(t) = \sin(at^2), \quad a > 0$

\F_c{sin(a*t^2)},
cos(ω^2/(4*a))-sin(ω^2/(4*a))

Fig. 18-61

Graphs of $f(t)$ and $F(\omega)$

Cal

$f(t) = \cos(at^2), \quad a > 0$

\F_c{cos(a*t^2)},
cos(ω^2/(4*a))+sin(ω^2/(4*a))

Fig. 18-62

Graphs of $f(t)$ and $F(\omega)$

$f(t) = t^{-2}\sin^2 at, \quad a > 0$ 5

\F_c{sin(a*t)^2/t^2},
(1-U_(2*a)(ω))*(a-ω/2)

Fig. 18-63

Graphs of $f(t)$ and $F(\omega)$

$f(t) = e^{-bt}\sin at, \quad a > 0, \; b > 0$ 2

\F_c{e^(-b*t)*sin(a*t)},
(a+ω)/(b^2+(a+ω)^2)+(a-ω)/(b^2+(a-ω)^2)

Fig. 18-64

Graphs of $f(t)$ and $F(\omega)$

$f(t) = -\ln(1 - e^{-t})$ ❹

\F_c{-ln(1-e^(-t))},
(π*ω*coth(π*ω)-1)/ω^2

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-65

f(t)=ln(1+a^2/t^2) ❹

\F_c{ln(1+a^2/t^2)},
(1-e^(-a*ω))/ω

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-66

f(t)=ln((a^2+t^2)/(b^2+t^2)) ❹

\F_c{ln((a^2+t^2)/(b^2+t^2))},
(e^(-b*ω)-e^(-a*ω))/ω

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-67

f(t)=1/cosh(a*t)

Cal

\F_c{1/cosh(a*t)},
1/cosh(π*ω/(2*a))

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-68

f(t)=sinh(a*t)/sinh(b*t) ❹

\F_c{sinh(a*t)/sinh(b*t)},
1/(cosh(π*ω/b)+cos(π*a/b))

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-69

f(t)=cosh(a*t)/cosh(b*t) ❹

\F_c{cosh(a*t)/cosh(b*t)},
cosh(π*ω/(2*b))/(cosh(π*ω/b)+cos(π*a/b))

Graphs of $f(t)$ and $F(\omega)$

Fig. 18-70

18.5 Multidimensional Fourier Transforms

In an n dimensional Euclidean space E^n with rectangular Cartesian coordinates $x, y, z, \ldots$, the position vector is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} + \cdots$. If $f(\mathbf{r})$ is a scalar function that satisfies for each $x, y, z, \ldots$ conditions similar to those of $f(t)$ in Sec. 18.1, then its *Fourier transform* is defined by the relation

F(s)=\F{f(r)}=(1/(2*π)^(n/2))*\I{...\I{f(r)*e^(i*r*s)*dx*dy*dz...}}

where $\mathbf{s} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} + \cdots$ and $\mathbf{r}\cdot\mathbf{s} = xu + yv + zw + \cdots$ is the scalar product of $\mathbf{r}$ and $\mathbf{s}$. The *inverse Fourier transform* is

f(r)=\F^(-1){F(s)}=(1/(2*π)^(n/2))*\I{...\I{F(s)*e^(-i*r*s)*du*dv*dw...}}

Besides $n = 1$ (see previous sections), the most usual cases are $n = 2$ and $n = 3$.

For $n = 2$, with $dS = dx\,dy$ and $dS' = du\,dv$, we have

F(s)=(1/(2*π))*\I_S{f(r)*e^(i*r*s)*dS},
f(r)=(1/(2*π))*\I_S'{F(s)*e^(-i*r*s)*dS'}

For $n = 3$, with $dV = dx\,dy\,dz$ and $dV' = du\,dv\,dw$, we have

F(s)=(1/(2*π)^(3/2))*\I_V{f(r)*e^(i*r*s)*dV},
f(r)=(1/(2*π)^(3/2))*\I_V'{F(s)*e^(-i*r*s)*dV'}

In each case, the integration extends over the whole space.

If $F(\mathbf{s}) = \mathcal{F}\{f(\mathbf{r})\}$ and $G(\mathbf{s}) = \mathcal{F}\{g(\mathbf{r})\}$, then for constants a, b, c and constant vector $\mathbf{p}$ we have the following properties:

$\mathcal{F}\{af(\mathbf{r}) + bg(\mathbf{r})\} = aF(\mathbf{s}) + bG(\mathbf{s})$, $\quad \mathcal{F}\{f^*(\mathbf{r})\} = F^*(-\mathbf{s})$, $\quad \mathcal{F}\{F(\mathbf{r})\} = f(-\mathbf{s})$

$\mathcal{F}\{f(a\mathbf{r})\} = a^{-n}F(\mathbf{s}/a)$, $\quad \mathcal{F}\{f(ax, by, cz)\} = (abc)^{-1}F(u/a, v/b, w/c)$, $\; a, b, c > 0$

$\mathcal{F}\{f(\mathbf{r} + \mathbf{p})\} = e^{-i\mathbf{p}\cdot\mathbf{s}}F(\mathbf{s})$, $\quad \mathcal{F}\{f(\mathbf{r})e^{i\mathbf{p}\cdot\mathbf{r}}\} = F(\mathbf{s} + \mathbf{p})$

$\mathcal{F}\{f(\mathbf{r})\cos(\mathbf{p}\cdot\mathbf{r})\} = \frac{1}{2}F(\mathbf{s} + \mathbf{p}) + \frac{1}{2}F(\mathbf{s} - \mathbf{p})$ Pro

\F{x^k*f(r)}=(-i)^k*(\d/\du)^k{F(s)}, \F{\d(f(r))/\dx}=-i*u*F(s), \F{\I[x,-\o,x]{f(r)}}=(i/u)*F(s)

Separation property: If $f(x, y) = g(x)h(y)$, then $\mathcal{F}\{f(x, y)\} = \mathcal{F}\{g(x)\}\mathcal{F}\{h(y)\}$.

The convolution is f**g=(1/(2*π)^(n/2))*\I{...\I{f(r')*g(r-r')*dx'*dy'*dz'... with

$\mathcal{F}\{f\star g\} = FG$, $\quad \mathcal{F}\{fg\} = F\star G$, $\quad f\star g = \mathcal{F}^{-1}\{FG\}$, $\quad fg = \mathcal{F}^{-1}\{F\star G\}$

Two dimensional Fourier transforms ($\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, $\mathbf{s} = u\mathbf{i} + v\mathbf{j}$)

$f(\mathbf{r}) = \delta(x-a)\,\delta(x-b)$		$F(\mathbf{s}) = (2\pi)^{-1} e^{i(au+bv)}$
f(r)=1:|x|<a;0:a<|x|		\F{1:|x|<a;0:a<|x|}, sin(a*u)*δ(v)/u
f(r)=1:|x|<a,|y|<b;0:other	Cal	\F{1:|x|<a,|y|<b;0:other}, sin(a*u)*sin(b*v)/(u*v)
$f(\mathbf{r}) = \exp(-ax^2 - by^2)$, $a > 0,\ b > 0$		\F{e^(-a*x^2-b*y^2)}, e^(-u^2/(4*a)-v^2/(4*b))
f(r)=1:ρ<a;0:a<ρ		\F{1:ρ<a;0:a<ρ}, J_1(a*s)/s
$f(\mathbf{r}) = e^{-k\rho}$, $k > 0$, $\rho = \lvert\mathbf{r}\rvert$	Ext	\F{e^(-k*ρ)}, 1/(s^2+k^2)^(3/2)
$f(\mathbf{r}) = \rho^{-1} e^{-k\rho}$, $k > 0$, $\rho = \lvert\mathbf{r}\rvert$		\F{e^(-k*ρ)/ρ}, 1/(s^2+k^2)^(1/2)

Three dimensional Fourier transforms ($\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{s} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$)

$f(\mathbf{r}) = \delta(x-a)\,\delta(x-b)\,\delta(z-c)$		$F(\mathbf{s}) = (2\pi)^{-3/2} e^{i(au+bv+cw)}$
f(r)=1:|x|<a,|y|<b,|z|<c;0:other		\F{1:|x|<a,|y|<b,|z|<c;0:other}, sin(a*u)*sin(b*v)*sin(c*w)/(u*v*w)
$f(\mathbf{r}) = \exp(-ax^2 - by^2 - cz^2)$, $a > 0,\ b > 0,\ c > 0$		\F{e^(-a*x^2-b*y^2-c*z^2)}, e^(-u^2/(4*a)-v^2/(4*b)-w^2/(4*c))
f(r)=1:r<a;0:a<r		\F{1:r<a;0:a<r}, (sin(a*s)-a*s*cos(a*s))/s^3
$f(\mathbf{r}) = e^{-kr}$, $k > 0$, $r = \lvert\mathbf{r}\rvert$	Ext	\F{e^(-k*r)}, 1/(s^2+k^2)^2
$f(\mathbf{r}) = r^{-1} e^{-kr}$, $k > 0$, $r = \lvert\mathbf{r}\rvert$		\F{e^(-k*r)/r}, 1/(s^2+k^2)
$f(\mathbf{r}) = r^{-a}$, $1 < a < 3$, $r = \lvert\mathbf{r}\rvert$	Cal	\F{1/r^a}, s^(a-3), $s = \lvert\mathbf{s}\rvert$

19 LAPLACE TRANSFORMS

19.1 Definitions

If $f(t)$ is a function defined in the interval $0 < t < \infty$, the function (if it exists)

F(s)=\L{f(t)}=\I[t,0\o]{f(t)*e^(-s*t)}

is the *Laplace transform* of $f(t)$. The parameter $s = \sigma + i\omega$ may be considered in general as an independent complex variable. If for some σ_0 the integral \I[t,0,\o]{|f(t)*e^(-σ_0*t)} exists, then the function $F(s)$ exists for every s with $\mathrm{Re}(s) = \sigma > \sigma_0$ [the integral that defines $F(s)$ may diverge because of the behavior of $f(t)$ at some finite t or as $t \to \infty$]. If $\mathcal{L}\{f(t)\} = F(s)$, then $f(t) = \mathcal{L}^{-1}\{F(s)\}$ is the *inverse Laplace transform* of $F(s)$.

The integral for the inverse transform is

f_I(t)=(1/(2*πi))*\I[s,c-i*\o,c+i*\o]{F(s)*e^(s*t)}=
(1/(2*πi))*lim[d,\o]{\I[s,c-i*d,c+i*d]{F(s)*e^(s*t)}}

Ext

where c is a real constant ($c > \sigma_0$) and all the singular points of $F(s)$ are located to the left of the straight line $\sigma = c$ in the (complex) s plane. The previous formula gives the value $\frac{1}{2}[f(t-0) + f(t+0)]$ to $f_I(t)$ at every point $t > 0$, the value $\frac{1}{2}f(0+0)$ at $t = 0$ and the value 0 for $t < 0$. At points where $f_I(t)$ is continuous, we have $f_I(t) = f(t)$.

19.2 Properties

If $F(s) = \mathcal{L}\{f(t)\}$, the following table gives the Laplace transforms of some functions obtained from $f(t)$. In essence, these are properties of the Laplace transformation, and they hold if the integral that defines $F(s)$ converges absolutely, i.e. if the integral \I[t,0,\o]{|f(t)|*e^(-σ*t)} exists for some $\mathrm{Re}(s) = \sigma = \sigma_0$. The parameters a and b are independent of t and s.

$f(t)$	$F(s)$	Property
$af(t) + bg(t)$	\L{a*f(t)+b*f(t)}=a*F(s)+b*G(s)	Linearity
$f(at) \quad (a > 0)$	\L{f(a*t)}=F(s/a)/a	Change of scale
$f(t-a)$	\L{f(t-a)}=e^(-a*s)*F(s)	Translation [assuming $f(t) = 0$ for $t \le 0$]
lim[a,a_0]{f(t,a)}	\L{lim[a,a_0]{f(t,a)}}, lim[a,a_0]{F(s,a)}	Continuity

f(t)/t	\L{f(t)/t}=\I[s,0,\o]{F(u)}	Division by t
e^(a*t)*f(t)	\L{e^(a*t)*f(t)}=F(s-a)	Multiplication by e^{at}
t^n*f(t)	\L{t^n*f(t)}=(-1)^n*F^((n))(s)	Multiplication by t^n $[F^{(n)}(s) = d^nF/ds^n]$
$f'(t)$ $f^{(n)}(t)$	\L{f'(t)}=s*F(s)-f(0+) \L{f^((n))(t)}=s^n*F(s)-\S[k,0,n-1]{s^(n-k-1)*f^((k))(0+)}	Derivative of order n [assuming $f^{(k)}(t)$ exist for $t > 0$] f^((k))(0+)=lim[ε,0]{f((k))(ε)}
\I[u,0,t]{f(u)}	\L{\I[u,0,t]{f(u)}}=F(s)/s	Integration [assuming $f'(t)$ exists for $t > 0$]
\I[t,0,t]{...\I[t,0,t]{f(t)}=\I[u,0,t]{(t-u)^(n-1)*f(u)/(n-1)!}	\L{\I[u,0,t]{(t-u)^(n-1)*f(u)/(n-1)!}}=F(s)/s^n	Repeated integration (integration n times)
\d(f(t,a))/\da	\L{\d(f(t,a))/\da}=\d(F(s,a))/\da	Differentiation with respect to a parameter
\I[u,a,b]{f(t,u)}	\L{\I[u,a,b]{f(t,u)}}=\I[u,a,b]{F(s,u)}	Integration with respect to a parameter
f**g=\I[u,0,t]{f(u)*g(t-u)}	\L{f**g}=F(s)*G(s)	Convolution
$f(t) = f(t+T)$	(1/(1-e^(-s*T)))*\I[u,0,T]{f(u)*e^(-s*u)}	Periodic function
$f(t)\sin\omega t$	\L{f(t)*sin(ω*t)}, F(s+i*ω)-F(s-i*ω)	Modulated sinusoidal functions
$f(t)\cos\omega t$	\L{f(t)*cos(ω*t)}, F(s+i*ω)+F(s-i*ω)	
(1/(π*t)^(1/2))*\I[u,0,\o]{e^(-u^2/(4*t))*f(u)}	\L{(1/(π*t)^(1/2))*\I[u,0,\o]{e^(-u^2/(4*t))*f(u)}}=F(s^(1/2))/s^(1/2)	
$f(t^2)$	\L{f(t^2)}	Ext

19.3 Tables of Laplace Transforms

Laplace transforms of some elementary functions Pro

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	\L{1}=1/s	$\sin at$	\L{sin(a*t)}=a/(s^2+a^2)
t	\L{t}=1/s^2	$\cos at$	\L{cos(a*t)}=s/(s^2+a^2)
$t^n, \quad n = 0, 1, \ldots$	\L{t^n}=n!/s^(n+1)	$\sinh at$	\L{sinh(a*t)}=a/(s^2-a^2)
$t^a, \quad a > -1$	\L{t^a}=Γ(a+1)/s^(a+1)	$\cosh at$	\L{cosh(a*t)}=s/(s^2-a^2)
e^{at}	\L{e^(a*t)}=1/(s-a)	$e^{at} \sin bt$	\L{e^(a*t)*sin(b*t)}=b/((s-a)^2+b^2)
$t^n e^{at}, \quad n = 0, 1, \ldots$	\L{t^n*e^(a*t)}=n!/(s-a)^(n+1)	$e^{at} \cos bt$	\L{e^(a*t)*cos(b*t)}=(s-a)/((s-a)^2+b^2)
$t^b e^{at}, \quad b > -1$	\L{t^b*e^(a*t)}=Γ(b+1)/(s-a)^(b+1)	U_a(t)=0:t<a;1:a<t	\L{U_a(t)}=e^(-a*s)/s
$\ln t$	\L{lnt}=-(γ+lns)/s	$\delta(t-a), \quad a \geq 0$ [delta function]	\L{$\delta(t-a)$}=e^{-as}

Inverse Laplace transforms

$F(s)$	$f(t)$	$F(s)$	$f(t)$
s^{-1}	\L^(-1){1/s}=1	$(s^2+a^2)^{-1}$	\L^(-1){1/(s^2+a^2)}=sin at/a
$s^{-n}, \quad n = 1, 2, \ldots$	\L^(-1){1/s^n}=t^{n-1}/(n - 1)!	$s(s^2+a^2)^{-1}$	\L^(-1){s/(s^2+a^2)}=cos at
$s^{-a}, \quad a > 0$	\L^(-1){1/s^a}=t^{a-1}/Γ(a)	$(s^2-a^2)^{-1}$	\L^(-1){1/(s^2-a^2)}=sinh at/a
$(s-a)^{-1}$	\L^(-1){1/(s-a)}=e^{at}	$s(s^2-a^2)^{-1}$	\L^(-1){s/(s^2-a^2)}=cosh at
$(s-a)^{-n}, \quad n = 1, 2, \ldots$	\L^(-1){1/(s-a)^n}=t^{n-1}e^{at}/(n - 1)!	$[(s-a)^2+b^2]^{-1}$	\L^(-1){1/((s-a)^2+b^2)}=e^{at}sin bt/b
$(s-a)^{-b}, \quad b > 0$	\L^(-1){1/(s-a)^b}=t^{b-1}e^{at}/Γ(b)	$s[(s-a)^2+b^2]^{-1}$	\L^(-1){s/((s-a)^2+b^2)}=e^{at}(b cos bt + a sin bt)/b

In many problems the final step is to find a function $f(t)$ (e.g. the solution of a differential equation) from its Laplace transform $F(s)$. For this last step [i.e. to find $f(t)$ from $F(s)$] we can use the following tables and the properties of Sec. 19.2.

Rational functions Inf

s^n/((s-a)*(s-b))	Cal	$n = 0$	\L^(-1){1/((s-a)*(s-b))}= (e^(a*t)-e^(b*t))/(a-b)
		$n = 1$	\L^(-1){s/((s-a)*(s-b))}= (a*e^(a*t)-b*e^(b*t))/(a-b)
s^n/(s^2+a^2)^2	Cal	$n = 0$	\L^(-1){1/(s^2+a^2)^2}= (sin(a*t)-a*t*cos(a*t))/(2*a^3)
		$n = 1$	\L^(-1){s/(s^2+a^2)^2}= t*sin(a*t)/(2*a)
		$n = 2$	\L^(-1){s^2/(s^2+a^2)^2}= (sin(a*t)+a*t*cos(a*t))/(2*a)
		$n = 3$	\L^(-1){s^3/(s^2+a^2)^2}= cos(a*t)-a*t*sin(a*t)/2
s^n/(s^2-a^2)^2	Cal	$n = 0$	\L^(-1){1/(s^2-a^2)^2}= (a*t*cosh(a*t)-sinh(a*t))/(2*a^3)
		$n = 1$	\L^(-1){s/(s^2-a^2)^2}= t*(sinh(a*t)/(2*a)
		$n = 2$	\L^(-1){s^2/(s^2-a^2)^2}= (sinh(a*t)+a*t*cosh(a*t))/(2*a)
		$n = 3$	\L^(-1){s^3/(s^2-a^2)^2}= cosh(a*t)+a*t*sinh(a*t)/2
s^n/(s^2+a^2)^3	Cal	$n = 0$	\L^(-1){1/(s^2+a^2)^3}= (3-a^2*t^2)*sin(a*t)-3*a*t*cos(a*t))/(8*a^5)
		$n = 1$	\L^(-1){s/(s^2+a^2)^3}= (t*sin(a*t)-a*t^2*cos(a*t))/(8*a^3)
		$n = 2$	\L^(-1){s^2/(s^2+a^2)^3}= ((1+a^2*t^2)*sin(a*t)-a*t*cos(a*t))/(8*a^3)
		$n = 3$	\L^(-1){s^3/(s^2+a^2)^3}= (3*t*sin(a*t)+a*t^2*cos(a*t))/(8*a)

s^n/(s^2+a^2)^3		$n = 4$	L^(-1){s^4/(s^2+a^2)^3}= ((3-a^2*t^2)*sin(a*t)+5*a*t*cos(a*t))/(8*a)
		$n = 5$	L^(-1){s^5/(s^2+a^2)^3}= ((8-a^2*t^2)*cos(a*t)-7*a*t*sin(a*t))/8
s^n/(s^2-a^2)^3	Cal	$n = 0$	L^(-1){1/(s^2-a^2)^3}= ((3+a^2*t^2)*sinh(a*t)-3*a*t*cosh(a*t))/(8*a^5)
		$n = 1$	L^(-1){s/(s^2-a^2)^3}= (a*t^2*cosh(a*t)-t*sinh(a*t))/(8*a^3)
		$n = 2$	L^(-1){s^2/(s^2-a^2)^3}= (a*t*cosh(a*t)+(a^2*t^2-1)*sinh(a*t))/(8*a^3)
		$n = 3$	L^(-1){s^3/(s^2-a^2)^3}= (3*t*sinh(a*t)+a*t^2+a*t^2*cosh(a*t))/(8*a)
		$n = 4$	L^(-1){s^4/(s^2-a^2)^3}= ((3+a^2*t^2)*sinh(a*t)+5*a*t*cosh(a*t))/(8*a)
		$n = 5$	L^(-1){s^5/(s^2-a^2)^3}= ((8+a^2*t^2)*cosh(a*t)+7*a*t*sinh(a*t))/8
s^n/(s^3+a^3)	Cal	$n = 0$	L^(-1){1/(s^3+a^3)}, e^(a*t/2), e^(-3*a*t/2), sin(3^(1/2)*a*t/2), cos(3^(1/2)*a*t/2)
		$n = 1$	L^(-1){s/(s^3+a^3)}, e^(a*t/2), e^(-3*a*t/2), sin(3^(1/2)*a*t/2), cos(3^(1/2)*a*t/2)
		$n = 2$	L^(-1){s^2/(s^3+a^3)}, e^(a*t/2), e^(-a*t/2), cos(3^(1/2)*a*t/2)
s^n/(s^4+a^4)	Cal	$n = 0$	L^(-1){1/(s^4+a^4)}, sin(a*t/2^(1/2)), cos(a*t/2^(1/2)), sinh(a*t/2^(1/2)), cosh(a*t/2^(1/2))
		$n = 1$	L^(-1){s/(s^4+a^4)}, sin(a*t/2^(1/2))*sinh(a*t/2^(1/2))
		$n = 2$	L^(-1){s^2/(s^4+a^4)}, sin(a*t/2^(1/2)), cos(a*t/2^(1/2)), sinh(a*t/2^(1/2)), cosh(a*t/2^(1/2))
		$n = 3$	L^(-1){s^3/(s^4+a^4)}, cos(a*t/2^(1/2)*cosh(a*t/2^(1/2))

s^n/(s^4-a^4) Cal	$n = 0$	\L^(-1){1/(s^4-a^4)}= (sinh(a*t)-sin(a*t))/(2*a^3)
	$n = 1$	\L^(-1){s/(s^4-a^4)}= (cosh(a*t)-cos(a*t))/(2*a^2)
	$n = 2$	\L^(-1){s^2/(s^4-a^4)}= (sinh(a*t)+sin(a*t))/(2*a)
	$n = 3$	\L^(-1){s^3/(s^4-a^4)}= (cosh(a*t)+cos(a*t))/2
P(s)/(s-a)^n	\L^(-1){P(s)/(s-a)^n}	$\begin{cases} P(s) = \text{polynomial} \\ \text{of degree} < n,\ P(a) \neq 0 \end{cases}$
P(s)/Q(s)	\L^(-1){P(s)/Q(s)}	$\begin{cases} P(s) = \text{polynomial of degree} < n, \\ Q(s) = \text{polynomial of degree } n \\ \text{with } n \text{ distinct roots } a_1, a_2, \ldots, a_n \end{cases}$

Irrational functions

(s-a)^(1/2)-(s-b)^(1/2)	\L^(-1){(s-a)^(1/2)-(s-b)^(1/2)}, (1/(2*π^(1/2))*(e^(b*t)-e^(a*t))/t^(3/2) Cal
1/((s+a)^(1/2)+(s+b)^(1/2))	\L^(-1){1/((s+a)^(1/2)+(s+b)^(1/2))}, (1/(2*(a-b)*π^(1/2)))*(e^(-b*t)-e^(-a*t))/t^(3/2)
s/(s-a)^(3/2)	\L^(-1){s/(s-a)^(3/2)}, (1/π^(1/2))*e^(a*t)*(1+2*a*t)
1/((s-a)^(1/2)+b)	\L^(-1){1/((s-a)^(1/2)+b)}, e^(a*t)*(1/(π*t)^(1/2)-b*e^(b^2*t)*erfc(b*t^(1/2)))
1/((s+a)*(s+b)^(1/2))	\L^(-1){1/((s+a)*(s+b)^(1/2))}, (1/(b-a)^(1/2))*e^(-a*t)*erf((b-a)^(1/2)*t^(1/2))
1/(s^(1/2)*(s+a^2))	\L^(-1){1/(s^(1/2)*(s+a^2))}, (2/(a*π^(1/2))*e^(-a^2*t)*\I[u,0,a*t^(1/2)]{e^(u^2)}
s^(1/2)/(s-a^2)	\L^(-1){s^(1/2)/(s-a^2)}, 1/(π*t)^(1/2)+a*e^(a^2*t)*erf(a*t^(1/2)) Cal
s^(1/2)/(s+a^2)	\L^(-1){s^(1/2)/(s+a^2)}, 1/(π*t)^(1/2)-(2*a/π^(1/2))*e^(-a^2*t)*\I[u,0,a*t^(1/2)]{e^(u^2)}

1/(s^(1/2)*(s^(1/2)+a)	\L^(-1){1/(s^(1/2)*(s^(1/2)+a)}= e^(a^2*t)*erfc(a*t^(1/2))
1/(s^2+a^2)^(1/2)	\L^(-1){1/(s^2+a^2)^(1/2)}= $J_0(at)$ Cal
1/(s^2-a^2)^(1/2)	\L^(-1){1/(s^2-a^2)^(1/2)}= $I_0(at)$
s^n/(s^2+a^2)^(3/2)	$n = 0$ \L^(-1){1/(s^2+a^2)^(3/2)}= $a^{-1}tJ_1(at)$
	$n = 1$ \L^(-1){s/(s^2+a^2)^(3/2)}= $tJ_0(at)$
	$n = 2$ \L^(-1){s^2/(s^2+a^2)^(3/2)}= $J_0(at) - atJ_1(at)$
s^n/(s^2-a^2)^(3/2)	$n = 0$ \L^(-1){1/(s^2-a^2)^(3/2)}= $a^{-1}tI_1(at)$
	$n = 1$ \L^(-1){s/(s^2-a^2)^(3/2)}= $tI_0(at)$
	$n = 2$ \L^(-1){s^2/(s^2-a^2)^(3/2)}= $I_0(at) + atI_1(at)$
((s^2+a^2)^(1/2)-s)^n/(s^2+a^2)^(1/2)	\L^(-1){((s^2+a^2)^(1/2)-s)^n/(s^2+a^2)^(1/2)}= $a^nJ_n(at)$
(s-(s^2-a^2)^(1/2))^n/(s^2-a^2)^(1/2)	\L^(-1){(s-(s^2-a^2)^(1/2))^n/(s^2-a^2)^(1/2)}= $a^nI_n(at)$

Exponential functions

e^(-a*s)/s^b	\L^(-1){e^(-a*s)/s^b}, (t-a)^(b-1)/Γ(b)
e^(-a/s)/s^(n+1)	\L^(-1){e^(-a/s)/s^(n+1)}, t^(n/2)*J_n(2*(a*t)^(1/2))
e^(a/s)/s^(n+1)	\L^(-1){e^(a/s)/s^(n+1)}, t^(n/2)*I_n(2*(a*t)^(1/2))
e^(-a*s^(1/2))	\L^(-1){e^(-a*s^(1/2))}, e^(-a^2/(4*t))/t^(3/2) Cal
e^(a/s)/s^(1/2)	\L^(-1){e^(a/s)/s^(1/2)}, cosh(2*(a*t)^(1/2))/t^(1/2)
e^(-a/s)/s^(1/2)	\L^(-1){e^(-a/s)/s^(1/2)}, cos(2*(a*t)^(1/2))/t^(1/2)
e^(a/s)/s^(3/2)	\L^(-1){e^(a/s)/s^(3/2)}, sinh(2*(a*t)^(1/2))

e^(-a/s)/s^(3/2)	\L^(-1){e^(-a/s)/s^(3/2)}= (1/(π*a)^(1/2))*sin(2*(a*t)^(1/2))	Cal
e^(-a*s^(1/2))/s	\L^(-1){e^(-a*s^(1/2))/s}= erfc(a/(2*t^(1/2)))	Cal
e^(-a/s^(1/2))/s^(n+1)	\L^(-1){e^(-a/s^(1/2))/s^(n+1)}	
e^(-a*s^(1/2))/s^(1/2)	\L^(-1){e^(-a*s^(1/2))/s^(1/2)}, e^(-a^2/(4*t))/t^(1/2)	
e^(-a*s^(1/2))/(s^(1/2)*(s^(1/2)+b))	\L^(-1){e^(-a*s^(1/2))/(s^(1/2)*(s^(1/2)+b))}, erfc(b*t^(1/2)+a/(2*t^(1/2)))	
e^(b*(s-(s^2+a^2)^(1/2)))/ (s^2+a^2)^(1/2)	\L^(-1){e^(b*(s-(s^2+a^2)^(1/2)))/(s^2+a^2)^(1/2)}, J_0(a*(t*(t+2*b))^(1/2))	
e^(-b*(s^2+a^2)^(1/2))/ (s^2+a^2)^(1/2)	\L^(-1){e^(-b*(s^2+a^2)^(1/2))/(s^2+a^2)^(1/2)}, J_0(a*(t^2-b^2)^(1/2))	
(e^(-a*s)-e^(-b*s))/s	\L^(-1){(e^(-a*s)-e^(-b*s))/s}=U_a(t) - U_b(t)	Graph of $f(t)$
1/(s*(1-e^(-a*s))	\L^(-1){1/(s*(1-e^(-a*s))}, \S[k,0,\o]{U(t-a*k)}	Graph of $f(t)$ Cal
1/(s*(1+e^(-a*s))	\L^(-1){1/(s*(1+e^(-a*s))}, \S[k,0,\o]{(-1)^k*U(t-a*k)}	Graph of $f(t)$
1/(s*(e^s-r))	\L^(-1){1/(s*(e^s-r))}, $\lfloor t \rfloor$ = maximum integer $\le t$	
π*a*(1+e^(-a*s))/ (a^2*s^2+π^2)	\L^(-1){π*a*(1+e^(-a*s))/ (a^2*s^2+π^2)}, sin(π*t/a)	Graph of $f(t)$ Cal
1/((s^2+1)*(1-e^(-π*s)))	\L^(-1){1/((s^2+1)*(1-e^(-π*s)))}, \S[k,0,\o]{(-1)^k*U(t-k*π)*sint}	Graph of $f(t)$ Cal
1/(a*s^2)+1/(s*(1-e^(a*s))	\L^(-1){1/(a*s^2)+1/(s*(1-e^(a*s))}, t/a-\S[k,1,\o]{U(t-a*k)}	Graph of $f(t)$

Logarithmic functions

lns/s	\L^(-1){lns/s}, -lnt-γ	Cal
lns^2/s	\L^(-1){lns^2/s}, (lnt+γ)^2-π^2/6	
π^2/(6*s)+(γ+lns)^2/s, \L{lnt^2}	\L^(-1){π^2/(6*s)+(γ+lns)^2/s}, lnt^2	
lns/(s-a)	\L^(-1){lns/(s-a)}, e^(a*t)*(lna+E_1(a*t))	
ln((s+a)/(s+b))	\L^(-1){ln((s+a)/(s+b))}, (e^(-b*t)-e^(-a*t))/t	
ln(1+s/a)/s	\L^(-1){ln(1+s/a)/s}, E_1(a*t), -Ei(-a*t)	Cal
lns/s^a	\L^(-1){lns/s^a}, t^(a-1)(ψ(a)-lnt)/Γ(a)	
lns/(s^2+1)	\L^(-1){lns/(s^2+1)}, cost*Si(t)-sint*Ci(t)	Cal
s*lns/(s^2+1)	\L^(-1){s*lns/(s^2+1)}, -sint*Si(t)-cost*Ci(t)	
ln(s^2+a^2)/s	\L^(-1){ln(s^2+a^2)/s}, 2*lna-2*Ci(a*t)	Cal
ln(s^2+a^2)/s^2	\L^(-1){ln(s^2+a^2)/s^2}, (2/a)*(a*t*lna+sin(a*t)-a*t*Ci(a*t))	
ln(1+a^2/s^2)	\L^(-1){ln(1+a^2/s^2)}, 2*(1-cos(a*t))/t	
ln(1-a^2/s^2)	\L^(-1){ln(1-a^2/s^2)}, 2*(1-cosh(a*t))/t	
ln((s^2+a^2)/(s^2+b^2))	\L^(-1){ln((s^2+a^2)/(s^2+b^2))}, 2*(cos(b*t)-cos(a*t))/t	
ln(((s+a)^2+c^2)/((s+b)^2+c^2))	\L^(-1){ln(((s+a)^2+c^2)/((s+b)^2+c^2))}, 2*cos(c*t)*(e^(-b*t)-e^(-a*t))/t	

Other functions

(1/s^2)*tanh(a*s/2)	\L^(-1){(1/s^2)*tanh(a*s/2)}, t*U(t)+2*\S[k,1,\o]{(-1)^k*(t-a*k)*U(t-a*k)}	Cal
(1/s)*tanh(a*s/2)	\L^(-1){(1/s)*tanh(a*s/2)}, U(t)+2*\S[k=1,\o]{(-1)^k*U(t-a*k)}	
(1/(s^2+a^2))*coth(π*s/(2*a))	\L^(-1){(1/(s^2+a^2))*coth(π*s/(2*a))}, |sin(a*t)|/a	Graph of $f(t)$
tan^(-1)(a/s)=cot^(-1)(s/a)	\L^(-1){tan^(-1)(a/s)}, sin(a*t)/t	Cal
(1/s)*tan^(-1)(a/s)=(1/s)*cot^(-1)(s/a)	\L^(-1){(1/s)*tan^(-1)(a/s)}, Si(a*t)	
erf(a/s^(1/2))	\L^(-1){erf(a/s^(1/2))}, (1/(π*t))*sin(2*a*t^(1/2))	Cal
(e^(a/s)/s^(1/2))*erfc((a/s)^(1/2))	\L^(-1){(e^(a/s)/s^(1/2))*erfc((a/s)^(1/2))}, e^(-2*(a*t)^(1/2))/(π*t)^(1/2)	
e^(a^2*s^2)*erfc(a*s)	\L^(-1){e^(a^2*s^2)*erfc(a*s)}, (1/(a*π^(1/2)))*e^(-t^2/(4*a^2))	Cal
(1/s)*e^(a^2*s^2)*erfc(a*s)	\L^(-1){(1/s)*e^(a^2*s^2)*erfc(a*s)}, erf(t/(2*a))	
e^(a*s)*erfc((a*s)^(1/2))	\L^(-1){e^(a*s)*erfc((a*s)^(1/2))}, (a^(1/2)/π)/(t^(1/2)*(t+a))	
(1/s^(1/2))*e^(a*s)*erfc((a*s)^(1/2))	\L^(-1){(1/s^(1/2))*e^(a*s)*erfc((a*s)^(1/2))}, 1/(π*(t+a))^(1/2)	Cal
e^(a*s)*E_1(a*s)=-e^(a*s)*Ei(-a*s)	\L^(-1){e^(a*s)*E_1(a*s), \L^(-1){-e^(a*s)*Ei(-a*s)}, 1/(t+a)	
(1/a)*(cos(a*s)*(π/2-Si(a*s))+sin(a*s)*Ci(a*s))	\L^(-1){(1/a)*(cos(a*s)*(π/2-Si(a*s))+sin(a*s)*Ci(a*s))}, 1/(t^2+a^2)	
sin(a*s)*(π/2-Si(a*s))-cos(a*s)*Ci(a*s))	\L^(-1){sin(a*s)*(π/2-Si(a*s))-cos(a*s)*Ci(a*s))}, t/(t^2+a^2)	
(1/s)*(cos(a*s)*(π/2-Si(a*s))+sin(a*s)*Ci(a*s))	\L^(-1){(1/s)*(cos(a*s)*(π/2-Si(a*s))+sin(a*s)*Ci(a*s))}, tan^(-1)(t/a)	
(1/s)*(sin(a*s)*(π/2-Si(a*s))+cos(a*s)*Ci(a*s))	\L^(-1){}, (1/s)*(sin(a*s)*(π/2-Si(a*s))+cos(a*s)*Ci(a*s)), (1/2)*ln(1+t^2/a^2)	

20 NUMERICAL ANALYSIS

20.1 Errors

If X is the exact value and x is an approximate value of a measurable entity, the *error* is $\varepsilon = X - x$ and the *relative error* is $\delta = (X - x)/X$, assuming $X \neq 0$. The *correction* is $-\varepsilon$. The absolute value of the error is $|\varepsilon|$ and is often called *absolute error*.

In numerical calculations, we have *input errors* (i.e. errors in the input data), *algorithmic errors* (i.e. *cutoff errors* due to the algorithm and *roundoff errors* due to the rounding of intermediate results), and *output errors* (i.e. errors in the results). **Inf**

For a quantity $y = y(z_1, z_2, \ldots, z_n)$, which depends on the variables $z_1, z_2, \ldots, z_n$, with exact values $X_1, X_2, \ldots, X_n$ and approximate values $x_1, x_2, \ldots, x_n$, the error is

ε_y=y(X_1,X_2,...,X_n)-y(x_1,x_2,...,x_n)\~\S[k,1,n]{(\dy/\dz_k)*ε_k}

where $\varepsilon_k = X_k - x_k$ and the partial derivatives (here and below) are evaluated at $z_k = x_k$. An upper bound for the absolute error can be estimated from the relation

|ε_y|\<\S[k,1,n]{|(\dy/\dz_k)|*|ε_k|}

If the errors ε_k are random, a more realistic estimation of the absolute error is given by the *standard error*

|ε_y|\~(\S[k,1,n]{(\dy/\dz_k)^2*ε_k^2})^(1/2)

20.2 Interpolation

Interpolation is used to calculate $y = f(x)$ from a set of known points (x_k, y_k).

Lagrange formula

A linear polynomial $p_1(x)$ of x passing through two points (x_0, y_0) and (x_1, y_1) is

p_1(x)=((x-x_1)/(x_0-x_1))*y_0+((x-x_0)/(x_1-x_0))*y_1

A polynomial $p_n(x)$ of degree n passing through $n+1$ points (x_k, y_k), $k = 0, 1, \ldots, n$, with $x_k \neq x_m$ for $k \neq m$ (usually called an *interpolating polynomial*) is **Exa**

p_n(x)=\S[k,0,n]{L_k(x)*y_k}

where the *Lagrange coefficients* are

L_k(x)=((x-x_0)*(x-x_1)...(x-x_n))/((x_k-x_0)*(x_k-x_1)...(x_k-x_n))

If the function $y = f(x)$ is approximated by $p_n(x)$, the error is

ε_n(x)=f(x)-p_n(x)=f^((n+1))(ξ)*(x-x_0)*(x-x_1)...(x-x_n)/(n+1)!

where $x_{\min} \le \xi \le x_{\max}$, $x_{\min}$ and $x_{\max}$ the minimum and the maximum of $x, x_0, \ldots, x_n$.

Formulas with divided differences

The previous polynomials $p_1(x), \ldots, p_n(x)$ can be obtained using *divided differences*:

First order f[x_0,x_1]=(f(x_1)-f(x_0))/(x_1-x_0)=(y_1-y_0)/(x_1-x_0)

$$p_1(x) = f(x_0) + f[x_0, x_1](x - x_0)$$

Second order f[x_0,x_1,x_2]=(f[x_1,x_2]-f[x_0,x_1])/(x_2-x_0)

$$p_2(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$ Exa

***n*th order** f[x_0,x_1,...,x_k]=(f[x_1,x_2,...,x_k]-f[x_0,x_1,...,x_(k-1)])/(x_k-x_0)

$$p_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \cdots, x_n](x - x_0)\cdots(x - x_{n-1})$$

Newton's formula with forward differences

Exa

For equally spaced $x_0, x_1, \ldots, x_n$, we have $x_{k+1} - x_k = h$, $x = x_0 + kh$ $(k \ge 0)$ and

$$y_k = f(x_k), \quad \Delta y_k = y_{k+1} - y_k, \quad \Delta^2 y_k = \Delta y_{k+1} - \Delta y_k, \quad \ldots, \quad \Delta^n y_k = \Delta^{n-1} y_{k+1} - \Delta^{n-1} y_k,$$

p_n(x)=p_n(x_0+k*h)=\S[i,0,n]{(k;i)*Δ^iy_0}

Newton's formula with backward differences

For equally spaced $x_0, x_{-1}, \ldots, x_{-n}$, we have $x_k - x_{k-1} = h$, $x = x_0 + kh$ $(k \le 0)$ and

$$y_k = f(x_k), \quad \nabla y_k = y_k - y_{k-1}, \quad \nabla^2 y_k = \nabla y_k - \nabla y_{k-1}, \quad \ldots, \quad \nabla^n y_k = \nabla^{n-1} y_k - \nabla^{n-1} y_{k-1},$$

p_n(x)=p_n(x_0+k*h)=...=\S[i,0,n]{(-1)^i*(-k;i)*∇^iy_0}

Cubic splines

Let $a = x_0 < x_1 < \cdots < x_n = b$ be $n + 1$ points in the interval $I = [a, b]$ with $x_{k+1} = x_k + h_k$, $h_k > 0$. A function $S_m(x)$ defined on I is a *spline of degree m* and represents a function $f(x)$ if (a) $S_m(x_k) = f(x_k)$ at all x_k, (b) in any subinterval $I_k = [x_k, x_{k+1}]$, $k = 0, 1, \ldots, n-1$, $S_m(x)$ is a polynomial $p_k(x)$ of degree m, and (c) $S_m(x)$ and its derivatives up to order $m - 1$ are continuous on I.

Most often, a *cubic spline* ($m = 3$) is used to represent a function $y = f(x)$ in I. In the kth interval $I_k = [x_k, x_{k+1}]$, $k = 0, 1, \ldots, n-1$, let the polynomial be

$$p_k(x) = a_k(x - x_k)^3 + b_k(x - x_k)^2 + c_k(x - x_k) + d_k$$

If $s_k = p_k''(x_k)$ are the values of the second order derivatives at $x = x_k$, the continuity conditions imply that for $k = 1, 2, \ldots, n-1$ we have

$$h_{k-1}s_{k-1} + 2(h_{k-1} + h_k)s_k + h_k s_{k+1} = 6(y_{k+1} - y_k)/h_k - 6(y_k - y_{k-1})/h_{k-1}$$

These are $n - 1$ linear equations with $n + 1$ unknowns s_k. They are supplemented by two boundary conditions at the end points x_0 and x_n.

If the boundary conditions are $s_0 = s_n = 0$, we have a system of $n + 1$ linear equations with $n + 1$ unknowns s_k that can be easily solved (only three diagonals have nonzero elements). Thus, we obtain the *natural spline* with ($k = 0, 1, \ldots, n-1$)

a_k=(s_(k+1)-s_k)/(6*h_k), b_k=s_k/2,
c_k=(y_(k+1)-y_k)/h_k-(2*h_k*s_k+h_k*s_(k+1))/6, d_k=y_k

Exa

Other boundary conditions [e.g. the values $f'(x_0)$, $f'(x_n)$] give other cubic splines.

20.3 Approximation of Functions

Often we know $n + 1$ pairs of values (x_k, y_k), $k = 0, 1, \ldots, n$, that may be approximate or exact values of x and y satisfying a known or unknown function $y = f(x)$, or may be not related by a functional relation. In any case, we can find a "simple" function $g(x)$ to use as an *approximate representation* of the relation between x and y.

Method of least squares

From $n + 1$ pairs of values (x_k, y_k), $k = 0, 1, \ldots, n$, the *method of least squares* defines the "best" approximation of $f(x)$ as a function $g(x)$ which minimizes the quantity

S=\S[k,0,n]{(y_k-g(x_k))^2

The method assumes a general form for $g(x)$ containing a number of parameters, and determines these parameters so that S becomes minimum.

The least-square line

If $g(x)$ is a first degree polynomial $ax + b$, then the coefficients of the *least-square line* $\bar{y} = ax + b$ are

a=(s_0*v_1-s_1*v_0)/(s_0*s_2-s_1^2),
b=(s_2*v_0-s_1*v_1)/(s_0*s_2-s_1^2)

where

$$s_0 = n + 1, \qquad s_1 = \Sigma x_k, \qquad s_2 = \Sigma x_k^2$$

$v_0 = \Sigma y_k$, $\quad v_1 = \Sigma x_k y_k$, $\quad \Sigma$= \S[k,0,n], $\quad n > 1$

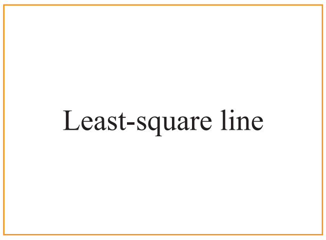

Fig. 20-1

The least-square parabola

If $g(x)$ is a second degree polynomial $ax^2 + bx + c$, then the coefficients of the *least-square parabola* $\bar{y} = ax^2 + bx + c$ satisfy the equations

$$s_0 c + s_1 b + s_2 a = v_0$$
$$s_1 c + s_2 b + s_3 a = v_1$$
$$s_2 c + s_3 b + s_4 a = v_2$$

Exa

where s_0, s_1, s_2, v_0, v_1 have the same expressions as above and

$s_3 = \Sigma x_k^3$, $\quad s_4 = \Sigma x_k^4$, $\quad v_2 = \Sigma x_k^2 y_k$, $\quad \Sigma$= \S[k,0,n], $\quad n > 2$

The least-square polynomial

If $g(x)$ is a polynomial of degree m, then the coefficients a_k of the *least-square polynomial* $a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$ satisfy the *normal equations*

$$s_0 a_0 + s_1 a_1 + \cdots + s_m a_m = v_0$$
$$s_1 a_0 + s_2 a_1 + \cdots + s_{m+1} a_m = v_1$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$s_m a_0 + s_{m+1} a_1 + \cdots + s_{2m} a_m = v_m$$

where

s_p=\S[k,0,n]{x_k^p}, v_q=\S[k,0,n]{x_k^q*y_k}

These equations are rarely used for $m > 5$, since the system is ill-conditioned for larger m (it still can be solved by special methods).

Continuous data

Instead of distinct pairs (x_k, y_k), we may have continuous data, i.e. pairs of (x, y) for any x in $I = [a, b]$ (e.g. when we obtain graphs from instruments or we want a simple representation of a known but complex analytical function). A linear transformation of x maps the interval $[a, b]$ to $[-1, 1]$. Usually, we write the polynomial approximation of $y = f(x)$ as a sum of orthogonal polynomials up to some degree m.

Using Legendre polynomials up to degree m, we write

$$g(x) = a_m P_m(x) + a_{m-1} P_{m-1}(x) + \cdots + a_0 P_0(x)$$ Exa

where $P_k(x)$ is the Legendre polynomial of degree k. The requirement that

S=\I[x,-1,1]{(y-g(x))^2}

becomes minimum gives the coefficients

a_k=((2*k+1)/2)*\I[x,-1,1]{f(x)*P_k(x)}

A similar procedure can be followed in the interval $[-1, 1]$ using the Chebyshev polynomials $T_k(x)$, which have a weight function $(1 - x^2)^{-1/2}$. The polynomial approximation of $f(x)$ is written in the form

$$h(x) = b_m T_m(x) + b_{m-1} T_{m-1}(x) + \cdots + b_0 T_0(x)$$ Exa

and the requirement that S becomes minimum gives the coefficients

b_0=(1/π)*\I[x,-1,1]{f(x)/(1-x^2)^(1/2)},
b_k=(2/π)*\I[x,-1,1]{f(x)*T_k(x)/(1-x^2)^(1/2)}

Approximating a function with a Chebyshev expansion in $[-1, 1]$ results in a smaller upper bound for the absolute error than a Taylor expansion.

Other methods

For discrete and/or continuous data, other methods for approximating a function have been developed based on different definitions of the "best" approximation. If we ask that the maximum of the absolute error $|f(x) - g(x)|$ be as small as possible [$g(x)$ is a polynomial in some form], we have the *minimax approximation* Ext. If $g(x)$ is a rational function, and we ask $f(x)$ and $g(x)$, and their derivatives (as many as needed) to be equal at $x = 0$, we have a *Padé approximation* Ext. Finally, a very good method for approximating a function (even discontinuous) is the *trigonometric approximation* obtained by truncating the corresponding Fourier series Inf.

20.4 Roots of Algebraic Equations

The roots (real or complex) of an algebraic or transcendental equation can be found using several methods. We assume we want to find a simple root r of $f(x) = 0$.

Interpolation methods

Interpolation methods start from an interval $[a, b]$ in which a single root r lies and then progressively narrow that interval.

Let $f(x)$ be a continuous function in $[a, b]$ and $f(a)f(b) < 0$. Then, the equation $f(x) = 0$ has (at least) a root in (a, b).

Method of bisecting the interval

(1) We set $c = \frac{1}{2}(a + b)$. If $f(c) = 0$, we have a root $x = c$.

If $f(c)f(b) < 0$, we replace a by c and go to (1).

If $f(a)f(c) < 0$, we replace b by c and go to (1). **Exa**

Thus, the interval containing the root is divided by 2 at each step. The process ends when $|a - b|$ becomes small enough (i.e. when the desired accuracy is achieved). Sometimes we require that $|f(c)|$ also be small enough.

Method of linear interpolation (false position or regula falsi)

(1) We set c=b-f(b)*(a-b)/(f(a)-f(b)). If $f(c) = 0$, we have a root $x = c$.

If $f(c)f(b) < 0$, we replace a by c and go to (1).

If $f(a)f(c) < 0$, we replace b by c and go to (1). **Exa**

The process ends when $|a - b|$ becomes small enough (i.e. when the desired accuracy is achieved). Sometimes we require that $|f(c)|$ be also small enough.

Iterative methods

We start from one or more values of x (estimations of the root) and generate a sequence $\{x_n\}$ that tends to the root r as $n \to \infty$. The convergence is concluded from the results. The process ends when $|x_{n+1} - x_n|$ and/or $f(x_n)$ becomes small enough.

Secant method

If x_0, x_1 are two values of x close to the root r of $f(x) = 0$, we define

x_(n+1)=x_n-f(x_n)*(x_(n-1)-x_n)/(f(x_(n-1))-f(x_n)) **Exa**

Newton's method (or Newton-Raphson method)

If x_0 is a value of x close to the root r of $f(x)$, we define

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$ **Exa**

In the simple cases of finding the square root or the root of order m of a number c (i.e. solving $x^2 - c = 0$ or $x^m - c = 0$), Newton's method gives, respectively, the formulas

$$x_{n+1}=\frac{x_n+c/x_n}{2}, \quad x_{n+1}=x_n-\frac{x_n^m-c}{m\,x_n^{m-1}}$$

Fixed-point iteration

To solve the equation $f(x) = 0$, we write it as $x = g(x)$, where $g(x)$ is a *contraction mapping*, i.e. a mapping from (a, b) to (a, b) with $|g(x) - g(y)| \le L|x - y|$, where $0 < L < 1$ for any x, y in (a, b). Then, starting with a value x_0 close to the root r, we generate the sequence $\{x_n\}$ with

$$x_{n+1} = g(x_n), \qquad n = 0, 1, \ldots$$ **Exa**

Convergence

In the fixed-point iteration method, the error ε_{n+1} at the $(n+1)$th step and the error $\varepsilon_n = |r - x_n|$ at the nth step satisfy the relations

$$\varepsilon_{n+1} \simeq g'(r)\varepsilon_n, \qquad |\varepsilon_{n+1}| \le \tau|\varepsilon_n|,$$

where τ is an upper bound of $|g'(x)|$ in a small interval containing the root r. The method converges *linearly* if $\tau < 1$.

The secant and the false position methods exhibit a faster convergence, since ε_{n+1} is proportional to the product $\varepsilon_n\varepsilon_{n-1}$. Newton's method converges *quadratically*, since ε_{n+1} and ε_n are related by

$$\varepsilon_{n+1} \simeq -\frac{f''(r)}{2f'(r)}\varepsilon_n^2$$

Other methods, many equations

There are variants or combinations of the above methods or other entirely new methods for solving $f(x) = 0$. Examples are *Muller's method* **Ext**, which uses a second degree polynomial for approximating $y = f(x)$ in the neighborhood of a root, *Bernoulli's method* for finding the largest simple root of a polynomial equation **Ext**, methods involving accelerated convergence, methods for multiple roots, and others. Many of the previous methods can be extended to two or more equations **Ext**.

20.5 Systems of Linear Equations

A system of linear equations may have more, less, or as many unknowns as the number of the independent equations. In all cases, the final step is the solution of a system of the form

\S[k,1,n]{a_(ik)*x_k}=b_i   or   **A**X=**B**

A=mat(a_(11),a_(12),...,a_(1n);
a_(21),a_(22),...,a_(2n);
...,...,...,...;
a_(n1),a_(n2),...,a_(nn))

X=mat(x_1;x_2;...;x_n), **B**=mat(b_1;b_2;...;b_n)

where a_{ik} are constants, b_i are constants or quantities considered known, and x_i are the n unknowns (i, $k = 1, 2, \dots, n$). It is emphasized that the determinant D of the matrix **A** is different than zero, since otherwise we can lower n until we get a similar system with $D \neq 0$.

A usual decomposition of **A** is **A** = **D** + **L** + **U**, where

D=mat(a_(11),0,...,0;0,a_(22),...,0;...,...,...,...;0,0,...,a_(nn)),
L=mat(0,0,...,0;a_(21),0,...,0;...,...,...,...;a_(n1),a_(n2),...,0),
U=mat(0,a_(12),...,a_(1n);0,0,...,a_(2n);...,...,...,...;0,0,...,0)

i.e. **D** may have nonzero elements along the main diagonal, **L** only below the main diagonal, and **U** only above the main diagonal.

The solution of the above system is (Sec. 2.6)

x_k=D_k/D

where D_k is the determinant obtained from D after replacing the elements a_{1k}, a_{2k}, …, a_{nk} of the kth column by b_1, b_2, …, b_n. In practice, the difficulty lies in the actual numerical calculation of D_k and D, especially in cases where $n > 10$ or even $n > 100$.

If D is very close to zero, a small change in the coefficients a_{ik}, b_i, or even small errors in the intermediate calculations, can substantially change the solution x_i. Such systems are called *ill-conditioned* or *unstable*, in contrast to *well-conditioned* or *stable* systems, in which small changes in the data or the intermediate calculations result in small changes in the solution. **Exa**

Direct methods

A method of solution of a linear system is called *direct*, if it gives the exact answer after a finite number of steps (assuming the intermediate calculations do not create errors).

Gaussian elimination

There are three actions by which a linear system can be transformed into an equivalent system that can be solved easily and without too many calculations:

(1) Multiplication of a row (i.e. an equation) by a constant.

(2) Addition of a multiple of a row to a multiple of another row.

(3) Interchange of two rows.

These actions can be performed on the *augmented matrix*

mat(a_(11),a_(12),...,a_(1n),b_1;
a_(21),a_(22),...,a_(2n),b_2;
...,...,...,...,...;
a_(n1),a_(n2),...,a_(nn),b_n)

and transform it to an *upper triangular* or *lower triangular* form respectively, i.e.

mat(a'_(11),a'_(12),...,a'_(1n),b'_1;
0,a'_(22),...,a'_(2n),b'_2;
...,...,...,...,...;
0,0,...,a'_(nn),b'_n)
	or	mat(a'_(11),0,...,0,b'_1;
		a'_(21),a'_(22),...,0,b'_2;
		...,...,...,...,...;
		a'_(n1),a'_(n2),...,a'_(nn),b_n)

The linear system with such a matrix can be solved easily: we start from the equation that contains only one unknown and find its value. We substitute this value into the equation that contains two unknowns and find the value of the second unknown. We continue with the equation that contains three unknowns, and so on.

In practice, assuming $a_{11} \neq 0$, we multiply the first equation by a_{21}/a_{11}, a_{31}/a_{11}, …, a_{n1}/a_{11} and subtract from the second, the third, …, the nth equation. The system thus obtained has no term with x_1, except in the first equation. We repeat the procedure for the system of the last $n-1$ equations and we continue until we obtain a triangular system. This is the *method of elimination of Gauss*. Exa

To improve the accuracy of the calculations, we can rearrange the equations and the unknowns so that a_{11} is the coefficient with the largest absolute value. Dividing by a larger value generates smaller round-off errors and gives better results. This is the *method of elimination of Gauss with pivoting* (a_{11} is the *pivoting* element).

Method of Gauss-Jordan

This is a variant of the Gaussian elimination method in which we eliminate an unknown not only from the subsequent equations but also from the previous ones at the same time. Thus we obtain a system of linear equations whose matrix $\mathbf{A}'$ is diagonal, i.e. $a'_{ik} = 0$ for $i \neq k$. Usually we make $\mathbf{A}'$ not only diagonal, but with each element equal to 1. Thus, the equations immediately give the solution. Exa

Indirect or iterative methods

Let $\mathbf{X} = [x_1, x_2, \ldots, x_n]^T$ be the exact solution of a linear system $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}$ is the matrix of the coefficients of the unknowns and $\mathbf{B} = [b_1, b_2, \ldots, b_n]^T$ ($\mathbf{X}$ and $\mathbf{B}$ are column vectors). If we can establish a recurrence relation of the form

$$\mathbf{X}^{(k+1)} = F(\mathbf{X}^{(k)})$$

we can start with an initial column vector $\mathbf{X}^{(0)}$, then find $\mathbf{X}^{(1)}$, $\mathbf{X}^{(2)}$, etc. and form a sequence $\mathbf{X}^{(k)}$ of column vectors, which under certain conditions converges to $\mathbf{X}$. Such a method is called *indirect* or *iterative*.

The Jacobi method

We solve the ith equation of the linear system with respect to the diagonal term $a_{ii}x_i$, divide by a_{ii} and replace x_i by $x_i^{(k+1)}$ and all other x_j by $x_j^{(k)}$. Thus, we obtain

x_i^((k+1))=(1/a_(ii))*(b_i-\S[j,1,i-1]{a_(ij)*x_j^((k))}-\S[j,i+1,n]{a_(ij)*x_j^((k))}) Exa

Starting with an initial column vector $\mathbf{X}^{(0)}$ we construct the sequence $\mathbf{X}^{(k)}$. If this sequence converges, then we have the required solution as $k \to \infty$.

With the decomposition $\mathbf{A} = \mathbf{D} + \mathbf{L} + \mathbf{U}$, the linear system $\mathbf{AX} = \mathbf{B}$ can be written $\mathbf{DX} = \mathbf{B} - (\mathbf{L} + \mathbf{U})\mathbf{X}$ or $\mathbf{X} = \mathbf{D}^{-1}\mathbf{B} - \mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})\mathbf{X}$ (assuming $\det \mathbf{D} \neq 0$). Thus, the Jacobi method in matrix form is

$$\mathbf{X}^{(k+1)} = \mathbf{D}^{-1}[\mathbf{B} - (\mathbf{L} + \mathbf{U})\mathbf{X}^{(k)}]$$

The Gauss-Seidel method

If $x_i^{(k)}$, $i = 1, 2, \ldots, n$, is an approximate solution of $\mathbf{AX} = \mathbf{B}$, then from the first equation we find a new value $x_1^{(k+1)}$ for x_1, from the second a new value $x_2^{(k+1)}$ for x_2, etc. with

x_i^((k+1))=(1/a_(ii))*(b_i-\S[j,1,i-1]{a_(ij)*x_j^((k+1))}-\S[j,i+1,n]{a_(ij)*x_j^((k))}) Exa

If the sequence $\mathbf{X}^{(k)}$ converges, we have a solution of the linear system. In matrix form, the Gauss-Seidel method can be written as

$$\mathbf{X}^{(k+1)} = \mathbf{D}^{-1}[\mathbf{B} - \mathbf{L}\mathbf{X}^{(k+1)} - \mathbf{U}\mathbf{X}^{(k)}]$$

Determinants and inversion of matrices

To find $D = \det \mathbf{A}$, we follow the elimination method of Gauss. Multiplying each row by a constant and adding to another row does not change the determinant of the matrix. Thus, **A** is transformed into a table with all the elements below the diagonal equal to zero (i.e. with $\mathbf{L} = \mathbf{0}$). The determinant D of this table is the product of the diagonal elements. Exa

Let $\mathbf{V} = \mathbf{A}^{-1}$ be the inverse of **A** (we assume it exists), $\mathbf{X}_i = [a_{1i}, a_{2i}, \ldots, a_{nn}]^T$ the column vector from the ith column of **V**, and $\mathbf{U}_i$ a column vector with only the ith element equal to 1 and the remaining elements equal to zero. Then $\mathbf{AX}_i = \mathbf{U}_i$, which means that **V** can be found by solving all the linear systems for all possible $\mathbf{U}_i$.

In practice, this can be done by writing **A** and **I** side-by-side as follows

mat(a_(11),a_(12),...,a_(1n);
a_(21),a_(22),...,a_(2n);
...,...,...,...;
a_(n1),a_(n2),...,a_(nn))
	mat(1,0,0,0;
	0,1,0,0;
	0,0,1,0;
	0,0,0,1)

Dividing the first row (of both **A** and **I**) by a_{11} we obtain 1 as the first element of the diagonal of **A**. Multiplying the (whole) first row successively by $a_{21}, a_{31}, \ldots, a_{n1}$ and subtracting from the subsequent rows, we make all the remaining elements of the first column equal to zero. Repeated applications of this procedure give all the diagonal elements of **A** equal to 1 and all other elements (above and below the diagonal) equal to zero. Then, at the left of the vertical line we have a unit matrix and at the right we have $\mathbf{A}^{-1}$. Essentially this is again the elimination method of Gauss. Exa

Eigenvalues and eigenvectors

For a square matrix **A**, there are some column vectors **Y** ($\neq 0$) such that

$$\mathbf{AY} = \lambda \mathbf{Y}$$

i.e. multiplication of **Y** by **A** is equivalent to multiplication of **Y** by a constant λ. The vector **Y** is an *eigenvector* of **A** and λ is the corresponding *eigenvalue*.

To find λ and **Y** for a given **A**, we have to find nonzero solutions of the linear system $(\mathbf{A} - \lambda \mathbf{I})\mathbf{Y} = \mathbf{0}$, which exist only for

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

This is an algebraic equation of degree n (if **A** is an $n \times n$ matrix) with respect to λ and is called the *characteristic equation* of **A**.

To calculate λ, we can use a variant of Gaussian elimination. We replace each equation of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{Y} = \mathbf{0}$ by a linear combination of all the equations so that λ is contained only in the last term of the equation. Then, following the method of elimination of Gauss, we obtain a system whose matrix is *upper triangular*, i.e. it has $\mathbf{L} = \mathbf{0}$. The determinant of such a matrix is the product of the diagonal coefficients. For $n > 5$ more efficient methods exist that give more accurate results. Exa

20.6 Differentiation Inf

From the definition of the derivative, we can approximately calculate the derivative $f'(x)$ of $y = f(x)$ using one or more of the following expressions:

```
f'(x)\~(f(x+h)-f(x))/h, f'(x)\~(f(x)-f(x-h))/h, f'(x)\~(f(x+h)-f(x-h))/(2*h)
```

For the second derivative, we can write

```
f"(x)\~(f(x+h)-2*f(x)+f(x-h))/h^2
```

To include more terms in the approximation, we can use an interpolating polynomial that approximates $f(x)$. For pairs (x_k, y_k), with equally spaced values of x, i.e. $x_{k+1} - x_k = h$, using Newton's formula with forward differences we have

```
f'(x)\~(1/h)*(Δy_0+((2*k-1)/2)*Δ^2y_0+...
```

If we use four points (x_k, y_k), $k = -2, -1, 1, 2$, $x_k = x_0 + kh$, the derivative at x_0 is

```
f'(x_0)\~(y_(-2)-8*y_(-1)+8*y_1-y_2)/(12*h)
```

Exa

Numerical differentiation using an interpolating polynomial involves all kinds of errors [$f(x) - p(x)$ may be small and still $f'(x) - p'(x)$ may be large]. Input errors are magnified since h appears in the denominator (contrary to numerical integration).

Other methods of representing discrete or continuous data (e.g. splines, trigonometric approximation, etc.) can be used for an approximate calculation of $f'(x)$. If we have many points (x_k, y_k), we may represent $f(x)$ by its least-squares or minimax approximation $g(x)$, and then compute the derivative using $f'(x) \simeq g'(x)$. For five points, fitting a parabola by the least-squares method leads to the formula

```
f'(x_0)\~(-2*y_(-2)-y_(-1)+y_1+2*y_
2)/(10*h)
```

20.7 Integration

The objective of numerical integration is the numerical calculation of a definite integral \I[x,a,b]{f(x)}. It is a process that can be carried out with great accuracy, even when very elementary methods are used. The data are either the values y_k of $y = f(x)$ at $x_k = x_0 + kh$, with $k = 0, 1, \ldots, n$, $x_0 = a$, $x_n = b$, or an analytic expression of $f(x)$ (whose indefinite integral usually cannot be computed analytically). The truncation error is defined by E=\I[x,a,b]{f(x)-p(x)}, where $p(x)$ is the polynomial or function that is used to calculate the integral instead of $f(x)$.

Newton-Cotes formulas

Trapezoidal rule

For 2 points, \I[x,x_0,x_1]{f(x)}\~(h/2)*(y_0+y_1)

\I[x,a,b]{f(x)}\~(h/2)*(y_0+2*y_1+2*y_2+...+2*y_(n-1)+y_n) **Exa**

$E = -(b-a)h^2 f''(\xi)/12$, where $a = x_0 < \xi < b = x_n$.

Simpson's rule (n even)

For 3 points, \I[x,x_0,x_2]{f(x)}\~(h/3)*(y_0+4*y_1+y_2)

\I[x,a,b]{f(x)}\~(h/3)*(y_0+4*y_1+2*y_2+...+4*y_(n-1)+y_n) **Exa**

$E = -(b-a)h^4 f^{(4)}(\xi)/180$, where $a = x_0 < \xi < b = x_n$.

Gaussian integration

To calculate a definite integral of a function $f(x)$ known analytically, whose indefinite integral cannot be found analytically, we can use a formula of the form

\I[x,a,b]{f(x)}\~\S[k,1,n]{c_k*f(x_k)} **Ext**

where x_k, c_k are constants to be determined and n is an arbitrarily chosen positive integer. We require this formula to be exact when $f(x)$ is any polynomial of degree 0, 1, 2, …, $2n-1$. This requirement gives $2n$ equations with $2n$ unknowns x_k, c_k, from which we find x_k, c_k. Then, we can use the formula for any given function $f(x)$ to find an approximate value of the integral.

Gauss-Legendre formulas

To determine c_k, x_k, we can use any set of $2n$ linearly independent polynomials. After transforming the interval $[a, b]$ to $[-1, 1]$, the integral formula becomes

\I[x,-1,1]{f(x)}\~\S[k,1,n]{c_k*f(x_k)} Exa

We now use the Legendre polynomials $P_k(x)$, $k = 0, 1, \ldots, 2n - 1$. Then, x_k are the zeros of $P_n(x)$ and

c_k=2*(1-x_k^2)/(n*P_(n-1)(x_k))^2)

The constants c_k, x_k are known and given in tables or by computer programs. Tab

An estimate of the error is given by Lanczos's formula

E\~(1/(2*n+1))*(f(-1)+f(1)-\S[k,1,n]{c_k*f(x_k)}-\S[k,1,n]{c_k*x_k*f'(x_k)})

Gauss-Laguerre formulas

The interval of integration is $[0, \infty]$ with a weight function $w(x) = e^{-x}$. The formula for approximate integration and the coefficients are

\I[x,0,\o]{e^(-x)*f(x)}\~\S[k,1,n]{c_k*f(x_k)},
c_k=(n!)^2/(x_k*(L'_n(x_k))^2) Tab

where x_k are now the zeros of the Laguerre polynomial $L_n(x)$.

Gauss-Hermite formulas

The interval of integration is $[-\infty, \infty]$ with a weight function $w(x) = \exp(-x^2)$. The formula for approximate integration and the coefficients are

\I[x,-\o,\o]{{e^(-x^2)*f(x)}\~\S[k,1,n]{c_k*f(x_k)},
c_k=2^(n+1)*n!*π^(1/2)/(H'_n(x_k))^2 Tab

where x_k are now the zeros of the Hermite polynomial $H_n(x)$.

Gauss-Chebyshev formulas

The interval of integration is $[-1, 1]$ with a weight function $w(x) = (1 - x^2)^{-1}$. The formula for approximate integration is

\I[x,-1,1]{f(x)/(1-x^2)^(1/2)}\~(π/n)*\S[k,1,n]{f(x_k)} Tab

where x_k are now the zeros of the Chebyshev polynomial $T_n(x)$.

20.8 Ordinary Differential Equations

Inf

A general problem in ODEs is to find $y = y(x)$ that satisfies

$$y' = f(x, y) \quad \text{with } y(x_0) = y_0$$

where x_0, y_0 are known constants and $f(x, y)$ is a known function of x and y. Assuming the solution $y = y(x)$ exists (its existence is guaranteed if the Lipschitz condition is satisfied), we can use a numerical method to find it (whether the solution can be found analytically or not). All methods determine a sequence of points (x_k, y_k) with $x_k = x_0 + kh$, thus approximating $y = y(x)$ using a curve passing through these points.

If one point is enough to determine the next point, we have a *single-step method*. If more points are required, we have a *multi-step method*. In some methods we use one formula, called the *predictor*, to find a value of y_{k+1} and then another formula, called the *corrector*, to find a better (more accurate) value of y_{k+1}.

Single-step methods

The Euler method Exa

$$y_{k+1} = y_k + hf(x_k, y_k)$$

$$\text{Error} = y(x_{k+1}) - y_{k+1} = y(x_k) - y_k + \tfrac{1}{2}h^2 y''(\xi_k), \qquad x_k < \xi_k < x_{k+1}$$

where $y(x_k) - y_k$ is the error from the previous steps and $\frac{1}{2}h^2 y''(\xi_k)$ is the local error.

Heun's method

$$y_{k+1} = y_k + \tfrac{1}{2}h[y'_k + f(x_{k+1}, y_k + hy'_k)]$$

Taylor series method

$$y_{k+1} = y_k + hy'_k + \tfrac{1}{2}h^2 y''_k + \tfrac{1}{6}h^3 y'''_k + \cdots$$

Runge-Kutta method of third order

$$k_1 = hf(x_k, y_k), \qquad k_2 = hf(x_k + \tfrac{1}{2}h, y_k + \tfrac{1}{2}k_1), \qquad k_3 = hf(x_k + h, y_k - k_1 + 2k_2),$$

$$y_{k+1} = y_k + \tfrac{1}{6}(k_1 + 4k_2 + k_3)$$

Runge-Kutta method of fourth order Exa

$$k_1 = hf(x_k, y_k), \qquad k_2 = hf(x_k + \tfrac{1}{2}h, y_k + \tfrac{1}{2}k_1),$$

$$k_3 = hf(x_k + \tfrac{1}{2}h, y_k + \tfrac{1}{2}k_2), \qquad k_4 = hf(x_k + h, y_k + k_3),$$

$$y_{k+1} = y_k + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

A simple predictor-corrector method Exa

Predictor $y_{k+1} = y_k + hy'_k$

Corrector $y_{k+1} = y_k + \frac{1}{2}h(y'_k + y'_{k+1})$

Multistep methods

Nystrom's method

$$y_{k+1} = y_{k-1} + 2hy'_k$$

The Milne method Ext

Predictor $y_{k+1} = y_{k-3} + (4/3)h(2y'_{k-2} - y'_{k-1} + 2y'_k)$

Corrector $y_{k+1} = y_{k-1} + \frac{1}{3}h(y'_{k-1} + 4y'_k + y'_{k+1})$

The Hamming method

Predictor $y_{k+1} = y_{k-3} + (4/3)h(2y'_{k-2} - y'_{k-1} + 2y'_k)$

Corrector $y_{k+1} = \frac{1}{8}(-y_{k-2} + 9y_k) + \frac{3}{8}h(-y'_{k-1} + 2y'_k + y'_{k+1})$

The Adams method

Predictor $y_{k+1} = y_k + h(-9y'_{k-3} + 37y'_{k-2} - 59y'_{k-1} + 55y'_k)/24$

Corrector $y_{k+1} = y_k + h(y'_{k-2} - 5y'_{k-1} + 19y'_k + 9y'_{k+1})/24$

20.9 Partial Differential Equations

The theoretical description of scalar or vector fields in physics often requires the integration of a partial differential equation (PDE) of second order. Such a PDE can be *elliptic* (e.g. Poisson's equation), *parabolic* (e.g. heat equation) or *hyperbolic* (e.g. wave equation). In each case, the solution u satisfies the PDE inside a region R of the space and certain conditions on the boundary S of R. Thus, we have a *boundary value problem* (*Dirichlet problem*, if u is given on S, or *Neumann problem*, if the derivative along the normal to S is given on S, or a *mixed problem* for mixed conditions).

The numerical solution of a boundary value problem in two or more dimensions usually involves the following basic steps:

(1) Choice of a set of points (*mesh points* or *nodes*) at which u is to be computed.

(2) Linear approximation of the value of u and its derivatives at each node in terms of the values at neighboring nodes, thus forming a system of linear equations.

(3) Solution of the linear system, i.e. computation of the value of u at each node.

Poisson's equation

In two dimensional Euclidean space with orthogonal Cartesian coordinates, Poisson's partial differential equation and the Dirichlet boundary conditions are

(\d/\dx)^2{u}+(\d/\dy)^2{u}=r(x,y):(x,y)\eR, u(x,y)=s(x,y):(x,y)\eS

In the simple case of a rectangular area $R = \{(x, y\} \mid a < x < b,\ c < y < d\}$, we divide R horizontally in m strips of width $h_x = (b - a)/m$ and vertically in n strips of width $h_y = (d - c)/n$. The lines $x = x_i = a + ih_x$ $(i = 0, 1, \ldots, m)$ and $y = y_j = c + jh_y$ $(j = 0, 1, \ldots, n)$ define a *mesh* or *grid* that covers R. The intersections of the lines define the *nodes* (mesh or grid points), at which the values of u are sought (Fig. 20-2).

In step (2), to approximate the derivatives of u in the PDE with finite differences, we use the expressions (and similarly for y)

(\d/\dx){u}\~(u(x+h_x,y)-u(x,y))/h_x

(\d/\dx)^2{u}\~(u(x+h_x,y)-2*u(x,y)+u(x-h_x,y))/h_x^2

A mesh for numerical integration

Fig. 20-2

If u_{ij} is the approximate value of $u(x_i, y_j)$, then at each interior node (red dot) the PDE is replaced by the linear equation $(i = 1, \ldots, m - 1$ and $j = 1, \ldots, n - 1)$

$$2[h_x^2 + h_y^2]u_{ij} - h_x^2(u_{i,j+1} + u_{i,j-1}) - h_y^2(u_{i+1,j} + u_{i-1,j}) = -h_x^2h_y^2r(x_i, y_i)$$ Exa

At the nodes on S (green dots), we have from the boundary conditions

$$u_{0,j} = s(x_0, y_j), \quad u_{m,j} = s(x_m, y_j), \quad u_{i,0} = s(x_i, y_0), \quad u_{i,n} = s(x_i, y_n)$$

We relabel the interior (red) nodes as 1, 2, 3, etc. (Fig. 20-2) and thus, we have $u_k = u_{i,j}$, where $k = i + (m - 1)(n - j - 1)$. Substituting in the previous equations, we obtain a linear system with $(m - 1)(n - 1)$ equations and unknowns. Only a few diagonals of the coefficient matrix contain non-zero elements (it is a *sparse* matrix). Thus, the linear system can be solved relatively easily (using Gaussian elimination or the Gauss-Seidel method). Numerical difficulties may arise as in linear systems.

Heat or diffusion equation

With one space variable x and time represented by t, the heat or diffusion equation with its boundary and initial conditions is

(\d/\dx)^2{u}=(1/c^2)*(\d/\dt){u}:0<x<a,t>0,

u(0,t)=u(a,t)=0:t>0, u(x,0)=g(x):0\<x\<a

To create a grid, we choose an integer m, a space step $h_x = a/m$, and a time step h_t. Then, the grid lines are $x_i = ih_x$ $(i = 0, 1, 2, \ldots, m)$ and $t_j = jh_t$. The *Crank-Nicolson method* (with mixed forward and backward differences) gives from the PDE

$$2(u_{i,j+1} - u_{ij}) - (c^2h_t/h_x^2)(u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}) = 0$$

and from the initial and boundary conditions $u_{i,0} = g(x_i)$, $u_{0,j} = u_{m,j} = 0$.

Wave equation

With one space variable x and time represented by t, the wave equation with its boundary and initial conditions is

(\d/\dx)^2{u}=(1/c^2)*(\d/\dt)^2{u}:0<x<a,t>0,

u(0,t)=u(a,t)=0:t>0, u(x,0)=g_1(x):0\<x\<a, (\d/\dt){u(x,0)}=g_2:0\<x\<a

To create a grid, we choose an integer m, a space step $h_x = a/m$, and a time step h_t. Then, the grid lines are $x_i = ih_x$ $(i = 0, 1, 2, \ldots, m)$ and $t_j = jh_t$. The PDE gives

$$u_{i,j+1} = 2(1 - \lambda^2)u_{i,j} + \lambda^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1} \quad \text{where } \lambda = ch_t/h_x$$

with $u_{0,j} = u_{m,j} = 0$ and $u_{i,0} = g_1(x_i)$ for $i = 1, \ldots, m - 1$ and any $j > 0$. To start, we can use for $i = 1, \ldots, m - 1$, and $j = 1$ the relation

$$u_{i,1} = (1 - \lambda^2)g_1(x_i) + \tfrac{1}{2}\lambda^2 g_1(x_{i+1}) + \tfrac{1}{2}\lambda^2 g_1(x_{i-1}) + h_t g_2(x_i)$$

20.10 Optimization

Optimization means finding the point(s), where a function $f(x)$ or $f(\mathbf{r})$ has an extremum. Analytically, as a first step we solve $f'(x) = 0$ or $f_x = f_y = 0$ (Sec. 6.6). Numerically, we plot $f(x)$ and $f(x, y)$ and then narrow repeatedly the interval or the area where a point of minimum or maximum lies (using bisection, representation by a fitted parabola, a speadsheet program, level curves, etc.).

In two dimensions, the **method of steepest descent** for finding a minimum of $f(x, y)$ consists of starting from $\mathbf{r}_0 = x\mathbf{i} + y\mathbf{j}$ and moving along the straight line $\mathbf{r} = \mathbf{r}_0 - t\nabla f(\mathbf{r}_0)$ until $f(\mathbf{r})$ becomes a minimum. Then we take $\mathbf{r}$ as a new $\mathbf{r}_0$, etc. Exa

In a problem of **linear programming** we have to minimize a linear function $y = a_1x_1 + \cdots + a_nx_n$ subject to $n + m$ constraints $x_i \geq 0$, $c_{j1}x_1 + \cdots + c_{jn}x_n \leq b_j$ $(j = 1, \ldots, m)$. At a *feasible point*, all the $n + m$ constraints are satisfied. At an *extreme feasible point* at least n constraints become equalities. Introducing the *slack variables* $x_{n+1}, \ldots, x_{n+m}$ with $x_{n+j} = b_j - (c_{j1}x_1 + \cdots + c_{jn}x_n)$, we convert the constraints to equalities. The **simplex method** starts at an extreme feasible point and proceeds to other such points until a point of minimum y is found. Exa

21 PROBABILITY AND STATISTICS

21.1 Introduction to Probabilities

An experiment which may give different results, if repeated, is called a *random* experiment. The reason for the possibility of a different outcome each time we perform a random experiment is that we cannot control all of the parameters on which the result depends. Examples are the toss of a coin or the throw of a die. **Inf**

The *sample space* S of an experiment is the set of all possible outcomes. A subset A of S is called *event*. S is the *certain* event and $\varnothing$ (the empty set) is the *impossible* event, since performing the experiment once certainly gives an element of S and never an element of $\varnothing$ (which has no elements).

In words, $A \cup B$ (the union of A and B) is the event "A or B or both", $A \cap B$ (the intersection of A and B) is the event "A and B". Also, $A' = S - A$ is the event "not A" (called the *complement* of A), while $A - B$ is the event "A but not B". Two events A and B are *mutually exclusive* if $A \cap B = \varnothing$.

The *probability* $P(A)$ of an event A is a real number such that

(1) $P(A) \ge 0$,   (2) $P(S) = 1$,   (3) if $A \cap B = \varnothing$, then $P(A \cup B) = P(A) + P(B)$.

Two events A and B are *independent* if $P(A \cap B) = P(A)P(B)$.

The *conditional probability* $P(B \mid A)$ is the probalility $P(B)$ provided A has occured. We have $P(A \cap B) = P(A)P(B \mid A)$, $P(A \cap B \cap C) = P(A)P(B \mid A)P(C \mid A \cap B)$.

If S is a union of n mutually exclusive events $A_1, A_2, \ldots, A_n$ with $P(A_1) = P(A_2) = \cdots = P(A_n)$, then $P(S) = P(A_1) + P(A_2) + \cdots + P(A_n) = 1$ and $P(A_k) = 1/n$. **Exa**

The basic theorems about probabilities are the following:

(1) $0 \le P(A) \le 1$,   (2) $A \subset B$ implies $P(A) \le P(B)$ and $P(B - A) = P(B) - P(A)$,

(3) $P(A') = 1 - P(A)$,   (4) $P(A \cup B) = P(A) + P(B) - P(A \cap \mathrm{B})$

Other properties: $P(\varnothing) = 0$, $P(A) = P(A \cap B) + P(A \cap B')$,

P(A_k|A)=P(A_k)*P(A|A_k)/\
S[i,1,n]{P(A_i)*P(A|A_i)}

Bayes's rule for n mutually exclusive events $A_1, A_2, \ldots, A_n$ with $A_1 \cup A_2 \cup \ldots \cup A_n = S$

Number of *permutations* or *arrangements* of r objects out of n (the order is important): =n!/(n-r)!, r\<n

Number of *combinations* of r objects out of n (the order is not important): =n!/(r!*(n-r)!), r\<n **Ext**

21.2 Distribution of Probabilities

Random variables and distributions

A *random variable* or *stochastic function* X is a function defined on a sample space S, i.e. a mapping of the elements of S to a set of values (usually numbers). A simple example is the mapping of the sample space of throwing a die (i.e. the results of throwing a die) to A, B, C, D, E, F or to 1, 2, 3, 4, 5, 6 (i.e. the values of X).

A *probability function* or *probability distribution* is a function $f(x)$, if

(a) for a **discrete** random variable X, the probability for X to have the value x is

P(X=x)=f(x) with f(x)\>0 and \S[x]{f(x)}=1

(b) for a **continuous** random variable X, the probability for X to have a value x in $[a, b]$ is P(a\<X\<b)=\I[x,a,b]{f(x)} with f(x)\>0 and \I[x,-\o,\o]{f(x)}=1.

For continuous X, the function $f(x)$ is also called the *probability density*.

A *distribution function* is a function $F(x)$ that gives the probability to have $X \leq x$.

(a) For a **discrete** random variable X, F(x)=P(X\<x)=\S[u\<x]{f(u)}.

(b) For a **continuous** random variable X, F(x)=P(X\<x)=\I[u,-\o,x]{f(u)}.

Parameters of a distribution

Ext

Expected or **mean value**

μ=E(X)=\S[x]{x*f(x)} for discrete X,
\I[x,-\o,\o]{x*f(x)} for continuous X

For a function $g(X)$, its *expected value* is

E(g(X))=\S[x]{g(x)*f(x)} or E(g(X))=\I[x,-\o,\o]{g(x)*f(x)}

Variance

σ^2=Var(X)=E((X-μ)^2)=E(X^2)-(E(X))^2
=\S[x]{(x-μ)^2*f(x)}=\S[x]{x^2*f(x)}-μ^2 for discrete X
=\I[x,-\o,\o]{(x-μ)^2*f(x)}=\I[x,-\o,\o]{x^2*f(x)}-μ^2 for continuous X

Standard deviation σ=Var(X)^(1/2)

Standardized random variable Z=(X-μ)/σ, $E(Z) = 0$, $\sigma_Z = 1$

Skewness a_3=E((X-μ)^3)/σ^3
=μ_3/σ^3

Kurtosis a_4=E((X-μ)^4)/σ^4
=μ_4/σ^4

Moments

Moment of order r about the mean or *central moment*

μ_r=E((X-μ)^r)=\S[x]{(x-μ)^r*f(x)} for discrete X
\I[x,-\o,\o]{(x-μ)^r*f(x)} for continuous X

Moment of order r about the origin ($\mu'_0 = 1, \mu'_1 = \mu$)

μ'_r=E(X^r)

Moment generating function (its series expansion gives μ'_r)

M_X(t)=E(e^(t*X))=\S[x]{e^(t*x)*f(x)} for discrete X
=\I[x,-\o,\o]{e^(t*x)*f(x)} for continuous X
=\S[k,0,\o]{μ'_k*t^k/k!}

Characteristic function

Φ_X(ω)=E(e^(i*ω*X))=\S[x]{e^(i*ω*x)*f(x)} for discrete X
\I[x,-\o,\o]{e^(i*ω*x)*f(x)} for continuous X
\S[k,0,\o]{i^k*μ'_k*ω^k/k!}

Two-dimensional distributions

A *joint probability function* is a function $f(x, y)$, if

(a) for **discrete** random variables X and Y, the probability for $X = x$ and $Y = y$ is
P(X=x,Y=y)=f(x,y) with f(x,y)\>0 and \S[x,y]{f(x,y)}=1

(b) for **continuous** random variables X and Y, the probability for X in $[a, b]$ and Y in $[c, d]$ is P(a\<X\<b,c\<Y\<d)=\I[x,a,b]{\I[y,c,d]{f(x,y)}} with f(x,y\>0
and \I[x,-\o,\o]{\I[y,-\o,\o]{f(x,y)}}=1.

The *joint distribution function* $F(x, y)$ gives the probability for $X \le x$ and $Y \le y$.

(a) For **discrete** random variables, F(x,y)=P(X\<x,Y\<y)=\S[u\<x,v\<y]{f(u,v)}.

(b) For **continuous** random variables,
F(x,y)=P(X\<x,Y\<y)=\I[u,-\o,x]{\I[v,-\o,y]{f(u,v)}}. **Ext**

X and Y are *independent* if $f(x, y) = f_1(x)f_2(y)$ or $F(x, y) = F_1(x)F_2(y)$.

Expected values

μ_X=E(X)=\S[x,y]{x*f(x,y)} for discrete X, Y
\I[y,-\o,\o]{\I[x,-\o,\o]{x*f(x,y)}} for continuous X, Y

Similarly for μ_Y.

Variances

σ_X^2=E((X-μ_X)^2)=\S[x,y]{(x-μ_X)^2*f(x,y)} (discrete)
\I[y,-\o,\o]{\I[x,-\o,\o]{(x-μ_X)^2*f(x,y)}} (continuous)

Similarly for σ_Y. Note: $\sigma_X > 0$, $\sigma_Y > 0$.

Covariance

σ_(XY)=Cov(X,Y)=E((X-μ_X)*(Y-μ_Y))=E(X*Y)-E(X)*E(Y)
=\S[x,y]{(x-μ_X)*(y-μ_Y)*f(x,y)} for discrete X, Y
=\I[y,-\o,\o]{\I[x,-\o,\o]{(x-μ_X)*(y-μ_Y)*f(x,y)}} for continuous X, Y

Correlation coefficient ρ=σ_(XY)/(σ_X*σ_Y), -1\<ρ\<1

Exa

General properties

$E(cX) = cE(X), \quad \mathrm{Var}(cX) = c^2\mathrm{Var}(X)$

$E(X + Y) = E(X) + E(Y)$

$(\sigma_{X\pm Y})^2 = \sigma_X{}^2 + \sigma_Y{}^2 \pm 2\sigma_{XY}, \quad |\sigma_{XY}| \le \sigma_X\sigma_Y$

For independent X, Y we have $E(XY) = E(X)E(Y)$, $\sigma_{XY} = 0$, $(\sigma_{X\pm Y})^2 = \sigma_X{}^2 + \sigma_Y{}^2$.

Chebyshev's inequality $P(|X - \mu| \ge \varepsilon) \le \sigma^2/\varepsilon^2$ (ε positive number)

Law of large numbers: For independent random variables $X_1, X_2, \ldots, X_n$ with the same mean value μ and variance σ^2, we have

lim[n,\o]{P(|(X_1+X_2+...+X_n)/n-μ|\>ε)}=0

Central limit theorem: If $X_1, X_2, \ldots, X_n$ are independent (discrete or continuous) random variables with the same probability function, a mean value μ, a variance σ^2, and $S_n = X_1 + X_2 + \cdots + X_n$, then the random variable $Z_n = (S_n - n\mu)/(\sigma\sqrt{n})$ "follows asymptotically" the standard normal distribution, i.e.

lim[n,\o]{P(a\<Z_n\<b)}=(1/(2*π)^(1/2))*\I[u,a,b]{e^(-u^2/2)}

21.3 Various Distributions

Normal distribution

The continuous random variable X has the probability (i.e. density) function

f(x)=(1/((2*π)^(1/2)*σ))*e^(-(x-μ)^2/(2*σ^2))

for $-\infty < x < \infty$ and distribution function

F(x)=(1/((2*π)^(1/2)*σ))*\I[u,-\o,x]{e^(-(u-μ)^2/(2*σ^2))}

Normal distribution

Fig. 21-1

Mean value μ, variance σ^2, skewness $a_3 = 0$,
kurtosis $a_4 = 3$, moment generating function $M(t) = \exp(\mu t + \sigma^2 t^2/2)$,
characteristic function $\Phi(\omega) = \exp(i\mu\omega - \sigma^2\omega^2/2)$

Setting $Z = (X - \mu)/\sigma$ we obtain the *standard normal distribution* with probability function and distribution function respectively

f(z)=(1/(2*π)^(1/2))*e^(-z^2/2),
F(z)=(1/(2*π)^(1/2))*\I[u,-\o,z]{e^(-u^2/2)}=(1/2)*(1+erf(z/2^(1/2)))

Tab

Binomial distribution

Let p be the probability for an event to happen (success) in performing a random experiment once (single trial) and $q = 1 - p$ be the probability for the same event not to happen (failure). If we repeat the experiment n times, then the probability for this event to happen exactly x times ($x = 0, 1, \ldots, n$) is given by the *binomial distribution*

f(x)=P(X=x)=(n;x)*p^x*q^(n-x)
=(n!/(x!*(n-x)!))*p^x*q^(n-x)

i.e. the coefficients of the binomial expansion

(p+q)^n=q^n+(n;1)*p*q^(n-1)+...+p^n

Binomial distribution

Fig. 21-2

Mean value $\mu = np$, variance $\sigma^2 = npq$,
skewness a_3=(q-p)/(n*p*q)^(1/2), kurtosis a_4=(3*(n-2)*p*q+1)/(n*p*q)
moment generating function $M(t) = (pe^t + q)^n$,
characteristic function $\Phi(\omega) = (pe^{i\omega} + q)^n$.

Ext

Multinomial distribution

Let the mutually exclusive events A_i, $i = 1, 2, \ldots, k$, be the possible results of a random experiment with probabilities p_i and $p_1 + p_2 + \cdots + p_k = 1$. We perform the experiment n times and let X_i represent the event of A_i appearing x_i times in total. Then, the joint probability function for $A_1, A_2, \ldots, A_k$ to occur $x_1, x_2, \ldots, x_k$ times, respectively, with $x_1 + x_2 + \cdots + x_k = n$ is

f(x_1,x_2,...,x_k)=P(X_1=x_1,...,X_k=x_k)=n!*p_1^(x_1)...p_k^(x_k)/(x_1!...x_k!)

This is the general term obtained by expanding $(p_1 + p_2 + \cdots + p_k)^n$.

Mean value of X_i $\mu_i = np_i$, variance of X_i $\sigma_i^2 = np_i(1 - p_i)$,

covariance of X_i and X_j $\mathrm{Cov}(X_i, X_j) = -np_i p_j$.

Hypergeometric distribution

A box contains N small identical spheres, except that b of them are black and those remaining are white. We take out one sphere at random, then another and we continue until we take out n spheres, *without replacing* any removed sphere in the box. The probability that we have exactly x black spheres among the n removed spheres is

f(x)=P(X=x)=(b;x)*((N-b);(n-x))/(N;n)

Mean value μ=n*b/N, variance σ^2=n*b*(N-b)*(N-n)/(N^2*(N-1)) **Ext**

Poisson distribution

A discrete random variable X takes on the values 0, 1, 2, … with probability function

f(x)=P(X=x)=λ^x*e^(-λ)/x!

where λ is a positive constant.

Mean value $\mu = \lambda$, variance $\sigma^2 = \lambda$,

skewness $a_3 = 1/$λ^(1/2), kurtosis $a_4 = 3 + 1/\lambda$,

moment generating function $M(t) = \exp[\lambda(e^t - 1)]$,

characteristic function $\Phi(\omega) = \exp[\lambda(e^{i\omega} - 1)]$

Poisson distribution

Fig. 21-3

Ext

Uniform distribution

We have a *discrete uniform distribution* when a discrete random variable X can take on any one of n values $x_1, x_2, \ldots, x_n$ with the same probability

f(x)=P(X=x)=1/n, x=x_1, x_2, ..., x_n

If $x_k = k$, $k = 1, 2, \ldots, n$, we have
mean value $\mu = (n+1)/2$, variance $\sigma^2 = (n^2-1)/12$,
moment generating function M(t)=e^t*(1-e^(n*t))/(n*(1-e^t)).

Discrete uniform distribution

Fig. 21-4

We have a *continuous uniform distribution* when a continuous random variable X can take on any value in an interval $[a, b]$ with a probability density

f(x)=1/(b-a):a\<x\<b;0,other

Mean value $\mu = (a+b)/2$, variance $\sigma^2 = (b-a)^2/12$,
moment generating function M(t)=(e^(b*t)-e^(a*t))/((b-a)*t).

Continuous uniform distribution

Fig. 21-5

Gamma and beta distributions

The gamma distribution has a probability (i.e. density) function

f_G(x)=x^(α-1)*e^(-x/β)/(β^α*Γ(α)):x>0;0:x\<0

with mean value $\mu = \alpha\beta$ and variance $\sigma^2 = \alpha\beta^2$.
Also, $M(t) = (1-\beta t)^{-\alpha}$ and $\Phi(\omega) = (1 - i\beta\omega)^{-\alpha}$.

Gamma distribution

Fig. 21-6

The beta distribution has a probability (i.e. density) function

f_B(x)=x^(α-1)*(1-x)^(β-1)/B(α,β):0<x<1;0:other

with mean value $\mu = a(a+b)^{-1}$ and
variance $\sigma^2 = ab(a+b)^{-2}(a+b+1)^{-1}$.

Beta distribution

Fig. 21-7

Chi-square distribution

A continuous random variable X with a probability (i.e. density) function

f(x)=x^((n-2)/2)*e^(-x/2)/(2^(n/2)*Γ(n/2)):x>0;0:x\<0

Chi-square distribution

Fig. 21-8

follows the *chi-square distribution* with n degrees of freedom (d.o.f.). Its mean value is $\mu = n$ and its variance $\sigma^2 = 2n$. Also $M(t) = (1 - 2t)^{-n/2}$. Tab

If X_i, $i = 0, 1, 2, \ldots, n$, are n random variables, each following the normal distribution with $\mu = 0$ and $\sigma = 1$, then the sum $X_1^2 + X_2^2 + \cdots + X_n^2$ follows the chi-square distribution with n degrees of freedom (d.o.f.) and is symbolized often by χ^2.

Student's *t*-distribution

A continuous random variable X with a probability (i.e. density) function

f(x)=(Γ(n/2+1/2)/((n*p)^(1/2)*Γ(n/2)))*(1+x^2/n)^(-(n+1)/2)

Student's *t*-distribution

Fig. 21-9

follows *Student's t-distribution* with n degrees of freedom (d.o.f.). Its mean value is $\mu = 0$ and its variance $\sigma^2 = n/(n-2)$ for $n > 2$. Tab

If Y and Z are independent random variables, Y is normally distributed with $\mu = 0$ and $\sigma = 1$, and Z is chi-square distributed with n degrees of freedom, then X=Y/(Z/n)^(1/2) follows the t-distribution with n degrees of freedom.

F distribution

A continuous random variable X with a probability (i.e. density) function

f(x)=Γ(m/2+n/2)*(m/n)^(m/2)*x^((m-2)/2)/(Γ(m/2)*Γ(n/2)*(1+(m/n)*x)^((m+n)/2))

F distribution

Fig. 21-10

has the *F distribution* with m and n degrees of freedom (d.o.f.). Mean value $\mu = n/(n-2)$ for $n > 2$ and variance $\sigma^2 = 2n^2(m+n-2)[m(n-2)^2(n-4)]^{-1}$ for $n > 4$. Tab

If Y and Z are independent random variables, and both are chi-square distributed with m and n degrees of freedom (d.o.f.), respectively, then the ratio $X = (Y/m)/(Z/n)$ follows the F distribution with m and n degrees of freedom. Ext

Geometric distribution

A discrete random variable X with a geometric distribution has probability (i.e. density) function, mean value, variance and moment generating function

$$f(x) = P(X = x) = pq^{x-1}, \quad x = 1, 2, \ldots, \quad q = 1 - p,$$

$$\mu = 1/p, \qquad \sigma^2 = q/p^2, \qquad M(t) = pe^t/(1 - qe^t)$$

If an event A has a probability p to appear in any single trial (Bernoulli trial), then X may represent the number of independent trials until the first appearance of A.

Exponential distribution

A continuous random variable X with an exponential distribution has a probability (i.e. density) function

f(x)=a*e^(-a*x):x>0;0:x\<0

The mean value, the variance, and the moment generating function are $\mu = a^{-1}$, $\sigma^2 = a^{-2}$, $M(t) = a/(a - t)$, respectively.

Cauchy distribution

A continuous random variable X with a Cauchy distribution has a probability (i.e. density) function

f(x)=a/(π*((x-x_0)^2+a^2))

Caushy distribution

Fig. 21-11

The integral that defines the mean value does not exist, but its principal value is equal to x_0. The variance and the moment generating functions do not exist, but the characteristic function is $\Phi(\omega) = e^{-a\omega}$.

Laplace distribution

A continuous random variable X with a Laplace distribution has a probability function, a mean value, a variance and a moment generating function

f(x)=(1/(2*b))*e^(-|x-x_0|/b), b>0, -\o<x<\o,
μ=x_0, σ^2=2*b^2, M(t)=e^(x_0*t)/(1-b^2*t^2)

Laplace distribution

Fig. 21-12

Maxwell distribution

A continuous random variable X with a Maxwell distribution has a probability (i.e. density) function, mean value and variance

f(x)=(2/π)^(1/2)*a^(3/2)*x^2*e^(-a*x^2/2):x>0;
0:x\<0

μ=2*(2/(π*a))^(1/2), $\sigma^2 = (3\pi - 8)/(\pi a)$.

Maxwell distribution

Fig. 21-13

Pascal distribution (negative binomial distribution)

A discrete random variable X with a Pascal distribution has a probability (i.e. density) function, mean value, variance and moment generating function

f(x)=P(X=x)=((x+r-1);(r-1))*p^r*q^x

$\mu = rq/p,\ \sigma^2 = rq/p^2,\ M(t) = p^r/(1 - qe^t)^r.$

Pascal distribution

Fig. 21-14

If an event A has a probability $p = 1 - q$ to appear in any single trial (Bernoulli trial), then X represents the number of independent trials until A appears r times.

Weibull distribution

A continuous random variable X with a Weibull distribution has a probability (i.e. density) function, mean value and variance

f(x)=a*b*x^(b-1)*e^(-a*x^b):x>0;0:x\<0
μ=(1/a^(1/b))*Γ(1+1/b),
σ^2=(1/a^(2/b))*(Γ(1+2/b)-Γ(1+1/b)^2)

Weibull distribution

Fig. 21-15

Two-dimensional normal distribution

Two continuous random variables X and Y with a two-dimensional normal distribution (or bivariate normal distribution) have a joint probability distribution

f(x,y)=(1/(2*π*σ_x*σ_y*(1-ρ^2)^(1/2)))*e^((1/(2*(ρ^2-1)))*(((x-μ_x)/σ_x)^2+((y-μ_y)/σ_y)^2-2*ρ*((x-μ_x)/σ_x)*((y-μ_y)/σ_y)))

where μ_x, μ_y are the mean values, σ_x, σ_y are the standard deviations, ρ is the correlation coefficient, and $-\infty < x < \infty$, $-\infty < y < \infty$ (note that $-1 \le \rho \le 1$). Ext

21.4 Statistical Samples

Sample parameters

Statistics deals with *random samples* taken from a *population*. The size of the population is denoted by N (integer > 0 or $N = \infty$), its mean value by μ, and its variance by σ^2. In a sample of size n ($\leq N$ for finite N and sampling without replacement), the independent random variables $X_1, X_2, \ldots, X_n$ take on the values $x_1, x_2, \ldots, x_n$ and a function $\Phi(X_1, X_2, \ldots, X_n)$, usually referred as a *statistic*, takes on the value $\Phi(x_1, x_2, \ldots, x_n)$.

Depending on the sampling procedure (i.e. on how we find the values $x_1, x_2, \ldots, x_n$), a *correction factor* λ is defined: (a) If the sample is taken from an infinite population ($N = \infty$) or from a finite population with replacement, $\lambda = 1$. (b) If the sample is taken from a finite population without replacement, $\lambda = (N - n)/(N - 1)$.

The *mean of a sample* or *sample mean* is the function $\bar{X} = (X_1 + X_2 + \cdots + X_n)/n$. For a specific sample, it has the value $\bar{x} = (x_1 + x_2 + \cdots + x_n)/n$. Its expected or mean value is

$$\mu_{\bar{X}} = E(\bar{X}) = \mu$$

and its variance is $$\sigma_{\bar{X}}^2 = E[(\bar{X} - \mu)^2] = \lambda\sigma^2/n$$

The *variance of a sample* or *sample variance* is defined in the literature as

S^2=(1/n)*\S[k,1,n]{(X_k-Xbar)^2}   or
Shut^2=(1/(n-1))*\S[k,1,n]{(X_k-Xbar)^2}

Ext

(which one it is clear from the symbol S or $\hat{S}$). For a specific sample, it has the value

s^2=(1/n)*\S[k,1,n]{(x_k-xbar)^2}   or
shut^2=(1/(n-1))*\S[k,1,n]{(x_k-xbar)^2}

The expected or mean values of S^2 and $\hat{S}^2$ are

μ_(S^2)=E(S^2)=(n-λ)*σ^2/n   or
μ_(Shut^2)=E(Shut^2)=(n-λ)*σ^2/(n-1)

Properties

(1) For samples from a normal population (mean value μ and variance σ^2), the sample mean follows a normal distribution with mean value μ and variance σ^2/n. The random variable nS^2/σ^2 has a chi-square distribution with $n - 1$ degrees of freedom.

(2) For samples from any population (with mean μ and variance σ^2), the standardized random variable $Z = (\bar{X} - \mu)/\sigma_{\bar{X}}$ "follows asymptotically" the normal distribution with a mean value of 0 and a variance of 1 (i.e. lim[n,\o]{P(Z\<z)}= (1/(2*π)^(1/2))*\I[u,-\o,z]{e^(-u^2/2)}).

(3) For samples from a binomial population, with p being the probability of success in the first trial, the proportion P of successes in n trials is a statistic that follows the *sampling distribution of proportions* with mean value $\mu_P = p$ and variance $\sigma_P{}^2 = \lambda pq/n$. For large n ($n \geq 30$), the distribution becomes approximately normal. Ext

(4) Let the statistic Φ_1 be computed for samples drawn from population 1 and have a sampling distribution with a mean value $\mu_{\Phi 1}$ and a variance $\sigma_{\Phi 1}{}^2$. Independently, let statistic Φ_2 be computed for samples drawn from population 2 and have a sampling distribution with a mean value $\mu_{\Phi 2}$ and a variance $\sigma_{\Phi 2}{}^2$. Then, the sampling distribution of $\Phi_1 \pm \Phi_2$ has mean value $\mu_{\Phi 1} \pm \mu_{\Phi 2}$ and variance $\sigma_{\Phi 1}{}^2 + \sigma_{\Phi 2}{}^2$.

(5) For samples of size n, drawn from a normal population, let the sample variance be S^2 or $\hat{S}^2$. Then the statistic $T = \frac{\bar{X}-\mu}{S/(n-1)^{1/2}} = \frac{\bar{X}-\mu}{\hat{S}/n^{1/2}}$ follows Student's t distribution with $n-1$ degrees of freedom.

(6) Let $\hat{S}_1{}^2$ be the sample variance for samples of size n_1 from a normal population 1 with variance $\sigma_1{}^2$. Independently, let $\hat{S}_2{}^2$ be the sample variance for samples of size n_2 from a normal population 2 with variance $\sigma_2{}^2$. Then, the statistic $F = \frac{\hat{S}_1{}^2/\sigma_1{}^2}{\hat{S}_2{}^2/\sigma_2{}^2}$ follows the F distribution with $n_1 - 1$, and $n_2 - 1$ degrees of freedom.

Estimation from samples

A statistic, i.e. a random function $\Phi(X_1, X_2, \ldots, X_n)$ from a sample, can give an approximate value or estimation of a parameter of the population. We obtain:

(1) A *point estimate*, if the estimation is expressed by a single value. Exa

(2) An *unbiased estimate* of the corresponding parameter of the population, if the expected value of Φ is equal to the corresponding parameter of the population.

(3) A *more efficient* estimate from Φ_1 than from Φ_2, if both have the same expected value and the variance of Φ_1 is smaller than the variance of Φ_2.

(4) An *interval estimate*, if the estimation is expressed by an interval in which the parameter of the population lies. Ext

(5) An r% *confidence interval* $[l, u]$, if we are r% confident that the parameter of the population lies in this interval. The *confidence limits* l and u are determined by r and the distribution of Φ [they may or may not be symmetric about $E(\Phi)$].

Some confidence intervals are given in the next page. In the simple case of a normal population with a mean μ, a sample mean value $\bar{X}$ and a sample variance $\sigma_{\bar{X}}{}^2$, the two-sided r% confidence interval is $\bar{X} - z_{\alpha/2}\sigma_{\bar{X}} \leq \mu \leq \bar{X} + z_{\alpha/2}\sigma_{\bar{X}}$, with $z_{\alpha/2}$ as follows:

r	50	68.27	80	90	95	95.45	96	98	99	99.73	99.8	99.9
$z_{\alpha/2}$	0.6745	1.00	1.28	1.645	1.96	2.00	2.05	2.33	2.58	3.00	3.09	3.29

Confidence intervals for one population

Parameter	Statistic	r% confidence interval, $\alpha = 1 - r/100$
μ (normal distribution or large sample, σ^2 known)	$\bar{X}$	two-sided: $\bar{x} - z_{\alpha/2}\sigma/\sqrt{n} \le \mu \le \bar{x} + z_{\alpha/2}\sigma/\sqrt{n}$ Exa lower: $\mu \le \bar{x} + z_\alpha \sigma/\sqrt{n}$ upper: $\bar{x} - z_\alpha \sigma/\sqrt{n} \le \mu$ $\alpha = 1 - F(z_\alpha)$ for normal distribution with $\mu = 0, \sigma = 1$
μ (large sample, σ^2 unknown)	$\bar{X}$	Replace σ by $\hat{s}$ in the above (approximate)
μ (normal distribution, σ^2 unknown)	$\bar{X}$	two-sided: $\bar{x} - t_{\alpha/2}\hat{s}/\sqrt{n} \le \mu \le \bar{x} + t_{\alpha/2}\hat{s}/\sqrt{n}$ Exa lower: $\mu \le \bar{x} + t_\alpha \hat{s}/\sqrt{n}$ upper: $\bar{x} - t_\alpha \hat{s}/\sqrt{n} \le \mu$ $\alpha = 1 - F(t_\alpha)$ for t distribution with $n-1$ d.o.f.
σ^2 (normal distribution)	S^2 or $\hat{S}^2$	two-sided: $ns^2/x_{\chi,\alpha/2} \le \sigma^2 \le ns^2/x_{\chi,1-\alpha/2}$ Exa lower: $\sigma^2 \le ns^2/x_{\chi,1-\alpha}$, upper: $ns^2/x_{\chi,\alpha} \le \sigma^2$, $x_{\chi,\alpha} = \chi_\alpha^2$ $\alpha = 1 - F(x_{\chi,\alpha})$ for chi-square distribution, $n-1$ d.o.f.
p (binomial distribution)	P	two-sided: $P - z_{\alpha/2}\sigma_P \le p \le P + z_{\alpha/2}\sigma_P$, σ_P=(P*Q/n)^(1/2) lower: $p \le P + z_\alpha \sigma_P$, upper: $P - z_\alpha \sigma_P \le p$ Exa $\alpha = 1 - F(z_\alpha)$ for normal distribution with $\mu = 0, \sigma = 1$

Confidence intervals for two populations

Parameter	Statistic	r% confidence interval, $\alpha = 1 - r/100$
$\mu_1 - \mu_2$ (normal distributions or large samples, σ_1^2, σ_2^2 known)	$\bar{X}_1 - \bar{X}_2$	two-sided: $\bar{x}_1 - \bar{x}_2 - z_{\alpha/2}\sigma_{1\text{-}2} \le \mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + z_{\alpha/2}\sigma_{1\text{-}2}$ lower: $\mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + z_\alpha \sigma_{1\text{-}2}$ Ext upper: $\bar{x}_1 - \bar{x}_2 - z_\alpha \sigma_{1\text{-}2} \le \mu_1 - \mu_2$ $\sigma_{1\text{-}2} = (\sigma_1^2/n_1 + \sigma_2^2/n_2)^{1/2}$
$\mu_1 - \mu_2$ (large samples, σ_1^2, σ_2^2 unknown)	$\bar{X}_1 - \bar{X}_2$	Replace σ by $\hat{s}$ in the above (approximate)
$\mu_1 - \mu_2$ (normal distributions, σ_1^2, σ_2^2 unknown)	$\bar{X}_1 - \bar{X}_2$	two-sided: $\bar{x}_1 - \bar{x}_2 - t_{\alpha/2}\sigma_{1\text{-}2} \le \mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + t_{\alpha/2}\sigma_{1\text{-}2}$ $\sigma_{1\text{-}2} = (\hat{s}_1^2/n_1 + \hat{s}_2^2/n_2)^{1/2}$ Ext $\alpha = 1 - F(t_\alpha)$ for t distribution with n d.o.f.
σ_1^2/σ_2^2 (normal distributions)	$\hat{S}_1^2/\hat{S}_2^2$	two-sided: $x_{F,\alpha/2}\hat{s}_1^2/\hat{s}_2^2 \le \sigma_1^2/\sigma_2^2 \le x_{F,1-\alpha/2}\hat{s}_1^2/\hat{s}_2^2$ Ext
$p_1 - p_2$ (binomial distributions))	$P_1 - P_2$	two-sided: $P_1 - P_2 - z_{\alpha/2}\sigma_{1\text{-}2} \le p_1 - p_2 \le P_1 - P_2 + z_{\alpha/2}\sigma_{1\text{-}2}$ Ext

21.5 Hypotheses and Tests

In statistics, we often make *hypotheses* (assumptions) about populations. Testing a hypothesis by using a sample from the population involves the risk of making two types of errors: (1) A *type I error* is made if we reject a true hypothesis. (2) A *type II error* is made if we fail to reject a false hypothesis.

H_0 is the *null hypothesis*, i.e. the assumption that we want to test, while H_1 is an alternative hypothesis, i.e. an assumption reasonable but incompatible with H_0. A *test* is a method that enables us to decide whether there is substantial reason to reject H_0 or not. A test may be *two-sided* or *one-sided*. **Ext**

Level of significance α of a test is the maximum probability for making a type I error. Usually this is chosen to be small ($\alpha = 0.05 = 5\%$ or $\alpha = 0.01 = 1\%$) in order to be confident with probability $1 - \alpha$ that we make the correct decision. **Exa**

Hypothesis testing goes as follows: (a) We assume H_0, define H_1, and choose α. (b) We find the $100(1 - \alpha)\%$ *region of acceptance* for a proper statistic, i.e. a statistic $\Phi(X_1, \ldots, X_n)$ that is an estimator of the population's parameter under testing. (c) From a single sample, we find the value of Φ. (d) If this value lies in the region of acceptance, we do not reject H_0 at significance level α. If this value lies in the *region of rejection* or *critical region* (i.e. outside the region of acceptance), we reject H_0.

Given the value Φ_1 of Φ from a single sample, the *P-value* is the probability that a value of Φ can be found beyond Φ_1 in the direction or directions of H_1. Given Φ_1 and using the appropriate test the P-value is the lowest α at which H_0 is rejected. **Ext**

The probability of making a type II error is denoted by β. To calculate β we need an alternative hypothesis H_1 with a specific value. The probability of rejecting H_0 when H_1 is true is called the *power* of the test and equals $1 - \beta$. **Exa**

Some frequently used cases of hypothesis testing are given in the tables of the next page. The symbols are explained in the following notes:

❶ z_c is the critical value for normal distribution ($\mu = 0, \sigma = 1$) with area c above z_c.

❷ t_c is the critical value for Student's t distribution with $n - 1$ d.o.f.

❸ $x_{\chi,c} = \chi_c^2$ is the critical value for chi-square distribution with $n - 1$ d.o.f.

❹ $x_{F,c}$ is the critical value for chi-square distribution with $n_1 - 1$, $n_2 - 1$ d.o.f.

❺ P is the sample value of p, $q_0 = 1 - p_0$, and c equal to α, $\alpha/2$, $1 - \alpha$ or $1 - \alpha/2$.

❻ For $\bar{X}_1 - \bar{X}_2$ we have $(\sigma_{1\text{-}2})^2 = \sigma_1^2/n_1 + \sigma_2^2/n_2$.

❼ For $P_1 - P_2$ we have $(\sigma_{1\text{-}2})^2 = PQ(n_1 + n_2)/(n_1 n_2)$, $P = 1 - Q = (n_1 P_1 + n_2 P_2)/(n_1 + n_2)$.

Hypotheses testing for one population

Statistic	Hypotheses		Region of acceptance
$\overline{X}$ (normal distribution, σ^2 known)	H_0: $\mu = \mu_0$	H_1: $\mu \neq \mu_0$	$\mu_0 - z_{\alpha/2}\sigma/\sqrt{n} \le \bar{x} \le \mu_0 + z_{\alpha/2}\sigma/\sqrt{n}$
		H_1: $\mu < \mu_0$	$\mu_0 - z_{\alpha}\sigma/\sqrt{n} \le \bar{x}$ ❶
		H_1: $\mu > \mu_0$	$\bar{x} \le \mu_0 + z_{\alpha}\sigma/\sqrt{n}$
Same as above but with σ^2 unknown			Replace $z_{\alpha/2}\sigma$ by $t_{\alpha/2}\hat{s}$ and $z_\alpha\sigma$ by $t_\alpha\hat{s}$ ❷
P (probability of success, binomial distribution, n large)	H_0: $p = p_0$	H_1: $p \neq p_0$	$p_0 - z_{\alpha/2}$ (p_0*q_0/n)^(1/2) $\le P \le p_0 + z_{\alpha/2}$ (p_0*q_0/n)^(1/2)
		H_1: $p < p_0$	$p_0 - z_{\alpha/2}$ (p_0*q_0/n)^(1/2) $\le P$ ❺
		H_1: $p > p_0$	$P \le p_0 + z_{\alpha/2}$ (p_0*q_0/n)^(1/2)
$\hat{s}^2$ (variance of normal population)	H_0: $\sigma^2 = \sigma_0^2$	H_1: $\sigma^2 \neq \sigma_0^2$	$x_{\chi,1-\alpha/2}\sigma_0^2/(n-1) \le \hat{s}^2 \le x_{\chi,\alpha/2}\sigma_0^2/(n-1)$
		H_1: $\sigma^2 < \sigma_0^2$	$x_{\chi,1-\alpha}\sigma_0^2/(n-1) \le \hat{s}^2$ ❸
		H_1: $\sigma^2 > \sigma_0^2$	$\hat{s}^2 \le x_{\chi,\alpha}\sigma_0^2/(n-1)$

Hypotheses testing for two populations

Statistic	Hypotheses		Region of acceptance
$\overline{X}_1 - \overline{X}_2$ (normal populations, σ_1^2, σ_2^2 known)	H_0: $\mu_1 = \mu_2$	H_1: $\mu_1 \neq \mu_2$	$-z_{\alpha/2}\sigma_{1\text{-}2} \le \bar{x}_1 - \bar{x}_2 \le z_{\alpha/2}\sigma_{1\text{-}2}$
		H_1: $\mu_1 < \mu_2$	$-z_{\alpha}\sigma_{1\text{-}2} \le \bar{x}_1 - \bar{x}_2$ ❻
		H_1: $\mu_1 > \mu_2$	$\bar{x}_1 - \bar{x}_2 \le z_{\alpha}\sigma_{1\text{-}2}$
$\overline{X}_1 - \overline{X}_2$ (same as above, same hypotheses but with σ_1^2, σ_2^2 unknown, n_1, n_2 large)			Same as above with σ_1^2, σ_2^2 replaced by $\hat{s}_1^2$, $\hat{s}_2^2$
$P_1 - P_2$ (probabilities of successes, binomial distributions, n_1, n_2 large)	H_0: $p_1 = p_2$	H_1: $p_1 \neq p_2$	$-z_{\alpha/2}\sigma_{1\text{-}2} \le P_1 - P_2 \le z_{\alpha/2}\sigma_{1\text{-}2}$
		H_1: $p_1 < p_2$	$-z_{\alpha}\sigma_{1\text{-}2} \le P_1 - P_2$ ❼
		H_1: $p_1 > p_2$	$P_1 - P_2 \le z_{\alpha}\sigma_{1\text{-}2}$
$\hat{s}_1^2/\hat{s}_2^2$ (ratio of variances of normal populations)	H_0: $\sigma_1^2 = \sigma_2^2$	H_1: $\sigma_1^2 \neq \sigma_2^2$	$x_{F,1-\alpha/2} \le \hat{s}_1^2/\hat{s}_2^2 \le x_{F,\alpha/2}$
		H_1: $\sigma_1^2 < \sigma_2^2$	$x_{F,1-\alpha} \le \hat{s}_1^2/\hat{s}_2^2$ ❹
		H_1: $\sigma_1^2 > \sigma_2^2$	$\hat{s}_1^2/\hat{s}_2^2 \le x_{F,\alpha}$

21.6 Simple Linear Regression and Correlation

The

Let a linear equation $Y_i = \beta_0 + \beta_1 x_i + E_i$ relate the random variable Y_i with the mathematical variable x_i and the random variable E_i, which has a mean equal to zero.

For a given sample of pairs (x_i, y_i), where y_i is the value of Y_i found for x_i, $i = 1, 2, \dots, n$, the values of β_0, β_1 that minimize the sum $\Sigma_i (y_i - \beta_0 - \beta_1 x_i)^2$ are

β_1=b_1=(xybar-xbar*ybar)/((x^2)bar-(xbar)^2),
β_0=b_0=ybar-b_1*xbar

where (Σ_i stands for summation from $i = 1$ to $i = n$)

$\bar{x} = n^{-1}\Sigma_i x_i$, $\bar{y} = n^{-1}\Sigma_i y_i$, (xy)bar $= n^{-1}\Sigma_i x_i y_i$, (x^2)bar $= n^{-1}\Sigma_i x_i^2$

The sample variances s_x^2 and s_y^2, covariance s_{xy}, and correlation coefficient r are

$$s_x^2 = n^{-1}\Sigma_i (x_i - \bar{x})^2, \quad s_y^2 = n^{-1}\Sigma_i (y_i - \bar{y})^2, \quad s_{xy} = n^{-1}\Sigma_i (x_i - \bar{x})(y_i - \bar{y}), \quad r = s_{xy}/(s_x s_y)$$

The (least-squares) *regression line of y on x* is $y = b_0 + b_1 x$ and gives an estimation of y from x, based on the assumption that there is an approximately linear relationship between x and y. The regression line of y on x can also be written as

y-ybar=(s_(xy)/s_x^2)*(x-xbar) or (y-ybar)/s_y=r*(x-xbar)/s_x,
since b_1=s_(xy)/s_x^2

Ext

Let $SSE = \Sigma_i (y_i - \hat{y}_i)^2$ = *error sum of squares* or *unexplained variation*,

$SSR = \Sigma_i (\hat{y}_i - \bar{y})^2$ = *regression sum of squares* or *explained variation*,

$SST = \Sigma_i (y_i - \bar{y})^2 = n s_y^2$ = *total corrected sum of squares* or *total variation*,

where $\hat{y}_i = b_0 + b_1 x_i$. It follows that $SST = SSE + SSR$ and $r^2 = SSR/SST$. The *standard error of estimate of y on x* is defined as (SSE/n)^(1/2) and expresses how much the points (x_i, y_i) are spread about the (least-squares) regression line $y = b_0 + b_1 x$.

Ext

If E_i has normal distribution with a mean value equal to zero and a variance σ^2, we can calculate the $100(1 - \alpha)\%$ confidence intervals for β_1, β_0, etc. as follows, where $t_{\alpha/2}$ is from the Student's t distribution with $n - 2$ d.o.f. and κ=(1/s_x)*(SSE/(n*(n-2)))^(1/2):

For β_1: b_1-κ*t_(a/2)<β_1<b_1+κ*t_(a/2)
For β_0: b_0-κ*t_(a/2)*((x^2)bar)^(1/2)<β_0<b_0+κ*t_(a/2)*((x^2)bar)^(1/2)
For μ_(Y|x_0): yhut_0-κ*t_(a/2)*(s_x^2+(x_0-xbar)^2)^(1/2)<μ_(Y|x_0)<yhut_0+κ*t_(a/2)*(s_x^2+(x_0-xbar)^2)^(1/2)

For a new value y_0 of y at $x = x_0$, the $100(1 - \alpha)\%$ confidence interval is

yhut_0-κ*t_(a/2)*((n+1)*s_x^2+(x_0-xbar)^2)^(1/2)<y_0<yhut_0+κ*t_(a/2)*((n+1)*s_x^2+(x_0-xbar)^2)^(1/2)

21.7 Analysis of Variance

From m populations with mean values μ_i, $i = 1, 2, \ldots, m$, and the same variance σ^2, we obtain m samples. The ith sample has values x_{ij} for its n_i random variables X_{ij}, $j = 1, 2, \ldots, n_i$. *Analysis of variance* (ANOVA) gives information about the mean values of the populations using variations within samples and between samples. We define:

(Mean of the ith sample or treatment) =xbar_i=(1/n_i)*\S[j,1,n_i]{x_(ij)}

(Overall mean)=xbar= (1/n)*\S[i,1,m]{\S[j,1,n_i]{x_ij)}}

(Variation within samples)=v_w= \S[i,1,m]{\S[j,1,n_i]{(x_(ij)-xbar_i)^2}}

(Variation between samples)=v_b= \S[i,1,m]{n_i*(xbar_i-xbar)^2}

Total variation=v=\S[i,1,m]{\S[j,1,n_i]{(x_(ij)-xbar)^2}} with $\upsilon = \upsilon_w + \upsilon_b$ The

In the literature, we find the notation $\upsilon_w = SSW$ = sums of squares within samples, $\upsilon_b = SSB$ = sum of squares between samples, and $\upsilon = SST$ = total sum of squares.

In a *linear* mathematical model for analysis of variance, we have the random variables $X_{ij} = \mu_i + E_{ij}$ that take on the values x_{ij}, where E_{ij} are independent normally distributed random variables. Setting $\mu = (\mu_1 + \mu_2 + \cdots + \mu_m)/m$ and $\alpha_i = \mu_i - \mu$, we have $X_{ij} = \mu + \alpha_i + E_{ij}$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_m = 0$. The null hypothesis for *one population* is H_0: $\mu_i = \mu$, $i = 1, 2, \ldots, m$ (true H_0 means that all x_{ij} are from one population).

The random variables V_w, V_b, V, that take on the values υ_w, υ_b, υ for the specific samples have mean values

E(V_w)=(n-m)*σ^2, E(V_b)=(m-1)*σ^2+\S[i,1,m]{n_i*α_i^2}, E(V)=(n-1)*σ^2+\S[i,1,m]{n_i*α_i^2}

Thus, we have the following:

(a) $\hat{S}_w^2 = V_w/(n-m)$ is an unbiased estimate of σ^2 always (for true or false H_0).

(b) $\hat{S}_b^2 = V_b/(m-1)$ and $\hat{S}^2 = V/(n-1)$ are unbiased estimates of σ^2, only if H_0 is true, i.e. when $\alpha_i = 0$.

The following three theorems can be proved:

(1) V_w/σ^2 is a chi-square distributed random variable with $n - m$ d.o.f.

(2) If H_0 is true, then V_b/σ^2 and V/σ^2 are chi-square distributed random variables with $m - 1$ and $n - 1$ d.o.f., respectively.

(3) The statistic $F = \hat{S}_b^2/\hat{S}_w^2$ has the F distribution with $m - 1$, $n - m$ d.o.f. (this enables us to check H_0 with a one-sided hypothesis test using the F distribution).

21.8 Nonparametric Statistics

If we assume only that a population distribution is continuous (without any assumption about its specific form), we can use *nonparametric statistics*, i.e. statistical methods that are not based on the population distribution. Such methods are useful especially for qualitative data (e.g. yes or no, order of preference, etc.).

Sign test

We want to test the median m of a continuous population (half of the population lies below m and half above). The null and the alternative hypotheses are H_0: $m = m_0$ and H_1: $m \neq m_0$. In a sample, the number of those less than m_0 or minuses is n_1 and the number of those larger than m_0 or pluses is n_2. At a significance level α, we use the binomial distribution with $p = 0.5$ (or its normal approximation) to decide whether to reject H_0 or not. The method can also be used for one-sided tests. Exa

In addition, the **Wilcoxon signed-rank test** uses the rank of the data. Ext

Mann-Witney-Wilcoxon test

To test whether or not two independent samples of sizes n_1, n_2 ($n_1 \leq n_2$), come from the same population, we do the following: (1) We rank all values of both samples. If two values are equal, we assign to each the average of their ranks. (2) We find the sums w_1 and w_2 of the ranks of each sample. (3) In a two-sided test, H_0: $\mu_1 = \mu_2$ is rejected if either w_1 or w_2 is less than or equal to w_α (obtained from table). (4) If n_1 and n_2 are at least 10, then the statistic $U = n_1 n_2 + \frac{1}{2} n_1(n_1 + 1) - w_1$ has an approximately normal distribution with a mean $\mu_U = \frac{1}{2} n_1 n_2$ and a variance $\sigma_U{}^2 = \frac{1}{12} n_1 n_2 (n_1 + n_2 + 1)$. Thus, the deviation of $z = (U - \mu_U)/\sigma_U$ from zero can be tested. Exa

A generalization to more samples is the **Kruskal-Wallis test**. Ext

Runs test

We consider a sequence made by repeated use of two symbols, e.g. a and b. Each maximum subset or subgroup of similar consecutive symbols is a *run*. Thus, in the sequence $\{a, a, b, b, b, a, b, b, b, b, a, a, a, b, b, a, a, a, a\}$ the first run is $\{a, a\}$, the second run is $\{b, b, b\}$, the third run is $\{a\}$, the fourth run is $\{b, b, b, b\}$, etc. The number of runs R is a statistic whose sampling distribution has mean and variance

$$\mu_R = 2 n_1 n_2 / n + 1, \quad \sigma_R^2 = 2 n_1 n_2 (2 n_1 n_2 - n)/(n^2 (n-1))$$

where n_1 is the total number of a's, n_2 is the total number of b's and $n = n_1 + n_2$. If both n_1 and n_2 are at least 8, the distribution of R is nearly normal and $z = (R - \mu_R)/\sigma_R$ can be considered a standardized normal variable. Exa

22 INEQUALITIES

22.1 Inequalities with Constants

All the constants and the functions used below are assumed to be real.

Triangle inequality

$$|a_1| - |a_2| \le |a_1 + a_2| \le |a_1| + |a_2|$$

$$|a_1 + a_2 + \cdots + a_n| \le |a_1| + |a_2| + \cdots + |a_n|$$

Inequalities for arithmetic, geometric and harmonic means

For positive numbers $a_1, a_2, \ldots, a_n$, we define the arithmetic mean A, the geometric mean G, and the harmonic mean H, respectively, by

$$A=(a_1+a_2+...+a_n)/n,\ G=(a_1*a_2*...*a_n)^{(1/n)},$$
$$1/H=(1/n)*(1/a_1+1/a_2+...+1/a_n)$$

Then, $H \le G \le A$. The equality holds only if $a_1 = a_2 = \cdots = a_n$. Ext

Cauchy-Schwarz inequality

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \le (a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2)(b_1{}^2 + b_2{}^2 + \cdots + b_n{}^2)$$ Pro

The equality holds only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

Hölder's inequality

If $a_i > 0$, $b_i > 0$, $i = 1, 2, \ldots, n$, then for any $p > 1$ and $p^{-1} + q^{-1} = 1$ (which implies $q > 1$), we have

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \le (a_1{}^p + a_2{}^p + \cdots + a_n{}^p)^{1/p}(b_1{}^q + b_2{}^q + \cdots + b_n{}^q)^{1/q}$$

The equality holds only if $a_1^{(p-1)}/b_1=a_2^{(p-1)}/b_2=...=a_n^{(p-1)}/b_n$.

For $p = q = 2$, we obtain the Cauchy-Schwarz inequality.

Minkowski's inequality

For positive $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ and $p > 1$ we have

$$\{(a_1 + b_1)^p + (a_2 + b_2)^p + \cdots + (a_n + b_n)^p\}^{1/p}$$
$$\le (a_1^p + a_2^p + \cdots + a_n^p)^{1/p} + (b_1^p + b_2^p + \cdots + b_n^p)^{1/p}$$

The equality holds only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

Chebyshev's inequality

If $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$, then

$$(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) \leq n(a_1b_1 + a_2b_2 + \cdots + a_nb_n)$$

Kantorovich inequality

Let $0 < x_1 < x_2 < \cdots < x_n$ and $\lambda_i \geq 0$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$. If $A = (x_1 + x_2)/2$ and $G = (x_1x_2)^{1/2}$, then

$$(\lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_nx_n)(\lambda_1/x_1 + \lambda_2/x_2 + \cdots + \lambda_n/x_n) = A^2/G^2$$

21.2 Inequalities with Integrals

If $f(x) \leq g(x)$ on an interval $a \leq x \leq b$, then

\I[x,a,b]{f(x)}\<\I[x,a,b]{g(x)}

If m is the maximum value of $|f(x)|$ on $a \leq x \leq b$, then

|\I[x,a,b]{f(x)}|\<\I[x,a,b]{|f(x)|}\<m*(b-a)

Cauchy-Schwarz inequality (or Buniakowsky-Schwarz inequality) Pro

(\I[x,a,b]{f(x)*g(x)})^2\<\I[x,a,b]{f(x)^2}*\I[x,a,b]{g(x)^2}

The equality holds only if $f(x) = c\,g(x)$, where c is a constant.

Hölder's inequality

For any $p > 1$ and $p^{-1} + q^{-1} = 1$ (which implies $q > 1$), we have

\I[x,a,b]{|f(x)*g(x)|}\<(\I[x,a,b]{|f(x)|^p})^(1/p)*(\I[x,a,b]{|g(x)|^q})^(1/q)

The equality holds only if $|f(x)|^{p-1} = |c\,g(x)|$, where c is a constant.

If $p = q = 2$, we obtain the Cauchy-Schwarz inequality for integrals.

Minkowski's inequality

If $p \geq 1$, then

(\I[x,a,b]{|f(x)+g(x)|^p})^(1/p)\<(\I[x,a,b]{|f(x)|^p})^(1/p)+(\I[x,a,b]{|g(x)|^p})^(1/p)

The equality holds only if $f(x) = c\,g(x)$, where c is a constant.

23 UNITS AND CONVERSIONS

23.1 SI Units

To measure and express physical quantities we need seven *base units*, two *supplementary units*, and many other *derived units*. In the International System of Units (Systeme International d'Unites) the following tables give these units, their names and symbols, and their interrelationships. In practice and everyday life, some additional, more convenient, units are used. **Ext**

SI: Base units

Quantity	Symbol of quantity	SI unit	Symbol of unit
Length	l	meter	m
Mass	m	kilogram	kg
Time	t	second	s
Electric current	I	ampere	A
Thermodynamic temperature	T	kelvin	K
Amount of substance	n	mole	mol
Luminous intensity	I_v	candela	cd

SI: Supplementary units

Quantity	Symbol of quantity	SI unit	Symbol of unit
Plane angle	-	radian	rad
Solid angle	-	steradian	sr

SI: Derived units with special names

Quantity	SI unit	Symbol	In terms of base units
Absorbed dose (of radiation)	gray	Gy	J/kg = m^2 s^{-2}
Activity (of a radionuclide)	becquerel	Bq	s^{-1}
Capacitance (electric)	farad	F	C/V = m^{-2} kg^{-1} s^4 A^2
Catalytic activity	katal	kat	s^{-1} mol

SI: Derived units with special names (continued)

Quantity	SI unit	Symbol	In terms of base units
Conductance (electric)	siemens	S	A/V = m^{-2} kg^{-1} s^3 A^2
Dose equivalent	sievert	Sv	J/kg = m^2 s^{-2}
	rem	rem	10^{-2} Sv
Electric charge	coulomb	C	s A
Electric potential	volt	V	W/A = m^2 kg s^{-3} A^{-1}
Electric resistance	ohm	Ω	V/A = m^2 kg s^{-3} A^{-2}
Energy	joule	J	N m = m^2 kg s^{-2}
Force	newton	N	m kg s^{-2}
Frequency	hertz	Hz	s^{-1}
Illuminance	lux	lx	lm/m^2 = m^{-2} cd sr
Inductance	henry	H	Wb/A = m^2 kg s^{-2} A^{-2}
Luminous flux	lumen	lm	cd sr
Magnetic flux	weber	Wb	V s = m^2 kg s^{-2} A^{-1}
Magnetic flux density	tesla	T	Wb/m^2 = kg $s^{-2}A^{-1}$
Power	watt	W	J/s = m^2 kg s^{-3}
Pressure	pascal	Pa	N/m^2 = m^{-1} kg s^{-2}

Other units defined from SI units

Quantity	Name of unit	Symbol	In terms of SI units
Time	minute	min	60 s
	hour	h	60 min = 3600 s
	day	d	24 h = 86400 s
Angle	degree	°	(π/180) rad
	minute	′	(π/10800) rad
	second	″	(π/648000) rad
Volume	liter	L	dm^3 = 10^{-3} m^3
Mass	tonne or metric ton	t	10^3 kg = Mg
Celsius temperature	degree Celsius	°C	K
Amount-of-substance concentration		M	10^3 mol m^{-3}

Other units with special names (some are abandoned slowly)

Quantity	Name of unit	Symbol	In SI units
Absorbed dose	rad	rad	10^{-2} Gy = 10^{-2} J kg^{-1}
Acceleration	gal	Gal	1 cm s^{-2} = 10^{-2} m s^{-2}
Activity	curie	Ci	3.7×10^{10} s^{-1}
Area	are	a	10^2 m^2
	hectare	ha	10^4 m^2
	barn	b	10^{-28} m^2
Dynamic viscosity	poise	P	10^{-1} Pa s
Energy	erg	erg	10^{-7} J
	thermochemical calorie	cal_{th}	4.184 J
	I.T. calorie	cal_{IT}	4.1868 J
Exposure	röntgen	R	2.58×10^{-4} C kg^{-1}
Force	dyne	dyn	10^{-5} N
	kilogram-force	kgf	9.80665 N
Kinematic viscosity	stokes	St	10^{-4} m^2 s^{-1}
Illuminance	phot	ph	10^4 lx
Length	ångström	Å	10^{-10} m
	micron	µm	10^{-6} m
Luminance	stilb	sb	10^4 cd m^{-2}
Magnetic flux density	gamma	γ	10^{-9} T
Pressure	atmosphere	atm	101325 Pa
	torr	Torr	(101325/760) Pa
	one mm of Hg	mmHg	13.5951×9.80665 Pa
	bar	bar	10^5 Pa
Temperature	degree Fahrenheit	°F	(5/9) K
	degree Rankine	°R	(5/9) K

Prefixes for SI units

Prefix	deka	hecto	kilo	mega	giga	tera	peta	exa	zetta	yotta
Symbol	da	h	k	M	G	T	P	E	Z	Y
Multiple	10	10^2	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}	10^{21}	10^{24}
Prefix	deci	centi	milli	micro	nano	pico	femto	atto	zepto	yocto
Symbol	d	c	m	µ	n	p	f	a	z	y
Multiple	10^{-1}	10^{-2}	10^{-3}	10^{-6}	10^{-9}	10^{-12}	10^{-15}	10^{-18}	10^{-21}	10^{-24}

23.2 Other Units and Conversions

Ext

Other units, customary or adopted to particular situations, are often used. A brief list is given below along with their relations to SI units.

Name of unit	In terms of SI units
astronomical unit	1 au $\simeq 1.49598 \times 10^{11}$ m
parsec	1 pc $\simeq 30857 \times 10^{11}$ m
unified atomic mass unit	1 u $\simeq 1.66054 \times 10^{-27}$ kg
electronvolt	1 eV $\simeq 1.60218 \times 10^{-19}$ J
kilowatt-hour	1 kWh = 3.6 MJ
nautical mile	1 nautical mile = 1852 m
knot (nautical mile per hour)	1 knot = 1852/3600 m s^{-1}
neper	1 Np = 1 (dimensionless)
bel	1 B = 0.5 ln 10 Np

United States customary units or English units

Unit with symbol and equivalent in SI units	
1 inch (in) = 25.40000 mm	1 fluid ounce (fl oz) = 29.57353 mL
1 foot (ft) = 12 in = 0.3048000 m	1 (liquid) pint (pt) = 473.1765 mL
1 yard (yd) = 3 ft = 0.9144000 m	1 (liquid) quart (qt) = 946.3529 mL
1 mile (mi) = 5280 ft = 1.609344 km	1 US gallon (gal) = 3.785412 L
1 square inch (sq in) = 6.451600 cm^2	1 oil barrel (bbl) = 117.3478 L
1 square foot (sq ft) = 0.092903 m^2	1 ounce = 28.34952 g
1 square yard (sq yd) = 0.836127 m^2	1 pound = 12 ounces = 0.4535924 kg
1 square mile (sq mi) = 2.589988 km^2	1 BTU = 1054 - 1060 J
1 acre = 4046.873 m^2	1 hp = 745.7 W
1 cubic inch (cu in) = 16.387024 mL	1 US ton = 907.18 kg
1 cubic foot = 0.02831685 m^3	1 lbf/in^2 (psi) = 6.89476 kPa

INDEX