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On ’t Hooft–Polyakov Monopole, Julia–Zee Dyon, and Higgs Field, throughout the Generalized Bogomoln’yi Equations

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Abstract In this paper, making use of the’tHooft–Polyakov–Julia–Zeeansatz for the SU(2) Yang–Mills–Higgs gauge field theory, we present the straightforward generalization of the Bogomoln’yi equations and its several consequences. Particularly, this is shown that this idea is able to generate new types of non-abelian both dyons and magnetic monopoles and, moreover, that within the new model the scalar field can be described through the Coulomb potential, whereas, upto aconstant, the non-abelian gauge field becomes the Wu–Yang monopole.

Keywords: *gauge field theories, SU (2), Yang–Mills–Higgs equations, ’ t Hooft–Polyakov monopole, Julia–Zee dyon, Bogomoln’yi equations, BPS limit, non-abelian dyons, non-abelian magnetic monopoles, Higgs field, Coulomb potential, Wu–Yang monopole*

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1. Yang–Mills–Higgs Equations

The Yang–Mills–Higgs theory is the non-abelian Yang–Mills theory of the gauge field A_ν^α [1] coupled to the Higgs field φ^a [2]. For the SU(N) gauge group, the Lagrangian has the form

$$L_{YMH} = -\frac{1}{4\mu_0}F^a_{\mu\nu}F_a^{\mu\nu} + \frac{\hbar c}{2}D_\mu\varphi^a D^\mu\varphi_a - \frac{\lambda\hbar c}{4}\left(\varphi^a\varphi_a - \varphi_0^2\right)^2, \quad (1)$$

where a = 1,...,N^2 -1 is a gauge group index, and

$$F^a_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + g f^a_{bc} A^b_\mu A^c_\nu, \quad D_\mu = \partial_\mu - igT_b A^b_\mu \quad (2)$$

are the strength tensor and the gauge-covariant derivative, respectively, while g is a coupling constant, and f^a_{bc} are the (antisymmetric) structure constants of the Lie algebra su (N) whose elements are the infinitesimal generators T_a of SU(N)

$$\left[T_a, T_b\right] = if^c_{ab}T_c, trT_a = 0, T_a^\dagger = T_a, \quad (3)$$

and SU(N) is produced by formal exponentiation of su(N). Furthermore,

$$\left\langle 0\left|\varphi^a\varphi_a\right|0\right\rangle = \varphi_0^2 = \frac{m^2c^2}{2\lambda\hbar^2}, \quad (4)$$

m is a mass parameter, and λ is a dimensionless constant. The energy density

$$H_{YMH} = \frac{\varepsilon_0}{2}E^a_k E^k_a + \frac{1}{2\mu_0}B^a_k B^k_a + \frac{\hbar c}{2}D_k\varphi^2 D^k\varphi_a + \frac{\lambda\hbar c}{4}\left(\varphi^a\varphi_a - \varphi_0^2\right)^2, \quad (5)$$

where $E^k_a = cF^{k0}_a$ is the electric field and $B^k_a = \frac{1}{2}\varepsilon^{kij}F^a_{ij}$ is the magnetic field induction, is minimized HYMH = 0 for the Higgs vacuum

$$\varphi^a\varphi_a = \varphi_0^2, D_\mu\varphi^a = 0, F^a_{\mu\nu} = 0, \quad (6)$$

which is degenerated, that is the set of vacuum expectation values of the Higgs field forms a sphere S^2_∞ on which all the points are equivalent because the gauge transformation connects them. Moreover, for $\varphi_0 \neq 0$ there is the spontaneous symmetry breakdown mechanism SU(N) → $U(1)^{N-1}$. The field equations for the Yang–Mills–Higgs theory (1) are

$$D^\mu D_\mu\varphi_a + \lambda\left(\varphi^c\varphi_c - \varphi_0^2\right)\varphi_a = 0 \quad (7)$$

$$\partial_\mu F^{\mu\nu}_a + g f_{abc}A^b_\mu F^{\mu\nu}_c = \mu_0\hbar c g f_{abc}\varphi^b D^\nu\varphi^c, \quad (8)$$

and the Bianchi identities

$$\partial_\mu G^{\mu\nu}_a + g f_{abc}A^b_\mu G^{\mu\nu}_c = 0. \quad (9)$$

are satisfied for the dual field strength tensor $G_a^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu k\lambda} F_{\kappa\lambda}^a$.

2. 't Hooft–Polyakov–Julia–Zee Ansatz

Dyon, a hypothetical object carrying a non-zero both electric and magnetic charges, was for the first time considered through Julian Seymour Schwinger [3,4], who applied this model as the phenomenological alternative to quarks to describe a particle analogous to the flavor-neutral J/ψ meson consisting of a charm quark and a charm antiquark, known as charmonium, discovered soon later [5,6]. Particularly, the Yang–Mills–Higgs system was considered for the case of the SU(2) gauge group, that is when the structure constants are the three-dimensional Levi–Civita symbols $\varepsilon_{ab}c$ and the infinitesimal generators in the fundamental representation are given by the Pauli matrices σ_a as $T_a = \frac{1}{2}\sigma_a$. In such gauge field theory, a smooth nonsingular solution to the Yang–Mills–Higgs equations, arising from the assumptions of spherical symmetry and time independence of both the non-abelian gauge field and the Higgs field

$$A_k^a = \frac{1-K(\xi)}{gr}\varepsilon_{ik}^a \frac{r^k}{r}, \varphi^a = \varphi_0 \frac{H(\xi)}{\xi}\frac{r^a}{r}, \quad \xi = \frac{\hbar}{\varepsilon} g\varphi_0 r, \tag{10}$$

known as the 't Hooft–Polyakov ansatz, was considered in the pioneering papers authored through Gerardus 't Hooft [7] and A.M. Polyakov [8,9], which with the boundary conditions

$$K(\xi \to 0) \to 1, H(\xi \to 0) < o(\xi), \quad K(\xi \to \infty) \to 0, H(\xi \to \infty) \sim \xi, \tag{11}$$

is known as the 't Hooft–Polyakov monopole. In the context of gauge field theories, dyons were for the first time considered as the solutions to the Yang–Mills–Higgs equations through Bernard Julia and Anthony Zee [10], who supplemented the 't Hooft–Polyakov ansatz by

$$A_0^a = \frac{J(\xi)}{gr}\frac{r^a}{r}, J(r \to 0) \to 0, J(r \to \infty) \to Cr, \tag{12}$$

where C is a constant. The't Hooft–Polyakov–Julia–Zee ansatz is compatible with the Lorentz gauge $\partial^\mu A_\mu^a = 0$ which gives

$$\varepsilon_{ki}^a r^k \nabla^i K = 0, \tag{13}$$

that is $\vec{r} \times \vec{\nabla} k = 0$, and, particularly, is satisfied through each spherically symmetric function K.

3. Bogomol'nyi–Prasad–Sommerfield Limit

The only known analytical solution occurs for the Bogomol'nyi–Prasad–Sommerfield (BPS) limit [11,12], that is $\lambda \to 0$. Then, the Higgs field is massless but the full gauge invariance remains broken due to its non-zero vacuum expectation value. It is easy to see that the energy calculated by (5)

$$\begin{aligned}\varepsilon_{YMH} &= \int d^3xV \\ &+ \sqrt{\frac{\hbar c}{\mu_0}}\left(\frac{1}{c}\sin a_c \int d^3x E_a^k D_k\varphi^a + \cos a_c \int d^3x B_a^k D_k\varphi^a\right) \\ &+ \frac{\varepsilon_0}{2}\int \frac{d^3x\left(E_a^k - c\sqrt{\mu_0\hbar c}\sin a_c D^k\varphi^a\right)}{\left(E_k^a - c\sqrt{\mu_0\hbar c}\sin a_c D_k\varphi^a\right)} \\ &+ \frac{1}{2\mu_0}\int \frac{d^3x\left(B_a^k - c\sqrt{\mu_0\hbar c}\sin a_c D^k\varphi_a\right)}{\left(B_k^a - c\sqrt{\mu_0\hbar c}\sin a_c D_k\varphi^a\right)}\end{aligned} \tag{14}$$

where $V = \frac{\lambda hc}{4}\left(\varphi^a\varphi_a - \varphi_0^2\right)^2$ is the scalar field potential., and

$$\tan\alpha_c = \frac{qc}{q_M}. \tag{15}$$

The minimum of (14) in the BPS limit occurs for the Bogomol'nyi equations

$$E_a^k = c\sqrt{\mu_0\hbar c}\sin\alpha_c D^k\varphi_a, \tag{16}$$

$$B_a^k = c\sqrt{\mu_0\hbar c}\cos\alpha_c D^k\varphi_a, \tag{17}$$

The Gauss law for electricity for $E = E^a\frac{\varphi_a}{\varphi_0}$ gives

$$\frac{1}{\varphi_0}\int d^3x E_a^k D_k\varphi^a = \int EdS = \frac{q}{\varepsilon_0}, \tag{18}$$

whereas the Gauss law for magnetism for $B = B^a\frac{\varphi_a}{\varphi_0}$ leads to

$$\frac{1}{\varphi_0}\int d^3x B_a^k D_k\varphi^a = \int BdS = \mu_0 q_M, \tag{19}$$

and, in this manner,

$$q = \frac{4\pi\varepsilon_0 c}{g}C = \frac{q_M}{c}C, C = \tan\alpha_c. \tag{20}$$

Consequently, one receives

$$\begin{aligned}&\varepsilon_{YMH=\hbar c\varphi_0\sqrt{4\pi\alpha}}\left(\frac{q}{e}\sin\alpha_c + \frac{q_M}{ec}\cos\alpha_c\right) \\ &+ \frac{\lambda\hbar c}{4}\int dV\left(\varphi^a\varphi_a - \varphi_0^2\right)^2\end{aligned} \tag{21}$$

where $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$ is the fine-structure constant. Through Einstein's relation $\varepsilon_{YMH} = Mc^2$, the effective mass M satisfies the Bogomol'nyi bound

$$M \geq \frac{\hbar\varphi_0}{c}\sqrt{4\pi\alpha}\left(\frac{q}{e}\sin a_c + \frac{q_M}{ec}\cos a_c\right), \tag{22}$$

And the equality holds in the BPS limit. For the't Hooft–Polyakov monopole, in the BPS limit, one has

$$M = \varphi_0 \sqrt{4\pi\alpha}\, \frac{q_M \hbar}{ec^2}, \tag{23}$$

the total energy density of such a solution is

$$H = \hbar c D_k \varphi^a D^k \varphi_a = \frac{1}{\cos^2 \alpha_c} \frac{B^2}{\mu_0} = \frac{\varepsilon_0}{\sin^2 \alpha_c} E^2, \tag{24}$$

and the Bogomol'nyi equations (17)-(16) become

$$E_k^a = 0, B_a^k = \sqrt{\mu_0 \hbar c} D^k \varphi_a. \tag{25}$$

The subject of non-abelian both magnetic monopoles and dyons has already met a certain interest, Cf. the Refs. [13-44] and references therein.

4. Generalization of Bogomol'nyi Equations

Let us generalize the BPS solution to any λ. Considering the Yang–Mills–Higgs energy in the form $\varepsilon_{\mathrm{YMH}} = \varepsilon_{\mathrm{YM}} + \varepsilon_{\mathrm{H}}$, where

$$\varepsilon_{YM} = \int d^3x \left(\frac{\varepsilon_0}{2} \tilde{E}_k^a \tilde{E}_a^k + \frac{1}{2\mu_0} \tilde{B}_k^a \tilde{B}_a^k \right), \tag{26}$$

where

$$\begin{aligned} &\tilde{E}_k^a = E_k^a - c\sqrt{\mu_0 \hbar c}\alpha_1 D_k \varphi^a - c\sqrt{\mu_0 V}\beta_1 \tilde{\delta}_k^a, \\ &\tilde{E}_a^k = \delta_{ab}\delta^{kj}\tilde{E}_j^b \end{aligned} \tag{27}$$

$$\begin{aligned} &\tilde{B}_k^a = B_k^a - c\sqrt{\mu_0 \hbar c}\alpha_2 D_k \varphi^a - c\sqrt{\mu_0 V}\beta_2 \tilde{\delta}'^a_k, \\ &\tilde{B}_a^k = \delta_{ab}\delta^{kj}\tilde{B}_j^b \end{aligned} \tag{28}$$

and

$$\tilde{\delta}_k^a = \delta_1 \delta_k^a + \delta_2 \frac{\varphi^a \varphi_k}{|\varphi|^2}, \tilde{\delta}_a^k = \delta_{ab}\delta^{kj}\tilde{\delta}_j^b, \tag{29}$$

$$\tilde{\delta}'^a_k = \delta'_1 \delta_k^a + \delta'_2 \frac{\varphi^a \varphi_k}{|\varphi|^2}, \tilde{\delta}'^k_a = \delta_{ab}\delta^{kj}\tilde{\delta}'^b_j, \tag{30}$$

and $\alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2, \delta_1', \delta_2'$ are dimensionless constants, and

$$\begin{aligned} \varepsilon_H = &\sqrt{\frac{\hbar c}{\mu_0}} \int d^3x \left(\alpha_1 \frac{E_k^a}{c} + \alpha_2 B_k^a \right) D^k \varphi_a + \\ &\int d^3x \left(\beta_1 \tilde{\delta}_a^k \frac{E_k^a}{c} + \beta_2 \tilde{\delta}'^k_a B_k^a \right) \sqrt{\frac{V}{\mu_0}} \\ &- \int d^3x \left(\alpha_1 \beta_1 \tilde{\delta}_a^k + \alpha_2 \beta_2 \tilde{\delta}'^k_a \right) \sqrt{\hbar c V} D_k \varphi^a, \end{aligned} \tag{31}$$

where the following consistency conditions hold

$$\alpha_1^2 + \alpha_2^2 = 1, \tag{32}$$

$$\frac{\beta_1^2}{2} \tilde{\delta}_a^k \tilde{\delta}_k^a + \frac{\beta_2^2}{2} \tilde{\delta}'^k_a \tilde{\delta}'^a_k = 1. \tag{33}$$

Applying the Gauss laws (18) and (19), one receives a

$$\begin{aligned} &\sqrt{\frac{\hbar c}{\mu_0}} \int d^3x \left(\alpha_1 \frac{E_k^a}{c} + \alpha_2 B_k^a \right) D^k \varphi_a = \\ &\hbar c \varphi_0 \sqrt{4\pi\alpha} \left(\alpha_1 \frac{q}{e} + \alpha_2 \frac{q_M}{ec} \right), \end{aligned} \tag{34}$$

and after careful calculations, one receives

$$\begin{aligned} &\varepsilon_{YMH} = \varepsilon_{YM} + \hbar c \varphi_0 \sqrt{4\pi\alpha} \left(\alpha_1 \frac{q}{e} + \alpha_2 \frac{q_M}{ec} \right) + \int d^3x V \\ &+ \int d^3x \left(\frac{\beta_1}{2} \tilde{E}_k^a \tilde{\delta}_a^k + \beta_2 \tilde{B}_k^a \tilde{\delta}'^k_a \right) \sqrt{\frac{V}{\mu_0}}. \end{aligned} \tag{35}$$

The minimum of (35) is not only the Higgs vacuum, but also $\tilde{E}_k^a = 0$ and $\tilde{B}_k^a = 0$, giving the BPS limit, or equivalently

$$E_k^a = c\sqrt{\mu_0} \left(\sqrt{\hbar c}\alpha_1 D_k \varphi^a + \sqrt{V}\beta_1 \tilde{\delta}_k^a \right), \tag{36}$$

$$\tilde{B}_k^a = \sqrt{\mu_0} \left(\sqrt{\hbar c}\alpha_2 D_k \varphi^a + \sqrt{V}\beta_2 \tilde{\delta}'^a_k \right) \tag{37}$$

and then the effective mass is

$$M = \frac{\hbar \varphi_0}{c} \sqrt{4\pi\alpha} \left(\alpha_1 \frac{q}{e} + \alpha_2 \frac{q_M}{ec} \right) + \frac{1}{c^2} \int d^3x V. \tag{38}$$

If, moreover, this mass identically vanishes, that is

$$\int d^3x V = -\hbar c \varphi_0 \sqrt{4\pi\alpha} \left(\alpha_1 \frac{q}{e} + \alpha_2 \frac{q_M}{ec} \right) \tag{39}$$

and

$$\frac{\alpha_1}{\alpha_2} = -\frac{q_M}{qc}, \tag{40}$$

then one has the condition

$$\int d^3x V = 0, \tag{41}$$

which includes, as the particular situation, the BPS limit, but in general V is not necessary zero. Similarly, for (38) the particular case

$$M = \frac{\hbar \varphi_0}{c} \sqrt{4\pi\alpha} \left(\alpha_1 \frac{q}{e} + \alpha_2 \frac{q_M}{ec} \right), \tag{42}$$

is true if and only if (41), also for the BPS limit. In general, the bound

$$M \geq \frac{\hbar \varphi_0}{c} \sqrt{4\pi\alpha} \left(\alpha_1 \frac{q}{e} + \alpha_2 \frac{q_M}{ec} \right), \tag{43}$$

which for α1 = sinαc, α2 = cosαc is the Bogomol'nyi bound, is true for

$$\int d^3x V \geq 0. \tag{44}$$

The condition (32) gives eight in equivalent solutions

$$\alpha_1 = \pm \sin\alpha_c \ \ \& \ \ \alpha_2 = \pm \cos\alpha_c, \tag{45}$$

$$\alpha_1 = \pm \cos\alpha_c \ \ \& \ \ \alpha_2 = \pm \sin\alpha_c, \tag{46}$$

whereas the condition (33) presented in the following form

$$\frac{\beta_1^2}{2}\left(D\delta_1^2+\delta_2^2+2\delta_1\delta_2\right)+\frac{\beta_2^2}{2}\left(D\delta_1^{'2}+\delta_2^{'2}+2\delta_1^{'}\delta_2^{'}\right)=1, \quad (47)$$

$$D=\delta_k^a\delta_a^k,$$

and parameterized as A +B = 1, also leads to eight solutions

$$\mathrm{A}=\pm\sin\beta_c \ \ \& \ \ \mathrm{B}=\pm\cos\beta_c, \quad (48)$$

$$\mathrm{A}=\pm\cos\beta_c \ \ \& \ \ \mathrm{B}=\pm\sin\beta_c, \quad (49)$$

where βc must not be related to αc in general. Then, one obtains

$$\begin{aligned}\delta_2&=\delta_1\left(-1\pm\sqrt{\frac{2A^2}{\beta_1^2\delta_1^2}+1-D}\right),\\ \delta_2^{'}&=\delta_1^{'}\left(1\pm\sqrt{\frac{2B^2}{\beta_2^2\delta_1^{'2}}+1-D}\right),\end{aligned} \quad (50)$$

$$\begin{aligned}\delta_1&=\frac{\delta_2}{D}\left(-1\pm\sqrt{\frac{2A^2}{\beta_2^2\delta_2^{'2}}+1-D}\right),\\ \delta_1^{'}&=\frac{\delta_2^{'}}{D}\left(1\pm\sqrt{\frac{2B^2}{\beta_2^2\delta_2^{'2}}+1-D}\right),\end{aligned} \quad (51)$$

what is consistent for

$$\begin{aligned}&\beta_1^2\delta_1^2\ge\frac{2A^2}{D-1},\beta_2^2\delta_1^{'2}\ge\frac{2B^2}{D-1},\\ &\beta_1^2\delta_2^{'2}\ge\frac{2A^2}{D-1},\beta_2^2\delta_2^2\ge\frac{2B^2}{D-1},\end{aligned} \quad (52)$$

The 't Hooft–Polyakov–Julia–Zee ansatz

$$\begin{aligned}A_\mu^a&=\frac{\hbar\varphi_0}{e}\left[\frac{J(\xi)}{\xi}\frac{r^a}{r},\frac{1-K(\xi)}{\xi}\frac{\varepsilon_{ik}^a r^k}{r}\right],\\ \varphi^a&=\varphi_0\frac{H(\xi)}{\xi}\frac{r^a}{r},\xi=\frac{\hbar}{e}g\varphi_0 r,\end{aligned} \quad (53)$$

leads to the relations

$$E_k^a=c\frac{\hbar^2\varphi_0^2}{e^2}\frac{g}{\xi^2}\left[\left(\xi\frac{dJ}{d\xi}-J(K+1)\right)\frac{r^a r_n}{r^2}+JK\delta_n^a\right], \quad (54)$$

$$B_k^a=\frac{\hbar^2\varphi_0^2}{e^2}\frac{g}{\xi^2}\left[\left(1-K^2+\xi\frac{dK}{d\xi}\right)\frac{r^a r_n}{r^2}-\xi\frac{dK}{d\xi}\delta_n^a\right], \quad (55)$$

$$D_i\varphi^a=\frac{\hbar\varphi_0^2}{e^2}\frac{g}{\xi^2}\left[\left(\xi\frac{dH}{d\xi}-KH-H\right)\frac{r^a r_n}{r^2}+\delta_i^a KH\right], \quad (56)$$

which, along with (36) and (37), lead to the system of equations for a dyon

$$\xi\frac{dJ}{d\xi}-J(K+1)=\alpha_1\sqrt{4\pi\alpha}\left(\begin{aligned}&\xi\frac{dH}{d\xi}-H(K+1)\\ &+\frac{\beta_1\delta_2 e}{\alpha_1\hbar g\varphi_0^2}\sqrt{\frac{V}{\hbar c}}\xi^2\end{aligned}\right), \quad (57)$$

$$JK=-\alpha_1\sqrt{4\pi\alpha}\left(KH+\frac{\beta_1\delta_1 e}{\alpha_1\hbar g\varphi_0^2}\sqrt{\frac{V}{\hbar c}}\xi^2\right), \quad (58)$$

$$\xi\frac{dK}{d\xi}-K^2+1=\alpha_2\sqrt{4\pi\alpha}\left(\begin{aligned}&\xi\frac{dH}{d\xi}-H(K+1)\\ &+\frac{\beta_2\delta_2^{'} e}{\alpha_2\hbar g\varphi_0^2}\sqrt{\frac{V}{\hbar c}}\xi^2\end{aligned}\right), \quad (59)$$

$$\xi\frac{dK}{d\xi}=-\alpha_2\sqrt{4\pi\alpha}\left(KH+\frac{\beta_2\delta_1^{'} e}{\alpha_2\hbar g\varphi_0^2}\sqrt{\frac{V}{\hbar c}}\xi^2\right) \quad (60)$$

For the 't Hooft –Polyakov monopole, J = 0, (57), (58), (59) and (60) are

$$\xi\frac{dH}{d\xi}-H(K+1)=-\frac{e}{\hbar g\varphi_0^2}\frac{\beta_1\delta_2}{\alpha_1}\sqrt{\frac{V}{\hbar c}}\xi^2, \quad (61)$$

$$KH=-\frac{e}{\hbar g\varphi_0^2}\frac{\beta_1\delta_1}{\alpha_1}\sqrt{\frac{V}{\hbar c}}\xi^2, \quad (62)$$

$$\begin{aligned}&\xi\frac{dK}{d\xi}-K^2+1\\ &=\alpha_2\sqrt{4\pi\alpha}\frac{e}{\hbar g\varphi_0^2}\left(\frac{\beta_2\delta_2^{'}}{\alpha_2}-\frac{\beta_1\delta_2}{\alpha_1}\right)\sqrt{\frac{V}{\hbar c}}\xi^2,\end{aligned} \quad (63)$$

$$\xi\frac{dK}{d\xi}=-\alpha_2\sqrt{4\pi\alpha}\frac{e}{\hbar g\varphi_0^2}\left(\frac{\beta_2\delta_1^{'}}{\alpha_2}-\frac{\beta_1\delta_1}{\alpha_1}\right)\sqrt{\frac{V}{\hbar c}}\xi^2 \quad (64)$$

and, therefore, one obtains the solutions

$$K=\pm\sqrt{1+\frac{G}{\varphi_0^2}\sqrt{\frac{V}{\hbar c}}\xi^2}, \quad (65)$$

$$H=\mp\frac{e}{\hbar g\varphi_0^2}\frac{\beta_1\delta_1}{\alpha_1}\frac{\sqrt{\frac{V}{\hbar c}}\xi^2}{\sqrt{1+\frac{G}{\varphi_0^2}\sqrt{\frac{V}{\hbar c}}\xi^2}} \quad (66)$$

Consequently, the Higgs field and the gauge field are

$$\varphi^a=\frac{\mp\frac{\beta_1\delta_1}{\alpha_1}\sqrt{\frac{V}{\hbar c}}}{\sqrt{1+\frac{G}{\varphi_0^2}\sqrt{\frac{V}{\hbar c}}\xi^2}}\tau^2, \quad (67)$$

$$A_\mu^a=\left[0,\left(1\mp\sqrt{1+\frac{G}{\varphi_0^2}\sqrt{\frac{V}{\hbar c}}\xi^2}\right)\varepsilon_{ik}^a\frac{r^k}{gr^2}\right], \quad (68)$$

whereas simultaneously the scalar field potential V satisfies the following differential equation

$$\frac{dV}{d\xi}=\frac{-4\pm4\frac{\delta_2+\delta_1}{\delta_1}\left(1+\frac{G}{\varphi_0^2}\sqrt{\frac{V}{\hbar c}}\xi^2\right)^{3/2}}{2+\frac{G}{\varphi_0^2}\sqrt{\frac{V}{\hbar c}}\xi^2}\frac{V}{\xi}, \quad (69)$$

where we have introduced the constant

$$G = \frac{e}{\hbar g}\sqrt{4\pi\alpha}\left(\frac{\alpha_2}{\alpha_1}\beta_1\left(\delta_2+\delta_1\right)-\beta_2\left(\delta_1^{'}+\delta_2^{'}\right)\right). \quad (70)$$

Interestingly, in the BPS limit, the 't Hooft–Polyakov monopole is described through H = 0 and K = ±1. Then, $\varphi^a = 0$, for K = 1 also $A_\mu^a = 0$, while for K = 1 k

$$A_\mu^a = 2\left(A_{\mu\nu}^a\right)_{WY}, \left(A_{\mu\nu}^a\right)_{WY} = \left[0, \varepsilon_{ik}^a \frac{r^k}{gr^2}\right], \quad (71)$$

where $\left(A_{\mu\nu}^a\right)_{WY}$ is the Wu–Yang monopole solving the SU(2) Yang–Mills theory without the Higgs field [45]. Nevertheless, one can also consider the situation wherein G = 0, or, equivalently

$$\frac{\beta_1}{\alpha_1}\left(\delta_2+\delta_1\right) = \frac{\beta_2}{\alpha_2}\left(\delta_1^{'}+\delta_2^{'}\right). \quad (72)$$

Then, the gauge field is either trivial or (71), while the Higgs field is

$$\varphi^a = \mp\frac{\beta_1\delta_1}{\alpha_1}\sqrt{\frac{V}{\hbar c}}\tau^a, \quad (73)$$

and, moreover, the equation (69) becomes

$$\frac{dV}{V} = 2\left(-1\pm\frac{\delta_2+\delta_1}{\delta_1}\right)\frac{d\xi}{\xi}, \quad (74)$$

and its straightforward integration gives two possible solutions δ

$$V_+\left(\xi\right) = V_+\left(\xi_0\right)\left|\frac{\xi}{\xi_0}\right|^{\delta_2/\delta_1}, \quad (75)$$

$$V_-\left(\xi\right) = V_-\left(\xi_0\right)\left|\frac{\xi}{\xi_0}\right|^{-2-\delta_2/\delta_1}, \quad (76)$$

where ξ_0 and $\mathrm{V}\left(\xi_0\right)$ are the integration constants. However, for the special case $\delta_2 = -\delta_1$ and $\mathrm{V}_+\left(\xi_0\right) = \mathrm{V}_-\left(\xi_0\right) = \mathrm{V}_0$, one obtained the unique solution

$$V\left(\xi\right) = V_0\left(\xi_0\right)\left|\frac{\xi}{\xi_0}\right|^{-1} \sim \frac{1}{r}, \quad (77)$$

which is the Coulomb potential. In such a situation, taking $\xi_0 = \frac{\hbar g}{e}\varphi_0 r_0$ and $V_0 = \frac{\lambda\hbar c}{4}\frac{1}{r_0^4}$, where r_0 is any initial value of r, which particularly can be identified with a minimal scale, the configuration of fields is established throughout the gauge field given through the solution (71), where as the Higgs field has the following form

$$\varphi^a = \mp\frac{\beta_1\delta_1}{2\alpha_1}\sqrt{\frac{\lambda}{r_0 r}}\frac{r^a}{r_0}. \quad (78)$$

5. Summary

We have seen that generalization of the Bogomoln'yi equations is realizable, and leads to potentially new results for non-abelian both dyons and magnetic monopoles. It should be emphasized that the idea of this paper was based on the author's monograph [46], and we hope for its further development.

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