

SUPERSTABILITY IN ABSTRACT ELEMENTARY CLASSES

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ABSTRACT. We prove that several definitions of superstability in abstract elementary classes (AECs) are equivalent under the assumption that the class is tame, has amalgamation, joint embedding, and arbitrarily large models. This partially answers questions of Shelah.

Theorem 0.1. Let K be a tame AEC with amalgamation, joint embedding, and arbitrarily large models. Assume K is stable. Then the following are equivalent:

- (1) For all high-enough λ , there exists $\kappa \leq \lambda$ such that there is a good λ -frame on the class of κ -saturated models in K_λ .
- (2) For all high-enough λ , K has a unique limit model of cardinality λ .
- (3) For all high-enough λ , K has a superlimit model of cardinality λ .
- (4) For all high-enough λ , the union of a chain of λ -saturated models is λ -saturated.
- (5) There exists θ such that for all high-enough λ , K is (λ, θ) -solvable.

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1. INTRODUCTION

In the context of classification theory for AECs, a notion analog to the first-order notion of *stability* exists: it is defined as one might expect (by counting Galois types). However it is still unclear what a parallel notion to superstability might be. Recall that for first-order theories we have the following:

Fact 1.1. Let T be a first-order complete theory, the following are equivalent

- (1) T is stable in every cardinal $\lambda \geq 2^{|T|}$.
- (2) For all λ the union of an increasing chain of λ -saturated models is λ -saturated.
- (3) $\kappa(T) = \aleph_0$ and T is stable.
- (4) There exists a cardinal $\mu \geq |T|$ such that T has a saturated model of cardinality λ for every $\lambda \geq \mu$.
- (5) T is stable and $D^n[\bar{x} = \bar{x}, L(T), \infty] < \infty$.
- (6) There does not exist a set of formulas $\Phi = \{\varphi_n(\bar{x}; \bar{y}_n) \mid n < \omega\}$ such that Φ can be used to code the structure $(\omega^{<\omega}, <, <_{lex})$

All the implications appear in Shelah's book [She90] with the exception of (2) \implies (6) which was established by Albert and Grossberg [AG90, Theorem 13.2].

In the last 30 years, in the context of classification theory for non elementary classes, several notions that generalize that of first-order superstability have been considered. See papers by Grossberg, Shelah, VanDieren, Vasey and Villaveces: [GS86, Gro88], [She99], [SV99], [Van06, Van13], [GVV], [Vasa, Vasb].

In [Shea, Discussion 2.9] Shelah mentions that part of the program of classification theory for AECs is to show that all the various notions of first-order saturation (limit, superlimit, or model-homogeneity, see Section 3) are equivalent under the assumption of superstability. A possible definition of superstability is *solvability*, which appears in the introduction to [She09a] and is hailed as a true counterpart to first-order superstability. Full justification is delayed to [Sheb] but [She09a,

Chapter IV] already uses it. Other definitions of superstability analog to the ones in Fact 1.1 can also be formulated. The main result of this paper is to accomplish that the above conjecture of Shelah holds for tame AECs, and that in addition several definitions of superstability that previously appeared in the literature are equivalent in this context.

Theorem 1.2 (Main Theorem). Let K be a tame AEC with amalgamation, joint embedding, and arbitrarily large models. Assume K is stable. Then the following are equivalent:

- (1) There exists $\mu_1 \geq \text{LS}(K)$ such that for every $\lambda \geq \mu_1$, for all $\delta < \lambda^+$, for all increasing continuous $\langle M_i : i \leq \delta \rangle$ in K_λ and all $p \in \text{gS}(M_\delta)$, if M_{i+1} is universal over M_i for all $i < \delta$, then there exists $i < \delta$ such that p does not λ -split over M_i .
- (2) There exists $\mu_2 \geq \text{LS}(K)$ such that for every $\lambda \geq \mu_2$, for some $\kappa \leq \lambda$, there is a good λ -frame on $K_\lambda^{\kappa\text{-sat}}$.
- (3) There exists $\mu_3 \geq \text{LS}(K)$ such that for every $\lambda \geq \mu_3$, K has uniqueness of limit models in cardinality λ .
- (4) There exists $\mu_4 \geq \text{LS}(K)$ such that for every $\lambda \geq \mu_4$, K has a superlimit model of cardinality λ .
- (5) There exists $\mu_5 \geq \text{LS}(K)$ such that for every $\lambda \geq \mu_5$, the union of a chain of λ -saturated models is λ -saturated.
- (6) There exists $\mu_6 \geq \text{LS}(K)$ such that for all $\lambda \geq \mu_6$, K is (λ, μ_6) -solvable.

Proof. Combine Theorem 4.8 and Theorem 5.28. □

At present, we do not know how to prove an analogs to the last two properties of Fact 1.1. More on this in Section 7.

Interestingly, the proof does not tell us that the threshold cardinals μ_ℓ above are equal. In fact, it uses tameness heavily to move from one cardinal to the next and use e.g. that one equivalent definition holds below λ to prove that another definition holds at λ . Showing equivalence of these definitions cardinal by cardinal, or even just showing that we can take $\mu_1 = \mu_2 = \dots = \mu_6$ seems much harder. In section 6, we show that the statements are still equivalent if we require $\mu_\ell < \beth_{(2^{\text{LS}(K)})^+}$, provided that the class is $(< \text{LS}(K))$ -tame. In a forthcoming paper [Vanb] VanDieren gives some relationships between versions of (3) and (5) in a single cardinal. This is done without assuming tameness using very different technologies than in this paper.

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2. PRELIMINARIES

We now review some of the basics of abstract elementary classes and fix some notation. The reader is advised to skim through this section quickly and go back to it as needed. We assume familiarity with a basic text on AECs such as [Bal09] or [Gro] and refer the reader to the preliminaries of [Vasc] for more details and motivations on the concepts used in this paper. Throughout this section, K is an AEC. For λ an infinite cardinal, define $h(\lambda) := \beth_{(2^\lambda)^+}$.

Shelah's program of classification theory for Abstract Elementary Classes started in 1977 with a circulation of a draft of [She87] (a revised version is [She09a, Chapter I]). As a full classification theory is impossible due to various counterexamples and immense technical difficulties of addressing some of the main conjectures, all known non-trivial results are obtained under some additional model-theoretic or even set-theoretic assumptions on the family of classes we try to develop structure/non-structure results for. On July 28 2001, Grossberg and VanDieren circulated a draft of a paper titled "Morley Sequences in Abstract Elementary Classes" (a revised version was published as [GV06b]). In that paper, they introduced the assumption of tameness as a useful assumption to prove existence of Morley sequence with respect to non-splitting in stable AECs and upward stability results.

Definition 2.1 (Definitions 3.2 in [GV06b]). Let χ be an infinite cardinal. K is $(< \chi)$ -tame if for any $M \in K$ and any $p \neq q$ in $\text{gS}(M)$, there exists $A \subseteq |M|$ such that $|A| < \chi$ and $p \upharpoonright A \neq q \upharpoonright A$. K is χ -tame if it is $(< \chi^+)$ -tame.

We say that K is tame provided there exists a cardinal χ such that the class K is $(< \chi)$ -tame.

In [GV06c] and [GV06a] Grossberg and VanDieren established several cases of Shelah's categoricity conjecture (which is still the best known 40 year old open problem in the field). At the time the main justification of the tameness assumption was that it appears in all known cases of structural results and it seems to be difficult to construct non-tame classes. It was proposed by Grossberg that perhaps almost-free non free abelian groups could be used for such an example. Within short time Baldwin and Shelah [BS08] used almost free non Whitehead groups in \aleph_1 to construct such a non \aleph_1 -tame class. In our opinion,

this result does not have interesting model-theoretic consequences, as the real notion is whether K is tame, not so much what the exact cardinal witnessing tameness (or its failure) is. In 2013, Will Boney [Bon14b] derived from the existence of a class of strongly compact cardinals that all AECs are tame. In a preprint from 2014 Lieberman and Rosický [LR] pointed out that this result of Boney follows from a 25 year old theorem of Makkai and Paré ([MP89, Theorem 5.5.1]). In a forthcoming paper Boney and Unger [BU] establish that if every AEC is tame then a proper class of large cardinals exists. Thus tameness (for all AECs) is a large cardinal axiom. We believe that this is evidence for the assertion that tameness is a new interesting model-theoretic property, a new dichotomy, that follows from categoricity in a “high-enough” cardinal.

A definition of superstability analog to $\kappa(T) = \aleph_0$ in first-order model theory has been studied in AECs (see [SV99, GVV, Van06, Van13, Vasa, Vasb]). Since it is not immediately obvious what forking should be in that framework, the simpler independence relation of μ -splitting is used for the purpose of the definition. Moreover in AECs, types over models are much better behaved than types over sets, so it does not make sense in general to ask for every type to not split over a finite set¹. Thus we require that every type over the union of a chain does not split over a model in the chain. For technical reasons (essentially because it makes it much easier to prove that the condition holds), we require the chain to be increasing with respect to universal extension. This gives a reformulation of (1) in Theorem 1.2:

Definition 2.2. Let $\lambda \geq \text{LS}(K)$. We say K has $(*)_\lambda$ if for any regular $\delta < \lambda^+$, any $\langle M_i : i < \delta \rangle$ in K_λ with $M_i <_{\text{univ}} M_{i+1}$ for all $i < \delta$, any $p \in \text{gS}(\bigcup_{i < \delta} M_i)$, there exists $i < \delta$ such that p does not λ -split over M_i .

Remark 2.3. In the notation² of [Vasb, Definition 3.14], $(*)_\lambda$ holds if and only if $\kappa_1(i_{\lambda\text{-ns}}(K_\lambda), <_{\text{univ}}) = \aleph_0$.

Definition 2.4 (Superstability).

- (1) For $M, N \in K$, say $M <_{\text{univ}} N$ (N is *universal over* M) if and only if $M < N$ and whenever we have $M' \geq M$ such that $\|M'\| \leq \|N\|$, then there exists $f : M' \xrightarrow[M]{\rightarrow} N$. Say $M \leq_{\text{univ}} N$ if and only if $M = N$ or $M <_{\text{univ}} N$.
- (2) K is μ -superstable if:

¹But see [Vasd, Theorem C.15].

²Of course, the κ notation has a long history, appearing first in [She70].

- (a) $\text{LS}(K) \leq \mu$.
- (b) There exists $M \in K_\mu$ such that for any $M' \in K_\mu$ there is $f : M' \rightarrow M$ with $f[M'] <_{\text{univ}} M$.
- (c) $(*)_\mu$ holds.

Remark 2.5. It is easy to check that Condition (2b) is equivalent to “ K_μ is nonempty, has amalgamation, joint embedding, no maximal models, and is stable in μ ”. Thus if K is nonempty, has amalgamation, joint embedding, and no maximal models, then for $\mu \geq \text{LS}(K)$, K is μ -superstable if and only if K is stable in μ and $(*)_\mu$ holds.

Remark 2.6. While Definition 2.4 makes sense in any AEC, here we focus on tame AECs with amalgamation, and will not study what happens to Definition 2.4 without these assumptions (although, as said above, the notion was first introduced in [SV99] without even amalgamation, and it has been further studied in [GVV] or even more generally [Van06, Van13], see also the forthcoming [Vana]).

For the convenience of the reader, we recall some facts about superstability for tame AECs with amalgamation.

Fact 2.7. Let K be an AEC with amalgamation.

- (1) [Vasb, Proposition 10.10] If K is μ -superstable, μ -tame, and $\mu' \geq \mu$, then K is μ' -superstable. In particular, $K_{\geq \mu}$ has joint embedding, no maximal models, and is stable in all cardinals.
- (2) [Vasb, Theorem 10.16]³ If K is $(< \kappa)$ -tame with $\kappa = \beth_\kappa > \text{LS}(K)$ and categorical in a $\lambda > \kappa$, then K is κ -superstable.

3. DEFINITIONS OF SATURATED

The search for a good definition of “saturated” in AECs is central. Perhaps the most natural one is:

Definition 3.1. Let $M \in K$ and let μ be a cardinal. M is μ -saturated if for any $N \geq M$, any $A \subseteq |M|$ of size less than μ , any $p \in \text{gS}(A; N)$ is realized in M . When $\mu = \|M\|$, we omit it.

We write $K^{\mu\text{-sat}}$ for the class of μ -saturated models in $K_{\geq \mu}$.

Remark 3.2. When $\mu = 0$, $K^{\mu\text{-sat}} = K$.

In [She01, Lemma 0.26] (see also [Gro02, Theorem 6.7] for a proof), it is observed that (under the amalgamation property) M saturated is

³The proof uses of [SV99, Theorem 2.2.1] and indeed it turns out that this theorem suffices to get an even stronger result, see Theorem 6.3.

equivalent to M model-homogeneous. This provides some justification for Definition 3.1 under amalgamation.

However when there is no amalgamation the following notion has played a central role (their study without amalgamation really started with [SV99] and was continued in [Van06, Van13]).

Definition 3.3 (Limit model).

- (1) $M \in K$ is *limit* over M_0 if $M_0 \leq M$ of the same cardinality and there exists a limit ordinal δ and an increasing continuous $\langle N_i : i \leq \delta \rangle$ such that $M_0 = N_0$, $M = N_\delta$, and $N_i <_{\text{univ}} N_{i+1}$ for all $i < \delta$. M is *limit* if it is limit over some M_0 .
- (2) We say K has *uniqueness of limit models in λ* if whenever $M_0 \in K_\lambda$ and M_1, M_2 are limit over M_0 in K_λ then $M_1 \cong_{M_0} M_2$. We say that K has *weak uniqueness of limit models in λ* if we only require $M_1 \cong M_2$.

Even with the amalgamation property, uniqueness of limit models is a key concept which is equivalent to superstability in first-order model theory (see [GVV, Theorem 6.1]). In fact, limit models are saturated precisely when this holds:

Fact 3.4. Let $\lambda > \text{LS}(K)$. Assume K_λ has amalgamation, joint embedding, no maximal models, and is stable in λ .

- (1) For any $M \in K_\lambda$, there exists $N \in K_\lambda$ such that $M <_{\text{univ}} N$. Thus there exists a limit model over M in K_λ .
- (2) K has weak uniqueness of limit models in λ if and only if any limit $M \in K_\lambda$ is saturated.

Proof.

- (1) See⁴ [She09a, Claim II.1.16] or [GV06b, Theorem 2.12].
- (2) This is folklore, so we include a proof. The right to left direction is by uniqueness of saturated models. For the left to right, assume weak uniqueness of limit models in K_λ and let $M \in K$ be limit. Let $A \subseteq |M|$ have size less than λ . Let $\delta := |A|^+$. Note that $\delta \leq \lambda$. By weak uniqueness of limit models, there exists $\langle M_i : i \leq \delta \rangle$ increasing continuous such that $M_\delta = M$ and $M_i <_{\text{univ}} M_{i+1}$ for all $i < \delta$. Pick $i < \delta$ such that $A \subseteq |M_i|$. Then any type over A is realized in M_{i+1} , as needed.

⁴The result first appeared without proofs in early versions of [She09a, Chapter II].

□

Another notion of saturation appears in [She87, Definition 3.1.1]⁵. The idea is to encode the fact that a union of saturated models should be saturated.

Definition 3.5. Let $M \in K$. We say M is a *superlimit in λ* if:

- (1) $M \in K_\lambda$.
- (2) M is “properly universal”: For any $N \in K_\lambda$, there exists $f : N \rightarrow M$ such that $f[N] < M$.
- (3) Whenever $\langle M_i : i < \delta \rangle$ is an increasing chain in K_λ , $\delta < \lambda^+$ and $M_i \cong M$ for all $i < \delta$, then $\bigcup_{i < \delta} M_i \cong M$.

Again we can ask when superlimits are saturated. The next lemma is a generalization of [Dru13, Corollary 2.3.12] (there $\chi = \lambda$).

Lemma 3.6. Assume K has amalgamation, joint embedding, and no maximal models. Let $\lambda > \text{LS}(K)$ be such that:

- There is a saturated model in K_λ .
- There exists a regular $\chi \leq \lambda$ such that for any increasing $\langle M_i : i < \chi \rangle$ in K_λ , if M_i is saturated for all $i < \chi$, then $\bigcup_{i < \chi} M_i$ is saturated.

The following are equivalent:

- (1) There is a superlimit model in K_λ .
- (2) In K_λ , the union of a chain of saturated models is saturated.

Proof. If in K_λ the union of a chain of saturated models is saturated, then the saturated model of size λ is a superlimit. Conversely, if K has a superlimit M in λ , it is enough to show that M is saturated. We build $\langle M_i : i < \chi \rangle, \langle N_i : i < \chi \rangle$ increasing in K_λ such that for all $i < \chi$, $M_i \leq N_i \leq M_{i+1}$, $M_i \cong M$ is superlimit, and N_i is saturated. In the end, $\bigcup_{i < \chi} M_i = \bigcup_{i < \chi} N_i$ is superlimit since it is a union of superlimit and saturated by definition of χ . Moreover, it is isomorphic to M , hence M is saturated. □

Thus (by the proof) under the assumptions of the lemma, superlimit and saturated coincide if chains of saturated models are saturated (another equivalent definition of superstability in the first-order case). In the remainder of this sections, we establish more implications between

⁵We use the definition in [She09a, Definition 2.4.4] which requires in addition that the model be universal.

uniqueness of limit models, union of saturated being saturated, and existence of a superlimit. We assume:

Hypothesis 3.7. K is a stable ($< \text{LS}(K)$)-tame AEC with amalgamation, joint embedding, and arbitrarily large models.

Results on uniqueness of limit models can be related to chains of saturated models as follows⁶:

Lemma 3.8. Let $\lambda > \text{LS}(K)$ be a limit cardinal. Assume that for unboundedly many $\mu < \lambda$, K is stable in μ and has weak uniqueness of limit models in μ . Then any increasing chain of saturated models in K_λ is saturated.

Proof. Let $\langle M_i : i < \delta \rangle$ be an increasing chain of λ -saturated models in K_λ with (without loss of generality) $\delta = \text{cf}(\delta) < \lambda$ and let $M_\delta := \bigcup_{i < \delta} M_i$. We want to see that M_δ is λ -saturated. Let $A \subseteq |M_\delta|$ have size less than λ . Let $\mu_0 := (\delta + \text{LS}(K) + |A|)^+$. Since λ is limit, $\mu_0 < \lambda$. Let $\mu \geq \mu_0$ be such that $\mu < \lambda$, K is stable in μ , and K has weak uniqueness of limit models in μ . Let $\langle M'_i : i \leq \delta \rangle$ be increasing continuous in K_μ such that for all $i < \delta$, $M'_i \leq M_i$, $(A \cap |M_i|) \subseteq |M'_i|$, and $M'_i <_{\text{univ}} M'_{i+1}$. Then M'_δ is (μ, δ) -limit, so by uniqueness of limit models is also (μ, μ_0) -limit. Also, M'_δ contains A , so by cofinality consideration, it must realize all types over A . As $M'_\delta \leq M_\delta$, M_δ realizes all types over A . \square

We now want to relate chains of saturated models and superlimit using Lemma 3.6. For this, we recall that the assumptions of this lemma hold in our context:

Fact 3.9 (Theorem 6.10 in [BV]). There exists $\chi < h(\text{LS}(K))$ such that if $\langle M_i : i < \delta \rangle$ is an increasing chain of λ -saturated models and:

- (1) $\text{cf}(\delta) \geq \chi$.
- (2) K is stable in unboundedly many $\mu < \lambda$.

Then $\bigcup_{i < \delta} M_i$ is λ -saturated.

Fact 3.10 (Theorem 4.13 in [Vasc]). There exists $\chi < h(\text{LS}(K))$ such that K is stable in any $\lambda = \lambda^{< \chi}$.

Lemma 3.11. There exists a regular $\chi < h(\text{LS}(K))$ and unboundedly many cardinals λ such that:

- (1) K is stable in λ

⁶The argument appears already as [Bal09, Theorem 10.22]

- (2) Any $M \in K_\lambda$ extends to a saturated $N \in K_\lambda$.
- (3) If $\langle M_i : i < \chi \rangle$ is an increasing chain of saturated models in K_λ , then $\bigcup_{i < \chi} M_i$ is saturated.

Proof. Fix $\chi < h(\text{LS}(K))$ regular satisfying the conclusions of both Fact 3.9 and Fact 3.10. Let $\lambda = \lambda^{<\chi}$ be such that $\mu^{<\chi} < \lambda$ for all $\mu < \lambda$. There are unboundedly many such limit λ by an easy “catching your tail” argument. Then K is stable in λ and in unboundedly many $\mu < \lambda$. Thus it is easy to check that any $M \in K_\lambda$ extends to a saturated model of size λ . \square

Theorem 3.12. Assume either of the following conditions hold:

- (1) K has weak uniqueness of limit models in all high-enough cardinals.
- (2) K has a superlimit model in all high-enough cardinals.

Then for unboundedly many λ , K is stable in λ and has a saturated superlimit model in λ .

Proof. If the first condition holds, then by Lemma 3.8, in any limit cardinal λ a chain of saturated models is saturated. In particular we can take λ as given by Lemma 3.11. Then K has a saturated model in λ and it is clearly superlimit. If the second condition holds, let again λ be high-enough satisfying the conclusion of Lemma 3.11. By Lemma 3.6, the union of a chain of saturated models in K_λ is saturated, and thus the saturated model in K_λ is superlimit. \square

In the next section we show how existence of a saturated superlimit implies superstability.

4. CHAIN LOCAL CHARACTER OVER SATURATED MODELS

Hypothesis 4.1. K is an AEC with amalgamation.

For background, we cite the following result, proven in [MS90, Proposition 4.12] for models of an $L_{\kappa,\omega}$ theory, κ strongly compact, and in [BG, Theorem 8.2.2] for AECs.

Fact 4.2. Let $\kappa > \text{LS}(K)$ be strongly compact. Let $\langle M_i : i \leq \delta \rangle$ be an increasing continuous chain of κ -saturated models (so M_i is κ -saturated also for limit i , including $i = \delta$). Let $p \in \text{gS}(M_\delta)$. If M_δ is κ -saturated, then there exists $i < \delta$ such that p is $(< \kappa)$ -satisfiable over M_i .

The assumption above that M_δ is κ -saturated is crucial (otherwise we would have proven superstability from just stability, which is impossible even in the first-order case). The proof uses the strongly compact to build an appropriate ultrafilter and taking an ultraproduct of the chain $\langle M_i : i \leq \delta \rangle$ in which p is realized. Here, we give a simpler proof that does not need that κ is strongly compact but only that κ is regular uncountable.

Lemma 4.3. Let $\kappa > \aleph_0$ be a regular cardinal. Let $\langle M_i : i \leq \delta \rangle$ be increasing continuous. Let $p \in \text{gS}^\alpha(M_\delta)$ with $|\alpha|^+ < \kappa$. If for all $i \leq \delta$, $p \upharpoonright M_i$ is $(< \kappa)$ -satisfiable over M_i and $\alpha < \text{cf}(\delta)$, then there exists $i < \delta$ such that p is $(< \kappa)$ -satisfiable over M_i .

Proof. Without loss of generality, $\delta = \text{cf}(\delta)$. Suppose for a contradiction that the conclusion fails, i.e. for every $i < \delta$, p is *not* $(< \kappa)$ -satisfiable over M_i . We consider two cases:

- Case 1: $\delta < \kappa$ Build $\langle A_i : i < \delta \rangle$ increasing such that for all $i < \delta$:
 - (1) $A_i \subseteq |M_\delta|$.
 - (2) $|A_i| < \kappa$.
 - (3) $p \upharpoonright A_i$ is not realized in M_i .

This is possible by the assumption on p and δ (we are also using that κ is regular to ensure that $|A_i| < \kappa$ is preserved at limit steps). This is enough: let $A := \bigcup_{i < \delta} A_i$. Note that $|A| < \kappa$ since $\delta < \kappa = \text{cf}(\kappa)$. As p is $(< \kappa)$ -satisfiable over M_δ , $p \upharpoonright A$ is realized in M_δ , say by \bar{b} . As $\ell(\bar{b}) = \alpha < \delta$, there exists $i < \delta$ such that $\bar{b} \in {}^\alpha |M_i|$. But then $p \upharpoonright A$, and therefore $p \upharpoonright A_i$, is realized in M_i by \bar{b} , contradicting (3).

- Case 2: $\delta \geq \kappa$ Let $\gamma := |\alpha|^+ + \aleph_0$. Note that γ is regular and (since κ is uncountable and $|\alpha|^+ < \kappa$), $\gamma < \kappa$. Build $\langle i_j : j \leq \gamma \rangle$ increasing continuous in δ such that for all $j < \gamma$, $p \upharpoonright M_{i_{j+1}}$ is *not* $(< \kappa)$ -satisfiable over M_{i_j} . This is possible by the assumption on p and δ (and the fact that whenever a type is not $(< \kappa)$ -satisfiable, there is a witness of size less than κ). This is enough: by construction, $p \upharpoonright M_{i_\gamma}$ is not $(< \kappa)$ -satisfiable over M_{i_j} for all $j < \gamma$. Since $\gamma = \text{cf}(\gamma) < \kappa$, this contradicts the first part.

□

Recall that in [Vasb, Definition 3.14] (or Definition 2.2), the locality cardinal for chain was defined without assuming that the union of the

chain was in the class. The above results shows that this is a necessary choice: Otherwise we could get strictly stable elementary classes in which $\kappa_1 = \aleph_0$. This also outlines the subtle difference between the chain and set local character cardinals, even in the elementary context. For example:

Corollary 4.4. Let T be a stable first-order theory. If $\langle M_i : i \leq \delta \rangle$ is an increasing continuous chain of \aleph_1 -saturated models (so M_i is \aleph_1 -saturated *also for limit i , including $i = \delta$*), $p \in S(M_\delta)$, then there exists $i < \delta$ such that p does not fork over M_i .

Proof. Set $\kappa = \aleph_1$ and $(K, \leq) = (\text{Mod}(T), \preceq)$ in Lemma 4.3. \square

Remark 4.5. This gives a quicker, more general, proof of [AG90, Theorem 13.2].

Question 4.6. Does Corollary 4.4 say anything nontrivial? For example, let T be a countable first-order theory and assume it is stable but not superstable. Let $\lambda \geq \aleph_1$. When can we build an increasing continuous chain $\langle M_i : i \leq \delta \rangle$ of \aleph_1 -saturated models of T of size λ ?

We can now prove most conditions in the main theorem. For a start, we point out that among the definitions in the statement of Theorem 1.2, the first one has been used as a starting hypothesis many times previously. In [Vasa], λ -superstability was shown to imply the existence of a good λ^+ -frame on the class of saturated models of size λ^+ , except that this class may not be an AEC. The result was later generalized in [Vasb]. Building on these papers, [BV] gave conditions under which a chain of saturated models is saturated, culminating in:

Fact 4.7 (Theorem 7.1 in [BV]). If K is μ -superstable (for some μ), then in all high-enough cardinals λ , $K_\lambda^{\lambda\text{-sat}}$ has a type-full good λ -frame, K has uniqueness of limit models, has a saturated superlimit model, and any chain of λ -saturated models is λ -saturated.

Thus the first condition in Theorem 1.2 implies all the other ones, except perhaps solvability (examined in Section 5).

We now restate and prove Theorem 1.2 with more conditions added (but without solvability):

Theorem 4.8. Let K be a nonempty tame AEC with amalgamation, joint embedding, and no maximal models. Assume K is stable. Then the following are equivalent:

- (1) For all high-enough λ , $(*)_\lambda$ holds.

- (2) For all high-enough cardinal λ , for some $\kappa \leq \lambda$, there is a good λ -frame on $K_\lambda^{\kappa\text{-sat}}$.
- (3) K has uniqueness of limit models in all high-enough cardinals.
- (4) K has a superlimit model in all high-enough cardinals.
- (5) For all high-enough cardinal λ , the union of a chain of λ -saturated models is λ -saturated.
- (6) For unboundedly many λ , K is stable in λ and has a saturated superlimit model in λ .

Proof. Without loss of generality, K is ($< \text{LS}(K)$)-tame. Note that with our background assumptions, $(*)_\lambda$ together with stability in λ is equivalent to K being λ -superstable (Remark 2.5).

First assume (1). Since K is stable, we can pick $\lambda \geq \text{LS}(K)$ such that K is stable in λ and satisfies $(*)_\lambda$, and hence is λ -superstable. By Fact 4.7, (2)-(6) all follow. Assume (2). By [She09a, Lemma II.4.8] (or see [Bon14a, Theorem 9.2]), (3) holds. Also, (5) directly implies (4). Moreover by Theorem 3.12, both (3) and (4) imply (6). To sum up, we have shown:

- (1) implies (5).
- (1) implies (2) implies (3) implies (6).
- (5) implies (4) implies (6).

Thus all the conditions imply (6) and (1) implies all the conditions, so it only remains to show (6) implies (1). Assume (6), fix a $\kappa > \text{LS}(K)$ such that $\kappa = \beth_\kappa$ and let $\lambda > \kappa$ witness (6). Let $\langle M_i : i < \delta \rangle$ be an increasing chain of saturated models in K_λ , $\delta < \lambda^+$. Then $M_\delta := \bigcup_{i < \delta} M_i$ is saturated. Therefore by Lemma 4.3 (with κ there standing for κ^+ here), any $p \in \text{gS}(M_\delta)$ is ($< \kappa^+$)-satisfiable (and hence ($< \kappa$)-satisfiable) over some M_i , $i < \delta$. By [Vasb, Fact 3.17.2d, Fact 3.20.3], any $p \in \text{gS}(M_\delta)$ does not λ -split over M_i for some $i < \delta$. Thus $(*)_\lambda$ holds so K is λ -superstable. By Fact 2.7, K is also λ' -superstable for all $\lambda' \geq \lambda$, so (1) holds. \square

Remark 4.9. In (2), we do *not* assume that the good frame is type-full (i.e. it may be that there exists some nonalgebraic types which are not basic, so fork over their domain). However if (1) holds, then the proof shows we can take the frame to be type-full. Therefore, in the presence of tameness, the existence of a good frame implies the existence of a *type-full* good frame (in a potentially much higher cardinal, and over more saturated models).

5. SOLVABILITY

Solvability appears as a possible definition of superstability for AECs in [She09a, Chapter IV]. In the introduction to the book, Shelah claims (without proof) that it is equivalent to first-order superstability. We give a proof here and actually show (assuming amalgamation, stability, and tameness) that solvability is equivalent to any of the definitions in the main theorem. This partially answers some questions of Shelah on [She09a, p. 56].

Definition 5.1. Let K be an AEC and let $\theta \leq \lambda$ be such that $\text{LS}(K) \leq \theta$.

- (1) [She09a, Definition IV.0.8.2] Let $\Upsilon_\theta[K]$ be the set of Φ proper for linear orders with:
 - (a) $|L(\Phi)| \leq \theta$.
 - (b) For I a linear order, $\text{EM}(I, \Phi)_{L(K)} \in K$.
 - (c) For $I \subseteq J$ linear orders, $\text{EM}(I, \Phi)_{L(K)} \leq \text{EM}(J, \Phi)_{L(K)}$.
- (2) [She09a, Definition IV.1.4.1] We say that Φ *witnesses* (λ, θ) -solvability if:
 - (a) $\Phi \in \Upsilon_\theta[K]$.
 - (b) If I is a linear order of size λ , then $\text{EM}(I, \Phi)_{L(K)}$ is superlimit of size λ .
 K is (λ, θ) -solvable if there exists Φ witnessing (λ, θ) -solvability.
- (3) K is *uniformly* (λ, θ) -solvable if there exists Φ such that for all $\mu \geq \lambda$, Φ witnesses (μ, θ) -solvability.

Remark 5.2. If K is uniformly (λ, θ) -solvable, then K is (μ, θ) -solvable for all $\mu \geq \lambda$.

Fact 5.3. Let K be an AEC and let $\theta \geq \text{LS}(K)$. Then K has arbitrarily large models if and only if $\Upsilon_\theta[K] \neq \emptyset$.

We start by giving some more manageable definitions of solvability. Shelah already mentions one of them on [She09a, p. 53] (but does not prove it is equivalent).

Proposition 5.4. Let K be an AEC and let $\text{LS}(K) \leq \theta \leq \lambda$. The following are equivalent.

- (1) K is [uniformly] (λ, θ) -solvable.
- (2) There exists $L' \supseteq L(K)$ with $|L'| \leq \theta$ and $\psi \in L'_{\theta^+, \omega}$ such that:
 - (a) ψ has arbitrarily large models.
 - (b) [For all $\mu \geq \lambda$], if $M \models \psi$ and $\|M\| = \lambda$ [$\|M\| = \mu$], then $M \upharpoonright L(K)$ is in K and is superlimit.

- (3) There exists $L' \supseteq L(K)$ and an AEC K' with $L(K') = L'$, $\text{LS}(K') \leq \theta$ such that:

(a) K' has arbitrarily large models.

For all $\mu \geq \lambda$, if $M \in K$ and $\|M\| = \lambda$ [$\|M\| = \mu$], then $M \upharpoonright L(K)$ is in K and is superlimit.

Proof.

- (1) implies (2): Let Φ witness (λ, θ) -solvability. Let $\Phi = \{p_n \mid n < \omega\}$. Let $L' := L(\Phi) \cup \{P, <\}$, where $P, <$ are symbols for a unary predicate and a binary relation respectively. Let $\psi \in L'_{\theta^+, \omega}$ say:
 - (1) $(P, <)$ is a linear order.
 - (2) For all $n < \omega$ and all $x_0 < \dots < x_{n-1}$ in P , $x_0 \dots x_{n-1}$ realizes p_n .
 - (3) For all y , there exists $n < \omega$, $x_0 < \dots < x_{n-1}$ in P , and τ an n -ary term of $L(\Phi)$ such that $y = \tau(x_0, \dots, x_{n-1})$.
 Then if $M \models \psi$, $M \upharpoonright L = \text{EM}(P^M, \Phi)$. Conversely, if $M = \text{EM}(I, \Phi)$, we can expand M to an L' -structure by letting $(P^M, <^M) := (I, <)$. Thus ψ is as desired.
- (2) implies (3): Given L' and ψ as given by (2), Let Ψ be a fragment of L' containing ψ of size θ and let K' be $\text{Mod}(\Psi)$ ordered by \preceq_Ψ . Then K' is as desired for (3).
- (3) implies (1): Directly from Fact 5.3.

□

Let K be an AEC and assume there exists θ such that K is (λ, θ) -solvable for all high-enough λ , then in particular K has a superlimit in all high-enough λ , so we obtain one of the conditions in the main theorem. We now work toward a converse.

Hypothesis 5.5. K is an $\text{LS}(K)$ -superstable AEC with amalgamation which is $(< \kappa)$ -tame, where $\kappa := \text{LS}(K)$.

We will use without much comments results about Galois-Morleyization and averages as defined in [Vasc, BV]. Still we have tried to give a syntax-free presentation. Unless otherwise noted, the definitions below all take place inside a fixed model $\mathcal{N} \in K$. The letters \mathbf{I}, \mathbf{J} will denote sequences of tuples of length less than κ . We will use the same conventions as in [BV, Section 5]. Note that the sequences *may be indexed by arbitrary linear orders*. Recall:

Definition 5.6 (Definition V.A.2.1 in [She09b]). \mathbf{I} is χ -convergent if $|\mathbf{I}| \geq \chi$ and for any $p \in \text{gS}^{<\kappa}(A)$, $|A| < \kappa$, the set of elements of \mathbf{I} realizing p either has fewer than χ elements or its complement has fewer than χ elements.

Definition 5.7 (Definition V.A.2.6 in [She09b]). For \mathbf{I} a sequence, χ an infinite cardinal such that $|\mathbf{I}| \geq \chi$, and A a set, define $\text{Av}_\chi(\mathbf{I}/A)$ to be the set of $p_0 \in \text{gS}^{<\kappa}(A_0)$ such that $A_0 \subseteq A$ has size less than κ and the set $\{\bar{b} \in \mathbf{I} \mid \bar{b} \not\models p_0\}$ has size less than χ . When there is a unique $p \in \text{gS}^{<\kappa}(A)$ such that $p \upharpoonright A_0$ is in $\text{Av}_\chi(\mathbf{I}/A)$ for all $A_0 \subseteq A$ of size less than κ , we identify the average with p .

Definition 5.8. $p \in \text{gS}^{<\kappa}(B)$ does not syntactically split over $A \subseteq B$ if it does not split in the Galois Morleyization. That is, for all $\bar{b}, \bar{b}' \in {}^{<\kappa}B$, if $\text{gtp}(\bar{b}/A)E_{<\kappa}\text{gtp}(\bar{b}'/A)$, then $(p \upharpoonright \bar{b})E_{<\kappa}(p \upharpoonright \bar{b}')$. Here, $q_1 E_{<\kappa} q_2$ if and only if $q_1 \upharpoonright A_0 = q_2 \upharpoonright A_0$ for all A_0 of size less than κ .

Remark 5.9. We are only assuming tameness for types of length one, so $E_{<\kappa}$ may not be equality for longer types.

It turns out that Morley sequences (defined below) are always convergent. The parameters represent respectively a bound on the size of the domain, the degree of saturation of the models, and the length of the sequence. They will be assigned default values shortly.

Definition 5.10 (Definition 5.14 in [BV]). We say $\langle \bar{a}_i : i \in I \rangle \curvearrowright \langle N_i : i \in I \rangle$ is a (χ_0, χ_1, χ_2) -Morley sequence for p over A if:

- (1) $\chi_0 \leq \chi_1 \leq \chi_2$ are infinite cardinals, I is a linear order, A is a set, p is a Galois type with $\ell(\bar{x}) < \kappa$, and there is $\alpha < \kappa$ such that for all $i < \delta$, $\bar{a}_i \in {}^\alpha \mathcal{N}$.
- (2) For all $i \in I$, $A \subseteq |N_i|$ and $|A| < \chi_0$.
- (3) $\langle N_i : i \in I \rangle$ is increasing, and each N_i is χ_1 -saturated.
- (4) For all $i \in I$, \bar{a}_i realizes⁷ $p \upharpoonright N_i$ and for all $j > i$ in I , $\bar{a}_i \in {}^\alpha N_j$.
- (5) $i < j$ in I implies $\bar{a}_i \neq \bar{a}_j$.
- (6) $|I| \geq \chi_2$.
- (7) For all $i < j$ in I , $\text{gtp}(\bar{a}_i/N_i) = \text{gtp}(\bar{a}_j/N_i)$.
- (8) For all $i \in I$, $\text{gtp}(\bar{a}_i/N_i)$ does not syntactically split over A .

When p or A is omitted, we mean “for some p or A ”. We call $\langle N_i : i \in I \rangle$ the *witnesses* to $\mathbf{I} := \langle \bar{a}_i : i \in I \rangle$ being Morley, and when we omit them we simply mean that $\mathbf{I} \curvearrowright \langle N_i : i \in I \rangle$ is Morley for some witnesses $\langle N_i : i \in I \rangle$.

⁷Note that $\text{dom}(p)$ might be smaller than N_i .

Remark 5.11 (Monotonicity). Let $\langle \bar{a}_i : i \in I \rangle \frown \langle N_i : i \in I \rangle$ be (χ_0, χ_1, χ_2) -Morley for p over A . Let $\chi'_0 \geq \chi_0$, $\chi'_1 \leq \chi_1$, and $\chi'_2 \leq \chi_2$. Let $I' \subseteq I$ be such that $|I'| \geq \chi'_2$, then $\langle \bar{a}_i : i \in I' \rangle \frown \langle N_i : i \in I' \rangle$ is $(\chi'_0, \chi'_1, \chi'_2)$ -Morley for p over A .

Remark 5.12. By the proof of [She90, Lemma I.2.5], a Morley sequence is indiscernible (this will not be used).

The next result is key in the treatment of average of [BV]:

Fact 5.13 (Theorem 5.21 in [BV]). Let $\chi_0 \geq 2^{\text{LS}(K)}$ be such that \mathcal{N} does not have the order property of length χ_0^+ . Let $\chi := (2^{2^{\chi_0}})^+$.

If \mathbf{I} is a $(\chi_0^+, \chi_0^+, \chi)$ -Morley sequence, then \mathbf{I} is χ -convergent.

Hypothesis 5.14. χ_0, χ are as in Fact 5.13. The default parameters for Morley sequences are $(\chi_0^+, \chi_0^+, \chi)$, χ is the default parameter for average and convergence.

Remark 5.15. We can take $\chi_0 < \chi < h(\text{LS}(K))$.

Fact 5.16 (V.A.1.12 in [She09b]). If $p \in \text{gS}(M)$ and M is χ_0^+ -saturated, there exists $A \subseteq |M|$ of size at most χ_0 such that p does not syntactically split over A .

Fact 5.17 (Lemma 6.9 in [BV]). Let $\lambda > \chi^+$:

- (1) If $\langle M_i : i < \delta \rangle$ is an increasing chain of λ -saturated models, then $\bigcup_{i < \delta} M_i$ is λ -saturated.
- (2) There is a good $(\geq \chi^{++})$ -frame with underlying class $K^{\chi^{++}\text{-sat}}$.

Therefore taking χ_0 bigger if necessary we also assume:

Hypothesis 5.18.

- (1) If $\lambda \geq \chi_0$ and $\langle M_i : i < \delta \rangle$ is an increasing chain of λ -saturated models, then $\bigcup_{i < \delta} M_i$ is λ -saturated.
- (2) There is a good $(\geq \chi_0)$ -frame with underlying class $K^{\chi_0\text{-sat}}$.

We obtain a characterization of forking that adds to those proven in [Vasb]. A form of it already appears in [She09a, Observation IV.4.6].

Lemma 5.19. Let $p \in \text{gS}(M)$ and let $M_0 \leq M$ be χ -saturated. The following are equivalent:

- (1) p does not fork over M_0 .
- (2) There exists $M'_0 \leq M_0$ of size χ_0 such that p does not syntactically split over M'_0 .

- (3) There exists \mathbf{I} a Morley sequence for p , with all the witnesses inside M_0 , such that $\text{Av}(\mathbf{I}/M) = p$.

Proof.

- (1) implies (2): By Fact 5.16, we can find $M'_0 \leq M_0$ such that $p \upharpoonright M_0$ does not syntactically split over M'_0 and $\|M'_0\| \leq \chi_0$. Taking M'_0 bigger if necessary, we can assume M'_0 is χ_0 -saturated and $p \upharpoonright M_0$ does not fork over M'_0 . Thus p does not fork over M'_0 . As in [BV, Lemma 6.9], there exists a splitting-like notion R such that p does not R -split over M'_0 . Let \mathbf{I} be a Morley sequence for $p \upharpoonright M_0$ over M'_0 inside M_0 . Then it is a Morley sequence for p over M'_0 and by [BV, Lemma 5.25], $\text{Av}(\mathbf{I}/M_0) = p$ so as \mathbf{I} is based on M'_0 , p does not syntactically split over M'_0 .
- (2) implies (3): As in the proof above (here, the splitting-like notion is just syntactic splitting).
- (3) implies (2): This is given by the proof above: \mathbf{I} is based on some $M'_0 \leq M_0$ of size χ_0 .
- (2) implies (1): Without loss of generality, we can choose M'_0 to be such that $p \upharpoonright M_0$ also does not fork over M'_0 . Now, build a Morley sequence \mathbf{I} for p over M'_0 inside M_0 . If q is the nonforking extension of $p \upharpoonright M_0$ to M , then \mathbf{I} is also a Morley sequence for q over M'_0 so by the previous parts we must have $\text{Av}(\mathbf{I}/M) = q$, but also $\text{Av}(\mathbf{I}/M) = p$, since p does not split over M'_0 . Thus $p = q$.

□

Thus taking χ_0 even bigger, we can assume:

Hypothesis 5.20. If $p \in \text{gS}(M)$ does not fork over $M_0 \leq M$, then p does not syntactically split over M_0 .

The advantage of considering Morley sequences indexed by arbitrary linear orders is that they are closed under unions:

Lemma 5.21. Let $\langle I_\alpha : \alpha < \delta \rangle$ be an increasing (with respect to substructure) sequence of linear orders and let $I_\delta := \bigcup_{\alpha < \delta} I_\alpha$. Let $p \in \text{gS}(M)$, $M_0 \leq M$, and for $\alpha < \delta$, let $\mathbf{I}_\alpha := \langle a_i : i \in I_\alpha \rangle$ together with $\langle N_i^\alpha : i \in I_\alpha \rangle$ be Morley for p over M_0 , with $N_i^\alpha \leq M$ for all $i < \delta$, $\alpha < \delta$. Let $\mathbf{I}_\delta := \langle a_i : i \in I_\delta \rangle$, $\langle N_i^\delta : i \in I_\delta \rangle$ be defined by continuity.

If p does not fork over M_0 , then $\mathbf{I}_\delta \cap \langle N_i^\delta : i \in I_\delta \rangle$ is Morley for p over M_0 .

Proof. By Hypothesis 5.20, p does not syntactically split over M_0 . Therefore the only problematic clauses in Definition 5.10 are (4) and (7). Let's check (4): let $i \in I_\delta$. By hypothesis, \bar{a}_i realizes $p \upharpoonright N_i^\alpha$ for all sufficiently high $\alpha < \delta$. By local character of forking, there exists $\alpha < \delta$ such that $\text{gtp}(\bar{a}_i/N_i^\delta)$ does not fork over N_i^α . Since $\text{gtp}(\bar{a}_i/N_i^\delta) \upharpoonright N_i^\alpha = p \upharpoonright N_i^\alpha$ and p does not fork over $M_0 \leq N_i^\alpha$, we must have that $p \upharpoonright N_i^\delta = \text{gtp}(\bar{a}_i/N_i^\delta)$. The proof of (7) is similar. \square

The next result is a version of [She90, Theorem III.3.10] in our context. It is implicit in the proof of [BV, Theorem 5.27].

Lemma 5.22. Let $M \in K^{\chi\text{-sat}}$. Let $\lambda \geq \chi$. The following are equivalent.

- (1) M is λ -saturated.
- (2) If $q \in \text{gS}(M)$ is not algebraic and does not syntactically split over $A \subseteq |M|$ with $|A| \leq \chi_0$, there exists a Morley sequence \mathbf{I} for q over A inside M with $|\mathbf{I}| = \lambda$.

Proof. (1) implies (2) is trivial using saturation. Now assume (2). Let $p \in \text{gS}(B)$, $|B| < \lambda$, $B \subseteq |M|$. Let $q \in \text{gS}(M)$ extend p . If q is algebraic, we are done so assume it is not. Let $A \subseteq |M|$ have size at most χ_0 such that q does not fork over A . By Hypothesis 5.20, it does not syntactically split over A . Now by (2), there exists a Morley \mathbf{I} for q over A of size λ inside M . Now by [BV, Lemma 5.25], $\text{Av}(\mathbf{I}/B) = p$ and it is realized by an element of \mathbf{I} . \square

We can now prove solvability (in fact even uniform solvability). We will use condition (3) in Proposition 5.4.

Definition 5.23. We define a class of models K' and a binary relation $\leq_{K'}$ on K' as follows.

- K' is a class of $L' := L(K')$ -structures, where:

$$L' := L(K) \cup \{N_0, N, F, R \mid i < \chi\}$$

and for all $i < \chi$:

- N_0, R , are binary relations symbols.
- N is a tertiary relation symbol.
- F is a binary function symbol.
- An L' -structure M is in K' if and only if:
 - (1) $M \upharpoonright L(K) \in K^{\chi\text{-sat}}$.
 - (2) R^M is a linear ordering on $|M|$. We write I for this linear ordering.

- (3) For⁸ all $a \in |M|$ and all $i \in I$, $N^M(a, i) \leq M \upharpoonright L(K)$ (where we see $N^M(a, i)$ as an $L(K)$ -structure. In particular, $N^M(a, i) \in K$), $N_0^M(a) \leq N^M(a, i)$, $N_0^M(a)$ is χ_0 -saturated.
- (4) There exists a map $a \mapsto p_a$ from $|M|$ onto the non-algebraic Galois types over $M \upharpoonright L(K)$ such that for all $a \in |M|$:
 - (a) p_a does not fork⁹ over $N_0^M(a)$.
 - (b) $\langle F^M(a, i) : i \in I \rangle \frown \langle N^M(a, i) : i \in I \rangle$ is a Morley sequence for p_a over $N_0^M(a)$.
- $M \leq_{K'} M'$ if and only if:
 - (1) $M \subseteq M'$.
 - (2) $M \upharpoonright L(K) \leq M' \upharpoonright L(K)$.
 - (3) For all $a \in |M|$, $N_0^M(a) = N_0^{M'}(a)$.

Before checking that K' is an AEC, we show that this accomplishes what we want:

Lemma 5.24. Let $\lambda \geq \chi$.

- (1) If $M \in K_\lambda$ is saturated, then there exists an expansion M' of M to L' such that $M' \in K'$.
- (2) If $M' \in K'$ has size λ , then $M' \upharpoonright L(K)$ is saturated.

Proof.

- (1) Let $R^{M'}$ be a well-ordering of $|M|$ of type λ . Identify $|M|$ with λ . By stability, we can fix a bijection $p \mapsto a_p$ from $\text{gS}(M)$ onto $|M|$. For each $p \in \text{gS}(M)$ which is not algebraic, fix $N_p \leq M$ such that p does not syntactically split over N_p and $\|N_p\| \leq \chi_0$ (possible by Fact 5.16). Then use saturation to build $\langle a_p^i : i < \lambda \rangle \frown \langle N_p^i : i < \lambda \rangle$ Morley for p over N_p (inside M). Let $N_0^{M'}(a_p) := N_p$, $N^{M'}(a_p, i) := N_p^i$, $F^{M'}(a, i) := a_p^i$. For p algebraic, pick $p_0 \in \text{gS}(M)$ nonalgebraic and let $N_0^{M'}(a_p) := N_0^{M'}(a_{p_0})$, $N^{M'}(a_p, i) := N^{M'}(a_{p_0}, i)$, $F^{M'}(a_p, i) := F^{M'}(a_{p_0}, i)$.
- (2) By Lemma 5.22.

□

Lemma 5.25. $(K', \leq_{K'})$ is an AEC with $\text{LS}(K') = \chi$.

⁸For a binary relation Q we write $Q(a)$ for $\{b \mid Q(a, b)\}$, similarly for a tertiary relation.

⁹With respect to the good frame introduced in Hypothesis 5.18.

Proof. It is straightforward to check that K' is an abstract class with coherence. Moreover:

- K' satisfies the chain axioms: Let $\langle M_i : i < \delta \rangle$ be increasing in K' . Let $M_\delta := \bigcup_{i < \delta} M_i$.
 - $M_0 \leq_{K'} M_\delta$, and if $N \geq_{K'} M_i$ for all $i < \delta$, then $N \geq_{K'} M_\delta$: Straightforward.
 - $M_\delta \in K'$: $M_\delta \upharpoonright L(K)$ is χ^{++} -saturated by Fact 5.17. Moreover, R^{M_δ} is clearly a linear ordering of M_δ . Write I_i for the linear ordering (M_i, R_i) . Condition 3 in the definition of K' is also easily checked. We now check Condition 4. Let $a \in |M_\delta|$. Fix $i < \delta$ such that $a \in |M_i|$. Without loss of generality, $i = 0$. By hypothesis, for each $i < \delta$, there exists $p_a^i \in \text{gS}(M_i \upharpoonright L(K))$ not algebraic such that $\langle F^{M_i}(a, j) \mid j \in I_i \rangle \frown \langle N^{M_i}(a, j) \mid j \in I_i \rangle$ is a Morley sequence for p_a^i over $N_0^{M_i}(a) = N_0^{M_0}(a)$. Clearly, $p_a^i \upharpoonright N_0^{M_0}(a) = p_a^0 \upharpoonright N_0^{M_0}(a)$ for all $i < \delta$. Moreover by assumption p_a^i does not fork over $N_0^{M_0}$. Thus for all $i < j < \delta$, $p_a^j \upharpoonright M_i = p_a^i \upharpoonright M_i$. By extension and uniqueness, there exists $p_a \in \text{gS}(M_\delta \upharpoonright L(K))$ that does not fork over $N_0^{M_0}(a)$ and we have $p_a \upharpoonright M_i = p_a^i$ for all $i < \delta$. Now by Lemma 5.21, $\langle F^{M_\delta}(a, j) \mid j \in I_\delta \rangle \frown \langle N^{M_\delta}(a, j) \mid j \in I_\delta \rangle$ is a Morley sequence for p_a over $N_0^{M_0}(a)$. Moreover, the map $a \mapsto p_a$ is onto the nonalgebraic Galois types over $M_\delta \upharpoonright L(K)$: let $p \in \text{gS}(M_\delta \upharpoonright L(K))$ be nonalgebraic. Then there exists $i < \delta$ such that p does not fork over M_i . Let $a \in |M_i|$ be such that $\langle F^{M_i}(a, j) \mid j \in I_i \rangle \frown \langle N^{M_i}(a, j) \mid j \in I_i \rangle$ is a Morley sequence for $p \upharpoonright M_i$ over $N_0^{M_i}(a)$. It is easy to check it is also a Morley sequence for p over $N_0^{M_i}(a)$. By uniqueness of the nonforking extension, we get that the extended Morley sequence is also Morley for p , as desired.
- $\text{LS}(K') = \chi$: An easy closure argument.

□

Corollary 5.26. K is uniformly (χ, χ) -solvable.

Proof. By Lemma 5.25, K' is an AEC with $\text{LS}(K') = \chi$. Now combine Lemma 5.24 and Proposition 5.4. Note that saturated models of size at least χ_0 are superlimit by Hypothesis 5.18, and K has arbitrarily large saturated models by superstability. □

For the next theorems, we drop our hypotheses.

Theorem 5.27. If K is $\text{LS}(K)$ -superstable with amalgamation and $(< \text{LS}(K))$ -tame, then there exists $\theta < h(\text{LS}(K))$ such that K is uniformly (θ, θ) -solvable.

Proof. By Remark 5.15 and Corollary 5.26 . □

Theorem 5.28. Let K be a stable tame AEC with amalgamation. The following are equivalent.

- (1) For all high-enough λ , K is λ -superstable.
- (2) There exists θ such that K is uniformly (θ, θ) -solvable.
- (3) There exists θ such that for all high-enough λ , K is (λ, θ) -solvable.

Proof. (1) implies (2) is Theorem 5.27, (2) implies (3) follows directly from the definition of solvability, and (3) implies (1) is because (3) implies that there is a superlimit in all high-enough λ and by Theorem 4.8, this implies superstability. □

6. SUPERSTABILITY BELOW THE HANF NUMBER

In this section, we show that we can require “ $\text{LS}(K) \leq \mu_\ell < h(\text{LS}(K))$ ” in Theorem 1.2 (provided the class is $(< \text{LS}(K))$ -tame). While this improves on some bounds e.g. in section 4, the arguments are harder.

Theorem 6.1. Let K be a $(< \text{LS}(K))$ -tame AEC with amalgamation, joint embedding, and arbitrarily large models. Assume K is stable. Then the following are equivalent:

- (1) There exists $\mu_1 < h(\text{LS}(K))$ such that for every $\lambda \geq \mu_1$, for all $\delta < \lambda^+$, for all increasing continuous $\langle M_i : i \leq \delta \rangle$ in K_λ and all $p \in \text{gS}(M_\delta)$, if M_{i+1} is universal over M_i for all $i < \delta$, then there exists $i < \delta$ such that p does not λ -split over M_i .
- (2) There exists $\mu_2 < h(\text{LS}(K))$ such that for every $\lambda \geq \mu_2$, for some $\kappa \leq \lambda$, there is a good λ -frame on $K_\lambda^{\kappa\text{-sat}}$.
- (3) There exists $\mu_3 < h(\text{LS}(K))$ such that for every $\lambda \geq \mu_3$, K has uniqueness of limit models in cardinality λ .
- (4) There exists $\mu_4 < h(\text{LS}(K))$ such that for every $\lambda \geq \mu_4$, K has a superlimit model of cardinality λ .
- (5) There exists $\mu_5 < h(\text{LS}(K))$ such that for every $\lambda \geq \mu_5$, the union of a chain of λ -saturated models is λ -saturated.
- (6) There exists $\mu_6 < h(\text{LS}(K))$ such that for all $\lambda \geq \mu_6$, K is (λ, μ_6) -solvable.

Proof sketch. By Theorem 5.27 (and being slightly more careful with the bounds, i.e. using [BV, Remark 6.12]), (1) implies (6). Moreover (6) implies (4) by definition. Thus it is enough to show the equivalence of (1)-(5).

We now revisit the proof of Theorem 4.8. Now the proofs of (1) implies (2)-(5) there show that we can take $\mu_2, \dots, \mu_5 < h(\text{LS}(K))$. Similarly for (2) implies (3) and (5) implies (4). Now we show that both (3) and (4) imply (5^-) , which is the following statement:

(5^-) For unboundedly many cardinals $\lambda < h(\text{LS}(K))$, the union of a chain of λ -saturated models is λ -saturated.

If (3), then Lemma 3.8 gives (5^-) . If (4), then let $\chi < h(\text{LS}(K))$ be as given by Fact 3.9. Without loss of generality, $\chi \geq \mu_4$. Let λ be a successor cardinal such that $\lambda = \lambda^{<\chi}$ (note that there are unboundedly many such λ below $h(\text{LS}(K))$). By Fact 3.10, K is stable in λ and in unboundedly many $\mu < \lambda$ (namely in its predecessor). By regularity, K must have a saturated model of size λ . By Lemma 3.6, we get (5^-) .

It remains to show that (5^-) implies (1). We follow section 5 and note that Facts 5.13 and 5.16 hold assuming only stability. Let χ_0, χ be as there. Without loss of generality, K is stable in χ and in any μ such that $\mu = \mu^{<\chi}$. By [BV, Fact 2.22], there is a forking-like notion on $K^{\chi^+ \text{-sat}}$ which satisfies base monotonicity, transitivity, uniqueness, and so that any type does not fork over a model of size χ . Now take $\lambda > \chi^+$ such that (5^-) holds. Restrict the independence relation to λ -saturated models of size at least $\lambda' := (\lambda^{<\chi})^+$. Note that by the choice of λ , $K^{\lambda \text{-sat}}$ is an AEC and it is easy to check that any type does not fork over a model of size λ' .

In other words, we have an analog of Fact 5.17 where we can still find a $(\geq \lambda')$ -frame with underlying class $K^{\lambda' \text{-sat}}$ that satisfies base monotonicity, transitivity, uniqueness, and so that any type does not fork over a model of size λ' . By the proof of [Vasb, Theorem 7.5], we also have a local version of extension: If $p \in \text{gS}(M)$, $M \leq N$ are saturated of the same size, then p has a nonforking extension to N . Taking χ_0 and λ bigger if necessary, the proof of Lemma 5.19 now goes through if M_0 and M are saturated of the same size (we use local extension in the proof).

Now let $\delta < (\lambda')^+$ be regular and let $\langle M_i : i < \delta \rangle$ be an increasing chain of saturated models in $K_{\lambda'}$. Let $M_\delta := \bigcup_{i < \delta} M_i$. Let $p \in \text{gS}(M_\delta)$. As in the proof of Theorem 4.8, it is enough to show that there exists $i < \delta$ such that p does not fork over M_i . If $\delta \geq \chi$ this is easy by construction

of the frame (again adjusting χ_0 or λ), so assume $\delta < \chi$. By definition of λ , we have that M_δ is λ -saturated. We also have that p does not fork over M_δ so by the proof of Lemma 5.19, there exists $M'_0 \leq M_0$ of size χ_0 and \mathbf{I} inside M_δ a Morley sequence for p over M'_0 with $|\mathbf{I}| \geq \chi$. Let $\mathbf{I}_i := M_i \cap \mathbf{I}$. Since $\delta < \chi$, there must exist $i < \delta$ such that $|\mathbf{I}_i| \geq \chi$. By [BV, Lemma 5.10] (where χ here is χ_0 there, again increasing χ_0 if necessary), I_i is based on a set of size χ_0 , i.e. $\text{Av}(\mathbf{I}_i/M_\delta)$ does not syntactically split over some $M'_i \leq M_i$ with $\|M'_i\| = \chi_0$. By the proof of Lemma 5.19, this means that p does not fork over M_i , as desired¹⁰. \square

The proof does not tell us if there is a Hanf number for superstability, namely:

Question 6.2. Let K be a ($< \text{LS}(K)$)-tame AEC with amalgamation which is λ -superstable for some $\lambda \geq h(\text{LS}(K))$. Is K μ -superstable for some $\mu < h(\text{LS}(K))$?

We end by improving Fact 2.7.(2). Recall that this tells us that (in tame AECs with amalgamation) superstability follows from categoricity in a high-enough cardinal. We give an improvement that does not use tameness and improves the bound on the categoricity cardinal. Even though all the ingredients are contained in [SV99], this has not appeared in print before.

Theorem 6.3 (The ZFC Shelah-Villaveces theorem). Let K be an AEC with arbitrarily large models and amalgamation in $\text{LS}(K)$ ¹¹. Let $\lambda > \text{LS}(K)$ be such that $K_{<\lambda}$ has no maximal models. If K is categorical in λ , then K is $\text{LS}(K)$ -superstable.

Proof. Set $\mu := \text{LS}(K)$. In the proof of [SV99, Theorem 2.2.1], in (c), ask that $\sigma = \chi$, where χ is the least cardinal such that $2^\chi > \mu$. The proof that (c) cannot happen goes through, and the rest only uses amalgamation in μ . \square

Corollary 6.4. Let K be an AEC with amalgamation in $\text{LS}(K)$. If K is categorical in a $\lambda \geq h(\text{LS}(K))$, then there exists $\lambda_0 < h(\text{LS}(K))$ such that K is μ -superstable in all $\mu \in [\lambda_0, \lambda)$ where K_μ has amalgamation.

Proof. Combine Theorem 6.3 with [Vasb, Proposition 10.13] (the argument uses only amalgamation in $\text{LS}(K)$). \square

¹⁰A similar argument already appears in the proof [She09a, Theorem IV.4.10].

¹¹In [SV99], this is replaced by GCH.

We can use the ZFC Shelah-Villaveces theorem to prove the following interesting result, showing that the solvability spectrum satisfies an analog of Shelah's categoricity conjecture in tame AECs (Shelah conjectures that this should hold in general, see Question 4.4 in the introduction to [She09a]). For notational purpose, we introduce one more definition:

Definition 6.5. K is λ -solvable if it is (λ, θ) -solvable for some $\theta < h(\text{LS}(K))$.

Theorem 6.6. Let K be a $(< \text{LS}(K))$ -tame AEC with amalgamation. If K is λ -solvable for some $\lambda \geq h(\text{LS}(K))$, then there exists $\theta < h(\text{LS}(K))$ such that:

- (1) K is θ -superstable.
- (2) K is (μ, θ) -solvable for all $\mu \geq \theta$.

Proof. Let $\theta_0 < h(\text{LS}(K))$ be such that K is (λ, θ_0) -solvable. First observe that K_λ has joint embedding, as any superlimit model is universal. Therefore (e.g. by [Vasb, Proposition 10.13]), there exists $\chi_0 < h(\text{LS}(K))$ such that $K_{\geq \chi_0}$ has joint embedding and no maximal models. Without loss of generality, $\chi_0 = \theta_0$. By the standard argument (see for example [Bal09, Theorem 8.21]), $K_{\geq \theta_0}$ is stable in all $\mu < \lambda$. By the proof of Theorem 6.3, K is θ_0 -superstable, and thus by Fact 2.7.(1) θ -superstable for any $\theta \geq \theta_0$. By Theorem 5.27 (we have to be slightly more careful with the bound on χ , see [BV, Remark 6.12]), there exists $\theta < h(\text{LS}(K))$ with $\theta \geq \theta_0$ such that K is uniformly (θ, θ) -solvable, hence by definition (μ, θ) -solvable for all $\mu \geq \theta$. \square

Remark 6.7. Since we required the starting solvability parameter θ_0 to be below $h(\text{LS}(K))$, this does not quite answer Question 6.2.

7. FUTURE WORK

While we managed to prove that some analogs of the conditions in Fact 1.1 are equivalent, much remains to be done. For example, even in tame AECs with amalgamation, we do not know whether stability on a tail of cardinals or having a saturated model on a tail of cardinals should imply superstability (although superstability certainly implies these).

Another direction would be to make precise what the analog to (5) and (6) in 1.1 should be in tame AECs. One possible definition for (6) would be:

Definition 7.1. Let λ and μ be cardinal numbers. We say that K has the (λ, μ) -tree property provided there exists $\{p_n(\mathbf{x}; \mathbf{y}_n) \mid n < \omega\}$ Galois-types such that $|\text{dom}(p_n)| < \mu$ and there exists $\{M_\eta \mid \eta \in {}^{<\omega}\lambda\}$ such that for all $n < \omega, \nu \in {}^n\lambda$ and every $\eta \in {}^\omega\lambda$:

$$\langle M_\eta, M_\nu \rangle \models p_n \iff \nu \text{ is an initial segment of } \eta.$$

We say that K has the tree property if it has it for all high-enough μ and all high-enough λ (where the “high-enough” quantifier on λ can depend on μ).

We can ask whether superstability implies that K does not have the tree property, or at least obtain many models from the tree property as in [GS86]. This is conjectured in [She99] (see the remark after Claim 5.5 there).

As for the D-rank in (5), perhaps a simpler analog would be the U -rank defined in terms of $(< \kappa)$ -satisfiability in [BG, Definition 7.2] (another candidate for a rank is Lieberman’s R -rank, see [Lie13])

Definition 7.2. Let K be a $(< \text{LS}(K))$ -tame AEC with amalgamation. Let $\kappa > \text{LS}(K)$ be least such that $\kappa = \beth_\kappa$ (for concreteness). We define a map U with domain a type over κ -saturated models and codomain an ordinal or ∞ inductively by, for $p \in \text{gS}(M)$:

- (1) Always, $U[p] \geq 0$.
- (2) For α limit, $U[p] \geq \alpha$ if and only if $U[p] \geq \beta$ for all $\beta < \alpha$.
- (3) $U[p] \geq \beta + 1$ if and only if there exists a κ -saturated $M' \geq M$ with $\|M'\| = \|M\|$ and an extension $q \in \text{gS}(M')$ of p such that q is not $(< \kappa)$ -satisfiable over M and $U[q] \geq \beta$.
- (4) $U[p] = \alpha$ if and only if $U[p] \geq \alpha$ and $U[p] \not\geq \alpha + 1$.
- (5) $U[p] = \infty$ if and only if $U[p] \geq \alpha$ for all ordinals α .

By [BG, Theorem 7.9], superstability implies that the U -rank is bounded but we do not know how to prove the converse. Perhaps it is possible to show that $U = \infty$ implies the tree property.

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