SPECTRAL PROPERTIES OF LIMIT-PERIODIC SCHRÖDINGER OPERATORS

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Abstract. We investigate the spectral properties of Schrödinger operators in $\ell 2(\mathbb{Z})$ with limit-periodic potentials. The perspective we take was recently proposed by Avila and is based on regarding such potentials as generated by continuous sampling along the orbits of a minimal translation of a Cantor group. This point of view allows one to separate the base dynamics and the sampling function. We show that for any such base dynamics, the spectrum is of positive Lebesgue measure and purely absolutely continuous for a dense set of sampling functions, and it is of zero Lebesgue measure and purely singular continuous for a dense G_δ set of sampling functions.

1. Introduction

Schrödinger operators H_ω acting in $\ell 2(\mathbb{Z})$ with dynamically defined potentials V_ω are given by

$$[H_\omega \psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n), \tag{1}$$

where

$$V_\omega(n) = f(T^n \omega), \quad \omega \in \Omega,\ n \in \mathbb{Z} \tag{2}$$

with a homeomorphism T of a compact space Ω and a continuous sampling function $f : \Omega \to \mathbb{R}$. It is often beneficial to study the operators $\{H_\omega\}_{\omega\in\Omega}$ as a family, as opposed to a collection of individual operators. This is especially true if a T-ergodic probability measure μ is chosen since the spectrum and the spectral type of H_ω are always μ-almost surely independent of ω due to ergodicity. Alternatively, in the topological setting, if T is minimal and f is continuous, then the spectrum and the absolutely continuous spectrum are independent of ω; however, the point spectrum and the singular continuous spectrum are in general not independent of ω.

In this paper we will study the class of limit-periodic potentials. As pointed out recently by Avila [1], these potentials are defined by continuous sampling along the orbits of a minimal translation of a Cantor group. We give the definitions of these terms now and defer the discussion of their relation to limit-periodic potentials to Section 2.

Indeed, promoting Avila's approach to limit-periodic potentials is part of our motivation for writing this paper because we believe that it has the potential to produce further insight into limit-periodic Schrödinger operators, in addition to the beautiful results obtained in [1]. Some developments in this direction can be found in [5].

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Definition 1.1. *We say that Ω is a Cantor group if it is a totally disconnected compact Abelian topological group with no isolated points. A map $T : \Omega \to \Omega$ is called a translation if it is of the form $T\omega = \omega + \alpha$ for some $\alpha \in \Omega$. A translation is called minimal if the orbit $\{T^n\omega : n \in \mathbb{Z}\}$ of every $\omega \in \Omega$ is dense in Ω.*

Given a minimal translation T of a Cantor group Ω and a continuous sampling function f, we consider the potentials generated by (2) and the associated operators given by (1). A major advantage of Avila's approach to limit-periodic potentials is that one can keep the base dynamics $T : \Omega \to \Omega$ fixed and vary the sampling function. This enables one to exhibit spectral properties that occur frequently, that is, for a dense set of sampling functions, or even a dense G_δ set of sampling functions. The following theorem describes such spectral properties that occur frequently; it makes statements about the Lebesgue measure of the spectrum, the structure of the gaps of the spectrum, and the type of the spectral measures.

Theorem 1.1. *Suppose Ω is a Cantor group and $T : \Omega \to \Omega$ is a minimal translation.*
(a) *For a dense set of $f \in C(\Omega, \mathbb{R})$ and every $\omega \in \Omega$, the spectrum of H_ω is a Cantor set of positive Lebesgue measure and H_ω has purely absolutely continuous spectrum.*
(b) *For a dense G_δ-set of $f \in C(\Omega, \mathbb{R})$ and every $\omega \in \Omega$, the spectrum of H_ω is a Cantor set of zero Lebesgue measure and H_ω has purely singular continuous spectrum.*

Remarks. (i) A Cantor set is by definition a closed set with empty interior and no isolated points. By general principles, the only property that needs to be addressed in the proof is the empty interior.

(ii) The statement in (b) can be strengthened as follows. If instead of f, one considers the one-parameter family $\{\lambda f\}_{\lambda>0}$, then the same conclusion holds for the family uniformly in λ. This is inferred easily from the proof.

(iii) The theorem above lists both known and new results for the sake of completeness. Let us explain which aspects are new. Part (a) is a version of results established in somewhat different form by various authors for continuum Schrödinger operators in the 1980's; compare [2, 3, 10]. It is intermediate between the existing denseness statements (again in the continuum setting) for all limit-periodic potentials with variation in Ω allowed and for specific families of limit-periodic potentials, where Ω is often fixed but the class of considered f's is restricted. Our proof is close in spirit to the work of Avron and Simon [2]. Most claims in part (b) are due to Avila [1]. What we add here is the generic absence of eigenvalues.

(iv) It would clearly be of interest to determine how often the third basic spectral type, pure point spectrum, occurs for limit-periodic potentials. Some examples were constructed by Pöschel [11]. Moreover, further results in this direction are stated in [9], and again in [4]. We plan to explore limit-periodic Schrödinger operators with pure point spectrum (both with and without exponentially decaying eigenfunctions) in a future work.

(v) In the families $\{H_\omega\}_{\omega\in\Omega}$ covered by Theorem 1.1, the spectral type of H_ω is independent of ω. It is known that such uniformity may fail in the more general class of almost periodic Schrödinger operators. For example, if one takes $\Omega = \mathbb{R}/\mathbb{Z}$,

$T\omega \mapsto \omega + \alpha$ with a Diophantine number α, and $f(\omega) = 3\cos(2\pi\omega)$, then the associated operator H_ω has pure point spectrum for Lebesgue almost every ω and purely singular continuous spectrum for a dense G_δ set of ω's. Thus, another interesting open problem is the following: can a limit-periodic family $\{H_\omega\}_{\omega\in\Omega}$ ever exhibit such a phenomenon or, on the contrary, is the spectral type of limit-periodic Schrödinger operators always independent of ω?

(vi) All families $\{H_\omega\}_{\omega\in\Omega}$ covered by Theorem 1.1 have Cantor spectrum and one may again ask whether this is always the case or not. In fact, it is known that this is not always the case. Again in Pöschel's paper [11], one may find examples of limit-periodic Schrödinger operators whose spectra have no gaps.

(vii) The present paper and [5] treat limit-periodic Schrödinger operators in two complementary regimes. The operators discussed here have zero Lyapunov exponents, while the operators discussed in [5] have positive Lyapunov exponents. The main result of [5] shows that for every Cantor group Ω and every minimal translation T on Ω, there exists a dense set $\mathcal{F} \subset C(\Omega, \mathbb{R})$ such that for every $f \in \mathcal{F}$ and every $\lambda \in \mathbb{R} \setminus \{0\}$, the following statements hold true for the operators H_ω with potentials $V_\omega(n) = \lambda f(T^n\omega)$: the spectrum of H_ω has zero Hausdorff dimension and all spectral measures are purely singular continuous (for every $\omega \in \Omega$), and the Lyapunov exponent is a positive continuous function of the energy.

2. Minimal Translations of Cantor Groups and the Description of Limit-Periodic Potentials

In this section we recall how the one-to-one correspondence between hulls of limit-periodic sequences and potential families generated by minimal translations of Cantor groups and continuous sampling functions exhibited by Avila in [1] arises.

Definition 2.1. *Let $S : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ be the shift operator, $(SV)(n) = V(n+1)$. A two-sided sequence $V \in \ell^\infty(\mathbb{Z})$ is called periodic if its S-orbit is finite and it is called limit-periodic if it belongs to the closure of the set of periodic sequences. If V is limit-periodic, the closure of its S-orbit is called the hull and denoted by* hull_V.

The first lemma (see [1, Lemma 2.1]) shows how one can write the elements of the hull of a limit-periodic function in the form (2) with a minimal translation T of a Cantor group and a sampling function $f \in C(\Omega, \mathbb{R})$:

Lemma 2.1. *Suppose V is limit-periodic. Then, $\Omega := \mathrm{hull}_V$ is compact and has a unique topological group structure with identity V such that $\mathbb{Z} \ni k \mapsto S^k V \in \mathrm{hull}_V$ is a homomorphism. Moreover, the group structure is Abelian and there exist arbitrarily small compact open neighborhoods of V in hull_V which are finite index subgroups.*

In particular, $\Omega = \mathrm{hull}_V$ is a Cantor group, $T = S|_\Omega$ is a minimal translation, and every element of Ω may be written in the form (2) with the continuous function $f(\omega) = \omega(0)$.

The second lemma (see [1, Lemma 2.2]) addresses the converse:

Lemma 2.2. *Suppose Ω is a Cantor group, $T : \Omega \to \Omega$ is a minimal translation, and $f \in C(\Omega, \mathbb{R})$. Then, for every $\omega \in \Omega$, the element V_ω of $\ell^\infty(\mathbb{Z})$ defined by $V_\omega(n) = f(T^n\omega)$ is limit-periodic and we have* $\mathrm{hull}_{V_\omega} = \{V_{\tilde\omega}\}_{\tilde\omega\in\Omega}$.

These two lemmas show that a study of limit-periodic potentials can be carried out by considering potentials of the form (2) with a minimal translation T of a Cantor group Ω and a continuous sampling function f. As shown for the first time in the context of limit-periodic potentials by Avila in [1], it is often advantageous to fix Ω and T and to vary f. This is what we will do in this paper and also in the forthcoming paper [5].

So fix a minimal translation T of a Cantor group Ω. Next, we discuss the set of periodic sequences that arise in the representation (2).

Definition 2.2. *A sampling function $f \in C(\Omega, \mathbb{R})$ is called periodic if there is a period $p \in \mathbb{N}$ such that for every $\omega \in \Omega$, we have $f(T^p\omega) = f(\omega)$.*

The next lemma (see [1, Subsubsection 2.3.2]) shows how to construct periodic sampling functions:

Lemma 2.3. *For every $f \in C(\Omega, \mathbb{R})$ and every compact subgroup $\Omega_0 \subset \Omega$ of finite index,*

$$f_{\Omega_0}(\omega) = \int_{\Omega_0} f(\omega + \tilde{\omega})\, d\mu_{\Omega_0}(\tilde{\omega})$$

is periodic. Here, μ_{Ω_0} denotes Haar measure on Ω_0.

Combining this construction with the fact that one can find compact subgroups of finite index in any neighborhood of the identity, we arrive at the following way of looking at the embedding of periodic sampling functions into $C(\Omega, \mathbb{R})$; compare [1, Section 3]:

Lemma 2.4. *There exists a decreasing sequence of Cantor subgroups $\Omega_k \subset \Omega$ with finite index p_k such that $\bigcap \Omega_k = \{0\}$. Let P_k be the sampling functions which are defined on Ω/Ω_k. Elements of P_k are periodic with period p_k. Conversely, every periodic sampling function belongs to some P_k. Thus, $P = \bigcup P_k$ is the set of all periodic sampling functions and it is dense in $C(\Omega, \mathbb{R})$.*

We refer the reader to [12] and [16] for further material and a comprehensive discussion of Cantor groups.

3. Some Features of Periodic Potentials

In this section we summarize some aspects of the spectral theory of Schrödinger operators with periodic potentials. Since limit-periodic potentials are obtained as uniform limits of periodic potentials, much of their spectral analysis is based on approximation by periodic potentials and hence a good understanding of the periodic theory is essential. We will focus here on the properties of periodic operators that are of immediate interest to us and refer the reader to [8, 13, 14, 15] for more details.

Consider a potential $V : \mathbb{Z} \to \mathbb{R}$ that is periodic with period p, that is,

$$V(n + p) = V(n) \quad \text{for every } n \in \mathbb{Z}.$$

This gives rise to a periodic Schrödinger operator in $\ell 2(\mathbb{Z})$, given by

$$[H\psi](n) = \psi(n + 1) + \psi(n - 1) + V(n)\psi(n). \tag{3}$$

We will link the spectral properties of H to properties of the solutions of the associated difference equation

$$u(n + 1) + u(n - 1) + V(n)u(n) = Eu(n). \tag{4}$$

These solutions are generated by the so-called transfer matrices

$$T_n^{(E,V)} = S_{n-1}^{(E,V)} \dots S_0^{(E,V)},$$

where

$$S_i^{(E,V)} = \begin{pmatrix} E - V(i) & -1 \\ 1 & 0 \end{pmatrix}.$$

The transfer matrix over one period, $T_p^{(E,V)}$, plays a special role as we will see below.

For $k \in \mathbb{R}$ and $l \in \mathbb{Z}$, we define

$$A_l^k = \begin{pmatrix} V(l) & 1 & & & & e^{-ikp} \\ 1 & V(l+1) & 1 & & & \\ & 1 & V(l+2) & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & 1 \\ e^{ikp} & & & & 1 & V(l+p-1) \end{pmatrix}.$$

These matrices are the restrictions of H to intervals of length p with suitable self-adjoint boundary conditions. The importance of this choice of boundary condition lies in its connection to the existence of special solutions of (4).

The following proposition summarizes several important results concerning the operator, the difference equation, and the matrices; see the references mentioned above for proofs. The results listed here are usually referred to as Floquet-Bloch theory.

Proposition 3.1. (a) *We have $E \in \sigma(H)$ if and only if (4) has a solution $\{u(n)\}$ obeying*

$$u(n+p) = e^{ikp} u(n)$$

for all n and some real number k. In this case, $\tilde{u} = \langle u(n) \rangle_{n=l}^{l+p-1}$ is an eigenvector of the matrix A_l^k corresponding to eigenvalue E.

(b) *The p eigenvalues of A_l^k are independent of l and*

$$\sigma(H) = \bigcup_k \sigma(A_l^k).$$

(c) *The characteristic polynomial of $A_l(k)$ obeys*

$$\det(E - A_l(k)) = \Delta(E) - 2\cos kp,$$

where $\Delta(E) = \operatorname{Tr} T_p^{(E,V)}$. We have

$$\sigma(H) = \{E : |\Delta(E)| \le 2\}.$$

The set $\sigma(H)$ is made of p bands such that on each band, $\Delta(E)$ is either strictly increasing or strictly decreasing.

(d) *If E is in the boundary of some band, we have $\Delta(E) = \pm 2$. Moreover, if two different bands intersect, then their common boundary point satisfies $T_p^{(E,V)} = \pm I$.*

The function Δ is called the discriminant associated with the periodic potential V. It is a polynomial of degree p with real coefficients.

The other important consequence of periodicity is the existence of a direct integral decomposition. This will be described next.

As we have seen above, we can treat $E \in \sigma(H)$ as a function of the variable $k \in [0, \frac{\pi}{p}]$. For each band, the association $k \mapsto E$ is one-to-one and onto. Moreover, if we consider energies in the interior of a band, that is, with $\Delta(E) \in (-2, 2)$ or $k \in (0, \frac{\pi}{p})$, then there are linearly independent solutions $\varphi^{\pm}(E)$ of (4) with

$$\varphi^{\pm}_{n+lp}(E) = e^{\pm ilkp} \varphi^{\pm}_{n}(E).$$

It is easy to see that one can normalize these solutions by requiring

$$\varphi^{\pm}_{0}(E) > 0$$

and

$$\sum_{j=0}^{p-1} \left| \varphi^{\pm}_{j}(E) \right|^2 = 1.$$

With this normalization, we have

$$\varphi^{-}(E) = \overline{\varphi^{+}(E)}.$$

Next, we define for $u = \{u_n\}_{n \in \mathbb{Z}}$ of finite support,

$$\hat{u}^{\pm}(E) = \sum_{n \in \mathbb{Z}} \overline{\varphi^{\pm}_{n}(E)} u_n.$$

We also define the measure $d\rho$ on $\sigma(H)$ by

$$d\rho(E) = \frac{1}{\pi} \left| \frac{dk}{dE}(E) \right| dE.$$

Then, we have the following result; see the references listed above for a proof.

Proposition 3.2. *The map $u \mapsto \hat{u}$ extends to a unitary map from $\ell 2(\mathbb{Z})$ to $L2(\sigma(H), d\rho; \mathbb{C}2)$. Its inverse is given by*

$$\left(\check{f}\right)_n = \frac{1}{2} \int_{\sigma(H)} \left[\varphi^{+}_{n}(E) f^{+}(E) + \varphi^{-}_{n}(E) f^{-}(E) \right] d\rho(E)$$

Moreover, we have that

$$\widehat{Hu}^{\pm}(E) = E \hat{u}^{\pm}(E).$$

Here, we use $f^{\pm}(E)$ for the two components of a $\mathbb{C}2$-valued function $f \in L2(\sigma(H), d\rho; \mathbb{C}2)$.

Proposition 3.2 shows that H has purely absolutely continuous spectrum (of multiplicity two). More precisely, the spectral measure associated with the operator H with periodic potential V and a finitely supported $u \in \ell 2(\mathbb{Z})$ is given by

$$d\mu_{V,u}(E) = g_{V,u}(E)\, dE \tag{5}$$

with density

$$g_{V,u}(E) = \frac{1}{2\pi} \left(|\hat{u}^{+}(E)|^2 + |\hat{u}^{-}(E)|^2 \right) \left| \frac{dk}{dE}(E) \right| \tag{6}$$

for $E \in \sigma(H)$ (we set $g_{V,u}(E)$ equal to zero outside of $\sigma(H)$).

Let us derive some consequences of these results.

Lemma 3.1. *For every $t \in (1, 2)$, there exists a constant $D = D(\|V\|_\infty, p, t)$ such that*

$$\int_{\sigma(H)} \left| \frac{dk}{dE}(E) \right|^t dE \leq D. \tag{7}$$

Proof. By Proposition 3.1, we have

$$\left|\frac{dk}{dE}(E)\right| = \left|\frac{\Delta'(E)}{2p\,\sin(kp)}\right|.$$

Since we can bound $|\Delta'(E)|$ by a $(\|V\|_\infty, p)$-dependent constant and $\int_0^\pi (\sin(x))^{1-t}\,dx < \infty$, we have the following estimates,

$$\int_{\sigma(H)} \left|\frac{dk}{dE}(E)\right|^t \,dE \lesssim \int_0^{\frac{\pi}{p}} \left|\frac{1}{2p\,\sin(kp)}\right|^{t-1} \,dk \lesssim \int_0^{\frac{\pi}{p}} |\sin(kp)|^{1-t}\,dk$$

and the last integral may be bounded by a t-dependent constant. □

Lemma 3.2. *Let $u \in \ell 2(\mathbb{Z})$ have finite support. Then, for every $t \in (1,2)$, there exists a constant $Q = Q(\|V\|_\infty, p, u, t)$ such that*

$$\int_{\sigma(H)} |g_{V,u}(E)|^t \,dE \le Q. \tag{8}$$

Proof. Since u has a finite support, we can find a constant $M = M(p,u)$ such that $|\hat{u}^\pm(E)|^2 \le M$. Thus, by (6) we have

$$\begin{aligned}\int_{\sigma(H)} |g_{V,u}(E)|^t \,dE &= \int_{\sigma(H)} \left[\frac{1}{2\pi}\left(|\hat{u}^+(E)|^2 + |\hat{u}^-(E)|^2\right)\left|\frac{dk}{dE}(E)\right|\right]^t \,dE \\ &\le \left[\frac{M}{\pi}\right]^t \int_{\sigma(H)} \left|\frac{dk}{dE}(E)\right|^t \,dE \\ &\le \left[\frac{M}{\pi}\right]^t D\end{aligned}$$

with the constant D from Lemma 3.1. □

Lemma 3.3. *Let $(X, d\mu)$ be a finite measure space, let $r > 1$ and let $f_n, f \in L^r$ with $\sup_n \|f_n\|_r < \infty$. Suppose that $f_n(x) \to f(x)$ pointwise almost everywhere. Then, $\|f_n - f\|_p \to 0$ for every $p < r$.*

Proof. This is [2, Lemma 2.6]. □

Lemma 3.4. *Suppose $u \in \ell 2(\mathbb{Z})$ has finite support and $V_n, V : \mathbb{Z} \to \mathbb{R}$ are p-periodic and such that $\|V_n - V\|_\infty \to 0$ as $n \to \infty$. Then, for any $t \in (1,2)$, we have*

$$\int_{\mathbb{R}} |g_{V_n,u}(E) - g_{V,u}(E)|^t \,dE \to 0$$

as $n \to \infty$.

Proof. By Lemmas 3.2 and 3.3 we only need to prove pointwise convergence. Given the explicit identity (6), pointwise convergence follows readily from the following two facts: the discriminant of the approximants converges pointwise to the discriminant of the limit and the matrices A_l^k associated with the approximants converge pointwise to those associated with the limit and therefore so do the associated eigenvectors. □

4. Cantor Spectrum

Theorem 4.1. *Let Ω be a Cantor group and let $T : \Omega \to \Omega$ be a minimal translation. Then there exists a dense G_δ set $\mathcal{C} \subseteq C(\Omega, \mathbb{R})$ such that for every $f \in \mathcal{C}$ and $\omega \in \Omega$, the spectrum of the operator H_ω given by (1) is a Cantor set.*

Fix Ω and T as in the theorem throughout this section. By minimality, for given $f \in C(\Omega, \mathbb{R})$, the spectrum of H_ω is independent of ω. For notational convenience, we will denote this set by $\Sigma(f)$. Let $P = \bigcup P_k$ be the set of periodic sampling functions in $C(\Omega, \mathbb{R})$ and denote the associated periods by p_k.

Lemma 4.1. $\mathfrak{N} := \{f \in C(\Omega, \mathbb{R}) : \Sigma(f)$ *has empty interior*$\}$ *is a G_δ set.*

Proof. This is essentially [2, Lemma 1.1]. □

Lemma 4.2. *For every $f \in P_k$, $k \in \mathbb{N}$ and every $\varepsilon > 0$, there exists $\tilde{f}$ in P_k satisfying $\|f - \tilde{f}\| < \varepsilon$ such that $\Sigma(\tilde{f})$ has exactly p_k components, that is, its $p_k - 1$ gaps are all open.*

Proof. This follows from the proof of [1, Claim 3.4]. For the reader's convenience, we provide a proof. Let $f \in P_k$ be given. By ω-independence of the spectrum, we may choose and fix an arbitrary $\omega \in \Omega$ for the purpose of this proof. Next, given $\varepsilon > 0$, let M be large enough so that $\frac{2p_k+1}{M} < \varepsilon$. Then, for $1 \le t \le 2p_k + 1$, there is $\tilde{f}_t \in P_k$ with

$$\tilde{f}_t(T^i\omega) = f(T^i\omega),\ 0 \le \ i \ \le p_k - 2 \ \text{ and } \ \tilde{f}_t(T^{p_k-1}\omega) = f(T^{p_k-1}\omega) + \frac{t}{M}.$$

Obviously, $\|\tilde{f}_t - f\| < \varepsilon$ and we claim that there exists some t in this range such that the spectrum associated with $\tilde{f}_t$ has exactly p_k components.

Suppose this claim fails. Then, for every t in this range, there exists by Proposition 3.1.(d) an energy $E_t \in \Sigma(\tilde{f}_t)$ with $T_{p_k}^{(E_t, \tilde{f}_t)} = \pm id$. That is,

$$\begin{pmatrix} E_t - f(T^{p_k-1}\omega) - \frac{t}{M} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E_t - f(T^{p_k-2}\omega) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E_t - f(\omega) & -1 \\ 1 & 0 \end{pmatrix} = \pm id.$$

Since f is p_k-periodic, we get from this

$$T_{p_k}^{(E_t, f)} = \pm \begin{pmatrix} 1 & \frac{t}{M} \\ 0 & 1 \end{pmatrix}. \tag{9}$$

Indeed, rewriting the identity above, we find that

$$\begin{aligned} T_{p_k}^{(E_t, f)} &= \pm id + \begin{pmatrix} \frac{t}{M} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E_t - f(T^{p_k-2}\omega) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E_t - f(\omega) & -1 \\ 1 & 0 \end{pmatrix} \\ &= \pm id + \begin{pmatrix} \frac{t}{M} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E_t - f(T^{p_k-1}\omega) & -1 \\ 1 & 0 \end{pmatrix}^{-1} T_{p_k}^{(E_t, f)} \end{aligned}$$

so that

$$
\begin{aligned}
T_{p_k}^{(E_t,f)} &= \pm\left(id - \begin{pmatrix} \frac{t}{M} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E_t - f(T^{p_k-1}\omega) & -1 \\ 1 & 0 \end{pmatrix}^{-1}\right)^{-1} \\
&= \pm\left(id - \begin{pmatrix} \frac{t}{M} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & E_t - f(T^{p_k-1}\omega) \end{pmatrix}\right)^{-1} \\
&= \pm\left(id - \begin{pmatrix} 0 & \frac{t}{M} \\ 0 & 0 \end{pmatrix}\right)^{-1} \\
&= \pm\begin{pmatrix} 1 & -\frac{t}{M} \\ 0 & 1 \end{pmatrix}^{-1} \\
&= \pm\begin{pmatrix} 1 & \frac{t}{M} \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

The relation (9) implies that if $t \neq t'$, we have $T_{p_k}^{(E_t,f)} \neq T_{p_k}^{(E_{t'},f)}$, and therefore $E_t \neq E_{t'}$. But there are at most $2p_k$ values of E for which $tr\ T_{p_k}^{(E,f)} = \pm 2$; contradiction. $\square$

Lemma 4.3. *Let $f \in P$ have period p. Then we have*
(1) *The measure of each band of $\Sigma(f)$ is at most $\frac{2\pi}{p}$.*
(2) *Let $C \geq 1$ be such that for every $E \in \Sigma(f)$, there exist $\omega \in \Omega$ and $k \geq 1$ such that $\|T_k^{(E,V_\omega)}\| \geq C$. Then, the total measure of $\Sigma(f)$ is at most $\frac{4\pi p}{C}$.*

Proof. This is [1, Lemma 2.4]. $\square$

Proof of Theorem 4.1. The proof is close in spirit to the proof of [2, Theorem 1]. By Lemma 4.1, we only need to prove that $\mathfrak{N}$ is dense. Since $P = \bigcup P_k$ is dense in $C(\Omega, \mathbb{R})$, it suffices to show that, given $f \in P$ and $\varepsilon > 0$, there is a potential $\tilde{f}$ such that $\|\tilde{f} - f\| < \varepsilon$ and $\Sigma(\tilde{f})$ is nowhere dense.

So let $f \in P$ and $\varepsilon > 0$ be given. Write f as $f = \sum_{j=0}^{N} a_j W_j$ with $W_j \in P_j$. We construct $s_0 = \sum_{i=0}^{N} a_n^{(0)} W_i$ so that $\|s_0\| < \varepsilon$ and $f_0 = f + s_0$ has all $n_N - 1$ gaps open. (This is possible due to Lemma 4.2.)

Suppose $s_0, s_1, \ldots, s_{k-1}$ are picked. Let α_{k-1} be the minimal gap size of f_{k-1} and define $\beta_k = \min\{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$. Applying Lemma 4.2, we pick $s_k = \sum_{i=0}^{N+k} a_i^{(k)} W_i$ so that

$$\|s_k\| < \frac{\varepsilon}{2^k}, \tag{10}$$

$$\|s_k\| < \frac{1}{3}\frac{\beta_k}{2^k}, \tag{11}$$

$$f_k = f + \sum_{j=0}^{k} s_j \text{ has all gaps open.} \tag{12}$$

The limit of f_k exists by (10), let $\tilde{f} = \lim_{k\to\infty} f_k$. By construction, we have $\|\tilde{f} - f\| < \varepsilon$. We claim that $\Sigma(\tilde{f})$ is nowhere dense; equivalently, its complement is dense.

Given $E \in \Sigma(\tilde{f})$ and $\tilde{\varepsilon} >> 0$, we can pick k large enough so that

$$\|\tilde{f} - f_k\| < \frac{\tilde{\varepsilon}}{3}, \tag{13}$$

$$\frac{2\pi}{n_{N+k}} < \frac{\tilde{\varepsilon}}{3}, \tag{14}$$

$$\frac{\varepsilon}{2^{k-1}} < \frac{\tilde{\varepsilon}}{3}. \tag{15}$$

By (13), there exists $E' \in \Sigma(f_k)$ such that $|E' - E| < \frac{\tilde{\varepsilon}}{3}$. Moreover, by Lemma 4.3 and (14), we can find $\tilde{E}$ in a gap of $\Sigma(f_k)$ such that $|E' - \tilde{E}| < \frac{\tilde{\varepsilon}}{3}$. Write this gap of $\Sigma(f_k)$ that contains $\tilde{E}$ as $(a - \delta, a + \delta)$. By definition, we have $2\delta \geq \beta_{k+1}$. By (11),

$$\|\tilde{f} - f_k\| = \left\| \sum_{j=k+1}^{\infty} s_j \right\| < \frac{\beta_{k+1}}{3} \left(\frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \cdots \right) \leq \frac{\delta}{3},$$

so we have $(a - \frac{\delta}{3}, a + \frac{\delta}{3}) \cap \Sigma(\tilde{f}) = \emptyset$.

We claim that there exists $\delta' \in [\frac{\delta}{3}, \delta)$ such that $(a - \delta', a + \delta') \cap \Sigma(\tilde{f}) = \emptyset$ and $|\delta' - \delta| < \frac{\varepsilon}{2^k}$. As we saw above, we may arrange for $\delta' \geq \frac{\delta}{3}$. Suppose that it is impossible to find such a δ' with $|\delta' - \delta| < \frac{\varepsilon}{2^k}$. Then, there will be a point $x \in \Sigma(\tilde{f})$ such that $[x - \frac{\varepsilon}{2^k}, x + \frac{\varepsilon}{2^k}] \subseteq (a - \delta, a + \delta)$. Then we get a contradiction to the already established fact $[x - \frac{\varepsilon}{2^k}, x + \frac{\varepsilon}{2^k}] \cap \Sigma(f_k) = \emptyset$ since (10) implies

$$\|\tilde{f} - f_k\| = \left\| \sum_{j=k+1}^{\infty} s_j \right\| < \frac{\varepsilon}{2^k}.$$

We can choose an energy $\hat{E}$ in the gap of $\Sigma(\tilde{f})$ that contains $(a - \delta', a + \delta')$ such that $|\hat{E} - \tilde{E}| \leq \frac{\varepsilon}{2^k} < \frac{\tilde{\varepsilon}}{3}$, where the second inequality follows from (15). Moreover, since we also have $|\tilde{E} - E'| < \frac{\tilde{\varepsilon}}{3}$ and $|E' - E| < \frac{\tilde{\varepsilon}}{3}$, it follows that $|\hat{E} - E| < \tilde{\varepsilon}$. This shows that $\mathbb{R} \setminus \Sigma(\tilde{f})$ is dense and completes the proof. □

Remark. The reader may notice that the statement of Theorem 4.1 is a consequence of [1, Corollary 1.2]. However, the main purpose of this section is the method of proof presented here, which is direct and flexible enough so that it can be used to also ensure absolutely continuous spectrum (this will be done in the next section). In particular, the Cantor spectra constructed here may have positive Lebesgue measure, whereas the Cantor spectra generated in the proof of [1, Corollary 1.2] always have zero Lebesgue measure.

5. Absolutely Continuous Spectrum

Theorem 5.1. *Let $T : \Omega \to \Omega$ be a minimal translation of a Cantor group. Then there exists a dense set of $f \in C(\Omega, \mathbb{R})$ so that for every $\omega \in \Omega$, the spectrum of H_ω is a Cantor set of positive Lebesgue measure and all spectral measures are purely absolutely continuous.*

Proof. The idea is to modify the construction from the proof of Theorem 4.1. Thus, we will again start with an arbitrarily small ball in $C(\Omega, \mathbb{R})$ and construct a point in this ball for which the associated Schrödinger operator has both Cantor spectrum and purely absolutely continuous spectrum. The presence of absolutely continuous spectrum then also implies that the Lebesgue measure of the spectrum is positive.

Fix $t \in (1,2)$ and let $u \in \ell 2(\mathbb{Z})$ have finite support. In going through the construction in the proof of Theorem 4.1, pick s_k so that in addition to the conditions above, we have

$$\left(\int_{-\infty}^{\infty} \left| g_u^{k-1}(E) - g_u^k(E) \right|^t \right)^{\frac{1}{t}} \le \frac{1}{2^k}, \tag{16}$$

where g_u^k is the density of the spectral measure associated with u and the periodic potential $n \mapsto f_k(T^n\omega)$, with the estimate above being uniform in $\omega \in \Omega$. This is possible due to Lemma 3.4.

By Lemma 3.2, there exists a constant $Q(u,t) < \infty$ such that $\int_{\mathbb{R}} \left| g_u^k(E) \right|^t dE \le Q(u,t)$.

Now fix any $\omega \in \Omega$. Let A be a finite union of open sets. If P_A^k is the spectral projection for the potential $n \mapsto f_k(T^n\omega)$ and P_A is the spectral projection for the potential $n \mapsto \tilde{f}(T^n\omega)$, it follows that $\langle u, P_A u\rangle \le \limsup_{k\to\infty} \langle u, P_A^{f_k} u\rangle$ since $\|f_k - \tilde{f}\|_\infty \to 0$ and hence the associated Schrödinger operators converge in norm.

Applying Hölder's inequality, we find

$$\langle u, P_A u\rangle \le \limsup_{k\to\infty} \int_A g_u^k(E)\, dE \le Q(u,t)|A|^{\frac{1}{q}},$$

where $\frac{1}{q} + \frac{1}{t} = 1$ and $|\cdot|$ denotes Lebesgue measure. This shows that the spectral measure associated with u and the Schrödinger operator with potential $n \mapsto \tilde{f}(T^n\omega)$ is absolutely continuous with respect to Lebesgue measure. Since this holds for every finitely supported u, it follows that this operator has purely absolutely continuous spectrum. □

Notice that this establishes part (a) of Theorem 1.1.

6. Absence of Point Spectrum

Definition 6.1. *A bounded function $V : \mathbb{Z} \to \mathbb{R}$ is called a Gordon potential if there are positive integers $q_k \to \infty$ such that*

$$\max_{1\le n\le q_k} |V(n) - V(n \pm q_k)| \le k^{-q_k}$$

for every $k \ge 1$.

Clearly, if V is a Gordon potential, then so is λV for every $\lambda \in \mathbb{R}$.

Lemma 6.1. *Suppose V is a Gordon potential. Then, the operator H given by* (3) *has empty point spectrum.*

This is essentially due to Gordon [7]; see [6] for the modification of the argument necessary to prove the result as stated.

Theorem 6.1. *Let Ω be a Cantor group and let $T : \Omega \to \Omega$ be a minimal translation. Then there exists a dense G_δ-set $\mathcal{G} \subseteq C(\Omega,\mathbb{R})$ such that for every $f \in \mathcal{G}$ and $\omega \in \Omega$, the potential V_ω given by* (2) *is a Gordon potential.*

Proof. We know the set of periodic potentials is dense in $C(\Omega,\mathbb{R})$. For $j,k \in \mathbb{N}$, let

$$\mathcal{G}_{j,k} = \left\{ f \in C(\Omega,\mathbb{R}) : \text{ there is a } j\text{-periodic } f_j \text{ such that } \|f - f_j\| < \frac{1}{2}(jk)^{(-jk)} \right\}.$$

Clearly, $\mathcal{G}_{j,k}$ is open. For $k \in \mathbb{N}$, let

$$\mathcal{G}_k = \bigcup_{j=1}^{\infty} \mathcal{G}_{j,k}.$$

The set $\mathcal{G}_k$ is open by construction and dense since it contains all periodic sampling functions. Thus,

$$\mathcal{G} = \bigcap_{k=1}^{\infty} \mathcal{G}_k$$

is a dense G_δ subset of $C(\Omega, \mathbb{R})$. We claim that for every $f \in \mathcal{G}$ and every $\omega \in \Omega$, the potential V_ω given by (2) is a Gordon potential.

So let $f \in \mathcal{G}$ and $\omega \in \Omega$ be given. Since $f \in A_k$ for every $k \in \mathbb{N}$, we can find j_k-periodic f_{j_k} satisfying

$$\|f - f_{j_k}\| < \frac{1}{2}(j_k k)^{-j_k k}.$$

Let $q_k = j_k k$, so that $q_k \to \infty$ as $k \to \infty$. Then, we have

$$\begin{aligned}
\max_{1 \le n \le q_k} \|V_\omega(n) - V_\omega(n \pm q_k)\| &= \max_{1 \le n \le q_k} \|f(T^n \omega) - f(T^{n \pm q_k} \omega)\| \\
&= \max_{1 \le n \le j_k k} \|f(T^n \omega) - f_{j_k}(T^n \omega) + f_{j_k}(T^{n \pm j_k k} \omega) - f(T^{n \pm j_k k} \omega)\| \\
&\le \max_{1 \le n \le j_k k} \|f(T^n \omega) - f_{j_k}(T^n \omega)\| + \max_{1 \le n \le j_k k} \|f(T^{n \pm j_k k} \omega) - f_{j_k}(T^{n \pm j_k k} \omega)\| \\
&< \frac{1}{2}(j_k k)^{-j_k k} + \frac{1}{2}(j_k k)^{-j_k k} \\
&\le k^{-j_k k} \\
&= k^{-q_k}.
\end{aligned}$$

It follows that V_ω is a Gordon potential. $\square$

Lemma 6.2. *Let $T : \Omega \to \Omega$ be a minimal translation of a Cantor group. For a dense G_δ set of $f \in C(\Omega, \mathbb{R})$, and for every $\lambda \neq 0$, the Schrödinger operator with potential $\lambda f(T^n \omega)$ has a spectrum of zero Lebesgue measure for every $\omega \in \Omega$.*

Proof. This is [1, Corollary 1.2.]. $\square$

We can now finish the proof of our main theorem.

Proof of Theorem 1.1.(b). Since the intersection of two dense G_δ sets is again a dense G_δ set and zero-measure spectrum precludes absolutely continuous spectrum, the result follows directly from Theorem 6.1 and Lemma 6.2. $\square$

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