Solution of a Class of Differential Equation with Variable Coefficients

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Abstract

In this paper, we obtain the formula of solution to the initial value problem for a hyperbolic partial differential equation with variable coefficient which is the modification of the famous D' Alembert formula.

Keywords: Differential equation with variable coefficient; Solution; D' Alembert formula

1. Introduction

The exact solutions are always not easy to find for differential equations, especially for differential equations with variable coefficients, nonlinear differential equations. Luckily[1], Euler equation as a ordinary differential equation with variable coefficients

$$a_k x^k \frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1} x^{k-1} \frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + a_1 x \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = 0$$

can be solved by variable transformation $y = \ln x$ which satisfies

$$x^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} = k(k-1)\cdots \frac{\mathrm{d}^k}{\mathrm{d}y^k}.$$

Gained enlightenment from Euler equation, the famous Black-Scholes equation[2]

$$\partial_t V + \sigma^2 S^2 \cdot \partial_s^2 V + rS \cdot \partial_S V - rV = 0,$$

was solved after changing it into heat conduct equation $\partial_\tau V - \sigma^2 \partial_T^2 V = 0$ by variable transformation $T = \ln S$ [3]. In this paper, we study the solutions of the following differential equation with variable coefficient:

$$\partial_t^2 u - a^2 (x^2 \partial_x^2 + x\partial_x) u = f(t,x) \tag{1}$$

which is similar to the Black-Scholes equation.

When $x \neq 0$, (1) is hyperbolic (since $\Delta = a^2 x^2 > 0$, $\forall\, x \neq 0$), the initial value problem of which includes two cases[4]:

$$\begin{cases} \partial_t^2 u - a^2 (x^2 \partial_x^2 + x\partial_x) u = f(t,x), & 0 < x < \infty, t > 0, \\ u(0,x) = \phi(x), & 0 < x < \infty, \\ u_t(0,x) = \psi(x), & 0 < x < \infty, \end{cases} \tag{2}$$

and

[1] See for example the book by Wang G X., Zhou Z M, Zhu SM (2007) .
[2] Black F, Scholes, M (1973) proposed this financial model when studying the pricing of options and corporate liabilities.
[3] Consult the book by Jiang L S (2008) for the detail.
[4] On the degenerate line $x = 0$, no boundary conditions are necessary to posed on it. Consult the book by E. DiBeneddetto (1993).

$$\begin{cases} \partial_t^2 u - a^2(x^2\partial_x^2 + x\partial_x)u = f(t,x), & -\infty < x < 0, t > 0, \\ u(0,x) = \phi(x), & -\infty < x < 0, \\ u_t(0,x) = \psi(x), & -\infty < x < 0. \end{cases} \tag{3}$$

When $x = 0$, (1) degenerate to 2-order ordinary differential equation, the initial value problem of which is:

$$\frac{d^2u}{dt^2} = f(t),\ u(0) = u_0,\ \frac{du}{dt}(0) = u_1\ . \tag{4}$$

By the variable transformation $y = \ln x$, the hyperbolic equation with variable coefficients $\partial_{tt}u - a^2(x^2\partial_{xx} + x\partial_x)u = 0$ can be convert into the string vibrating equation $\partial_t^2 u - a^2\partial_y^2 u = 0$. Thus, the solution of (2), (3) can be found by applying the famous D' Alembert formula of the string vibrating equation $\partial_t^2 u - a^2\partial_y^2 u = 0$.

2. Solutions of the Initial Value Problem

Let's recall D' Alembert formula first which is exact solution for initial value problem of string vibrating equation (see i.g. J. Smoller (1994)):

Lemma 1. If $\varphi \in C^2(-\infty,\infty)$, $\psi \in C^1(-\infty,\infty)$ and $f \in C^1[(-\infty,\infty)\times(0,\infty)]$. The initial value problem

$$\begin{cases} \partial_t^2 u - a^2\partial_x^2 u = f(t,x), & -\infty < x < \infty,\ t > 0, \\ u(0,x) = \varphi(x), & -\infty < x < \infty, \\ \partial_t u(0,x) = \psi(x), & -\infty < x < \infty \end{cases}$$

has the unique solution

$$u(t,x) = \frac{\varphi(x-at) + \varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi + \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau\ . \tag{5}$$

Based on (5), we can establish the solution of (2) as follows:

Theorem 2. If $\varphi \in C^2(0,\infty)$, $\psi \in C^1(0,\infty)$ and $f \in C^1[(0,\infty)\times(0,\infty)]$. Then the initial value problem (2) has unique solution

$$u(t,x) = \frac{\phi(e^{\ln x - at}) + \phi(e^{\ln x + at})}{2} + \frac{1}{2a}\int_{\ln x - at}^{\ln x + at}\psi(e^{\xi})d\xi + \frac{1}{2a}\int_0^t\int_{\ln x - a(t-\tau)}^{\ln x + a(t-\tau)} f(e^{\xi},\tau)d\xi d\tau\ . \tag{6}$$

Proof. By the variable transformation $y = \ln x$, we obtain

$$x\partial_x = \partial_y,\ \ x^2\partial_x^2 = \partial_y^2 - \partial_y\ .$$

Applying the above formula to

$$\partial_t^2 u - a^2(x^2\partial_x^2 + x\partial_x)u = 0,\ t > 0,\ 0 < x < \infty,$$

we obtain

$$\partial_t^2 u - a^2\partial_y^2 u = 0,\ t > 0,\ -\infty < y < \infty\ .$$

Thus, initial value problem (2) is converted to

$$\begin{cases} \partial_t^2 u - a^2\partial_y^2 u = f(t,e^y), & -\infty < y < \infty, t > 0, \\ u(0,y) = \phi(e^y), & -\infty < y < \infty, \\ u_t(0,y) = \psi(e^y), & -\infty < y < \infty. \end{cases} \tag{7}$$

Applying the formula (5) to (7), we obtain

$$u(t,y)=\frac{\phi(e^{y-at})+\phi(e^{y+at})}{2}+\frac{1}{2a}\int_{y-at}^{y+at}\psi(e^{\xi})d\xi+\frac{1}{2a}\int_0^t\int_{y-a(t-\tau)}^{y+a(t-\tau)}f(e^{\xi},\tau)d\xi d\tau\,.$$

Since $y=\ln x$, finally we obtain the solution of (2)

$$u(t,x)=\frac{\phi(e^{\ln x-at})+\phi(e^{\ln x+at})}{2}+\frac{1}{2a}\int_{\ln x-at}^{\ln x+at}\psi(e^{\xi})d\xi+\frac{1}{2a}\int_0^t\int_{\ln x-a(t-\tau)}^{\ln x+a(t-\tau)}f(e^{\xi},\tau)d\xi d\tau\,.\quad\square$$

In order to find the solution of (3), we'll establish the relation between (2) and (3) first. To this end, apply transformation $y=-x$ to (3). Then, we have

$$x\partial_x=y\partial_y\,,\quad x^2\partial_x^2=y^2\partial_y^2$$

which implies that (3) can be converted into

$$\begin{cases}\partial_t^2u-a^2(y^2\partial_x^2+y\partial_x)u=f(t,-y), & 0<y<\infty,\ t>0,\\ u(0,y)=\phi(-y), & 0<y<\infty,\\ u_t(0,y)=\psi(-y), & 0<y<\infty.\end{cases}\tag{8}$$

Applying (6) to (8), obtain

$$u(t,y)=\frac{\phi(-e^{\ln y-at})+\phi(-e^{\ln y+at})}{2}+\frac{1}{2a}\int_{\ln y-at}^{\ln y+at}\psi(-e^{\xi})d\xi+\frac{1}{2a}\int_0^t\int_{\ln y-a(t-\tau)}^{\ln y+a(t-\tau)}f(-e^{\xi},\tau)d\xi d\tau\,.$$

Put back $y=-x$ into the above equation, we obtain the formula which we desired:

Theorem 3. If $\varphi\in C^2(-\infty,0)$, $\psi\in C^1(-\infty,0)$ and $f\in C^1[(0,\infty)\times(-\infty,0)]$. Then the initial value problem (1.3) has unique solution

$$u(t,x)=\frac{\phi(-e^{\ln(-x)-at})+\phi(-e^{\ln(-x)+at})}{2}+\frac{1}{2a}\int_{\ln(-x)-at}^{\ln(-x)+at}\psi(-e^{\xi})d\xi+\frac{1}{2a}\int_0^t\int_{\ln(-x)-a(t-\tau)}^{\ln(-x)+a(t-\tau)}f(-e^{\xi},\tau)d\xi d\tau.\tag{9}$$

□

Finally, (4) is the initial value problem of second order ordinary differential equation, its solution is as follows:

$$u(t)=u_1t+u_0+\int_0^t\int_0^{\xi}f(\eta)d\eta d\xi\,.$$

3. Applications

Example 4. Find the solution of the initial value problem：

$$\begin{cases}u_{tt}-a^2(x^2u_{xx}+xu_x)=t+x, & 0<x<\infty,t>0,\\ u(0,x)=x, & 0<x<\infty,\\ u_t(0,x)=1nx, & 0<x<\infty.\end{cases}\tag{10}$$

Let $\phi(x)=x$, $\psi(x)=1nx$, $f(t,x)=t+x$. Apply the formula (6) to (10), we obtain：

$$u(t,x)=\frac{e^{1nx+at}+e^{1nx-at}}{2}+t\cdot 1nx+\frac{1}{6}t^3+\frac{x}{2a^2}(e^{at}+e^{-at}-2)\,.\quad\square$$

Example 5. Find the solution of the initial value problem:

$$\begin{cases}u_{tt}-a^2(x^2u_{xx}+xu_x)=t+x, & -\infty<x<0,t>0,\\ u(0,x)=\sin x, & -\infty<x<0,\\ u_t(0,x)=x, & -\infty<x<0.\end{cases}\tag{11}$$

Let $\phi(x)=\sin x$, $\psi(x)=x$, $f(t,x)=t+x$. Apply formula (9) to (11), we obtain：

$$u(t,x) = -\frac{\sin(e^{\ln(-x)+at}) + \sin(e^{\ln(-x)-at})}{2} - \frac{e^{\ln(-x)+at} - e^{\ln(-x)-at}}{2a} + \frac{t^3}{6} + \frac{x(e^{at} + e^{-at} - 2)}{2a^2}. \quad \square$$

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