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Symmetric Bi-multipliers on *d-algebras*

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Abstract In this study, we introduce the notion of symmetric bimultipliers in *d-algebras* and investigate some related properties. Among others kernels and sets of fixed points of a *d-algebra* are characterized by symmetric bi-multipliers.

Keywords: *d- algebras, multipliers, fixed set,kernel*

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1. Introduction

Imai and Iski introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [2] and [3]. The class of BCK-algebras is a proper subclass of the class of BCI-algebras. It is known that the notion of BCI-algebras is a generalization of BCK-algebras. J. Neggers and H. S. Kim [5] introduced the class of *d-algebras* which is a generalization of BCK-algebras, and investigated relations between *d-algebras* and BCK-algebras.

A partial multiplier on a commutative semigroup $(A,\cdot)$ has been introduced in [4] as a function F from a nonvoid subset DF of A into A such that $F(x)\cdot y = x\cdot F(y)$ for all $x, y \in D_F$. The notion of multipliers on lattices was introduced and studied by [6,7] and it was generalized to the partial multipliers on partially ordered sets in [8,9]. Muhammad Anwar Chaudhry and Faisal Ali defined the notion of multipliers on d-algebras in [1].

In this paper the notion of symmetric bi-multipliers in d-algebras are given and properties of these multipliers are researched. Also, kernels and set of fixed points of a d-algebra are characterized by symmetric bi-multipliers.

2. Preliminaries

Definition 2.1. [5] A d-algebra is a non-empty set X with a constant 0 and a binary operation denoted by * satisfying the following axioms for all $x, y \in X$:

(I) $x*x = 0,$

(II) $0*x = 0,$

(III) $x*y = 0,$ *and* $y*x = 0$ *imply* $x = y$ *for all* $x, y \in X.$

Definition 2.2. **[5]** Let S be a non-empty subset of a *d-algebra* X, then S is called subalgebra of X if $x*y \in S$ for all $x, y \in S$.

Definition 2.3. Let X be a d-algebra and I be a subset of X, then I is called an ideal of X if it satisfies the following conditions:

(1) $0 \in I$

(2) $x*y \in I$ *and* $y \in I$ *imply* $x \in I$.

Definition 2.4. Let X be a d-algebra and I be a non-empty subset of X, then I is called a d-ideal of X if it satisfies the following conditions:

(1) $x*y \in I$ *and* $y \in I$ *imply* $x \in I$ *and*

(2) $x \in I$ and $y \in X$ imply $x*y \in I$. From condition (2) it is obvious that for $x \in I \subseteq X;\ \ 0 = x*x \in I$.

3. Symmetric Bi-multipliers on *d-algebras*

The following Definition introduces the notion of symmetric bi-multiplier for a *d-algebra*. In what follows, let X denote a *d-algebra* unless otherwise specified.

Definition 3.1. Let X be a *d-algebra*. A mapping $f(.,.): X \times X \to X$ is called symmetric if $f(x, y) = f(y, x)$ for all $x, y \in X$.

Definition 3.2. Let X be a *d-algebra* and let $f(.,.): X \times X \to X$ be a symmetric mapping. We call f a *symmetric bi-multiplier* on X if it satisfies;

$$f(x, y*z) = f(x, y)*z \quad \textit{for all } x, y, z \in X.$$

Example 3.1. *Let* $X = \{0, a, b\}$, *and with the binary operation * defined by* :

*	0	a	b
0	0	0	0
a	a	0	0
b	b	a	0

Then X is a d-algebra.

The mapping $f(.,.): X \times X \to X$ *defined by*

$$f(x, y) = \begin{cases} a, & \textit{if } x = y = b, \\ 0, & \textit{otherwise}. \end{cases}$$

Then we can see that f is a symmetric bi-multiplier on X.

Remark 3.1. *If X is a d-algebra with a binary operation *, then we can define a binary operation* $\leq$ *on X by;*

$x \le y$ *if and only if* $x * y = 0$ *for all* $x, y \in X$.

Proposition 3.3. *Let X be a d-algebra and f be the symmetric bi-multiplier on X. Then the followings hold for all* $x, y, z \in X$ *:*

i) $f(0,0) = 0,$

ii) $f(0,x) \le x,$

iii) If $x \le y$ *then* $f(0,x) \le y.$

Proof: Let *X* be a *d-algebra* and *f* be the symmetric bi-multiplier on *X*.

i) By using the definition of symmetric bi-multiplier on *X* and (I) we have the following:

$$f(0,0) = f(0,0 * f(0,0)) = f(0,0) * f(0,0) = 0$$

Therefore, $f(0,0) = 0$.

ii) By i) we have

$$0 = f(0,0) = f(0, x * x) = f(0,x) * x$$

Therefore, we have $f(0,x) * x = 0$ and hence $f(0,x) \le x$.

iii) Let x, y be elements in *X* and $x \le y$.

$$0 = f(0,0) = f(0, x * y) = f(0,x) * y$$

Therefore, we get $0 = f(0,x) * y$ and hence $f(0,x) \le y$.

Definition 3.4. [1] A *d-algebra X* is said to be positive implicative if

$$(x * y) * z = (x * z) * (y * z)$$

for all $x, y, z \in X$.

Let *S(X)* be the collection of all symmetric bi-multipliers on *X*. It is clear that $O(.,.) : X \times X \to X$ defined by $O(x, y) = 0$ for all $(x, y) \in X \times X$ and $P(.,.) : X \times X \to X$ defined by $P(x, y) = x$ for all $(x, y) \in X \times X$ are in *S(X)*. Therefore, *S(X)* is not empty.

Definition 3.5. Let *X* be a positive implicative *d-algebra* and *S(X)* be the collection of all symmetric bi-multipliers on *X*. We define a binary operation * on *S(X)* by

$$(f * g)_{x,y} = f(x, y) * g(x, y)$$

$(x, y) \in X \times X$ and $f, g \in S(X)$.

Theorem 3.6. *Let X be a positive implicative d-algebra. Then* $(S(X), *, 0)$ *is a positive implicative d-algebra*.

Proof: Let *X* be a positive implicative *d-algebra* and let $g, f \in S(X)$. Then

$$\begin{aligned}(g * f)_{(x,y)*(z,t)} &= (g * f)_{(x,z)*(y,t)} \\ &= (g(x * z, y * t)) * (f(x * z, y * t)) \\ &= (g(x * z, y) * t) * (f(x * z, y) * t) \\ &= (g(x * z, y)) * (f(x * z, y)) * t \\ &= (g * f)_{(x*z,y)} * t\end{aligned}$$

So, $g * f \in S(X)$.

Let $f \in S(X)$. Then

$$(O * f)_{(x,y)} = O(x, y) * f(x, y) = 0 * f(x, y) = 0 = O(x, y)$$

for al $(x, y) \in X \times X$. So $O * f = O$ for all $f \in S(X)$.

Now let $f \in S(X)$, we have

$$(f * f_{(x,y)}) = f(x, y) * f(x, y) = 0 = O(x, y)$$

for all $(x, y) \in X \times X$. So $f * f = O$.

Let $f, g \in S(X)$ such that $f * g = O$ and $g * f = O$. This implies that $(f * g)_{(x,y)} = 0$ and $(g * f)_{(x,y)} = 0$ for all $(x, y) \in X \times X$. That is $f_{(x,y)} * g_{(x,y)} = 0$ and $g_{(x,y)} * f_{(x,y)} = 0$ which implies that $f_{(x,y)} = g_{(x,y)}$ for all $(x, y) \in X \times X$. Thus $f = g$. Hence *S(X)* is a d-algebra.

Now we need to show that it is positive implicative. Let $f, g, h \in S(X)$. Then

$$\begin{aligned}((f * g) * h)_{(x,y)} &= (f * g)_{(x,y)} * h(x, y) \\ &= (f(x, y) * g(x, y)) * h(x, y) \\ &= (f(x, y) * h(x, y)) * (g(x, y) * h(x, y)) \\ &= ((f * h)_{(x,y)}) * ((g * h)_{(x,y)}) \\ &= ((f * h) * (g * h))_{(x,y)}\end{aligned}$$

for all $(x, y) \in X \times X$. Hence $(f * g) * h = (f * h) * (g * h)$ for all $f, g, h \in S(X)$. Therefore *S(X)* is an implicative *d-algebra*.

Definition 3.7. Let *f* be a symmetric bi-multiplier on *X*. We define *Ker(f)* by

$$Ker(f) = \{x \in X \mid f(0, x) = 0\}$$

for all $x \in X$.

Proposition 3.8. *Let X be a d-algebra and f be the symmetric bi-multiplier on X. Then Ker(f) is a subalgebra of X.*

Proof: Let *X* be a *d-algebra* and f be the symmetric bi-multiplier on *X*. Let $x, y \in Ker(f)$. Then we have $f(0, x) = 0$ and $f(0, y) = 0$. So $f(0, x * y) = f(0, x) * y = 0 * y = 0$. Thus $x * y \in Ker(f)$. Therefore, $Ker(f)$ is a subalgebra of *X*.

Definition 3.9. [1] A *d-algebra X* is called commutative if $x * (x * y) = y * (x * y)$ for all $(x, y) \in X$.

Proposition 3.10. *Let X be a commutative d-algebra satisfying* $x * 0 = 0,$ $x \in X$ *and f be the symmetric bi-multiplier on X. If* $x \in Ker(f)$ *and* $y \le x$ *then* $y \in Ker(f)$.

Proof: Let $x \in Ker(f)$ and $y \le x$. Then we have $f(0, x) = 0$ and $y * x = 0$. And then

$$f(0,y) = f(0,y*0) = f(0,y*(y*x))$$
$$= f(0,x*(x*y))$$
$$= f(0,x)*(x*y)$$
$$= 0*(x*y)$$
$$= 0$$

Therefore, $x \in Ker(f)$.

Definition 3.11. Let X be a *d-algebra* and *f* be the symmetric bi-multiplier on *X*. Then the set

$$Fix(f) = \{x \in X \mid f(0,x) = x\}$$

for all $x \in X$ is called the set of fixed points of f.

Proposition 3.12. *Let X be a d-algebra and f be the symmetric bi-multiplier on X. Then Fix(f) is a subalgebra of X.*

Proof: Let X be a *d-algebra* and *f* be the symmetric bi-multiplier on X.

Since $f(0,0) = 0$ $Fix(f)$ is non-empty. Let $x, y \in Fix(f)$. Then we have $f(0,x) = x,\ f(0,y) = y$. Then

$$f(0,x*y) = f(0,x)*y = x*y$$

Therefore, $x*y \in Fix(f)$. Hence, *Fix(f)* is a subalgebra of *X*.

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