

Scale Free Small World Networks and the Structure of Quantum Space-Time

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Abstract

We report on parallel observations in two seemingly unrelated areas of dynamical network research. The one is the so-called small world phenomenon and/or the observation of scale freeness in certain types of large (empirical) networks and their theoretical analysis. The other is a discrete cellular network approach to quantum space-time physics on the Planck scale we developed in the recent past. In this context we formulated a kind of geometric renormalisation group or coarse graining process in order to construct some fixed point which can be associated to our macroscopic space-time (physics). Such a fixed point can however only emerge if the network on the Planck scale has very peculiar critical geometric properties which strongly resemble the phenomena observed in the above mentioned networks. A particularly noteworthy phenomenon is the appearance of translocal bridges or short cuts connecting widely separated regions of ordinary space-time and which we expect to become relevant in various of the notorious quantum riddles.

1 Introduction

In this paper we want to report on parallel observations in two, at first glance, quite unrelated fields of current research. The one is a discrete network approach to quantum space-time physics, the other is the analysis of large complicated information networks of interacting agents, displaying the *small world phenomenon* and/or the *synchronisation* or *phase locking* among, for example, coupled non-linear oscillators as they occur in biological and related model systems. As the literature concerning the latter phenomenon is huge and as we are planning to discuss the relevance of this particular phenomenon of synchronisation for our own approach to quantum gravity elsewhere, we mention only very few but typical papers ([5], [6], [7]) and the book [8].

In the following we will rather concentrate on the parallels between, on the one hand, the *small world phenomenon* and/or the emergence of *scale free networks* ([1], [2], [9] [10], [3], [4]) and, on the other hand, our dynamical network approach to quantum space-time physics ([21], [16], [17], [18], [19], [20], [22]).

The small world phenomenon was, for the first time, observed in empirical networks while in the past, most of the network modelling exploited the *random graph concept*. In this latter framework links are drawn (practically) independently of each other according to a certain *edge probability*. As a consequence, in general no particular *near-* or *far-order* does exist. Therefore it came as quite a surprise that in the real world networks do occur, which seem to encode a certain (hidden) principle which combines a sparse wiring (usually implying a large *typical node distance* and a low *local clustering*) with both a surprisingly high *local clustering* and a relatively small *average node distance*.

On the other hand, (sparse) random graphs, more precisely, random graphs with a comparable edge probability, have typically both a low local clustering and, at least in general, a small *diameter* or average node distance (see section 3). This observation suggests that a particular principle is at work, amalgamating these two seemingly antagonistic properties.

Analysing the underlying laws which may lead to such an interesting structure, Barabasi et al contributed the concept of *scale free networks*, having, for example, a power law *vertex degree distribution* (in contrast to random graphs in which the vertex degree is binomially or Poisson distributed). This notion calls to mind concepts like critical behavior or self-similarity. While the small world effect is attributed to the existence of *short cuts* or *hubs*, scale freeness is a much stronger property and points to the existence of a certain *hierarchical organisation* of the network.

This latter observation establishes the link to our own research in the field of quantum space-time physics. In the recent past, being unaware of the possible connections to the small world phenomenon, we were led to the conclusion that the *geometric renormalisation group* or *coarse graining process* we developed within our discrete network approach, leads only to an interesting large

scale fixed point, we finally want to associate with our macroscopic or mesoscopic space-time (compared to the primordial Planck scale regime), if the initial network of relations is in a very particular *critical state*.

Geometrically this implies that the wiring of the primordial network is in a certain sense organised in a hierarchical, scale free manner, which strongly resembles the features described above. It is perhaps particularly noteworthy that in our case this scale freeness is accomplished by a certain *translocal* character of the wiring which emerges in the renormalisation process (existence of a hierarchy of short cuts). This provides also strong clues as to certain mysteries of the quantum world.

We conclude this introduction with some interdisciplinary speculation. For biological and many other networks the central objective is that they function. That is, their organisation should make them robust and resistant to (at least) local and/or stochastic failures. If one observes the particular kind of (intricate) organisation in quite a few real networks we mentioned above, one is led to surmise that properties like scale freeness or small world behavior will in fact make them more robust (cf. also [4]).

If the parallels we invoked regarding a possible similar organisation of microscopic space-time do not turn out to be unfounded, the same conclusion can be drawn here. That is, given that microscopic space-time is a highly complicated dynamical network, wildly fluctuating on small scales, the organisation principles we are going to describe in the following may be crucial in preventing such a structure from becoming chaotic or simply disintegrating into incoherent pieces. An important role in this context is expected to be played by the alluded translocal bridges on microscopic scales between macroscopically widely separated lumps of space-time.

2 Notions from Graph Theory

In this section we introduce some terminology and concepts employed in graph theory and fix the notation. As to the general context see for example [12]

Definition 2.1 *A simple, countable, labelled, undirected graph, G , consists of a countable set of nodes or vertices, V , and a set of edges, E , each connecting two of the nodes. There exist no multiple edges (i.e. edges, connecting the same pair of nodes) or elementary loops (an edge, starting and ending at the same node). In this situation the edges can be described by giving the corresponding set of unordered pairs of nodes. The members of V are denoted by x_i , the edges by e_{ij} , connecting the nodes x_i and x_j .*

Remarks: We could also admit a non-countable vertex set. The above restriction is only made for technical convenience. From a physical point of view one may argue that the *continuum* or uncountable sets are idealisations, anyhow. The

notions *vertex*, *node* or *edge*, *link* or *bond* are used synonymously. Furthermore, the labeling of the nodes is only made for technical convenience (to make some discussions easier) and does not carry a physical meaning. As in general relativity, all models being invariant under *graph isomorphisms* (i.e. relabelling of the nodes and corresponding bonds) are considered to be physically equivalent.

In the above definition the edges are not directed (but oriented; see below). In certain cases it is also useful to deal with directed graphs.

Definition 2.2 *A directed graph is a graph as above, with E consisting now of directed bonds or ordered pairs of nodes. In this case we denote the edge, pointing from x_i to x_j by d_{ij} . There may now also exist the opposite edge, denoted by d_{ji} .*

Observation 2.3 *An undirected graph, as in definition 2.1, can be considered as a particular directed graph with e_{ij} corresponding to the pair of directed edges, d_{ij}, d_{ji} .*

In the following we deal, for reasons of simplicity, with *connected* graphs. We denote the number of vertices and edges by n , m , respectively. The maximal possible number of edges over n vertices is $N = n(n - 1)/2$. n , m , are called *order* and *size* of the graph, G . The number of edges, being incident with a node, x_i , is called its *vertex degree*, k_i . An *edge sequence* or *walk* is a sequence of consecutive edges or nodes,

$$(e_{i_1, i_2}, e_{i_2, i_3}, \dots, e_{i_{k-1}, i_k}) \text{ or } (x_{i_1}, \dots, x_{i_k}) \quad (1)$$

where the edges or nodes need not be distinct. A *path* is an edge sequence where no x_{i_l} occurs twice with the possible exception of x_{i_1}, x_{i_k} . In the latter case the path is called a *cycle*.

Between each pair of nodes, x_i, x_k , there exists a path, γ , of minimal length, l , with $l(\gamma) = \#(\text{edges})$, connecting x_i and x_k . This path is called a *geodesic path* and defines a distance function or metric, $d(x, y)$ on the graph. We have

$$d(x_i, x_k) = d(x_k, x_i) > 0 \text{ for } i \neq k \quad (2)$$

$$d(x_i, x_k) \leq d(x_i, x_l) + d(x_l, x_k) \quad (3)$$

This metric is called the canonical graph metric. There exist of course other interesting distance concepts on graphs, cf. for example [16], where we compared, among other things, the canonical with the Connes-metric on graphs.

The above metric allows us to introduce a neighborhood concept. We denote by $\Gamma_l(x_0)$ the set of nodes having exactly distance l from the reference node x_0 and by $U_l(x_0)$ the set of nodes with distance $d(x_0, x_i) \leq l$. The cardinality of $\Gamma_1(x_0)$ is just the vertex degree of x_0 . In [17] we studied systematically the scaling

behavior of $|\Gamma_l(x)|$ and $|U_l(x)|$ with l (that is, the number of nodes in $\Gamma_l(x), U_l(x)$) and related it to dimensional concepts of, typically, fractal type. We argued that it encodes the kind of dimension which turns out to be relevant in many physical systems.

In [15] the sequence of $d_l(x_0) := |\Gamma_l(x_0)|$ $l = 1, 2, \dots$ is called the *distance degree sequence* relative to node x_0 , and is denoted by $dds(x_0)$. Tabulating this for the whole graph, G , we get the *distance distribution*

$$dd(G) = (D_1, D_2, \dots) \quad (4)$$

with D_l the number of pairs, (x_i, x_j) , having distance equal to l . We have of course

$$2D_l = \sum_V d_l(x_i) \quad (5)$$

Whereas $dd(G)$ derives from the ensemble of $dds(x_i)$, it is frequently easier to handle and gives a more compact characteristic of the global wiring structure of the graph under discussion.

From the above a particularly important graph characteristic can be derived which is heavily employed in the small world context.

Definition 2.4 *The mean distance, $L(G)$, of a connected graph is given by the average of the distances between the pairs of nodes of G .*

$$L(G) := N^{-1} \cdot \sum l \cdot D_l, \quad N = n(n-1)/2 \quad (6)$$

If graphs become very large it is frequently very difficult to envisage the essentials of the geometric structure of a given graph. So it is useful to develop more concepts which allow us to encode typical characteristics of the graph under discussion. A more subtle concept is the *clique structure* or *clique distribution*.

Definition 2.5 (Subsimplices and Cliques) *With G a given fixed graph and V_i a subset of its vertex set V , the corresponding induced subgraph over V_i (that is, its edges being the corresponding edges, occurring in G) is called a subsimplex or a complete subgraph, if all its pairs of nodes are connected by a bond. In this class, which is in fact partially ordered, the order being given by graph inclusion, there exist certain maximal subsimplices, that is, subsimplices so that every addition of another node of the underlying graph (together with the respective bonds existing in G and pointing to other nodes of the selected subset destroys this property. These maximal simplices are usually called cliques in combinatorics (we like to call them also lumps as they are the candidates for our construction of physical points).*

It has been described in detail in e.g. section 4 of [19] how these cliques can be constructed in an algorithmic way, starting from an arbitrary node. Note in

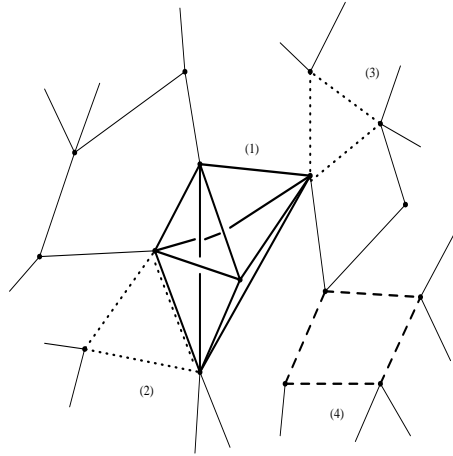


Figure 1:

particular that a given node will, in general belong to many different (overlapping) cliques or lumps. The situation is illustrated by the following picture: In this picture we have drawn a subgraph of a larger graph. (1) denotes a clique, i.e. a maximal subsimplex. Subsets of nodes of such a clique support subsimplices (called faces in algebraic topology), the clique being the maximal element in this partial ordered set. (2) and (3) are other, smaller cliques overlapping with (1) in a common bond or node. (4) is an example of a subgraph which is not a clique or subsimplex. Evidently, each node or bond lies in at least one clique. The smallest possible cliques which can occur in a connected graph consist of two nodes and the corresponding edge. The detailed investigation of the clique structure of graphs was one of the main topics of [18] and earlier in [19] and [20].

Remark: Note that our definition of a clique (which conforms with Bollobas') deviates slightly from the one employed by other authors. Our cliques are the maximal! members in the ascending chains of complete subgraphs while sometimes the complete subgraphs themselves are called cliques. The typical order of cliques turns out to be a very interesting random variable in random graphs (see below).

Another interesting notion is the *cluster coefficient*, $C_i = C(x_i)$. It is defined by

$$C_i := |E(\Gamma_1(x_i))| / \binom{k_i}{2} \quad (7)$$

Here $\binom{k_i}{2}$ is the maximal possible number of edges in $\Gamma_1(x_i)$ and $|E(\Gamma_1(x_i))|$ the actual number. Henceforth, C denotes the average over the C_i 's. Further useful concepts are the *average vertex degree* and the *vertex degree distribution*

$$k := \langle k \rangle = n^{-1} \cdot \sum k_i = 2m/n \quad (8)$$

$$P(k) := \text{prob}(k_i = k) \quad (9)$$

where either the probability is taken over a certain ensemble of graphs or is calculated in a given fixed graph (cf. the end of the next section).

In this way one can construct a whole bunch of interesting graph characteristics which are more or less related to each other and their combination supplying a relatively complete picture of the local and global structure of a graph. These concepts become particularly powerful if we combine them with true probabilistic or statistical concepts. This leads to the definition of a *random graph*.

3 The Concept of a Random Graph

One kind of probability space is constructed as follows. Take all possible labeled graphs over n nodes as probability space \mathcal{G} (i.e. each graph represents an elementary event). The maximal possible number of bonds is $N := \binom{n}{2}$, which corresponds to the unique *simplex graph* (denoted usually by K_n). Give each bond the *independent probability* $0 \leq p \leq 1$, (more precisely, p is the probability that there is a bond between the two nodes under discussion). Let G_m be a graph over the above vertex set, V , having m bonds. Its probability is then

$$\text{pr}(G_m) = p^m \cdot q^{N-m} \quad (10)$$

where $q := 1 - p$. There exist $\binom{N}{m}$ different labeled graphs G_m , having m bonds, and the above probability is correctly normalized, i.e.

$$\text{prob}(\mathcal{G}) = \sum_{m=0}^N \binom{N}{m} p^m q^{N-m} = (p + q)^N = 1 \quad (11)$$

This probability space is sometimes called the space of *binomially random graphs* and denoted by $\mathcal{G}(n, p)$. Note that the number of edges is binomially distributed, i.e.

$$\text{prob}(m) = \binom{N}{m} p^m q^{N-m} \quad (12)$$

and

$$\langle m \rangle = \sum m \cdot \text{prob}(m) = N \cdot p \quad (13)$$

The really fundamental observation made already by Erdős and Rényi (a rigorous proof of this deep result can e.g. be found in [12]) is that there are what physicists would call *phase transitions* in these *random graphs*. To go a little bit more into the details we have to introduce some more graph concepts.

Definition 3.1 (Graph Properties) Graph properties are certain particular random variables (*indicator functions of so-called events*) on the above probability space \mathcal{G} . I.e., a graph property, Q , is represented by the subset of graphs of the sample space having the property under discussion.

To give some examples: i) connectedness of the graph, ii) existence and number of certain particular subgraphs (such as subsimplices etc.), iii) other geometric or topological graph properties etc.

In this context Erdős and Rényi made the following important observation.

Observation 3.2 (Threshold Function) *A large class of graph properties (e.g. the monotone increasing ones, cf. [11] or [13]) have a so-called threshold function, $m^*(n)$, with $m^*(n) := N \cdot p^*(n)$, so that for $n \rightarrow \infty$ the graphs under discussion have property Q almost surely for $m(n) > m^*(n)$ and almost surely not for $m(n) < m^*(n)$ or vice versa (more precisely: for $m(n)/m^*(n) \rightarrow \infty$ or 0 ; for the details see the above cited literature). That is, by turning on the probability p , one can drive the graph one is interested in beyond the phase transition threshold belonging to the graph property under study. Note that, by definition, threshold functions are only unique up to “factorization”, i.e. $m_2^*(n) = O(m_1^*(n))$ is also a threshold function.*

We briefly illustrate the effects of randomisation on some of the concepts introduced above. We take for example the vertex degree as a random variable. The probability of a vertex, x_i , having vertex degree k is

$$\text{prob}(k_i = k) = \binom{n-1}{k} \cdot p^k (1-p)^{n-1-k} \quad (14)$$

(the mean value being $p \cdot (n-1)$). In the asymptotic regime of large n , small k , $n \cdot p = O(1)$, the vertex degree is Poisson-distributed (see, for example, [3]). For a clean discussion of the asymptotic case of the Poisson distribution see [14]. Barabasi et al contrasted this kind of distribution with so-called *scale free* degree distributions which are sometimes found in empirical networks ([3],[4]).

For the clustering coefficient, defined above, we have in the context of random graphs the following simple result. As all the edges are independently distributed with probability p over the random graph, the distribution is the same for each subclass of nodes, that is, we have

$$C_{\text{rand}} = p = k/(n-1) = \langle m \rangle / N \quad (15)$$

In contrast to that typical small world networks may be globally sparse, i.e. $m/N \ll 1$, but, nevertheless, $C \gg m/N$. That is, in contrast to randomly wired networks they may display a certain local order which differs from the order viewed on a more global level.

Very interesting is the behavior of the average length, $L(G)$, in a random graph. This random variable, together with the clustering coefficient, is the pair of graph properties which is primarily employed to contrast the behavior of random graphs with the wiring diagrams of so-called small world networks (see [1] and the other literature cited above).

It is a remarkable (perhaps a little bit counter-intuitive) property of random graphs that for a large portion of values in the parameter space, given by the pair (n, p) , that is, p not too small, a typical random graph has diameter less than or equal to two! (for the details see e.g. [11], some estimates are also given in [19], p.2053f). To have at all some (weak) scaling with n , $p(n)$ has to vanish for $n \rightarrow \infty$ sufficiently strongly without destroying the connectedness of the graph.

To underpin this qualitative statement, we take over the central result of chapt. X.2 of [11] and calculate its asymptotic behavior for the regime; n large and $p \cdot n \rightarrow \text{const.}$. We have for the diameter the approximative result

$$\text{diam} \approx \text{const} \cdot \log n / \log pn \quad (16)$$

where $p(n-1) = k$ in a random graph, hence

$$\text{diam} \approx \text{const} \cdot \log n / \log k \quad (17)$$

A similar result holds for the average distance, which is, however, not so easily accessible in general. Strogatz and Watts ([1],[2]) contrast this weak scaling with the scaling of, for example, lattice graphs, which are occupying exactly the opposite end with respect to order or randomness.

The average distance in a d -dimensional lattice graph (with, for simplicity, periodic boundary conditions) can easily be calculated as follows (cf. also [17] for more general scenarios). The $|\Gamma_l(x_0)|$ of the $dds(x_0)$ of some arbitrary node scale as $\sim l^{d-1}$. For the average distance relative to x_0 we hence have

$$L_{x_0}(G) \sim (n-1)^{-1} \cdot \sum_1^{j_n} l \cdot l^{d-1} \quad (18)$$

with $n \sim j_n^d$. This behaves in leading order as $j_n^{-d} \cdot j_n^{d+1} / (d+1) \sim j_n$. That is, $L(G)$ scales linearly with the diameter j_n .

It is perhaps interesting to compare the two notions, clustering coefficient and clique order, introduced above, in a small network and a random graph. As cliques are subgraphs of the graphs generated by vertices, x , and their one-neighborhoods, $\Gamma_1(x)$, it is reasonable to calculate the edge probability only with respect to the induced subgraphs formed by $\Gamma_1(x)$ and x . From the clustering coefficient, C , and the average vertex degree, V , we get a corresponding local edge probability:

$$p_{loc} = (C \cdot \binom{v}{2} + v) / \binom{v+1}{2} = C - ((2C-2)/(v+1)) \approx C \quad (19)$$

for v sufficiently large.

With this p_{loc} we can now calculate the typical clique order, r_{loc} , if we treat the above subgraphs as random graphs. We get

$$r_{loc} \approx 2 \log(v+1) / \log(p_{loc}^{-1}) \quad (20)$$

(cf. section 5.1). That is, the clique order scales with C roughly as $r_{loc} \sim (\log C^{-1})^{-1}$.

We conclude that, as in a small world network C is larger than C_{rand} , where C_{rand} is the clustering coefficient in a random graph with the same *global* edge probability, we have correspondingly $p_{loc} > p_{rand}$ and hence $r_{loc} > r_{rand}$. Therefore both the clustering coefficient *and* the clique order are larger in a small world network than in a true random graph with the same global edge probability.

We close this section with a brief remark about the statistical framework, as certain points in this connection are sometimes glossed over in the literature. We actually are dealing with two kinds of statistics in this enterprise. For one, if we have very large networks or graphs, we can apply (practical) statistics within the concretely given individual system, that is, perform certain averages over nodes, edges and the like. On the other hand, we may prefer to study a full ensemble of such graphs, formed according to certain statistical or probabilistical principles, an example being the above probability space of (binomially distributed) random graphs.

In principle these are different statistical frameworks, but in practice they are frequently intermixed. We note in passing that a similar philosophy underlies the foundations of statistical mechanics. If a system is both sufficiently large and sufficiently typical or generic, the differences are expected to be negligible. But in any case, it may be wise to remember these (frequently only implicit) statistical preassumptions. We made some more detailed (physical) remarks about the *statistical hypothesis* in sect. 3.1 of [18].

4 Protogeometry and Protodynamics

We briefly want to motivate why we are modelling the underlying fabrique of space-time or the quantum vacuum as a relational network of nodes and links, the geometrical aspects of which can be dealt with in the context of graphs.

On the one side we have a working philosophy which is similar to the one, expounded by 't Hooft in e.g. [24] to [26]. That is, we entertain the idea that for example quantum theory may well emerge as an *effective (continuum) theory* on the mesoscopic scale of an underlying discrete more microscopic theory. As we want our underlying *(pre)geometry* to *coevolve* with the patterns living in this substratum, we developed the above mentioned generalisation of the more regular cellular automata.

Another essential property of such *discrete dynamical systems* is, while the basic ingredients and elementary building blocks are reasonably simple, their potential for the emergence of very complex behavior on the more macroscopic scales, thus supporting the speculation that such systems may be capable of generating viable continuum theories.

It is now suggestive to regard the edges between pairs of points as describing

their (direct) interaction. This becomes more apparent if we impose dynamical network laws on these graph structures so that they become a particular class of discrete dynamical systems. Henceforth we denote such a dynamical network, which is supposed to underly our continuous space-time manifold, by QX (“*quantum space*”). We want to make the general remark that the *cellular networks*, introduced in the following, can either be regarded as mere models of a perhaps more hypothetical character, encoding, or rather simulating, some of the expected features of a surmised *quantum space-time* or, on the other hand, as a faithful realisation of the primordial substratum, underlying our macroscopic space-time picture. So far this is a matter of taste.

For the time being we choose, to keep matters reasonably simple, a discrete overall clock-time (not to be confused with the *physical time* which is rather supposed to be an emergent and intrinsic characteristic, related to the evolution of quasi-macroscopic patterns in such large and intricately wired networks). In principle the clock-time can also be made into a local dynamical variable. Cf. also the complex of investigations grouped around the phenomenon of synchronization in large populations of coupled oscillators (a small selection of the existing literature being for example [1], [5], [6]). Furthermore, we assume the node set of our initial network to be fixed and being independent of clock-time (in contrast to the links). This property may however change if we apply the renormalisation process which we described in [18]. That is, on the higher levels, the class of *lumps* or *meta-nodes* may become dependent on time.

On this network we now define a dynamical law or a (clock-) time evolution. We assume that each node, x_i , or bond, e_{ik} , carries an internal (for simplicity) discrete *state space*, the internal states being denoted by s_i or J_{ik} . In simple examples we chose for instance:

$$s_i \in q \cdot \mathbb{Z} \quad , \quad J_{ik} \in \{-1, 0, +1\} \quad (21)$$

with q an elementary quantum of information and

$$e_{ki} = -e_{ik} \Rightarrow J_{ki} = -J_{ik} \quad (22)$$

In most of the studied cellular automata systems even simpler internal state spaces are chosen like e.g. $s_i \in \{0, 1\}$. This is at the moment not considered to be a crucial point. The above choice is only an example.

In our approach the bond states are dynamical degrees of freedom which, a fortiori, can be switched off or on (see below). Therefore the *wiring*, that is, the pure *geometry* (of relations) of the network is a clock-time dependent, dynamical property and is *not* given in advance. Consequently, the nodes and bonds are typically not arranged in a more or less regular array, a regular lattice say, with a fixed near-/far-order. This implies that *geometry* will become to some degree a *relational* (Machian) concept and is no longer a static background.

As in cellular automata, the node and bond states are updated (for convenience) in discrete clock-time steps, $t = z \cdot \tau$, $z \in \mathbb{Z}$ and τ being an elementary clock-time interval. This updating is given by some *local* dynamical law (examples are given below). In this context *local* means that the node/bond states change at each clock time step according to a prescription with input the overall state of a certain neighborhood (in some topology) of the node/bond under discussion.

A simple example of such a local dynamical law we are having in mind is given in the following definition (first introduced in [21]).

Definition 4.1 (Example of a Local Law) *At each clock time step a certain quantum q is exchanged between, say, the nodes x_i , x_k , connected by the bond e_{ik} such that*

$$s_i(t + \tau) - s_i(t) = q \cdot \sum_k J_{ki}(t) \quad (23)$$

(i.e. if $J_{ki} = +1$ a quantum q flows from x_k to x_i etc.)

The second part of the law describes the back reaction on the bonds (and is, typically, more subtle). We assume the existence of two critical parameters $0 \leq \lambda_1 \leq \lambda_2$ with:

$$J_{ik}(t + \tau) = 0 \quad \text{if} \quad |s_i(t) - s_k(t)| =: |s_{ik}(t)| > \lambda_2 \quad (24)$$

$$J_{ik}(t + \tau) = \pm 1 \quad \text{if} \quad 0 < \pm s_{ik}(t) < \lambda_1 \quad (25)$$

with the special proviso that

$$J_{ik}(t + \tau) = J_{ik}(t) \quad \text{if} \quad s_{ik}(t) = 0 \quad (26)$$

On the other side

$$J_{ik}(t + \tau) = \begin{cases} \pm 1 & J_{ik}(t) \neq 0 \\ 0 & J_{ik}(t) = 0 \end{cases} \quad \text{if} \quad \lambda_1 \leq \pm s_{ik}(t) \leq \lambda_2 \quad (27)$$

In other words, bonds are switched off if local spatial charge fluctuations are too large or switched on again if they are too small, their orientation following the sign of local charge differences, or remain inactive.

Another interesting law arises if one exchanges the role of λ_1 and λ_2 in the above law, that is, bonds are switched off if the local node fluctuations are too small and are switched on again if they exceed λ_2 .

We make the following observation:

Observation 4.2 (Gauge Invariance) *The above dynamical law depends nowhere on the absolute values of the node “charges” but only on their relative differences. By the same token, charge is nowhere created or destroyed. We have*

$$\Delta\left(\sum_{QX} s(x)\right) = 0 \quad (28)$$

(Δ denoting the change in total charge of the network between two consecutive clocktime steps). To avoid artificial ambiguities we can e.g. choose a fixed reference level, taking as initial condition at $t = 0$ the following constraint

$$\sum_{QX} s(x) = 0 \quad (29)$$

We resume what we consider to be the crucial ingredients of network laws, we are interested in

1. As in gauge theory or general relativity, our evolution law should implement the mutual interaction of two fundamental substructures, put a little bit vaguely : “*geometry*” acting on “*matter*” and vice versa, where in our context “*geometry*” is assumed to correspond in a loose sense to the local and/or global array of bond states and “*matter*” to the structure of the node states.
2. By the same token the alluded *selfreferential* dynamical circuitry of mutual interactions is expected to favor a kind of *undulating behavior* or *selfexcitation* above a return to some uninteresting *equilibrium state* (being devoid of stable structural details), as is frequently the case in systems consisting of a single component which directly acts back on itself. This propensity for the *autonomous* generation of undulation patterns is in our view an essential prerequisite for some form of “*protoquantum behavior*” we hope to recover on some coarse grained and less primordial level of the network dynamics.
3. In the same sense we expect the large scale pattern of switching-on and -off of bonds to generate a kind of “*protogravity*”.

Remark: The above dynamical law shows that bonds with $J_{ik} = 0$ at clock time t do not participate in the dynamics in the next time step. We hence may consider them as being temporally inactive. The shape of the network, neglecting all the internal states of the nodes and bonds together with the inactive bonds we call the *wiring diagram*.

If one concentrates solely on this *wiring diagram*, figure 2 (below) describes one clocktime step in the life of a *dynamic graph*. In the picture only a small subgraph is shown and the deletion and creation of edges (that is, elementary interactions among nodes or possible information channels). The new bonds are represented as bold lines. It should be emphasized that the graph is *not* assumed to be a triangulation of some preexisting smooth manifold. This is emphasized by the existence of edges, connecting nodes which are not necessarily close with respect to e.g. the euclidean distance. We recently observed that similar ideas have been entertained within the framework of cellular automata (see e.g. [27] and [28]), the models being called *structurally dynamic cellular automata* or SDCA.

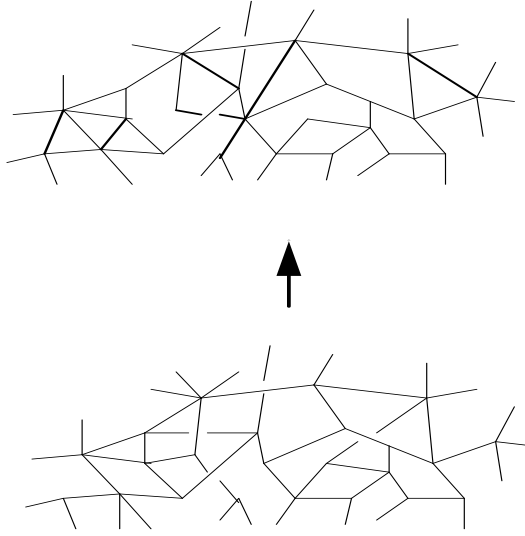


Figure 2:

As far as we can see at the moment, the adopted technical framework is not exactly the same but we think, a comparison of both approaches should turn out to be profitable.

Remark 4.3 *We conjecture that such discrete dynamical laws as introduced above may be discrete protoforms of the dynamical laws governing the arrays of coupled nonlinear oscillators in the papers cited previously. This is corroborated by computer simulations on arrays of several thousand nodes, performed by us in the past (cf. [23]) which clearly exhibited an amalgamation or superposition of statistical behavior and a collective undulation pattern. The second law, described above, has in particular extremely short transients, reaching a periodic attractor in a very short time, having the further remarkable property that, given the huge accessible phase space and the complexity of the network states, it has a period of only six.*

5 The Translocal Depth-Structure of (Quantum) Space-Time

In this section we want to prepare the stage which will allow us to relate our own approach to quantum space-time structure with the small-world network view, being expounded in, on the surface, quite distinct areas of research. But in order to keep the exposition of the partly quite intricate technical details within reasonable length, we will mainly refer, as to the technical details, to the two papers, [18], [22], and try to give here only the general ideas.

The central picture is that, what we experience as a practically continuous

space-time manifold, will turn out, under sufficient magnification, as a network of overlapping local clusters or lumps, being superposed by a, in a measure theoretic sense meager or sparse second network connecting these local clusters or lumps in a basically translocal manner. That means, this second network connects local clusters which may be quite a distance apart with respect to the metrical structure of the network of underlying lumps. We tried to make more explicit in [18], [19], [22] how this double structure is expected to go over, on the macroscopic scale of ordinary space-time physics, into, on the one hand, a smooth local causal space-time plus, on the other hand, a classically almost hidden nonlocal network structure which is, due to the weaker and fluctuating connections, of a more stochastic nature and which we expect to make an effect in many of the notorious quantum phenomena (note again the parallels to observations in certain social networks mentioned below).

This overall picture can be made more precise as follows. To construct this underlying network of lumps, our main tool will be a kind of *geometric renormalisation group* or systematic *coarse graining procedure* which we developed in [18]. To put it briefly, we regard the ordinary space or space-time as a medium having a rich internal nested fine structure, which is however largely hidden on the ordinary macroscopic scales due to the usually low level of resolution of space-time processes as compared to e.g. the Planck scale. In the process of *coarse graining*, described in the following, the resolution of the details of space-time is steadily scaled down from the Planck level to the level of ordinary continuum physics.

On the deepest level, that is, the Planck scale, proto space-time is supposed to be a wildly fluctuating network of dynamic relations or exchange of pieces of information among a given set of nodes. At each fixed clock-time step there exist certain subclusters of nodes in this initial network which are particularly densely entangled and the whole graph can be covered by this uniquely given set of subclusters of nodes and the respective induced subgraphs (edges given by overlap of clusters). We dealt with these distinguished clusters of nodes (called cliques or lumps) in quite some detail in e.g. [19] or [20] and define them in the introductory section on graph concepts. We emphasize the interesting relations to earlier ideas of Menger, Rosen et al, which have been discussed in [20].

It is fascinating that a similar picture was developed quite some time ago in mathematical sociology (cf. [1] p.14 f and [31]). These people developed networks consisting of two kinds of *ties*, they called (as we did in, for example, [22]) *weak* and *strong* ties. The strong ties define closely knitted clusters of friends they call *clumps* (similar to our cliques or lumps) while the weak ties form (nonlocal) *bridges* to acquaintances who are usually not friends of each other but are lying in local clumps of their own, with these clumps non-overlapping with each other. In this work the role of the weak ties is particularly emphasized, playing a role very similar to the translocal web in our framework. As to the geometric correspondences compare this picture with a very similar picture we developed in section 5.2 (cf. in particular figure 3), being completely unaware of

the above work.

Technically we need a general principle which allows us to lump together subsets of nodes, living on a certain level of resolution of space-time, to construct the building blocks of the next level of coarse graining (see below). After a series of such coarse graining steps we will wind up with a nested structure of lumps, containing smaller lumps and so forth, which, after appropriate *rescaling*, may yield in the end some quasi-continuous but nested structure. This principle is provided by the following mathematical prescription.

5.1 The Geometric Renormalization Process

It is an important observation that in a generic random graph of order n and edge probability p the order of cliques is concentrated with very high probability in a relatively small interval $I = (r_0/2, r_0)$;

$$r_0 \approx 2 \log(n) / \log(p^{-1}) + O(\log \log(n)) \quad (30)$$

see [11],[18],[19]. That is, each renormalization or coarse graining step consists of the following pieces.

- Starting from a given fixed graph $G = G_0$, defining the level zero, pick the (generic) *cliques*, S_i , in G , their order lying in the above mentioned interval, $(r_0/2, r_0)$.
- These cliques form the new nodes of the *clique-graph*, $G_{cl} = G_1$ of $G = G_0$. The corresponding new edges are drawn between cliques, having a (sufficient degree of) overlap. Size, overlap and distribution of cliques in a generic (*random*) graph have been analyzed in [18],[19].
- That is, both *marginal* (i.e. very small) cliques (if they do exist at all) and, more importantly (as they are more numerous), *marginal* overlaps are deleted. In this respect a coarse-graining step includes also a certain *purification* of the graph structure.

What is considered to be a “sufficient overlap” depends of course on the physical context and the general working philosophy. A particular node will in general belong to several, and in the case of densely entangled graphs to many, cliques. The minimal possible overlap is given by a single common node. If, on the other hand, the cliques on a certain level of coarse graining are comparatively large, comprising, say, typically several hundred nodes, it may be reasonable to neglect *marginal*, i.e. too small, overlaps as physically irrelevant and define a sufficient degree of overlap to consist of an appreciable fraction of the typical clique order. The numerical effect of such choices have been studied in sect.5 of [18].

Definition 5.1 We call the graph, defined above, the (purified) clique graph, G_{cl} , constructed from the initial graph, G .

We note in passing that the robustness of, say, the graph property *connectedness* under these coarse graining steps has been dealt with in sect. 5 of [18]. We emphasize that our coarse graining procedure is *universal* in the sense that the *same* principles are applied on every level of the renormalization procedure, as the transition from a graph to its clique graph is always a well defined prescription. In the end, after some rescaling of the length unit, we hope to arrive at a (quasi-)continuous manifold, displaying, under appropriate magnification, an intricate internal fine structure. This should (or rather, can only expected to) happen if the original network has been in a *critical state* as will be described in the following.

On each level of coarse-graining, that is, after each renormalisation step, labelled by $l \in \mathbb{Z}$, we get, as in the block spin approach to critical phenomena, a new level set of cliques or lumps, S_i^l , (i labelling the cliques on renormalisation level l), consisting on their sides of $(l - 1)$ -cliques which are the l -nodes of level l , starting from the level $l = 0$ with $G =: G_0$. That is, we have

$$S_j^l = \bigcup_{i \in j} S_i^{(l-1)}, \quad S_i^{(l-1)} = \bigcup_{k \in i} S_k^{(l-2)} \text{ etc.} \quad (31)$$

($i \in j$ denoting the $(l - 1)$ -cliques, belonging, as meta nodes, to the l -clique, S_j). These cliques form the meta nodes in the next step.

Definition 5.2 The cliques, S_i^0 , of $G =: G_0$ are called zero-cliques. They become the one-nodes, x_i^1 , of level one, i.e. of G_1 . The one-cliques, S_i^1 , are the cliques in G_1 . They become the 2-nodes, x_i^2 , of G_2 etc. Correspondingly, we label the other structural elements, for example, 1-edges, 2-edges or the distance functions, $d_l(x_i^l, x_j^l)$. These higher-level nodes and edges are also called meta-nodes, -edges, respectively.

Remark 5.3 The above construction may lead to the wrong impression that the network becomes sparser after each step. Quite to the contrary, the number of cliques in G_{cl} may be much larger than the number of nodes in the original graph, G (cf. the table in section 3 of [18]). This happens if there is an appreciable overlap among the occurring cliques, that is, a given node may belong to many different cliques. On the other hand, after several renormalisation steps, the picture usually seems to become stable in the generic case (see subsect. 5.2 of [18]).

We illustrate the preceding remarks with a couple of numerical results. Starting, as in [18], from an initial network with the parameters, $n = 10^{100}$, $p = 0.7$, which implies $r_0 = 1291$ and taking as sufficient overlap of cliques a value of, say,

fifty, we get after the first coarse graining step the new parameters on level one:

$$l = 1: n_1 \approx 10^{10^4}, p_1 \approx 10^{-7 \cdot 10^3}, r_1 = 3, \text{ average vertex degree } k \approx 10^{0.3 \cdot 10^4}$$

We see that after only one step the typical cliques are already very small. Therefore, in the next step, an overlap greater or equal to one is appropriate, that is we can use the ordinary clique graph instead of the purified clique graph and get on the second level:

$$l = 2: n_2 \approx 10^{10^4} \text{ (number of cliques on level one)}, p_2 \approx p_1 \approx 10^{-7 \cdot 10^3}, r_2 = 3$$

We convinced ourselves that the qualitative picture will no longer change under further renormalisation steps.

Remark: We note that the above choice of parameters is not really crucial. The qualitative picture will essentially remain the same for other choices (cf. section 5 of [18]).

We briefly resume the picture we tried to convey in this subsection. We argued that, what we regard as the building blocks of our physical space-time continuum and what we dubbed *physical points* in previous work, have actually a nested internal structure which is built up, starting from the Planck scale, via the renormalisation steps, described above.

On the other hand, we want to emphasize that such a continuum as a *fixed point* or limit state is far from being a quasi automatic consequence of our procedure. Quite to the contrary, the initial network has to be in a very peculiar *critical state*, see sect.8 of [18], embodying a kind of seemingly *scale free translocal order* as being described, for example, in a different context by Barabasi et al ([3]).

5.2 The Translocal Network

This subsection mainly refers to [22], but we deal primarily with the (random) graph aspects which can be found in sect.6 of [22]. We assume that on a certain scale, l , of our renormalisation process, the network of lumps, G_l , is sufficiently close to the continuous limit manifold, M . We take its meta-nodes, i.e. the cliques, $S_i^{(l-1)}$, of level $(l-1)$ as an approximation of the physical points of M .

This clique graph of level l (i.e. after l coarse graining steps have been performed) carries a natural graph metric, $d_l(S_i, S_j)$, which is given by the natural distance between the meta nodes (or cliques), S_i, S_j (with edges given by sufficient overlap). If one wants to, one can relate this integer-valued (grainy) distance functional to a continuous distance function between the corresponding (fuzzy) physical points, P_i, P_j of the associated continuous manifold (which, in this approach, may be viewed rather as a mental construct) with

$$d_l(s_i, s_j) \sim d_{man}(P_i, P_j) \tag{32}$$

For more details as to these aspects see [20]. We want to focus our attention in the following on another somewhat hidden but very important aspect of the construction.

By assumption we draw an edge between a pair of cliques, S_i, S_j , if they have a certain degree of overlap of common nodes. This has the effect that all edges which may exist between nodes lying in S_i, S_j , respectively, are deleted in the next step if these cliques have empty or too marginal overlap. That is, these cliques, occurring as meta nodes of the following level are now unrelated in the next step whereas there may still exist a certain (limited) amount of information exchange on the preceding, more fine grained levels. In this sense information will also be coarse grained with only sufficiently robust information surviving the process.

In subsection 5.2 of [18] we made a detailed (numerical) analysis of these effects of renormalisation. Among other things we calculated, depending on the edge probability p , the typical cardinality of the *local group* of a given clique, S_0 , i.e. the number of cliques having an overlap with S_0 bigger than some prescribed number, the typical cardinality of the cliques on the consecutive levels, the level-dependent edge probability and vertex degree and so on. We made the important observation that already after a few steps the whole picture becomes relatively stationary, implying that the idea of a stationary limit phase is perhaps not so far-fetched.

What we have discussed so far is the *local structure* or *near order* aspect of the network which becomes more and more apparent as a consequence of the consecutive coarse graining steps. We showed also in [18] that this process leads to an unfolding of the initially densely entangled network towards a network having a large average distance or diameter similar to a local network in contrast to a typical random graph.

We infer that in contrast to the initial graph, $G = G_0$, in which a large portion of the vertices is directly connected, most of the cliques of G_0 , i.e. nodes of the first level G_1 , are no longer directly connected. On the other hand, many of the nodes lying *in*, say, the non-overlapping cliques, S_i, S_j are connected by edges of the initial graph G_0 .

Another important consequence of our analysis is that practically all occurring cliques are lying in the above interval $(r_0/2, r_0)$ with $r_0 = 1291$ in our example. This implies that essentially every edge in G_0 belongs to at least one such large clique. Put differently, with S_i, S_j two cliques, having a large distance in G_1 , and e_{ij} some edge connecting two nodes in S_i, S_j respectively, e_{ij} almost certainly belongs to another large clique of roughly the same order. This conclusion makes the global picture both intricate and interesting.

We try to express this situation in the following picture. The circles denote some generic cliques which are assumed to have sufficient overlap with (some of) their neighbors. A part of another clique (denoted by (1)), assumed to be of more or less the same order, but having only *weak* bonds (i.e., weak overlap)

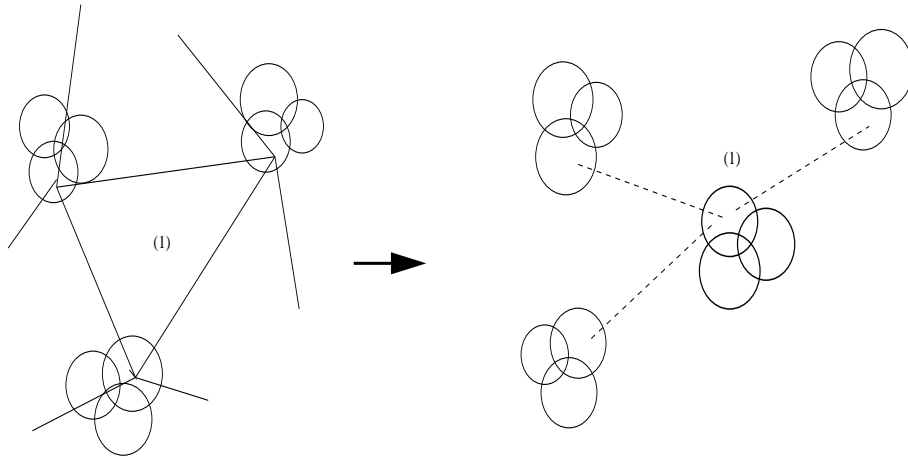


Figure 3:

with these possibly widely separated local neighborhood clusters, is represented by thin lines.

On the other hand, we learned from our numerical estimates (see also below) that, typically, also clique (1) will have its own local group, that is, cliques with strong overlap. In other words, this particular clique (1) is *almost shurely* the member of another local group of roughly the same shape, but lying in another region of the manifold, M . This situation is described in the picture on the right, with the weak bonds between the three local groups, represented on the left and the clique (1) depicted by dashed lines. The clique (1) is now represented as a member of another local group of generic cliques.

Summing up our observations we can conclude that, even if we start from a densely entangled random graph, we typically arrive after only a few renormalisation steps at a coarse grained network displaying both a markedly local behavior in form of strongly coupled local clusters and a superposed sparser network of a different, translocal character, with links spreading like a spider web over the whole underlying local network. We note that these findings exhibit a strong resemblance to the small world scale free networks discussed previously. We underpin this speculation with a variety of analytic results in the following section.

6 Scale-Free Critical Network States

As we remarked above, random graphs have a vertex degree distribution which is of a binomial, and, in a certain limit, of Poisson type. In contrast to such graphs, Barabasi et al observed that certain classes of concrete networks are not of this type but instead are of scale-free type ([3],[4]). This means, the vertex degrees

are (asymptotically) distributed according to a *power law*

$$P(k) \sim k^{-\gamma} \quad (33)$$

Geometrically such a distribution is related to the existence of a certain portion of *hubs* in the networks, that is, nodes having an unusually large number of links. However, this is not sufficient in general. The network has to be hierarchically organised so that these short cuts do exist on consecutive levels of coarse graining as described by us in the preceding section. One therefore may suppose that scale-free networks represent a subclass of small world networks with clusters in clusters and so on. In the following we will corroborate this hypothesis with a number of analytic results.

An important conceptual tool in the analysis of the large scale behavior of graphs of order, n , near or at infinity, is the *distance degree sequence*, $dds(x)$, relative to an arbitrary but fixed node, x ([17],[18]). In [17] we gave arguments that its scaling behavior is closely related to a geometric characteristic of spaces or, rather, systems (in physics) which may be identified with the concept of *intrinsic dimension*.

Remark 6.1 *We use the adjective ‘intrinsic’ to distinguish the concept from the more common concept of ‘embedding dimension’. In contrast to the latter it encodes the intrinsic geometrical or relational organisation of the system itself and not the lesser important structure of the ambient space.*

Definition 6.2 (Internal Scaling Dimension) *Let x be an arbitrary node of G . Let $\#(U_l(x))$ denote the number of nodes in $U_l(x)$. We consider the sequence of real numbers $D_l(x) := \frac{\ln(\#(U_l(x)))}{\ln(l)}$. We say $\underline{D}_S(x) := \liminf_{l \rightarrow \infty} D_l(x)$ is the lower and $\overline{D}_S(x) := \limsup_{l \rightarrow \infty} D_l(x)$ the upper internal scaling dimension of G starting from x . If $\underline{D}_S(x) = \overline{D}_S(x) =: D_S(x)$ we say G has internal scaling dimension $D_S(x)$ starting from x . Finally, if $D_S(x) = D_S \forall x$, we simply say G has internal scaling dimension D_S .*

Definition 6.3 (Connectivity Dimension) *Let x again be an arbitrary node of G . Let $\#(\partial U_l(x))$ denote the number of nodes in the boundary of $U_l(x)$ (in our previous notation $\Gamma_l(x) = \partial U_l(x)$). We set $\tilde{D}_l(x) := \frac{\ln(\#(\partial U_l(x)))}{\ln(l)} + 1$ and define $\underline{D}_C(x) := \liminf_{l \rightarrow \infty} \tilde{D}_l(x)$ as the lower and $\overline{D}_C(x) := \limsup_{l \rightarrow \infty} \tilde{D}_l(x)$ as the upper connectivity dimension. If lower and upper dimension coincide, we say G has connectivity dimension $D_C(x) := \overline{D}_C(x) = \underline{D}_C(x)$ starting from x . If $D_C(x) = D_C$ for all x we call D_C simply the connectivity dimension of G .*

The two definitions are not strictly equivalent but coincide in the more regular situations. In the following, for the sake of brevity, we only use the first notion. Like *fractal dimension* the above definitions coincide with the usual (embedding) dimension for the more regular situations like e.g. lattice graphs.

It is remarkable that our concept of graph or network dimension is stable under a variety of graph transformations or deformations, in particular local ones ([17],[18]). In the following we want to concentrate on one particular aspect, namely the relevance of a dimensional analysis in connection with *critical network states*. These we will associate later with small world scale-free networks.

In a first step we compare the dimension of a graph, G , with its (unpurified) clique graph, G_{cl} , in order to exhibit the importance of *coarse graining*.

Theorem 6.4 *Assuming that G has dimension D and globally bounded node degree, $v_i \leq v < \infty$, we have that D_{cl} also exists and it holds*

$$D_{cl} = D \quad (34)$$

Note that this result does hold for the ordinary clique graph, viz. arbitrary overlap, viz., no purification. In other words, under these assumptions, the renormalisation steps do not change the graph dimension.

The (longer) proof can be found in [18].

This result is reminiscent of a similar observation in statistical mechanics where the non-coarse-grained Gibbsian entropy happens to be a *constant of motion*. The same happens here. In the ordinary clique graph each original bond occurs in at least one clique, i.e. there is no real (or, more precisely, not enough) coarse graining.

Note that there are two important assumptions underlying the above result. First, the node degree is assumed to be globally bounded. Second, there has been no coarse graining. If we allow for purification, we have the following weaker result.

Corollary 6.5 *For the purified clique graph, with overlaps exceeding a certain fixed number, l_0 , we can only prove*

$$D_{cl} \leq D \quad (35)$$

Having for example the picture in mind, frequently invoked by Wheeler and others, of a space-time foam, with a concept of dimension depending on the scale of resolution (see e.g. Box 44.4 on p.1205 in [29]), we infer from our above observations that this may turn out to be both an interesting and not entirely trivial topic. We have to analyze under what specific conditions the dimension can actually shrink under coarse-graining, so that we may start from a very erratic network on, say, the Planck scale, and arrive in the end at a smooth macroscopic space-time having perhaps an integer dimension of, preferably, value 4 or so.

While the above corollary seems to allow in principle that space- or network dimension may become smaller under coarse graining, the following remarkable result shows that this is not so easily achieved. On the other hand, it gives

strong clues as to the kind of (critical) network states which actually do admit a decrease of dimension under purification.

In [17] we investigated the effect of additional edge insertions in a graph on its dimension.

Proposition 6.6 *Additional insertions of bonds between arbitrarily many nodes, y, z , having original graph distance, $d(y, z) \leq k$, $k \in \mathbb{N}$ arbitrary but fixed, do not change $\underline{D}(x)$ or $\overline{D}(x)$.*

From this we learn the following. Phase transitions in graphs, changing the dimension, have to be intrinsically *non-local*. That is, they necessarily involve nodes, having an arbitrarily large distance in the original graph. We think, this is a crucial observation from the physical point of view. It shows that systems have to be *critical* in a peculiar way, that is, having a lot of distant correlations or, rather, correlations on all scales (cf. also Smolins's discussion in e.g. [30] and elsewhere).

In the preceding proposition we made the transition from a graph, G , to a graph G' living on the same node set but having more edges with the special proviso that edge insertions take only place between pairs of nodes, (x, y) , having

$$d_G(x, y) \leq k \quad (36)$$

In the purification process we are rather interested in edge deletions! These two processes are however not! strictly symmetric.

Under edge insertions the distance between nodes does not increase, i.e.

$$G \rightarrow G' \Rightarrow d_{G'}(x, y) \leq d_G(x, y) \quad (37)$$

On the other hand, edge deletions may lead to

$$G' \rightarrow G \Rightarrow d_G(x, y) \geq d_{G'}(x, y) \quad (38)$$

If we want to employ the above proposition also in the case of edge deletions, we have to guarantee that edges in G' between nodes, x, y , may be deleted under the proviso that $d_G(x, y) \leq k$. The condition $d_{G'}(x, y) \leq k$ would not suffice.

If we apply these findings to our renormalisation steps, that is, passing from a graph to its associated (purified) clique graph, this implies the following. We saw that assuming a network or graph, G , having a dimension, D , the unpurified clique graph still has

$$D_{cl} = D \quad (39)$$

On the other hand, denoting for the moment the purified clique graph by \hat{G}_{cl} , we have the estimate

$$\hat{D}_{cl} \leq D_{cl} = D \quad (40)$$

The transition from G_{cl} to \hat{G}_{cl} consists of the deletion of marginal overlaps among cliques (with the necessary criteria provided by the physical context). That is, \hat{G}_{cl} lives on the same node set (the set of cliques) but has fewer (meta)bonds. The above proposition shows that this does *not* automatically guarantee that we really have

$$\hat{D}_{cl} < D_{cl} \quad (41)$$

Quite to the contrary, we learned that this can only be achieved if the bond deletions happen in a very specific way.

On G_{cl} we have, as on any graph, a natural distance or neighborhood structure, given by the canonical graph metric, $d_{cl}(S_i, S_j)$. We thus infer that edge deletions in G_{cl} between cliques which are not very far apart in the final purified graph \hat{G}_{cl} cannot alter the final dimension of \hat{G}_{cl} . More precisely, only edge deletions between cliques having distances in \hat{G}_{cl} which approach infinity in a specific way, can have an effect.

The preceding observation fits into the picture one invokes in the context of critical behavior and scale freeness in, for example, statistical mechanics. Furthermore it seems to be closely related to the kind of scale freeness of networks as observed by Barabasi et al. In the last section of [18] we gave a simple but, as we think, instructive example in which the effects of edge deletions on all scales as the origin of dimensional change can explicitly be studied. The example consists of the one-dimensional (discrete) line, \mathbb{Z}_1 , being embedded in \mathbb{Z}_2 in a particular way, so that the corresponding edge deletions lead to a dimensional change from \mathbb{Z}_2 to \mathbb{Z}_1 .

References

- [1] D.J.Watts: “Small Worlds, The Dynamics of Networks between Order and Randomness”, Princt.Univ.Pr., Princeton 1999
- [2] D.J.Watts,S.H.Strogatz: “Collective Dynamics of ‘Small World’ Networks”, Nature 393(1998)440
- [3] R.Albert,A.-L.Barabasi: “Statistical Mechanics of Complex Networks”, Rev.Mod.Phys. 74(2002)47, cond-mat/0106096
- [4] A.-L.Barabasi,E.Bonabeau: “Scale-Free Networks”, Sci.Am. May 2003 p.50
- [5] S.Strogatz. “From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators”, Physica D 143(2000)1
- [6] s.H.Strogatz,R.E.Mirollo: “Phase Locking and Critical Phenomena in Lattices of Coupled Nonlinear Oscillators”, Physica D 31(1988)143
- [7] S.H.Strogatz: “Exploring Complex Networks”, Nature 410(2001)268

- [8] S.H.Strogatz: “Sync: The Emerging Science of Spontaneous Order”, Penguin, N.Y. 2003
- [9] M.E.J.Newman: “Models of the Small World”, J.Stat.Phys. 101(2000)819
- [10] M.E.J.Newman,D.J.Watts: “Renormalisation Group Analysis of the Small World Nertwork Model”, cond-mat/9903357
- [11] B.Bollobas: “Random Graphs”, Acad.Pr., N.Y.1985
- [12] B.Bollobas: “Modern Graph Theory”, Springer, N.Y.1998
- [13] B.Bollobas: “Combinatorics”, Cambridge Univ.Pr., London 1986
- [14] A.Papoulis: “Probability, Random Variables, and Stochastic Processes”, McGraw-Hill, N.Y. 1965
- [15] F.Buckley,F.Harary: “Distances in Graphs”, Addison-Wesley, N.Y. 1990
- [16] M.Requardt: “Dirac Operators and the Calculation of the Connes Metric on arbitrary (infinite) Graphs”, J.Phys.A:Math.Gen. 35(2002)759, math-ph/0001026
- [17] T.Nowotny,M.Requardt: “Dimension Theory on Graphs and Networks”, J.Phys.A:Math.Gen. 31(1998)2447, hep-th/9707082
- [18] M.Requardt: “A Geometric Renormalisation Group in Discrete Quantum Space-Time”, to appear JMP, gr-qc/0110077
- [19] M.Requardt: “(Quantum) Space-Time as a Statistical Geometry of Lumps in Random Networks”, Class.Quant.Grav. 17(2000)2029, gr-qc/9912059
- [20] M.Requardt,S.Roy: “(Quantum) Space-Time as a Statistical Geometry of Fuzzy Lumps and the Connection with Random Metric Spaces”, Class.Quant.Grav. 18(2001)3039, gr-qc/0011076
- [21] M.Requardt: “Cellular Networks as Models for Planck-Scale Physics”, J.Phys.A:Math.Gen. 31(1998)7997, hep-th/9806135
- [22] M.Requardt: “The Translocal Depth-Structure of Space-Time ”, hep-th/0205168
- [23] Th.Nowotny: Diploma Thesis, Goettingen 1998, a compilation of some of the extensive computer simulations cab be found on his web site: <http://inls.ucsd.edu~novotny/>
- [24] G.’t Hooft: “Deterministic and Quantum Mechanical Systems”, J.Stat.Phys. 53(1988)323
- [25] G.’t Hooft: “Quantisation of Discrete Deterministic Theories”, Nucl.Phys. B342(1990)471

- [26] G.'t Hooft: "Quantum Gravity and Dissipative Deterministic Systems", gr-qc/9903084
- [27] A.Ilachinski: "Cellular Automata, A Discrete Universe" sect.8.8 and sect.12, World Scientific, Singapre 2001
- [28] A.Ilachinski,P.Halpern: "Structurally Dynamic Cellular Automata", Complex Systems 1(1987)503
- [29] Ch.W.Misner,K.S.Thorne,J.A.Wheeler: "Gravitation", Freeman, San Francisco 1973
- [30] S.Kauffman,L.Smolín: "Combinatorial Dynamics in Quantum Gravity", "Towards Quantum Gravity", Proc.Polanica, Ed. J.Kowalski-Glikman, Springer Lect.N.Phys., Springer, N.Y.2000
- [31] M.S.Granovetter: "The Structure of Weak Ties", Sociological Theory 1(1983)203