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A Uniformly Convergent Scheme for A System of Two Coupled Singularly Perturbed Reaction-Diffusion Robin Type Boundary Value Problems with Discontinuous Source Term

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Abstract In this paper, a uniformly convergent scheme for a system of two coupled singularly perturbed reaction-diffusion Robin type mixed boundary value problems (MBVPs) with discontinuous source term is presented. A fitted mesh method has been used to obtain the difference scheme for the system of MBVPs on a piecewise uniform Shishkin mesh. A cubic spline scheme is used for Robin boundary conditions and the classical central difference scheme is used for the differential equations at the interior points. An error analysis is carried out and numerical results are provided to show that the method is uniformly convergent with respect to the singular perturbation parameter which supports the theoretical results.

Keywords: *singular perturbation problem, weakly coupled system, discontinuous source term, Robin boundary conditions, Shishkin mesh, fitted mesh method, uniform convergence*

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1. Introduction

Singular perturbation problems (SPPs) arise in various fields of science and engineering which include fluid mechanics, fluid dynamics, quantum mechanics, control theory, semiconductor device modeling, chemical reactor theory, elasticity, hydrodynamics, gas porous electrodes theory, etc. SPPs are characterized by the presence of a small parameter $(0<\varepsilon\ll 1)$ that multiplies the highest derivative term. This leads to boundary and/or interior layers in the solution of such problems. A much attention has been drawn on these problems to obtain good approximate solutions for the past few decades. Since classical numerical methods fail to produce good approximations for these equations, it is inevitable to go for non-classical methods. There are several articles available at the literature but they are mainly based on singularly perturbed problems containing one equation. Some authors have developed robust numerical methods for a system of singularly perturbed convection-reaction-diffusion problems on smooth data. Very few researchers can be seen for problems with non-smooth data which frequently arises in electro analytic chemistry, predator-prey population dynamics, etc. as a perfect application. Oseen equations form a convection-diffusion system where as linearized Navier-Stokes equations yield a reaction-diffusion system at large Reynolds number.

For a parameter-uniform methods pertaining to singular perturbation problems, one can refer the books [1,2,3]. A standard finite difference method is proved uniformly convergent on a fitted piece wise uniform Shishkin mesh for a single equation reaction-diffusion problem [2]. The same approach for coupled system of two singularly perturbed reaction-diffusion problems, with diffusion coefficients $\varepsilon_1,\varepsilon_2$ was originally proposed by Shishkin [4] and identified three different cases $(i)\, 0<\varepsilon_1=\varepsilon_2\ll 1; (ii)\, 0<\varepsilon_1\ll\varepsilon_2=1; (iii)\, 0<\varepsilon_1\le\varepsilon_2\ll 1.$ For case-(i), Matthews et al. [5] proved almost first order convergence using classical finite difference scheme on Shishkin mesh for a system of singularly perturbed reaction-diffusion equations subject to Dirichlet boundary conditions. Tamilselvan et al. [6] developed a numerical method using fitted piecewise uniform Shishkin mesh for the coupled system of singularly perturbed reaction-diffusion equations for case-(i) with discontinuous source term subject to Dirichlet boundary conditions and obtained almost first order uniform convergence. Singularly perturbed linear second order ordinary differential equations of reaction-diffusion type with discontinuous source term subject to Dirichlet boundary conditions having diffusion parameters with different magnitudes was studied by Paramasivam et al. [7]. In that paper, the authors constructed a numerical method using classical finite difference scheme on Shishkin mesh with first order parameter-uniform accuracy. Using mesh equidistribution technique, Das and Natesan [8] studied the singularly perturbed system of reaction-diffusion problems subject to Dirichlet boundary conditions on smooth data having diffusion parameters with different magnitudes. In that article, the central difference scheme is used to discretize the problem on adaptively generated mesh and obtained an optimal second order parameter uniform convergence.

In recent years, system of singularly perturbed Robin type reaction-diffusion problems has attracted a lot of attention for many researchers. Das and Natesan [9] achieved perfect second order accuracy for a single second order Robin type reaction-diffusion problems using adaptively generated grid for smooth case. In that article, the authors proposed the cubic spline difference scheme for mixed boundary conditions and the classical central difference scheme for the differential equation at the interior points to get second order parameter uniform convergence. Das and Natesan [10] also proposed an efficient hybrid numerical scheme, which uses cubic spline difference scheme in the inner region and central difference scheme in the outer region, for singularly perturbed system of Robin type reaction-diffusion problems on Shishkin meh for smooth case. It has been shown that the scheme is ε-uniform convergent with almost second order accuracy. Two hybrid difference schemes on the Shishkin mesh were constructed by Mythili Priyadharshini and Ramanujam [11] for solving the singularly perturbed coupled system of convection-diffusion equations with mixed type boundary conditions on smooth data which generate ε -uniform convergent numerical approximations to the solution. Recently, Mahabub Basha and Shanthi [12] have considered a numerical method for singularly perturbed coupled system of convection-diffusion Robin type boundary value problems with discontinuous source term. Motivated by the above works, in this article, we have developed a uniformly convergent numerical method on the Shishkin mesh for a system of two coupled singularly perturbed reaction-diffusion Robin type boundary value problems with discontinuous source term.

This paper is organized as follows: In Section-2, some analytical results of the solution of singularly perturbed MBVP with discontinuous source term are presented. The numerical method is described in Section-3. Error analysis is carried out in Section-4. Numerical examples are provided in Section-5 and conclusions are given in Section-6.

Throughout this paper, C denotes a generic positive constant independent of the singular perturbation parameter ε, the nodal points x_i and the number of mesh intervals N which may not be same at each occurrence. Let $y: D=[a,b]\to\mathbb{R}$. The norm which is suitable for studying the convergence of numerical solution to the exact solution of the singular perturbation problem is the maximum norm $\|y\|_D = \sup_{x\in D}|y(x)|$. Further, $|\bar{y}(x)| = (|y_1(x)|,|y_2(x)|)^T$ and $\|\bar{y}\| = \max_{x\in D}\{\|y_1\|,\|y_2\|\}$.

2. Continuous Problem

2.1. Statement of the Problem

Find $y_1, y_2 \in Y \equiv C^0(\bar{\Omega})\cap C^2(\Omega^-\cup\Omega^+)$ such that

$$P_1\bar{y}(x) \equiv -\varepsilon y_1''(x)+a_{11}(x)y_1(x)+a_{12}(x)y_2(x)=f_1(x),\quad \forall x\in\Omega^-\cup\Omega^+, \tag{1}$$

$$P_2\bar{y}(x) \equiv -\varepsilon y_2''(x)+a_{21}(x)y_1(x)+a_{22}(x)y_2(x)=f_2(x),\quad \forall x\in\Omega^-\cup\Omega^+, \tag{2}$$

with the boundary conditions

$$B_{10}y_1(0)\equiv\alpha_1 y_1(0)-\varepsilon\beta_1 y_1'(0)=p, \tag{3}$$

$$B_{11}y_1(1)\equiv\gamma_1 y_1(1)+\varepsilon\delta_1 y_1'(1)=q, \tag{4}$$

$$B_{20}y_2(0)\equiv\alpha_2 y_2(0)-\varepsilon\beta_2 y_2'(0)=r, \tag{5}$$

$$B_{21}y_2(1)\equiv\gamma_2 y_2(1)+\varepsilon\delta_2 y_2'(1)=s, \tag{6}$$

where ε is a small parameter $(0<\varepsilon\ll 1)$, $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\delta_1,\delta_2>0$ and $a_{11}(x),a_{12}(x),a_{21}(x),a_{22}(x)$ are such that

$$a_{12}(x)\le 0,\ a_{21}(x)\le 0, \tag{7}$$

$$a_{11}(x)>|a_{12}(x)|,a_{22}(x)>|a_{21}(x)|,\forall x\in\bar{\Omega} \tag{8}$$

and also

$$|[f_1](d)|\le C,|[f_2](d)|\le C. \tag{9}$$

Here $\Omega=(0,1),\Omega^-=(0,d),\Omega^+=(d,1),d\in\Omega$ and $\bar{y}=(y_1,y_2)^T$.

It is also assumed that the source terms f_1, f_2 are sufficiently smooth on $\bar{\Omega}\setminus\{d\}$; At the point $d\in\Omega$, the functions f_1, f_2 have jump discontinuity. In general, this discontinuity gives rise to interior layers in the solution of the problem. Since f_1, f_2 are discontinuous at d, the solution $\bar{y}$ of (1)-(6) does not necessary to have a continuous second order derivative at the point d. i.e., $y_1, y_2\notin C^2(\Omega)$. But the first derivative of the solution exists and is continuous.

The above system (1)-(6) can be written in matrix form as

$$P\bar{y}\equiv\begin{pmatrix}P_1\bar{y}\\ P_2\bar{y}\end{pmatrix}\equiv\begin{pmatrix}-\varepsilon\dfrac{d^2}{dx^2} & 0\\ 0 & -\varepsilon\dfrac{d^2}{dx^2}\end{pmatrix}\bar{y}+A(x)\bar{y}=\bar{f}(x),$$

$$\equiv-\varepsilon\bar{y}''(x)+A(x)\bar{y}(x)=\bar{f}(x),\forall x\in\Omega^-\cup\Omega^+$$

with the boundary conditions

$$\begin{pmatrix} B_{10}y_1(0) \\ B_{20}y_2(0) \end{pmatrix} = \begin{pmatrix} p \\ r \end{pmatrix}, \begin{pmatrix} B_{11}y_1(1) \\ B_{21}y_2(1) \end{pmatrix} = \begin{pmatrix} q \\ s \end{pmatrix},$$

where $A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}$ and $\bar{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$.

The jump at d is denoted in any function ω with $[\omega](d) = \omega(d^+) - \omega(d^-)$.

Remark-1: The presence of ε multiplying the derivative terms in the mixed boundary conditions amplifies the significance of the boundary layers at both ends. In the absence of ε, the layers are sufficiently weak [9,13].

2.2. Some Analytical Results

In this section, the existence of a solution, the maximum principle and stability result are established for the MBVP (1)-(6).

Theorem-1: The MBVP (1)-(6) has a solution $\bar{y} = (y_1, y_2)^T$ with $y_1, y_2 \in Y$.

Proof: The proof is by construction. Let $\bar{y}^-$ and $\bar{y}^+$ be the particular solutions of the following system of equations

$$-\varepsilon\left(\bar{y}^-\right)''(x) + A(x)\bar{y}^-(x) = \bar{f}(x), x \in \Omega^-,$$

and

$$-\varepsilon\left(\bar{y}^+\right)''(x) + A(x)\bar{y}^+(x) = \bar{f}(x), x \in \Omega^+$$

respectively.

Also let $\bar{\phi}$ and $\bar{\psi}$ be the solutions of the following MBVPs:

$$-\varepsilon\bar{\phi}''(x) + A(x)\bar{\phi}(x) = \bar{0}, x \in \Omega,$$
$$\bar{\alpha}\bar{\phi}(0) - \varepsilon\bar{\beta}\bar{\phi}'(0) = \bar{1}, \bar{\gamma}\bar{\phi}(1) - \varepsilon\bar{\delta}\bar{\phi}'(1) = \bar{0},$$

and

$$-\varepsilon\bar{\psi}''(x) + A(x)\bar{\psi}(x) = \bar{0}, x \in \Omega,$$
$$\bar{\alpha}\bar{\psi}(0) - \varepsilon\bar{\beta}\bar{\psi}'(0) = \bar{0}, \bar{\gamma}\bar{\psi}(1) - \varepsilon\bar{\delta}\bar{\psi}'(1) = \bar{1}$$

respectively.

Here $\bar{\phi} = (\phi_1, \phi_2), \bar{\psi} = (\psi_1, \psi_2), \bar{\alpha} = (\alpha_1, \alpha_2),$ $\bar{\beta} = (\beta_1, \beta_2), \bar{\gamma} = (\gamma_1, \gamma_2), \bar{\delta} = (\delta_1, \delta_2), \bar{0} = (0,0)^T$ and $\bar{1} = (1,1)^T$.

Then $\bar{y}$ can be written as

$$\bar{y}(x) = \begin{cases} \left(\bar{y}^-(x)\right) + \begin{pmatrix} B_{10}^* & 0 \\ 0 & B_{20}^* \end{pmatrix}(\bar{\phi}) + K(\bar{\psi}), x \in \Omega^- \\ \left(\bar{y}^+(x)\right) + \begin{pmatrix} B_{11}^* & 0 \\ 0 & B_{21}^* \end{pmatrix}(\bar{\psi}) + K^*(\bar{\phi}), x \in \Omega^+ \end{cases}$$

where

$$B_{10}^* = \left(\alpha_1 y_1(0) - \varepsilon\beta_1 D^+ y_1(0)\right) - \left(\alpha_1 y_1^-(0) - \varepsilon\beta_1 D^+ y_1^-(0)\right),$$
$$B_{20}^* = \left(\alpha_2 y_2(0) - \varepsilon\beta_2 D^+ y_2(0)\right) - \left(\alpha_2 y_2^-(0) - \varepsilon\beta_2 D^+ y_2^-(0)\right),$$
$$B_{11}^* = \left(\alpha_1 y_1(1) - \varepsilon\beta_1 D^- y_1(1)\right) - \left(\alpha_1 y_1^+(1) - \varepsilon\beta_1 D^- y_1^+(1)\right),$$
$$B_{21}^* = \left(\alpha_2 y_2(1) - \varepsilon\beta_2 D^- y_2(1)\right) - \left(\alpha_2 y_2^+(1) - \varepsilon\beta_2 D^- y_2^+(1)\right),$$

K and K^* are matrices with constant entries.

On $(0,1), \bar{0} < \bar{\phi}, \bar{\psi} < \bar{1}$ and $\bar{\phi}, \bar{\psi}$ cannot have internal maximum or minimum [14].

Hence $\bar{\phi}' < \bar{0}, \bar{\psi}' > \bar{0}, x \in (0,1)$.

Choose the matrices $K = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ and $K^* = \begin{pmatrix} k_1^* & 0 \\ 0 & k_2^* \end{pmatrix}$ so that $y_1, y_2 \in C^1(\Omega)$. i.e., we impose the conditions $\bar{y}(d^-) = \bar{y}(d^+)$ and $\bar{y}'(d^-) = \bar{y}'(d^+)$.

For the matrices K and K^* to exist it requires

$$\begin{vmatrix} \psi_1(d) & 0 & -\phi_1(d) & 0 \\ 0 & \psi_2(d) & 0 & -\phi_2(d) \\ \psi_1'(d) & 0 & -\phi_1'(d) & 0 \\ 0 & -\psi_2'(d) & 0 & -\phi_2'(d) \end{vmatrix} \neq 0.$$

This implies $\left(\psi_2'\phi_2 - \psi_2\phi_2'\right)\left(\phi_1\psi_1' - \psi_1\phi_1'\right) > 0.$

Theorem-2: *(Maximum principle)* Suppose $y_1, y_2 \in Y$. Further suppose that $\bar{y} = (y_1, y_2)^T$ satisfies $B_{10}y_1(0) \geq 0, B_{20}y_2(0) \geq 0, B_{11}y_1(1) \geq 0,$ $B_{21}y_2(1) \geq 0, P_1\bar{y}(x) \geq 0, P_2\bar{y}(x) \geq 0$ and $\left[\bar{y}'\right](d) \leq \bar{0}$. Also let $a_{12}(x) \leq 0$ and $a_{21}(x) \leq 0$ on $\bar{\Omega}$. Then if there exists a function $\bar{t} = (t_1, t_2)^T, t_1, t_2 \in Y$, such that $B_{10}t_1(0) \geq 0, B_{20}t_2(0) \geq 0, B_{11}t_1(1) \geq 0,$ $B_{21}t_2(1) \geq 0, P_1\bar{t}(x) \geq 0, P_2\bar{t}(x) \geq 0$ and $\left[\bar{t}'\right](d) \leq \bar{0}$, then $\bar{y}(x) \geq \bar{0}, \forall x \in \bar{\Omega}$.

Proof: Define

$$\eta = \max\left\{\max_{x\in\bar{\Omega}}\left(\frac{-y_1}{t_1}\right), \max_{x\in\bar{\Omega}}\left(\frac{-y_2}{t_2}\right)\right\}.$$

Assume that the theorem is not true.

Then $\eta > 0$ and there exists a point x_0 such that $\left(\frac{-y_1}{t_1}\right)(x_0) = \eta$ or $\left(\frac{-y_2}{t_2}\right)(x_0) = \eta$ or both.

Further, $x_0 \in \Omega^- \cup \Omega^+$ or $x_0 = d$. Also $(y_i + \eta t_i)(x) \geq 0, i = 1,2, x \in \bar{\Omega}$.

Case-(i): $(y_1 + \eta t_1)(x_0) = 0$, for $x_0 = 0$. It implies that $(y_1 + \eta t_1)$ attains a minimum at x_0. Therefore, $0 < B_{10}(y_1 + \eta t_1)(x_0) = \alpha_1(y_1 + \eta t_1)(x_0) - \beta_1(y_1 + \eta t_1)'(x_0) \leq 0,$ which is a contradiction.

Case-(ii): $\left(\frac{-y_1}{t_1}\right)(x_0) = \eta, x_0 \in \Omega^- \cup \Omega^+,$ i.e., $(y_1 + \eta t_1)(x_0) = 0$. Therefore,

$$0 < P_1(y_1 + \eta t_1)(x_0) =$$
$$-\varepsilon(y_1 + \eta t_1)''(x_0) + a_{11}(x_0)(y_1 + \eta t_1)(x_0)$$
$$+a_{12}(x_0)(y_2 + \eta t_2)(x_0) \le 0,$$

since ($(y_1 + \eta t_1)$ attains a minimum at x_0, which is a contradiction.

Case-(iii): $\left(\frac{-y_1}{t_1}\right)(x_0) = \eta, x_0 = d,$ i.e., $(y_1 + \eta t_1)(x_0) = 0$. Since $(y_1 + \eta t_1)$ attains a minimum at x_0, then

$0 \le \left[(y_1 + \eta t_1)'\right](x_0) = \left[y_1'\right](d) + \eta\left[t_1'\right](d) < 0,$ which is a contradiction.

Case-(iv): $(y_1 + \eta t_1)(x_0) = 0,$ for $x_0 = 1$. It implies that $(y_1 + \eta t_1)$ attains a minimum at x_0, Therefore,

$0 < B_{11}(y_1 + \eta t_1)(x_0) = \gamma_1(y_1 + \eta t_1)(x_0) - \delta_1(y_1 + \eta t_1)'(x_0) \le 0,$ which is a contradiction.

Case-(v): $(y_2 + \eta t_2)(x_0) = 0, x_0 \in \Omega^- \cup \Omega^+$. Similar to Case-(ii), it leads to a contradiction.

Case-(vi): $(y_2 + \eta t_2)(x_0) = 0, x_0 = d$. Similar to Case-(iii), it leads to a contradiction.

Case-(vii): $(y_2 + \eta t_2)(x_0) = 0, x_0 = 0$. Similar to Case-(i), it leads to a contradiction.

Case-(viii): $(y_2 + \eta t_2)(x_0) = 0, x_0 = 1$. Similar to Case-(iv), it leads to a contradiction.

Hence, $\bar{y}(x) \ge 0, \forall \bar{x} \in \bar{\Omega}$.

Corollary-1: Consider the differential equations (1)-(2) subject to the conditions (7)-(8). Let $\bar{t} = (t_1, t_2)^T$ where

$$t_1 = \begin{cases} \frac{1}{4} - \frac{x}{8} + \frac{d}{8}, x \in \Omega^- \cup \{0, d\}, \\ \frac{1}{4} - \frac{x}{4} + \frac{d}{4}, x \in \Omega^+ \cup \{1\}, \end{cases}$$

and

$$t_2 = \begin{cases} \frac{1}{2} - \frac{x}{8} + \frac{d}{8}, x \in \Omega^- \cup \{0, d\}, \\ \frac{1}{2} - \frac{x}{4} + \frac{d}{4}, x \in \Omega^+ \cup \{1\}. \end{cases}$$

Then the above maximum principle is true for the MBVP (1)-(6).

Remark-2: The MBVP (1)-(6) has a solution and it is unique.

Theorem-3: *(Stability result)* Consider the differential equations (1)-(2) subject to the quasi-monotonicity and diagonally dominant conditions (7)-(8). If $y_1, y_2 \in Y$, then

$$|y_i(x)| \le C[\max\{|B_{10}y_1(0)|, |B_{11}y_1(1)|, |B_{20}y_2(0)|,$$
$$|B_{21}y_2(1)|, \|P_1\bar{y}\|_{\Omega^- \cup \Omega^+}, \|P_2\bar{y}\|_{\Omega^- \cup \Omega^+}\}], x \in \bar{\Omega}, i = 1, 2.$$

Proof: Let $R = C[\max\{|B_{10}y_1(0)|, |B_{11}y_1(1)|, |B_{20}y_2(0)|, |B_{21}y_2(1)|, \|P_1\bar{y}\|_{\Omega^- \cup \Omega^+}, \|P_2\bar{y}\|_{\Omega^- \cup \Omega^+}\}]$.

Define the functions $\bar{\varpi}^{\pm}(x) = \left(\omega_1^-(x), \omega_2^-(x)\right)^T$, where $\omega_1^{\pm}(x) = Rt_1(x) \pm y_1(x), \omega_2^{\pm}(x) = Rt_2(x) \pm y_2(x)$. It is easy to prove that $\bar{\alpha}\bar{\varpi}^{\pm}(0) - \varepsilon\bar{\beta}\bar{\varpi}'^{\pm}(0) \ge \bar{0}, \bar{\gamma}\bar{\varpi}^{\pm}(1) + \varepsilon\bar{\delta}\bar{\varpi}'^{\pm}(1) \ge \bar{0},$ and $P_1\bar{\varpi}^{\pm}(x) \ge \bar{0}, P_2\bar{\varpi}^{\pm}(x) \ge \bar{0},$ $\left[\bar{\varpi}'\right](d) \le \bar{0}$, by a proper choice of C. Therefore, by the maximum principle the required result follows.

Remark-3: The MBVP (1)-(6) is well-posed. i.e., the problem has a unique stable solution.

2.3. Derivative Estimates

In this section, the derivative estimates for the MBVP (1)-(6) are provided.

Theorem-3: Let $\bar{y}$ be the solution of the MBVP (1)-(6). Then for k=1, 2 and $\forall x \in \bar{\Omega} \setminus \{d\}, \left|\bar{y}^{(k)}\right| \le C(1 + \varepsilon^{-\frac{k}{2}})$ and $\left|\bar{y}^{(3)}\right| \le C\varepsilon^{-\frac{3}{2}}$.

Proof: This theorem can be proved by using the results of [10] and [15].

Remark-4: The sharper bounds on the derivatives of the solution are obtained by decomposing the solution $\bar{y}$ into smooth and singular components as $\bar{y} = \bar{v} + \bar{w}$, where the smooth component $\bar{v}$ is given by

$$P_k\bar{v}(x) = f_k(x), x \in \Omega^- \cup \Omega^+, k = 1, 2,$$
$$\bar{\alpha}\bar{v}(0) - \varepsilon\bar{\beta}\bar{v}'(0) = A^{-1}(0)\bar{f}(0),$$
$$\bar{v}(d-) = A^{-1}(d)\bar{f}(d-), \bar{v}(d+) = A^{-1}(d)\bar{f}(d+),$$
$$\bar{\gamma}\bar{v}(1) - \varepsilon\bar{\delta}\bar{v}'(1) = A^{-1}(1)\bar{f}(1)$$

and the singular component $\bar{w}$ is given by

$$P_k\bar{w}(x) = 0, x \in \Omega^- \cup \Omega^+, k = 1, 2,$$
$$\left[\bar{w}(d)\right] = \left[\bar{v}(d)\right], \left[\bar{w}'(d)\right] = -\left[\bar{v}'(d)\right],$$
$$\bar{\alpha}\bar{w}(0) - \varepsilon\bar{\beta}\bar{w}'(0) = \left(\bar{\alpha}\bar{y}(0) - \varepsilon\bar{\beta}\bar{y}'(0)\right)$$
$$-\left(\bar{\alpha}\bar{v}(0) - \varepsilon\bar{\beta}\bar{v}'(0)\right),$$
$$\bar{\gamma}\bar{w}(1) + \varepsilon\bar{\delta}\bar{w}'(1) = \left(\bar{\gamma}\bar{y}(1) + \varepsilon\bar{\delta}\bar{y}'(1)\right)$$
$$-\left(\bar{\gamma}\bar{v}(1) + \varepsilon\bar{\delta}\bar{v}'(1)\right),$$

where

$$\bar{\alpha} = (\alpha_1, \alpha_2), \bar{\beta} = (\beta_1, \beta_2), \bar{\gamma} = (\gamma_1, \gamma_2), \bar{\delta} = (\delta_1, \delta_2)$$

The solution $\bar{w}$ can be constructed by the procedure given in [16]. Therefore, the singular component is well defined.

Theorem-4: The smooth and singular components $\bar{v}$ and $\bar{w}$ of $\bar{y}$ satisfy the bounds

$$\left|v_i^{(k)}(x)\right| \le \begin{cases} C\left(1+\varepsilon^{\left(1-\frac{k}{2}\right)}\mathbb{B}_l(x,\rho)\right), x\in\Omega^- \\ C\left(1+\varepsilon^{\left(1-\frac{k}{2}\right)}\mathbb{B}_r(x,\rho)\right), x\in\Omega^+ \end{cases}$$

and

$$\left|w_i^{(k)}(x)\right| \le \begin{cases} C\varepsilon^{-\frac{k}{2}}\mathbb{B}_l(x,\rho), x\in\Omega^- \\ C\varepsilon^{-\frac{k}{2}}\mathbb{B}_r(x,\rho), x\in\Omega^+ \end{cases}, i=1,2,$$

where

$$\mathbb{B}_l(x,\rho) = e^{-x\sqrt{\rho/\varepsilon}} + e^{-(d-x)\sqrt{\rho/\varepsilon}},$$

$$\mathbb{B}_r(x,\rho) = e^{-(x-d)\sqrt{\rho/\varepsilon}} + e^{-(1-x)\sqrt{\rho/\varepsilon}}$$

and

$$\rho = \min_{x\in\bar{\Omega}}\{\rho_1,\rho_2\}, \rho_1 = \min_{x\in\bar{\Omega}}\{a_{11}(x)+a_{12}(x)\},$$

$$\rho_2 = \min_{x\in\bar{\Omega}}\{a_{21}(x)+a_{22}(x)\}.$$

Proof: This theorem can be proved by using the results of [15,16,17] and by following the technique of [16,17,18]. Note that $v_1, v_2, w_1, w_2 \notin C^0(\Omega)$, but $v_1+w_1, v_2+w_2 \in C^1(\Omega)$.

3. Discrete Problem

A fitted mesh method for problem (1)-(6) is now described. On $\Omega^-\cup\Omega^+$ a piecewise uniform mesh of N mesh intervals is constructed as follows:

The interval $\bar{\Omega}^-$ is subdivided into three subintervals $[0,\tau_1],[\tau_1,d-\tau_1]$ and $[d-\tau_1,d]$ for some τ_1 that satisfies $0<\tau_1\le\frac{d}{4}$. On $[0,\tau_1]$ and $[d-\tau_1,d]$ a uniform mesh with $\frac{N}{8}$ mesh intervals is placed while $[\tau_1,d-\tau_1]$ has a uniform mesh with $\frac{N}{4}$ mesh intervals. The subintervals $[d,d+\tau_2],[d+\tau_2,1-\tau_2]$ and $[1-\tau_2,1]$ of $\bar{\Omega}^+$ are treated analogously for some τ_2 satisfying $0<\tau_2\le\frac{(1-d)}{4}$. The interior points of the mesh are denoted by

$$\Omega_\varepsilon^N = \left\{x_i : 1\le i\le\frac{N}{2}-1\right\}\cup\left\{x_i : \frac{N}{2}+1\le i\le N-1\right\}.$$

Clearly $x_{\frac{N}{2}} = d$ and $\bar{\Omega}_\varepsilon^N = \{x_i\}_0^N$. Note that this mesh is a uniform mesh when $\tau_1 = \frac{d}{4}$ and $\tau_2 = \frac{1-d}{4}$. The transition parameters τ_1 and τ_2 are functions of N and ε and are chosen as $\tau_1 = \min\left\{\frac{d}{4}, 2\sqrt{\varepsilon/\rho}\ln N\right\}$ and $\tau_2 = \min\left\{\frac{1-d}{4}, 2\sqrt{\varepsilon/\rho}\ln N\right\}$.

The six mesh widths are given by $h_1 = h_3 = \frac{8\tau_1}{N}, h_2 = \frac{4(d-2\tau_1)}{N}, h_4 = h_6 = \frac{8\tau_2}{N}$, $h_5 = \frac{4(1-d-2\tau_2)}{N}$.

On the piecewise uniform mesh $\bar{\Omega}_\varepsilon^N$, a cubic spline scheme is used for Robin boundary conditions and the classical central difference scheme is used for the differential equations at the interior points. Then the fitted mesh method for MBVP (1)-(6) is:

$$P_1^N\bar{y}_i \equiv -\varepsilon\delta^2\ y_{1,i} + a_{11}(x_i)y_{1,i} + a_{12}(x_i)y_{2,i} = f_1(x_i), \forall x_i\in\Omega_\varepsilon^N, \tag{10}$$

$$P_2^N\bar{y}_i \equiv -\varepsilon\delta^2\ y_{2,i} + a_{21}(x_i)y_{1,i} + a_{22}(x_i)y_{2,i} = f_2(x_i), \forall x_i\in\Omega_\varepsilon^N, \tag{11}$$

i.e., $\bar{P}^N\bar{y} \equiv \left(-\varepsilon\delta^2\ + A(x)\right)\bar{y} = \bar{f}$

and at $x_{\frac{N}{2}} = d$, the scheme is given by [19,20]

$$\bar{P}^N\bar{y}(d) \equiv -\varepsilon\delta^2\ \bar{y}(d) + A(d)\bar{y}(d) = \bar{f}(d),$$

where

$$\bar{f}(d) = \frac{f_{j1}\left(x\left(\frac{N}{2}-1\right)\right) + f_{j2}\left(x\left(\frac{N}{2}+1\right)\right)}{2}, j=1,2, \tag{12}$$

with the boundary conditions

$$B_{10}^N y_1(0) \equiv \alpha_1 y_1(0) - \varepsilon\beta_1 S^+ y_1(0) = p, \tag{13}$$

$$B_{11}^N y_1(1) \equiv \gamma_1 y_1(1) + \varepsilon\delta_1 S^- y_1(1) = q, \tag{14}$$

$$B_{20}^N y_2(0) \equiv \alpha_2 y_2(0) - \varepsilon\beta_2 S^+ y_2(0) = r, \tag{15}$$

$$B_{21}^N y_2(1) \equiv \gamma_2 y_2(1) + \varepsilon\delta_2 S^- y_2(1) = s, \tag{16}$$

$$\delta^2 y_i = \frac{\left(D^+y_i - D^-y_i\right)}{\bar{h}_i}, D^+y_i = \frac{y_{i+1}-y_i}{h_i},$$

where $D^-y_i = \frac{y_i - y_{i-1}}{h_{i-1}}, h_i = x_{i+1}-x_i, h_{i-1} = x_i - x_{i-1}$,

$$\bar{h}_i = \frac{(h_i + h_{i-1})}{2}$$

and $S^+y_j(0), S^-y_j(1)$ can be obtained from the one sided limits

$$S'\left(x_i^+\right)=-\frac{h_i}{3}M_{j,i}-\frac{h_i}{6}M_{j,i+1}+\frac{y_j\left(x_{i+1}\right)-y_j\left(x_i\right)}{h_i} \quad (17)$$

$$S'\left(x_i^-\right)=\frac{h_{i-1}}{6}M_{j,i-1}-\frac{h_{i-1}}{3}M_{j,i}+\frac{y_j\left(x_i\right)-y_j\left(x_{i-1}\right)}{h_{i-1}} \quad (18)$$

for j=1,2, respectively of the first order derivatives of cubic spline function, given in [9,10]. Substituting $M_{1,i}, M_{2,i}$ from $-\varepsilon M_{1,i}+a_{11}\left(x_i\right)y_{1,i}+a_{12}\left(x_i\right)y_{2,i}=f_1(x_i)$ and $-\varepsilon M_{2,i}+a_{21}\left(x_i\right)y_{1,i}+a_{22}\left(x_i\right)y_{2,i}=f_2(x_i)$ to (17) and (18), we get the approximation of the one sided first order derivatives at both boundary points. Hence, the discretization of the Robin boundary conditions of (13)-(16) reduce to

$$\begin{aligned}&\left[\frac{3\varepsilon}{h_0}\left(\alpha_1+\frac{\varepsilon\beta_1}{h_0}\right)+a_{11}\left(x_0\right)\varepsilon\beta_1\right]y_{1,0}\\&+\left[-\frac{3\varepsilon^2\beta_1}{h_0^2}+\frac{a_{11}\left(x_1\right)\varepsilon\beta_1}{2}\right]y_{1,1}\\&+\varepsilon\beta_1 a_{12}\left(x_0\right)y_{2,0}+\frac{\varepsilon\beta_1 a_{12}\left(x_1\right)}{2}y_{2,1}\\&=\frac{3\varepsilon p}{h_0}+\varepsilon\beta_1 f_{11}\left(x_0\right)+\frac{\varepsilon\beta_1}{2}f_{11}\left(x_1\right),\end{aligned} \quad (19)$$

$$\begin{aligned}&\left[-\frac{3\varepsilon^2\delta_1}{h_{N-1}^2}+\frac{a_{11}\left(x_{N-1}\right)\varepsilon\delta_1}{2}\right]y_{1,N-1}\\&+\left[\frac{3\varepsilon}{h_{N-1}}\left(\gamma_1+\frac{\varepsilon\delta_1}{h_{N-1}}\right)+a_{11}\left(x_N\right)\varepsilon\delta_1\right]y_{1,N}\\&+\frac{\varepsilon\delta_1 a_{12}\left(x_{N-1}\right)}{2}y_{2,N-1}+\varepsilon\delta_1 a_{12}\left(x_N\right)y_{2,N}\\&=\frac{3\varepsilon q}{h_{N-1}}+\varepsilon\delta_1 f_{12}\left(x_N\right)+\frac{\varepsilon\delta_1}{2}f_{12}\left(x_{N-1}\right),\end{aligned} \quad (20)$$

$$\begin{aligned}&\left[\frac{3\varepsilon}{h_0}\left(\alpha_2+\frac{\varepsilon\beta_2}{h_0}\right)+a_{22}\left(x_0\right)\varepsilon\beta_2\right]y_{2,0}\\&+\left[-\frac{3\varepsilon^2\beta_2}{h_0^2}+\frac{a_{22}\left(x_1\right)\varepsilon\beta_2}{2}\right]y_{2,1}+\varepsilon\beta_2 a_{21}\left(x_0\right)y_{1,0}\\&+\frac{\varepsilon\beta_2 a_{21}\left(x_1\right)}{2}y_{1,1}=\frac{3\varepsilon r}{h_0}+\varepsilon\beta_2 f_{21}\left(x_0\right)+\frac{\varepsilon\beta_2}{2}f_{21}\left(x_1\right),\end{aligned} \quad (21)$$

$$\begin{aligned}&\left[-\frac{3\varepsilon^2\delta_2}{h_{N-1}^2}+\frac{a_{22}\left(x_{N-1}\right)\varepsilon\delta_2}{2}\right]y_{2,N-1}\\&+\left[\frac{3\varepsilon}{h_{N-1}}\left(\gamma_2+\frac{\varepsilon\delta_2}{h_{N-1}}\right)+a_{22}\left(x_N\right)\varepsilon\delta_2\right]y_{2,N}\\&+\frac{\varepsilon\delta_2 a_{21}\left(x_{N-1}\right)}{2}y_{1,N-1}+\varepsilon\delta_2 a_{21}\left(x_N\right)y_{1,N}\\&=\frac{3\varepsilon s}{h_{N-1}}+\varepsilon\delta_2 f_{22}\left(x_N\right)+\frac{\varepsilon\delta_2}{2}f_{22}\left(x_{N-1}\right),\end{aligned} \quad (22)$$

The following discrete maximum principle and discrete stability result can be proved analogous to the continuous results stated in Theorem-2 and Theorem-3.

Theorem-5: *(Discrete maximum principle)* For any mesh function $\bar{\Psi}\left(x_i\right)$, assume that $B_{10}^N\Psi_1\left(x_0\right)\ge 0, B_{20}^N\Psi_2\left(x_0\right)\ge 0, B_{11}^N\Psi_1\left(x_N\right)\ge 0,$ $B_{21}^N\Psi_2\left(x_N\right)\ge 0, P_1^N\bar{\Psi}\left(x_i\right)\ge 0, P_2^N\bar{\Psi}\left(x_i\right)\ge 0, \forall x_i\in\Omega_\varepsilon^N$ and $D^+\bar{\Psi}_{\frac{N}{2}}-D^-\bar{\Psi}_{\frac{N}{2}}\le\bar{0}.$ Also let $a_{12}\left(x_i\right)\le 0, a_{21}(x_i)\le 0.$ Then if there exists a mesh function $\bar{t}_i$ such that $\begin{matrix}B_{10}^N t_0>0, B_{20}^N t_0>0, B_{21}^N t_N>0,\\ P_1^N\bar{t}_i>0, P_2^N\bar{t}_i>0\forall x_i\in\Omega_\varepsilon^N\end{matrix}$ and $D^+\bar{t}_{\frac{N}{2}}-D^-\bar{t}_{\frac{N}{2}}\le\bar{0},$ then $\bar{\Psi}\left(x_i\right)\ge\bar{0}\forall x_i\in\bar{\Omega}_\varepsilon^N.$

Corollary-2: Consider the discrete problem (10)-(15) subject to the conditions (7)-(8). Let $\bar{t}_i=\left(t_{1,i},t_{2,i}\right)^T$ where

$$t_{1,i}=\begin{cases}\frac{1}{4}-\frac{x_i}{8}+\frac{d}{8}, 0\le i\le\frac{N}{2},\\ \frac{1}{4}-\frac{x_i}{4}+\frac{d}{4},\left(\frac{N}{2}\right)+1\le i\le N,\end{cases}$$

and

$$t_{2,i}=\begin{cases}\frac{1}{2}-\frac{x_i}{8}+\frac{d}{8}, 0\le i\le\frac{N}{2},\\ \frac{1}{2}-\frac{x_i}{4}+\frac{d}{4},\left(\frac{N}{2}\right)+1\le i\le N.\end{cases}$$

Then the above discrete maximum principle is true for (10)-(15).

Theorem-6: *(Discrete stability result)*If $\bar{Z}\left(x_i\right)=\left(Z_1\left(x_i\right),Z_2\left(x_i\right)\right)^T$ is any mesh function, then

$$\begin{aligned}&\forall x_i\in\bar{\Omega}_\varepsilon^N,\\&j=1,2,\left|Z_j\left(x_i\right)\right|=C[\max\{\left|B_{10}y_1\left(0\right)\right|,\left|B_{11}y_1\left(1\right)\right|,\\&\left|B_{20}y_2\left(0\right)\right|,\left|B_{21}y_2\left(1\right)\right|P_1\bar{y}_{\Omega^-\cup\Omega^+},P_2\bar{y}_{\Omega^-\cup\Omega^+}\}]\end{aligned}$$

4. Error Analysis

Using the results of Theorem-4, the procedure adopted in [10,17] and the basic ideas of the proofs of some theorems presented in [16] for the derivation of estimates for the truncation error, the following inequalities can be derived for the MBVP (1)-(6):

$$\begin{aligned}&\left(P_k^N-P_k\right)\bar{y}(x_i)\le C(N^{-1}\ln N)^2,\\&x_i\in\Omega_\varepsilon^N, k=1,2.\end{aligned} \quad (23)$$

At the point $x_{\frac{N}{2}}=d$, using the procedure adopted in [19,20] with appropriate barrier functions, it is easy to see that

$$\begin{aligned}&\left(P_k^N-P_k\right)\bar{y}(d)\le C(N^{-1}\ln N)^2,\\&k=1,2\end{aligned} \quad (24)$$

The truncation errors of the solution y at boundary points x=0,1, for the discrete problem (10)-(16), where Robin boundary conditions are discretized by using spline approximation from (17)-(22), lead to the following estimates [9]:

$$\left(P_k^N - P_k\right)\bar{y}\left(x_i\right) \le CN^{-2}, \quad i = 0, N, k = 1, 2. \tag{25}$$

Theorem-7: The error of the numerical scheme (10)-(16) at inner grid points $x_i \in \Omega_\varepsilon^N$ satisfies

$$\left|y_k\left(x_i\right) - y_{k,i}\right| \le C(N^{-1}\ln N)^2, \quad k = 1, 2, \tag{26}$$

for N sufficiently large.

Proof: Using (23), (24) and (25), the desired result follows.

Remark-5: *(Adjoint system*) Consider the MBVP (1)-(6).

Suppose that the quasi-monotonicity condition (7) is not satisfied by the system. Then the following system is adjoined to (1)-(2):

$$-\varepsilon\hat{y}_1''\left(x\right) + a_{11}\left(x\right)\hat{y}_1\left(x\right) - a_{12}^+\left(x\right)\hat{y}_4\left(x\right) + a_{12}\left(x\right)\hat{y}_2\left(x\right) = -f_1\left(x\right), \tag{27}$$

$$-\varepsilon\hat{y}_2''\left(x\right) - a_{21}^+\left(x\right)\hat{y}_3\left(x\right) + a_{21}^-\left(x\right)\hat{y}_1\left(x\right) + a_{22}\left(x\right)\hat{y}_2\left(x\right) = -f_2\left(x\right), \forall x \in \Omega^- \cup \Omega^+, \tag{28}$$

$$-\varepsilon\hat{y}_3''\left(x\right) + a_{11}\left(x\right)\hat{y}_3\left(x\right) - a_{12}^+$$

$$-\varepsilon\hat{y}_3''\left(x\right) + a_{11}\left(x\right)\hat{y}_3\left(x\right) - a_{12}^+\left(x\right)\hat{y}_2\left(x\right) + a_{12}^-\left(x\right)\hat{y}_4\left(x\right) = f_1\left(x\right), \tag{29}$$

$$-\varepsilon\hat{y}_4''\left(x\right) - a_{21}^+\left(x\right)\hat{y}_1\left(x\right) + a_{21}^-\left(x\right)\hat{y}_3\left(x\right) + a_{22}\left(x\right)\hat{y}_4\left(x\right) = f_2\left(x\right), \forall x \in \Omega^- \cup \Omega^+, \tag{30}$$

$$\alpha_1\hat{y}_1\left(0\right) - \varepsilon\beta_1\hat{y}_1'\left(0\right) = -p, \gamma_1\hat{y}_1\left(1\right) + \varepsilon\delta_1\hat{y}_1'\left(1\right) = -q, \tag{31}$$

$$\alpha_2\hat{y}_2\left(0\right) - \varepsilon\beta_2\hat{y}_2'\left(0\right) = -r, \gamma_2\hat{y}_2\left(1\right) + \varepsilon\delta_2\hat{y}_2'\left(1\right) = -s, \tag{32}$$

$$\alpha_1\hat{y}_3\left(0\right) - \varepsilon\beta_1\hat{y}_3'\left(0\right) = p, \gamma_1\hat{y}_3\left(1\right) + \varepsilon\delta_1\hat{y}_3'\left(1\right) = q, \tag{33}$$

$$\alpha_2\hat{y}_4\left(0\right) - \varepsilon\beta_2\hat{y}_4'\left(0\right) = r, \gamma_2\hat{y}_4\left(1\right) + \varepsilon\delta_2\hat{y}_4'\left(1\right) = s, \tag{34}$$

and

$$a_{12}^+\left(x\right) = \begin{cases} a_{12}\left(x\right), \textit{if } a_{12}\left(x\right) \ge 0 \\ 0, \textit{otherwise} \end{cases},$$

$$a_{12}^- = a_{12}\left(x\right) - a_{12}^+(x)$$

and

$$a_{21}^+\left(x\right) = \begin{cases} a_{21}\left(x\right), \textit{if } a_{21}\left(x\right) \ge 0 \\ 0, \textit{otherwise} \end{cases},$$

$$a_{21}^- = a_{21}\left(x\right) - a_{21}^+\left(x\right).$$

Since $\bar{y} = \left(y_1, y_2\right)^T$ is a solution of (1)-(6), $\bar{\hat{y}} = (-y_1, -y_2, y_1, y_2)$ is a solution of the above adjoint system (27)-(34). The results derived for (1)-(6) still hold good even if the quasi-monotonicity condition is not met.

5. Numerical Results

In this section, two examples are given to illustrate the computational methods discussed in this paper.

Consider the following singularly perturbed Robin type boundary value problems with discontinuous source term:

Example-1:

$$-\varepsilon y_1''\left(x\right) + 2y_1\left(x\right) - y_2\left(x\right) = f_1\left(x\right), x \in \Omega^- \cup \Omega^+,$$

$$-\varepsilon y_2''\left(x\right) - y_1\left(x\right) + 2y_2\left(x\right) = f_2\left(x\right), x \in \Omega^- \cup \Omega^+,$$

$$3y_1\left(0\right) - \varepsilon y_1'\left(0\right) = 0, 2y_1\left(1\right) + \varepsilon y_1'\left(1\right) = 1,$$

$$3y_2\left(0\right) - \varepsilon y_2'\left(0\right) = 2, 2y_2\left(1\right) + \varepsilon y_2'\left(1\right) = 2,$$

where $f_1\left(x\right) = \begin{cases} 1, 0 \le x \le 0.5 \\ 0.8, 0.5 < x \le 1 \end{cases}$ and $f_2\left(x\right) = \begin{cases} 2, 0 \le x \le 0.5 \\ 1.8, 0.5 < x \le 1. \end{cases}$

Example-2:

$$-\varepsilon y_1''\left(x\right) + 2\left(x+1\right)^2 y_1\left(x\right) - \left(1 + x^3\right)y_2\left(x\right) = f_1\left(x\right), x \in \Omega^- \cup \Omega^+,$$

$$-\varepsilon y_2''\left(x\right) - 2\cos\left(\frac{\pi}{4}x\right)y_1\left(x\right) + 2.2e^{1-x}y_2\left(x\right) = f_2\left(x\right), x \in \Omega^- \cup \Omega^+,$$

$$y_1\left(0\right) - \varepsilon y_1'\left(0\right) = 0, 2y_1\left(1\right) + \varepsilon y_1'\left(1\right) = 1,$$

$$y_2\left(0\right) - 3\varepsilon y_2'\left(0\right) = 0, y_2\left(1\right) + \varepsilon y_2'\left(1\right) = 1,$$

where $f_1\left(x\right) = \begin{cases} 2e^x, 0 \le x \le 0.5 \\ 1, 0.5 < x \le 1 \end{cases}$ and $f_2\left(x\right) = \begin{cases} 10x + 1, 0 \le x \le 0.5 \\ 2, 0.5 < x \le 1. \end{cases}$

The maximum errors and the orders of convergence for the solution of the above two examples are presented for various values of and in the Table 1- Table 2 and Table 3- Table 4 respectively. For a finite set of values $\varepsilon = \left\{10^{-1}, 10^{-2}, \ldots, 10^{-15}\right\}$, maximum point-wise errors $E_{\varepsilon,j}^N$ are computed as $E_{\varepsilon,j}^N = \max_{x_i \in \bar{\Omega}_\varepsilon^N}\left|y_j^N - \tilde{y}_j^{8192}\right|$ for $j = 1, 2$, where $\tilde{y}_j^{8192}$ is the piecewise linear interpolant of the mesh function y_j^{8192} onto $\left[0,1\right]$. From these values, the $\varepsilon-$ uniform maximum error is calculated by $E_j^N = \max_\varepsilon E_{\varepsilon,j}^N, j = 1, 2.$ Further, the order of convergence is computed by $p_j^N = \log_2\left(\frac{E_j^N}{E_j^{2N}}\right), j = 1, 2$.

Table 1. Maximum point-wise errors $E_{\varepsilon,1}^{N}$. $\varepsilon-$uniform error E_1^N and $\varepsilon-$ uniform order of convergence p_1^N for different values of the mesh points N for the solution y_1 of Example-1

ε	Number of mesh points N						
	64	128	256	512	1024	2048	4096
10^{-1}	4.5229E-03	2.2575E-03	1.1143E-03	5.4000E-04	2.5219E-04	1.0812E-04	3.6047E-05
10^{-2}	1.4428E-02	7.4183E-03	3.7173E-03	1.8153E-03	8.5103E-04	3.6557E-04	1.2200E-04
10^{-3}	3.8752E-02	2.1588E-02	1.1278E-02	5.6259E-03	2.6660E-03	1.1514E-03	3.8528E-04
10^{-4}	6.4210E-02	4.4804E-02	2.8419E-02	1.6701E-02	8.1900E-03	3.5958E-03	1.2133E-03
10^{-5}	6.4016E-02	4.4504E-02	2.8031E-02	1.6250E-02	8.6693E-03	4.1064E-03	1.4761E-03
10^{-6}	6.4016E-02	4.4504E-02	2.8031E-02	1.6250E-02	8.6693E-03	4.1064E-03	1.4761E-03
10^{-7}	6.4016E-02	4.4504E-02	2.8031E-02	1.6250E-02	8.6693E-03	4.1064E-03	1.4761E-03
10^{-8}	6.4016E-02	4.4504E-02	2.8031E-02	1.6250E-02	8.6693E-03	4.1064E-03	1.4761E-03
10^{-9}	6.4016E-02	4.4504E-02	2.8031E-02	1.6250E-02	8.6693E-03	4.1064E-03	1.4761E-03
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10^{-15}	6.4016E-02	4.4504E-02	2.8031E-02	1.6250E-02	8.6693E-03	4.1064E-03	1.4761E-03
E_1^N	6.4210E-02	4.4804E-02	2.8419E-02	1.6701E-02	8.6693E-03	4.1064E-03	1.4761E-03
p_1^N	0.52	0.66	0.77	0.95	1.08	1.48	-

Table 2. Maximum point-wise errors $E_{\varepsilon,2}^{N}$. $\varepsilon-$uniform error E_2^N and $\varepsilon-$ uniform order of convergence p_2^N for different values of the mesh points N for the solution y_2 of Example-1

ε	Number of mesh points N						
	64	128	256	512	1024	2048	4096
10^{-1}	4.5062E-03	2.2534E-03	1.1132E-03	5.3974E-04	2.5213E-04	1.0811E-04	3.6040E-05
10^{-2}	1.4426E-02	7.4179E-03	3.7172E-03	1.8152E-03	8.5103E-04	3.6557E-04	1.2200E-04
10^{-3}	3.8752E-02	2.1588E-02	1.1278E-02	5.6259E-03	2.6660E-03	1.1514E-03	3.8528E-04
10^{-4}	6.4210E-02	4.4804E-02	2.8419E-02	1.6701E-02	8.1900E-03	3.5958E-03	1.2133E-03
10^{-5}	6.4016E-02	4.4504E-02	2.8031E-02	1.6250E-02	8.6693E-03	4.1064E-03	1.4761E-03
10^{-6}	6.4016E-02	4.4504E-02	2.8031E-02	1.6250E-02	8.6693E-03	4.1064E-03	1.4761E-03
10^{-7}	6.4016E-02	4.4504E-02	2.8031E-02	1.6250E-02	8.6693E-03	4.1064E-03	1.4761E-03
10^{-8}	6.4016E-02	4.4504E-02	2.8031E-02	1.6250E-02	8.6693E-03	4.1064E-03	1.4761E-03
10^{-9}	6.4016E-02	4.4504E-02	2.8031E-02	1.6250E-02	8.6693E-03	4.1064E-03	1.4761E-03
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10^{-15}	6.4016E-02	4.4504E-02	2.8031E-02	1.6250E-02	8.6693E-03	4.1064E-03	1.4761E-03
E_2^N	6.4210E-02	4.4804E-02	2.8419E-02	1.6701E-02	8.6693E-03	4.1064E-03	1.4761E-03
p_2^N	0.52	0.66	0.77	0.95	1.08	1.48	-

Table 3. Maximum point-wise errors $E_{\varepsilon,1}^{N}$. $\varepsilon-$ uniform error E_1^N and $\varepsilon-$ uniform order of convergence p_1^N for different values of the mesh points N for the solution y_1 of Example-2

ε	Number of mesh points N						
	64	128	256	512	1024	2048	4096
10^{-1}	3.3958E-02	1.7143E-02	8.5109E-03	4.1366E-03	1.9348E-03	8.3011E-04	2.7686E-04
10^{-2}	9.9367E-02	5.2230E-02	2.6474E-02	1.3004E-02	6.1144E-03	2.6304E-03	8.7847E-04
10^{-3}	2.4631E-01	1.4540E-01	7.8468E-02	3.9818E-02	1.9036E-02	8.2580E-03	2.7695E-03
10^{-4}	3.8266E-01	2.9741E-01	2.0468E-01	1.1431E-01	5.7489E-02	2.5598E-02	8.6990E-03
10^{-5}	3.7988E-01	2.9480E-01	2.0143E-01	1.2316E-01	6.7860E-02	3.2738E-02	1.1889E-02
10^{-6}	3.7929E-01	2.9458E-01	2.0137E-01	1.2315E-01	6.7856E-02	3.2737E-02	1.1889E-02
10^{-7}	3.7910E-01	2.9451E-01	2.0135E-01	1.2314E-01	6.7854E-02	3.2736E-02	1.1889E-02
10^{-8}	3.7904E-01	2.9449E-01	2.0134E-01	1.2314E-01	6.7854E-02	3.2736E-02	1.1889E-02
10^{-9}	3.7902E-01	2.9449E-01	2.0134E-01	1.2314E-01	6.7854E-02	3.2736E-02	1.1889E-02
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10^{-15}	3.7901E-01	2.9448E-01	2.0134E-01	1.2314E-01	6.7854E-02	3.2736E-02	1.1889E-02
E_1^N	3.8266E-01	2.9741E-01	2.0468E-01	1.2316E-01	6.7860E-02	3.2738E-02	1.1889E-02
p_1^N	0.36	0.54	0.73	0.86	1.05	1.46	-

Table 4. Maximum point-wise errors $E_{\varepsilon,2}^N$. $\varepsilon-$uniform error E_2^N and $\varepsilon-$ uniform order of convergence p_2^N for different values of the mesh points N for the solution y_2 of Example-2

ε	Number of mesh points N						
	64	128	256	512	1024	2048	4096
10^{-1}	5.8034E-02	2.9453E-02	1.4658E-02	7.1346E-03	3.3396E-03	1.4334E-03	4.7816E-04
10^{-2}	1.7003E-01	9.0019E-02	4.5787E-02	2.2528E-02	1.0602E-02	4.5626E-03	1.5241E-03
10^{-3}	4.2076E-01	2.5056E-01	1.3576E-01	6.9016E-02	3.3024E-02	1.4332E-02	4.8077E-03
10^{-4}	6.5436E-01	5.1220E-01	3.5386E-01	1.9808E-01	9.9726E-02	4.4428E-02	1.5102E-02
10^{-5}	6.5586E-01	5.1023E-01	3.4915E-01	2.1365E-01	1.1777E-01	5.6831E-02	2.0642E-02
10^{-6}	6.5681E-01	5.1066E-01	3.4932E-01	2.1372E-01	1.1779E-01	5.6838E-02	2.0644E-02
10^{-7}	6.5711E-01	5.1079E-01	3.4937E-01	2.1374E-01	1.1780E-01	5.6840E-02	2.0644E-02
10^{-8}	6.5721E-01	5.1084E-01	3.4939E-01	2.1374E-01	1.1780E-01	5.6840E-02	2.0645E-02
10^{-9}	6.5724E-01	5.1085E-01	3.4940E-01	2.1374E-01	1.1780E-01	5.6840E-02	2.0645E-02
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10^{-15}	6.5725E-01	5.1086E-01	3.4940E-01	2.1375E-01	1.1780E-01	5.6841E-02	2.0645E-02
E_2^N	6.5725E-01	5.1220E-01	3.5386E-01	2.1375E-01	1.1780E-01	5.6841E-02	2.0645E-02
p_2^N	0.36	0.53	0.73	0.86	1.05	1.46	-

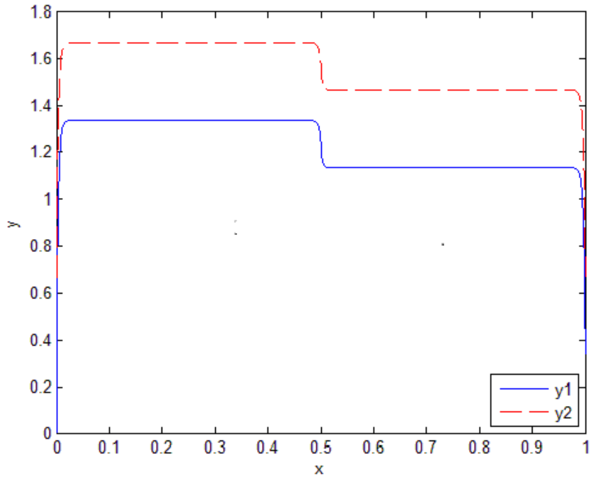

Figure 1. Numerical solutions y_1, y_2 of Example-1 for $\varepsilon = 10^{-5}, N = 256$

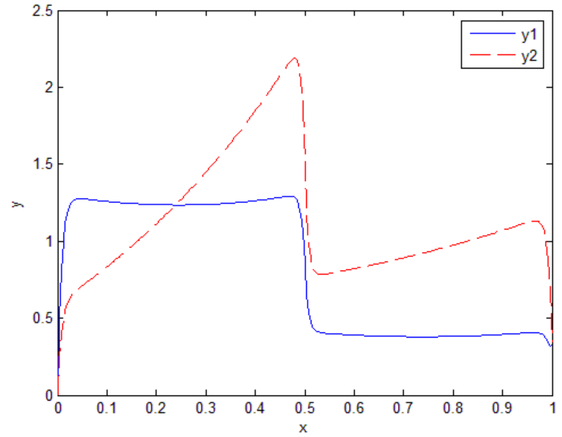

Figure 2. Numerical solutions y_1, y_2 of Example-2 for $\varepsilon = 10^{-4}, N = 256.$

6. Conclusions

A system of two coupled singularly perturbed reaction-diffusion Robin type boundary value problem with discontinuous source term was examined. A difference scheme using fitted mesh method on piecewise uniform Shishkin mesh was constructed for solving the problem which gives $\varepsilon-$ uniform convergence. A cubic spline scheme is used for Robin boundary conditions and the classical central difference scheme is used for the differential equations at the interior points. From the obtained numerical results, it is noted that the rate of convergence is approaching to almost the second order as N increases and are in agreement with the theoretical results.

Remark-6: The authors are in the process of extending the same analysis for convection-diffusion problems considered in [21].

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