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Properties of the Distributions Generated by Mixing Weibull and Inverse Weibull Distributions with Zero Truncated Poisson

Salah H Abid*, Sajad H Mohammed

Mathematics department, Education College, Al-Mustansiriya University, Baghdad, Iraq
*Corresponding author: abidsalah@gmail.com

Abstract In reliability analysis, a lot of failure distributions are used to represent lifetime data. Recently, new distributions are derived to extend some of well-known families of distributions, such that the new distributions are more flexible than the others to model real data. In this paper, properties of Weibull-Poisson distribution (WPD) and inverse Weibull-Poisson distribution (IWPD) will be considered. We provide forms for characteristic function, *r*th raw moment, mean, variance, median, Shannon entropy function, Rényi entropy function and Relative entropy function. This paper deals also with the determination of R = P[Y < X] when X and Y are two independent WPD (IWPD) distributions with different parameters.

Keywords: *Weibull-Poisson distribution, inverse Weibull-Poisson distribution, Shannon entropy, Rényi entropy, Relative entropy, stress-strength reliability*

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1. Introduction

Statistical lifetime distributions are used extensively in data modeling. They are widely applied in areas such as reliability engineering, survival analysis, social sciences and a huddle of other applications. The Weibull distribution is a very widespread model in reliability and it has been exceedingly used for analyzing lifetime data. Several new models have been derived, either from or in some way are related to the Weibull distribution. When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, it does not provide a reasonable parametric fit for modeling phenomenon with non-monotone failure rates such as the bathtub shaped and the unimodal failure rates that are common in reliability and biological studies.

Recently, various methods of generating new distributions have been studied in the statistical literature. Among these methods, the compounding of some discrete and important lifetime distributions has been in the foreword of lifetime modeling. So, several families of distributions were proposed by compounding some useful lifetime and truncated discrete distributions.

In this paper, properties of Weibull-Poisson distribution (WPD) and inverse Weibull-Poisson distribution (IWPD) will be considered. We provide forms for characteristic function, *r*th raw moment, mean, variance, median, Shannon entropy function, Rényi entropy function and Relative entropy function. This paper deals also with the determination of R = P[Y < X] when X and Y are two independent WPD (IWPD) distributions with different parameters.

2. Weibull-Poisson Distribution

In 2009 [3], DeMorais introduced Weibull Poisson distribution (WPD). He assumed that Z has a truncated Poisson distribution with parameter $\lambda > 0$ and probability mass function given by,

$$p(z) = e^{-\lambda}\lambda^{z}\Gamma^{-1}(z+1)\left(1-e^{-\lambda}\right)^{-1}, z = 1,2,\ldots \qquad (1)$$

Where, $\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx\ (for\ p>0)$ is the gamma function. DeMorais assumed also that $\{W_i\}_{i=1}^{Z}$ to be independent and identically random variable having the Weibull density function defined by,

$$\pi(w) = \propto \beta\, w^{\propto-1}e^{-\beta w^{\propto}}, w > 0$$

Where $\propto > 0$ is the shape parameter and $\beta > 0$ is the scale parameter.

If the random variables Z and $W's$ are independent, then the random variable $X = Min\{W_1, W_2, \ldots, W_Z\}$, Will distributed as Weibull Poisson with the following probability density function,

$$f(x) = \frac{\propto \beta\lambda\, e^{-\lambda}}{1-e^{-\lambda}} x^{\propto-1}e^{-\beta x^{\propto}}e^{\lambda\, e^{-\beta x^{\propto}}}, x > 0 \qquad (2)$$

In this paper we will refer to Weibull Poisson distribution by $X \sim WP(\propto, \beta, \lambda)$, which is mean that the random variable X follow Weibull Poisson distribution with parameters $\propto, \beta$ and λ.

The WPD is well-stimulated for industrial and biological implementations. As an example, consider the time to recrudesce of tumor under the first-activation scheme. Suppose that Z is the number of cells which Causing tumor for an individual left active after the initial treatment follows a truncated Poisson distribution and let W_i be the time spent for the *i*th cell to produce a detectable tumor mass, for $i \geq 1$. If $\{W_i\}_{i\geq 1}$ is a sequence of independent and identically distributed (iid) Weibull random variables independent of Z, then the time to recrudesce of tumor of a squeamish individual can be modeled by the WPD. Another example considers that the failure of a device occurs due to the presence of an unknown number Z of initial defects of the same type, which can be distinguishable only after causing failure and are repaired completely. Let W_i be the time to the failure of the system owing to the ith defect, for $i \geq 1$, and $W_i's$ are iid Weibull random variables independent of Z, which is a truncated Poisson random variable, then the time to the first failure is fittingly represented by the WPD. In reliability analysis, the distributions for $X = Min\{W_1, W_2, \dots, W_Z\}$ and $Y = Max\{W_1, W_2, \dots, W_Z\}$ can be used in serial and parallel systems with identical components, which appear in many industrial and biological implementations. The first stimulation scheme may be queried by certain diseases. Consider that the number Z of latent factors that must all be activated by failure follows a truncated Poisson distribution and assume that W represents the time of impedance to a disease appearance owing to the *i*th latent factor has the Weibull distribution. In the last-stimulation scheme, the failure occurs after all Z factors have been activated. So, the WPD is suitable to fit the time of failure under last- stimulation scheme.

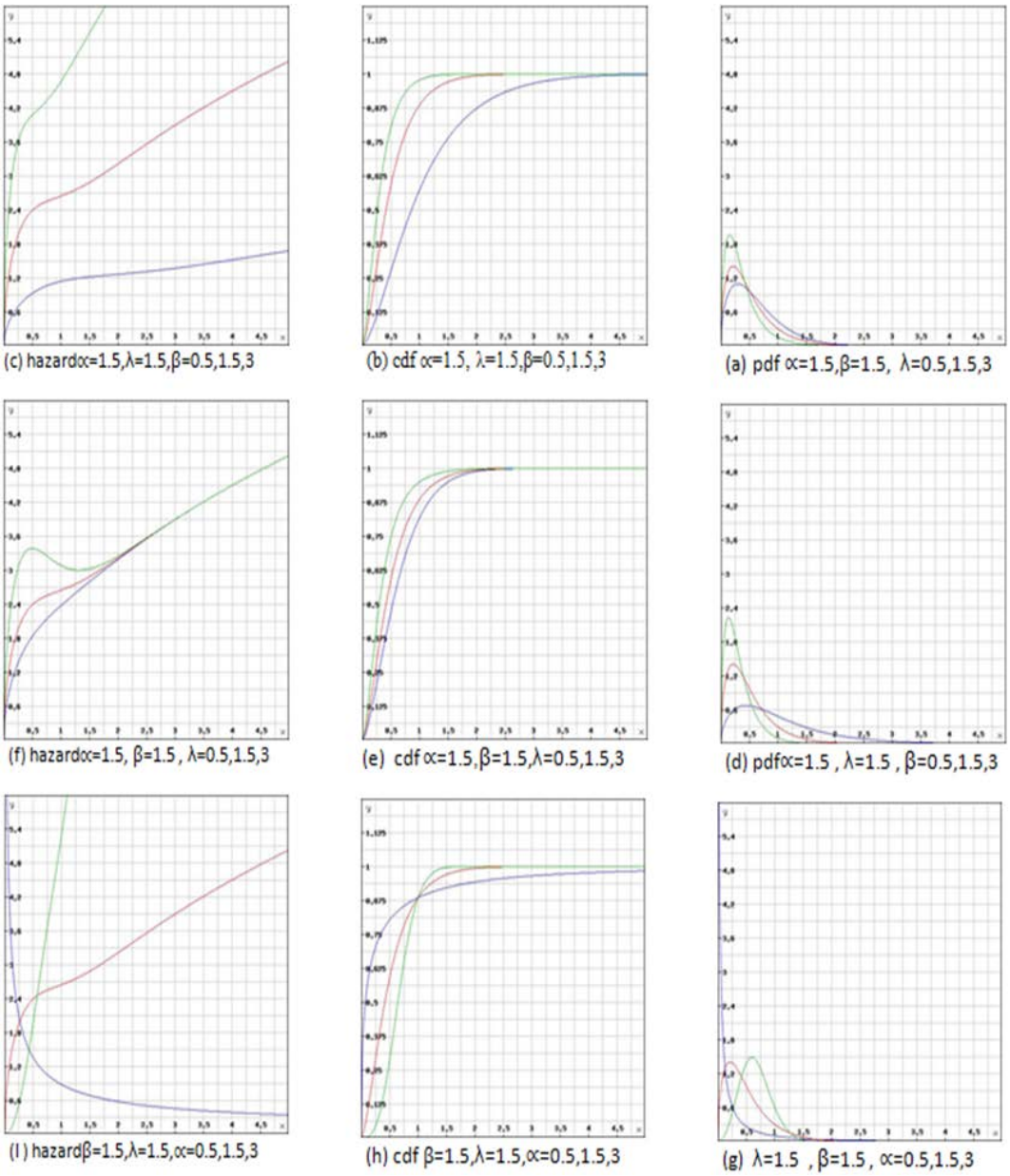

Figure 1. Pdf, cdf and hazard functions of the WPD for some values of the parameters

Percontini, et al. in 2013 [5], proposed new five-parameter distribution by compounding the Weibull Poisson and beta distributions. They called it the beta Weibull Poisson distribution.

The cdf of WPD, corresponding to (2) is obtained by [3],

$$F(x)=\int_0^x \frac{\propto \beta\lambda\, e^{-\lambda}}{1=e^{-\lambda}} x^{\propto-1}e^{-\beta x^{\propto}} e^{\lambda e^{-\beta x^{\propto}}}\,dx$$

$$=\frac{-e^{-\lambda}}{1-e^{-\lambda}}\int_0^x -\propto \beta\lambda\, x^{\propto-1}e^{-\beta x^{\propto}} e^{\lambda\, e^{-\beta x^{\propto}}}\,dx \qquad (3)$$

$$=\frac{1}{1-e^{\lambda}}[e^{\lambda\, e^{-\beta x^{\propto}}}]_0^x=\frac{1}{1-e^{\lambda}}[e^{\lambda\, e^{-\beta x^{\propto}}}-e^{\lambda}]$$

So, the reliability function of WPD is,

$$R(x)=1-F(x)=1-\frac{1}{1-e^{\lambda}}[e^{\lambda\, e^{-\beta x^{\propto}}}-e^{\lambda}]$$

$$=\frac{1-e^{\lambda}-e^{\lambda e^{-\beta x^{\propto}}}+e^{\lambda}}{1-e^{\lambda}}=\left(1-e^{\lambda e^{-\beta x^{\propto}}}\right)(1-e^{\lambda})^{-1}. \qquad (4)$$

And the Hazard function is,

$$h(x)=\frac{f(x)}{R(x)}=\frac{\propto \beta\lambda\left(1-e^{\lambda}\right)e^{-\lambda}x^{\propto-1}e^{-\beta x^{\propto}} e^{\lambda\, e^{-\beta x^{\propto}}}}{\left(1-e^{-\lambda}\right)(1-e^{\lambda\, e^{-\beta x^{\propto}}})}$$

$$h(x)=\frac{\propto \beta\lambda\left(1-e^{\lambda}\right)x^{\propto-1}e^{-\beta x^{\propto}} e^{\lambda\, e^{-\beta x^{\propto}}}}{\left(e^{\lambda}-1\right)(1-e^{\lambda\, e^{-\beta x^{\propto}}})}$$

$$h(x)=\frac{\propto \beta\lambda\, x^{\propto-1}e^{-\beta x^{\propto}} e^{\lambda\, e^{-\beta x^{\propto}}}}{(e^{\lambda\, e^{-\beta x^{\propto}}}-1)}. \qquad (5)$$

Figure 1 plots some shapes of the pdf, cdf and hazard functions of WPD.

2.1. The Moments

The *r*th raw moment of the WP random variable X is,

$$E(X^r)=\int_0^{\infty} x^r \frac{\propto \beta\lambda\, e^{-\lambda}}{1-e^{-\lambda}} x^{\propto-1}e^{-\beta x^{\propto}} e^{\lambda\, e^{-\beta x^{\propto}}}\,dx$$

Since $e^{\lambda e^{-\beta x^{\propto}}}=\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^u(-\beta u)^s}{u!s!}x^{\propto s}$, then,

$$E(X^r)=\frac{\propto \beta\lambda}{e^{\lambda}-1}\int_0^{\infty}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^u(-\beta u)^s}{u!s!}x^{\propto s}x^r x^{\propto-1}e^{-\beta x^{\propto}}\,dx$$

$$=\frac{\propto \beta\lambda}{e^{\lambda}-1}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^u(-\beta u)^s}{u!s!}\int_0^{\infty}x^{r+\propto s+\propto-1}e^{-\beta x^{\propto}}\,dx$$

$$=\frac{\propto \beta\lambda}{e^{\lambda}-1}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^u(-\beta u)^s}{u!s!}\frac{1}{\propto}\frac{\Gamma(\frac{r+\propto s+\propto}{\propto})}{\beta^{\frac{r+\propto s+\propto}{\propto}}} \qquad (6)$$

$$=\frac{1}{e^{\lambda}-1}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^{u+1}(-u)^s}{u!s!}\frac{\Gamma(\frac{r}{\propto}+s+1)}{\beta^{\frac{r}{\propto}}}.$$

The mean and variance of WP variable X are respectively,

$$E(x)=\frac{1}{e^{\lambda}-1}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^{u+1}(-u)^s}{u!s!}\frac{\Gamma(\frac{1}{\propto}+s+1)}{\beta^{\frac{1}{\propto}}} \qquad (7)$$

$$Var(x)=E(x^2)-(E(x))^2$$

$$=\frac{1}{e^{\lambda}-1}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^{u+1}(-u)^s}{u!s!}\frac{\Gamma(\frac{2}{\propto}+s+1)}{\beta^{\frac{2}{\propto}}} \qquad (8)$$

$$-(\frac{1}{e^{\lambda}-1}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^{u+1}(-u)^s}{u!s!}\frac{\Gamma\left(\frac{1}{\propto}+s+1\right)}{\beta^{\frac{1}{\propto}}})^2.$$

The characteristic function of $X\sim WP(\propto,\beta,\lambda)$ is,

$$Q_x(t)=E(e^{itx})=E\left(\sum_{r=0}^{\infty}\frac{(itx)^r}{r!}\right)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}E(x^r),$$

$$\text{since } e^{itx}=\sum_{r=0}^{\infty}\frac{(itx)^r}{r!} \qquad (9)$$

$$=\frac{1}{e^{\lambda}-1}\sum_{r=0}^{\infty}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{(it)^r\lambda^{u+1}(-u)^s}{r!u!s!}\frac{\Gamma(\frac{r}{\propto}+s+1)}{\beta^{\frac{r}{\propto}}}$$

2.2. Shannon Entropy, Rènyi Entropy and Kullback–Leibler Divergence

An entropy of a random variable X is a measure of variation of the uncertainty. The Shannon entropy (SE) of $WP(\propto,\beta,\lambda)$ random variable X can be found as follows,

$$SE=E\{-\ln f(x)\}$$

$$=-\ln\frac{\propto \beta\lambda e^{-\lambda}}{1-e^{-\lambda}}\int_0^{\infty}f(x)dx+\beta\int_0^{\infty}x^{\propto}f(x)dx$$

$$-(\propto-1)\int_0^{\infty}\ln x f(x)dx-\lambda\int_0^{\infty}e^{-\beta x^{\propto}}f(x)dx$$

T0 solve

$$I_1=\int_0^{\infty}x^{\propto}\frac{\propto \beta\lambda\, e^{-\lambda}}{1-e^{-\lambda}}x^{\propto-1}e^{-\beta x^{\propto}} e^{\lambda\, e^{-\beta x^{\propto}}}\,dx$$

$$=\frac{\propto \lambda\beta e^{-\lambda}}{1-e^{-\lambda}}\int_0^{\infty}x^{2\propto-1}e^{-\beta x^{\propto}} e^{\lambda\, e^{-\beta x^{\propto}}}\,dx$$

$$=\frac{\propto \lambda\beta}{e^{\lambda}-1}\int_0^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^r(-\beta r)^s}{r!s!}x^{\propto s}x^{2\propto-1}e^{-\beta x^{\propto}}\,dx$$

$$=\frac{\propto \lambda\beta}{e^{\lambda}-1}[\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^r(-\beta r)^s}{r!s!}\int_0^{\infty}x^{2\propto+\propto s-1}e^{-\beta x^{\propto}}\,dx]$$

$$=\frac{\propto \lambda\beta}{e^{\lambda}-1}[\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^r(-\beta r)^s}{r!s!}\frac{1}{\propto}\frac{\Gamma(s+2)}{\beta^{s+2}}]$$

T0 solve

$$I_2 = \int_0^{\infty} (\ln x) f(x) dx$$

$$= \int_0^{\infty} \frac{\propto \beta\lambda e^{-\lambda}}{1-e^{-\lambda}} x^{\propto -1} e^{-\beta x^{\propto}} e^{\lambda e^{-\beta x^{\propto}}} \ln x dx$$

$$= \frac{\propto \beta\lambda e^{-\lambda}}{1-e^{-\lambda}} \int_0^{\infty} x^{\propto -1} e^{-\beta x^{\propto}} e^{\lambda e^{-\beta x^{\propto}}} \ln x dx$$

Since $e^{\lambda e^{-\beta x^{\propto}}} = \sum_{r=0}^{\infty} \frac{\lambda^r e^{-\beta r x^{\propto}}}{r!}$, then,

$$I_2 = \frac{\propto \lambda\beta}{e^{\lambda}-1} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \int_0^{\infty} \ln(x) x^{\propto -1} e^{-\beta(r+1)x^{\propto}} dx$$

Let $y=x^{\propto}, x = y^{\frac{1}{\propto}}, dx = \frac{1}{\propto} y^{\frac{1}{\propto}-1} dy$

$$\int_0^{\infty} \ln(x) x^{\propto -1} e^{-\beta(r+1)x^{\propto}} dx$$

$$= \int_0^{\infty} \frac{1}{\propto} \ln(y) y^{1-\frac{1}{\propto}} e^{-\beta(r+1)y} \frac{1}{\propto} y^{\frac{1}{\propto}-1} dy$$

$$= \frac{1}{\propto^2} \int_0^{\infty} \ln(y) e^{-\beta(r+1)y} dy$$

Since $\int_0^{\infty} x^{s-1} e^{-\mu x} \ln(x)\, dx = \mu^{-s} \Gamma(s)\{\Psi(s) - \ln \mu\}$

Where $\Psi(1) = -\gamma$, $\gamma = 0.57721$, and $\Gamma(t+1) = t\,\Gamma(t)$, $\Gamma(n+1) = n!$

Put s=1, $\mu = \beta(r+1)$, then

$$\int_0^{\infty} \ln(x)\, x^{\propto -1} e^{-\beta(r+1)x^{\propto}}\, d\, \frac{1}{\propto^2} \frac{1}{\beta(r+1)} \{-\gamma - \ln(\beta(r+1))\}$$

$$I_2 = \frac{1}{\propto(1-e^{\lambda})} \sum_{r=0}^{\infty} \frac{\lambda^{r+1}}{(r+1)!} \{\gamma + \ln(\beta(r+1))\}$$

To solve

$$I_3 = \int_0^{\infty} e^{-\beta x^{\propto}} f(x) dx$$

$$= \int_0^{\infty} e^{-\beta x^{\propto}} \frac{\propto \beta\lambda e^{-\lambda}}{1-e^{-\lambda}} x^{\propto -1} e^{-\beta x^{\propto}} e^{\lambda e^{-\beta x^{\propto}}} dx$$

$$= \frac{\propto \beta\lambda}{e^{\lambda}-1} \int_0^{\infty} x^{\propto -1} e^{-2\beta x^{\propto}} e^{\lambda e^{-\beta x^{\propto}}} dx$$

$$= \frac{\propto \lambda\beta}{e^{\lambda}-1} \int_0^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\lambda^r (-\beta r)^s}{r!s!} x^{\propto s} x^{\propto -1} e^{-2\beta x^{\propto}} dx$$

$$= \frac{\propto \lambda\beta}{e^{\lambda}-1} [\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\lambda^r (-\beta r)^s}{r!s!} \int_0^{\infty} x^{\propto s+\propto -1} e^{-2\beta x^{\propto}} dx]$$

$$= \frac{\propto \lambda\beta}{e^{\lambda}-1} \left[\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\lambda^r (-\beta r)^s}{r!s!} \frac{1}{\propto} \frac{\Gamma(s+1)}{(2\beta)^{s+1}} \right].$$

So, the Shannon entropy is,

$$SE = \ln\left(\frac{e^{\lambda}-1}{\propto\lambda\beta}\right) + \frac{1}{e^{\lambda}-1} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\lambda^{r+1}(-r)^s}{r!s!} \Gamma(s+2) - \frac{(\propto -1)}{\propto(1-e^{\lambda})} \sum_{r=0}^{\infty} \frac{\lambda^{r+1}}{(r+1)!} \{\gamma + \ln(\beta(r+1))\} - \frac{1}{e^{\lambda}-1} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\lambda^{r+2} (\frac{-r}{2})^s}{2(r!)} \quad (10)$$

Rènyi entropy of $WP(\propto, \beta, \lambda)$ random variable X can be found as follows,

$$RE = \frac{1}{1-w} \ln \int_0^{\infty} (f(x))^w dx$$

$$= \frac{1}{1-w} \ln \int_0^{\infty} (\frac{\propto \beta\lambda e^{-\lambda}}{1-e^{-\lambda}} x^{\propto -1} e^{-\beta x^{\propto}} e^{\lambda e^{-\beta x^{\propto}}})^w dx$$

$$= \frac{1}{1-w} \ln\{(\frac{\propto \lambda\beta}{e^{\lambda}-1})^w \int_0^{\infty} x^{\propto w - w} e^{-\beta w x^{\propto}} e^{\lambda w e^{-\beta x^{\propto}}} dx\}$$

$$= \frac{1}{1-w} \ln\left\{ (\frac{\propto \lambda\beta}{e^{\lambda}-1})^w \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{} \frac{(\lambda w)^r (-\beta r)^s}{r!s!} x^{\propto s} x^{\propto w - w} e^{-\beta w x^{\propto}} dx \right\}$$

$$= \frac{1}{1-w} \ln\left\{ (\frac{\propto \lambda\beta}{e^{\lambda}-1})^w \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\lambda w)^r (-\beta r)^s}{r!s!} \int_0^{\infty} x^{\propto s + \propto w - w} e^{-\beta w x^{\propto}} dx \right\}$$

$$= \frac{1}{1-w} \ln\left\{ (\frac{\propto \lambda\beta}{e^{\lambda}-1})^w \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\lambda w)^r (-\beta r)^s}{r!s!} \frac{1}{\propto} \frac{\Gamma(s+w+\frac{(1-w)}{\propto})}{(\beta w)^{s+w+\frac{(1-w)}{\propto}}} \right\}$$

$$= \frac{1}{1-w} \left\{ \begin{array}{l} \ln(\frac{\propto^{w-1}}{(e^{\lambda}-1)^w}) \\ + \ln \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\lambda^{r+w} w^{r-s-w-\frac{(1-w)}{\propto}} (-r)^s}{r!s! \beta^{\frac{(1-w)}{\propto}}} \\ \Gamma(s+w+\frac{(1-w)}{\propto}) \end{array} \right\}$$

$$= \frac{1}{1-w} \left\{ \begin{array}{l} \ln\left(\frac{\propto^{w-1}}{(e^{\lambda}-1)^w}\right) - \frac{1-w}{\propto} \ln(\beta w) + w \ln\left(\frac{\lambda}{w}\right) \\ + \ln \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\lambda^r w^{r-s} (-r)^s \Gamma(s+w+\frac{(1-w)}{\propto})}{r!s!} \end{array} \right\} \quad (11)$$

The Kullback–Leibler divergence (KL) (or the relative entropy) is a measure of the difference between two probability distributions G and G^*. It is not symmetric in G and G^*. In applications, G typically represents the "true" distribution of data, observations, or a precisely calculated theoretical distribution, while G^* typically represents a theory, model, description, or approximation of G. Specifically, the Kullback–Leibler divergence of G^* from G, denoted $D_{KL}(G \| G^*)$, is a measure of the information gained when one revises ones beliefs from the prior probability distribution G^* to the posterior

probability distribution G. More exactly, it is the amount of information that is *lost* when G^* is used to approximate G ,defined operationally as the expected extra number of bits required to code samples from G using a code optimized for G^* rather than the code optimized for G.

Let $f(x)= \frac{\propto\beta\lambda\, e^{-\lambda}}{1-e^{-\lambda}} x^{\propto-1}e^{-\beta x^{\propto}}\, e^{\lambda e^{-\beta x^{\propto}}}$ and $g(x) = \frac{abc\, e^{-c}}{1-e^{-c}}\, x^{a-1}\, e^{-bx^{a}} e^{ce^{-bx^{a}}}$

Then, $\frac{f(x)}{g(x)} = \frac{\propto\beta\lambda(e^{c}-1)}{abc(e^{\lambda}-1)} \frac{x^{\propto-1}e^{-\beta x^{\propto}}\, e^{\lambda e^{-\beta x^{\propto}}}}{x^{a-1}\, e^{-bx^{a}} e^{ce^{-bx^{a}}}}$, and,

$$\ln\frac{f(x)}{g(x)} = \ln(\frac{\propto\beta\lambda(e^{c}-1)}{abc(e^{\lambda}-1)}) - \beta x^{\propto} + (\propto -1)\ln(x) + \lambda e^{-\beta x^{\propto}} + bx^{a} - (a-1)\ln(x) - ce^{-bx^{a}}$$

The Kullback–Leibler divergence can be found as follows,

$$KL = \int_0^{\infty} f(x)\ln\frac{f(x)}{g(x)}dx$$

$$= \ln(\frac{\propto\beta\lambda(e^{c}-1)}{abc(e^{\lambda}-1)}) - \beta\int_0^{\infty} x^{\propto} f(x)dx + (\propto -a)\int_0^{\infty}\ln(x) f(x)dx + b\int_0^{\infty} x^{a} f(x)dx - c\int_0^{\infty} e^{-bx^{a}} f(x)dx + \lambda\int_0^{\infty} e^{-\beta x^{\propto}} f(x)dx.$$

Since $\int_0^{\infty} x^{\propto} f(x)dx = \frac{\propto\lambda\beta}{e^{\lambda}-1}[\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^{r}(-\beta r)^{s}}{r!s!}\frac{1}{\propto}\frac{\Gamma(s+2)}{\beta^{s+2}}]$ and $\int_0^{\infty}(\ln x)\, f(x)\, dx = \frac{1}{\propto(1-e^{\lambda})}\sum_{r=0}^{\infty}\frac{\lambda^{r+1}}{(r+1)!}\{\gamma + \ln(\beta(r+1))\}$ and

$$\int_0^{\infty} x^{a} f(x)dx = \int_0^{\infty} x^{a}\frac{\propto\beta\lambda\, e^{-\lambda}}{1-e^{-\lambda}} x^{\propto-1}e^{-\beta x^{\propto}}\, e^{\lambda\, e^{-\beta x^{\propto}}}\, dx$$

$$= \frac{\propto\lambda\beta}{e^{\lambda}-1}\int_0^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^{r}(-\beta r)^{s}}{r!s!} x^{\propto s} x^{a} x^{\propto-1}e^{-\beta x^{\propto}}\, dx$$

$$= \frac{\propto\lambda\beta}{e^{\lambda}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^{r}(-\beta r)^{s}}{r!s!}\int_0^{\infty} x^{\propto s+\propto+a-1}e^{-\beta x^{\propto}}\, dx$$

$$= \frac{\propto\beta}{e^{\lambda}-1}\sum_{r=0}^{\infty}\sum_{r=0}^{\infty}\frac{\lambda^{r}(-\beta r)^{s}}{r!s!}\frac{1}{\propto}\frac{\Gamma\left(s+1+\frac{a}{\propto}\right)}{\beta^{s+1+\frac{a}{\propto}}}$$

and

$$\int_0^{\infty} e^{-bx^{a}} f(x)dx = \int_0^{\infty} e^{-bx^{a}}\frac{\propto\beta\lambda\, e^{-\lambda}}{1-e^{-\lambda}} x^{\propto-1}e^{-\beta x^{\propto}}\, e^{\lambda\, e^{-\beta x^{\propto}}}\, dx$$

$$= \frac{\propto\lambda\beta}{e^{\lambda}-1}\int_0^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^{r}(-\beta r)^{s}}{r!s!} x^{\propto s} x^{\propto-1}e^{-\beta x^{\propto}} e^{-bx^{a}}\, dx$$

Since $e^{-bx^{a}} = \sum_{u=0}^{\infty}\frac{(-b)^{u}}{u!} x^{au}$, then,

$$\int_0^{\infty} e^{-bx^{a}} f(x)dx = \frac{\propto\lambda\beta}{e^{\lambda}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\frac{\lambda^{r}(-\beta r)^{s}(-b)^{u}}{r!s!u!}\int_0^{\infty} x^{\propto s+au+\propto-1}e^{-\beta x^{\propto}}dx$$

$$= \frac{\propto\lambda\beta}{e^{\lambda}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\frac{\lambda^{r}(-\beta r)^{s}(-b)^{u}}{r!s!u!}\frac{1}{\propto}\frac{\Gamma(\frac{au}{\propto}+s+1)}{\beta^{\frac{au}{\propto}+s+1}}$$

and

$$\int_0^{\infty} e^{-\beta x^{\propto}} f(x)d = \frac{\propto\lambda\beta}{e^{\lambda}-1}\left[\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^{r}(-\beta r)^{s}}{r!s!}\frac{1}{\propto}\frac{\Gamma(s+1)}{(2\beta)^{s+1}}\right]$$

Then

$$KL = \ln(\frac{\propto\beta\lambda(e^{c}-1)}{abc(e^{\lambda}-1)}) - \frac{1}{e^{\lambda}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(s+1)\lambda^{r+1}(-r)^{s}}{r!} + \frac{(\propto-a)}{\propto(1-e^{\lambda})}\sum_{r=0}^{\infty}\frac{\lambda^{r+1}}{(r+1)!}\{\gamma + \ln(\beta(r+1))\} + \frac{b}{e^{\lambda}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^{r+1}(-r)^{s}}{r!s!}\frac{\Gamma\left(\frac{a}{\propto}+s+1\right)}{\beta^{\frac{a}{\propto}}} - \frac{c}{e^{\lambda}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\frac{\lambda^{r+1}(-\beta)^{s}(-b)^{u}}{r!s!u!}\frac{\Gamma(\frac{au}{\propto}+s+1)}{\beta^{\frac{au}{\propto}}} + \frac{1}{2(e^{\lambda}-1)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^{r+2}(-\frac{r}{2})^{s}}{r!} \quad (12)$$

2.3. Stress-Strength Reliability

Inferences about R = P[Y < X], where X and Y are two independent random variables, is very common in the reliability literature. For example, if X is the strength of a component which is subject to a stress Y, then R is a measure of system performance and arises in the context of mechanical reliability of a system. The system fails if and only if at any time the applied stress is greater than its strength.

Let Y and X be the stress and the strength random variables, independent of each other, follow respectively $WP(\propto,\beta,\lambda)$ and $WP(a,b,c)$, then, the Stress-Strength reliability is,

$$R = \int_0^{\infty} f(x)F(x)dx = \int_0^{\infty}\left[\left(\frac{\propto\beta\lambda\, e^{-\lambda}}{1-e^{-\lambda}} x^{\propto-1}e^{-\beta x^{\propto}}\, e^{\lambda e^{-\beta x^{\propto}}}\right)\left(\frac{1}{1-e^{c}}\left[e^{ce^{-bx^{a}}} - e^{c}\right]\right)\right]dx$$

$$= \frac{\propto\beta\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)\left(1-e^{c}\right)}\int_0^{\infty}\left(x^{\propto-1}e^{-\beta x^{\propto}}\, e^{\lambda e^{-\beta x^{\propto}}}\, e^{ce^{-bx^{\propto}}}\right)dx - \frac{e^{c}}{1-e^{c}}.$$

Since $e^{\lambda e^{-\beta x^{\propto}}} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\lambda^{r}(-\beta r)^{s}}{r!s!} x^{\propto s}$ and $e^{ce^{-\beta x^{\propto}}} = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{c^{u}(-bu)^{v}}{u!v!} x^{av}$

$$R=\frac{\propto \beta\lambda}{\left(e^{\lambda}-1\right)(1-e^{c})}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\left(\frac{\lambda^{r}(-\beta r)^{s}}{r!s!}\frac{c^{u}(-bu)^{v}}{u!v!}\int_{0}^{\infty}x^{\propto s+av+\propto-1}e^{-\beta x^{\propto}}\right)dx$$

$$=\frac{\propto \beta\lambda}{\left(e^{\lambda}-1\right)(1-e^{c})}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\left[\left(\frac{\lambda^{r}(-\beta r)^{s}}{r!s!}\frac{c^{u}(-bu)^{v}}{u!v!}\right)\left(\frac{1}{\propto}\frac{\Gamma(\frac{\propto s+av+\propto}{\propto})}{\beta^{\frac{\propto s+av+\propto}{\propto}}}\right)-\frac{e^{c}}{1-e^{c}}\right]$$

$$=\frac{1}{\left(e^{\lambda}-1\right)(1-e^{c})}$$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\left(\frac{\lambda^{r+1}c^{u}(-1)^{s+v}r^{s}(bu)^{v}}{r!s!u!v!}\frac{\Gamma(\frac{av}{\propto}+s+1)}{\beta^{\frac{av}{\propto}}}\right)-\frac{e^{c}}{1-e^{c}} \quad (13)$$

3. Inverse Weibull-Poisson Distribution

In 2015 [2], Bera introduced Inverse Weibull-Poisson distribution (IWPD). He assumed that Z has a truncated Poisson distribution with parameter $\theta>0$ with probability mass function which is defined in (1). Bera assumed also that $\{W_i\}_{i=1}^{Z}$ to be independent and identically random variable having the Inverse Weibull density function defined by,

$$\pi(w)=\propto \beta\, w^{-\beta-1}e^{-\propto w^{-\beta}},\ w>0$$

Where $\propto>0$ is the scale parameter and $\beta>0$ is the shape parameter.

If the random variables Z and $W's$ are independent, then the random variable $X=Max\{W_1,W_2,\dots,W_Z\}$, Will distributed as Inverse Weibull-Poisson with the following probability density function,

$$f(x)=\frac{\theta \propto \beta}{e^{\theta}-1}x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}},\ x>0. \quad (14)$$

In this paper we will refer to Inverse Weibull-Poisson distribution by $X\sim IWP(\propto,\beta,\theta)$, which is mean that the random variable X follow Inverse Weibull-Poisson distribution with parameters $\propto,\beta$ and θ.

Hassan, et al. in 2016 [4], studied Exponentiated Inverse Weibull-Power Series Family of Distributions by compounding the Inverse Weibull and Power Series distributions. In fact, they studied inclusively Inverse Weibull-Poisson distribution, since the Poisson distribution is special case from Power Series distribution.

The cdf of WPD, corresponding to (14) is obtained by,

$$F(x)=\int_{0}^{x}\frac{\theta \propto \beta}{e^{\theta}-1}x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}}dx$$

$$\frac{1}{e^{\theta}-1}\int_{0}^{x}\theta \propto \beta x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}}dx$$

$$=\frac{1}{e^{\theta}-1}\left[e^{\theta e^{-\propto x^{-\beta}}}\right]_{0}^{x} \quad (15)$$

$$=\frac{1}{e^{\theta}-1}\left[e^{\theta e^{-\propto x^{-\beta}}}-1\right]$$

$$=\frac{1-e^{\theta e^{-\propto x^{-\beta}}}}{1-e^{\theta}}.$$

So, the reliability function of IWPD is,

$$R(x)=1-F(x)=1-\frac{1-e^{\theta e^{-\propto x^{-\beta}}}}{1-e^{\theta}}$$

$$=\frac{1-e^{\theta}-1+e^{\theta e^{-\propto x^{-\beta}}}}{1-e^{\theta}} \quad (16)$$

$$=\frac{e^{\theta e^{-\propto x^{-\beta}}}-e^{\theta}}{1-e^{\theta}}.$$

And the Hazard function is,

$$h(x)=\frac{f(x)}{R(x)}=\frac{\theta \propto \beta x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}}}{\left(e^{\theta}-1\right)\left(\frac{e^{\theta}-e^{\theta e^{-\propto x^{-\beta}}}}{e^{\theta}-1}\right)}$$

$$=\frac{\theta \propto \beta x^{-\beta-1}e^{-\propto x^{-\beta}}}{\left(e^{\theta-\theta e^{-\propto x^{-\beta}}}-1\right)} \quad (17)$$

$$=\frac{\theta \propto \beta x^{-\beta-1}e^{-\propto x^{-\beta}}}{e^{\theta(1-e^{-\propto x^{-\beta}})}-1}$$

Figure 2 plots some shapes of the pdf, cdf and hazard functions of IWPD.

3.1. The Moments

The rth raw moment of the IWP random variable X is,

$$E(X^{r})=\int_{0}^{\infty}x^{r}\,f(x)dx$$

$$=\int_{0}^{\infty}x^{r}\frac{\theta \propto \beta}{e^{\theta}-1}x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}}dx.$$

Since, $e^{\theta e^{-\propto x^{-\beta}}}=\sum_{0}^{\infty}\frac{(\theta e^{-\propto x^{-\beta}})^{u}}{u!}=\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^{u}(-\propto u)^{s}}{u!s!}x^{-\beta s}$,

then,

$$E\left(X^r\right)=\frac{\theta \propto \beta}{e^{\theta}-1}\int_0^{\infty}\left(\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^u(-\propto u)^s}{u!s!}x^{-\beta s}\right)dx$$

$$x^r\ x^{-\beta-1}e^{-\propto x^{-\beta}}$$

$$=\frac{\theta \propto \beta}{e^{\theta}-1}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^u(-\propto u)^s}{u!s!}$$

$$\int_0^{\infty}x^{-(\beta s+\beta-r)-1}e^{-\propto x^{-\beta}}dx \qquad (18)$$

$$=\frac{\theta \propto \beta}{e^{\theta}-1}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^u(-\propto u)^s}{u!s!}\frac{1}{\beta}\frac{\Gamma(\frac{\beta s+\beta-r}{\beta})}{\propto^{\frac{\beta s+\beta-r}{\beta}}}$$

$$=\frac{1}{e^{\theta}-1}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s\theta^{u+1}u^s\propto^{\frac{r}{\beta}}}{u!s!}\Gamma(s-\frac{r}{\beta}+1)$$

So, the mean and variance of WP variable X are respectively,

$$E\left(x\right)=\frac{1}{e^{\theta}-1}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s\theta^{u+1}u^s\propto^{\frac{1}{\beta}}}{u!s!}\Gamma(s-\frac{1}{\beta}+1)$$

$$Var\left(x\right)=E(x^2)-(E\left(x\right))^2$$

$$Var\left(x\right)=\frac{1}{e^{\theta}-1}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s\theta^{u+1}u^s\propto^{\frac{2}{\beta}}}{u!s!}\Gamma(s-\frac{2}{\beta}+1) \qquad (19)$$

$$-(\frac{1}{e^{\theta}-1}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s\theta^{u+1}u^s\propto^{\frac{1}{\beta}}}{u!s!}\Gamma\left(s-\frac{1}{\beta}+1\right))^2$$

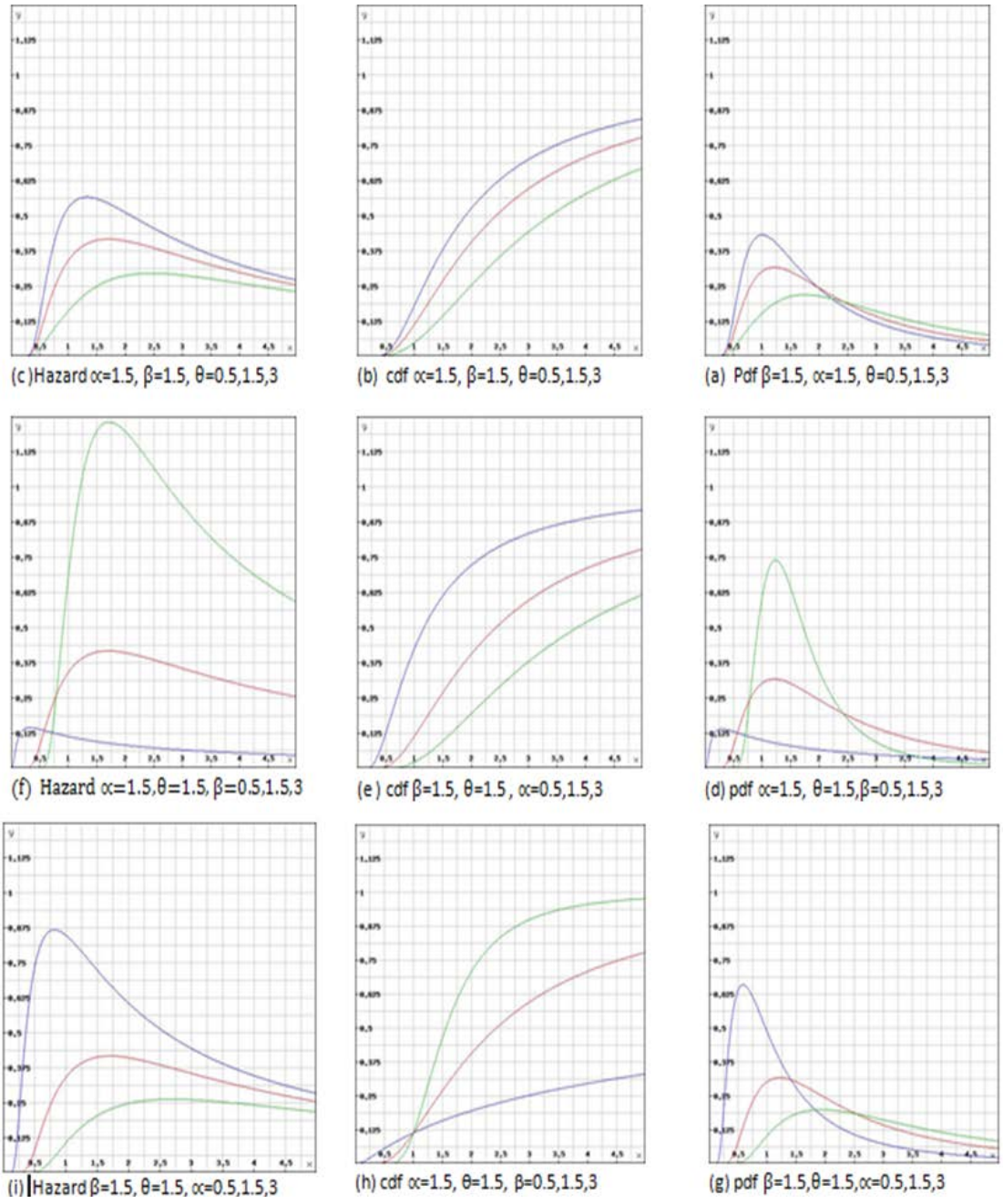

Figure 2. Pdf, cdf and hazard functions of the IWPD for some values of the parameters

The characteristic function of $X \sim IWP(\propto, \beta, \theta)$ is,

$$Q_x(t) = E\left(e^{itx}\right) = E\left(\sum_{r=0}^{\infty}\frac{(itx)^r}{r!}\right) = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}E\left(x^r\right)$$

$$= \frac{1}{e^{\theta}-1}\sum_{r=0}^{\infty}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\left(\frac{(-1)^s\theta^{u+1}u^s(it)^r}{r!u!s!\ \propto^{\frac{r}{\beta}}\Gamma(s-\frac{r}{\beta}+1)}\right). \quad (20)$$

3.2. The entropy

The Shannon entropy (SE) of $IWP(\propto, \beta, \theta)$ random variable X can be found as follows,

$$SE = E\{-\ln f(x)\}$$

$$= \int_0^{\infty}[-\ln(\frac{\theta \propto \beta}{e^{\theta}-1}x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}})]f(x)dx$$

$$= \int_0^{\infty}\left(\begin{matrix}-\ln\frac{\theta \propto \beta}{e^{\theta}-1}+(\beta+1)\ln(x)\\ +\propto x^{-\beta}-\theta e^{-\propto x^{-\beta}}\end{matrix}\right)f(x)dx$$

$$= -\ln\frac{\theta \propto \beta}{e^{\theta}-1}+(\beta+1)\int_0^{\infty}\ln(x)f(x)dx$$

$$+\propto\int_0^{\infty}x^{-\beta}f(x)dx-\theta\int_0^{\infty}e^{-\propto x^{-\beta}}f(x)dx.$$

To solve $I_1 = \int_0^{\infty}\ln(x)\frac{\theta \propto \beta}{e^{\theta}-1}x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}}dx$

Since $e^{\theta e^{-\propto x^{-\beta}}} = \sum_{r=0}^{\infty}\frac{\theta^r e^{-\propto r x^{-\beta}}}{r!}$

$$I_1 = \frac{\theta \propto \beta}{e^{\theta}-1}\int_0^{\infty}\ln(x)x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}}dx$$

$$= \frac{\theta \propto \beta}{e^{\theta}-1}\sum_{r=0}^{\infty}\frac{\theta^r}{r!}\int_0^{\infty}\ln(x)x^{-\beta-1}e^{-\propto(r+1)x^{-\beta}}dx.$$

Let $y=x^{-\beta}$, $x = y^{\frac{-1}{\beta}}$, $dx = \frac{-1}{\beta}y^{-\frac{1}{\beta}-1}$ dy

$$I_1 = \frac{\theta \propto \beta}{e^{\theta}-1}\sum_{r=0}^{\infty}\frac{\theta^r}{r!}\int_0^{\infty}\left(\begin{matrix}\frac{-1}{\beta}\ln(y)y^{\frac{-1}{\beta}(-\beta-1)}\\ e^{-\propto(r+1)y}\frac{1}{\beta}y^{\frac{-1}{\beta}-1}\end{matrix}\right)dy$$

$$= \frac{\theta \propto \beta}{e^{\theta}-1}\sum_{r=0}^{\infty}\frac{\theta^r}{r!}\frac{-1}{\beta^2}\int_0^{\infty}\ln(y)e^{-\propto(r+1)y}dy$$

By $\int_0^{\infty}x^{s-1}\ e^{-\mu x}\ln(x)\,dx = \mu^{-s}\Gamma(s)\{\Psi(s)-\ln\mu\}$
Where $\Psi(1)=-\gamma, \gamma = 0.57721$,and s=1, $\mu = \propto(r+1)$

$$I_1 = \frac{\theta \propto \beta}{e^{\theta}-1}\sum_{r=0}^{\infty}\frac{\theta^r}{r!}\frac{-1}{\beta^2}\frac{1}{\propto(r+1)}\{-\gamma-\ln(\propto(r+1))\}$$

$$= \sum_{r=0}^{\infty}\frac{\theta^{r+1}}{\beta(e^{\theta}-1)(r+1)!}\{\gamma+\ln(\propto(r+1))\}$$

To solve

$$I_2 = \int_0^{\infty}x^{-\beta}f(x)dx$$

$$= \int_0^{\infty}x^{-\beta}\frac{\theta \propto \beta}{e^{\theta}-1}x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}}dx$$

$$= \frac{\theta \propto \beta}{e^{\theta}-1}\int_0^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^r(-\propto r)^s}{r!s!}x^{-\beta s}x^{-2\beta-1}e^{-\propto x^{-\beta}}dx$$

$$= \frac{\theta \propto \beta}{e^{\theta}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^r(-\propto r)^s}{r!s!}\int_0^{\infty}x^{-(\beta s+2\beta)-1}e^{-\propto x^{-\beta}}dx$$

$$= \frac{\theta \propto \beta}{e^{\theta}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^r(-\propto r)^s}{r!s!}\frac{1}{\beta}\frac{\Gamma(s+2)}{\propto^{s+2}}.$$

To solve

$$I_3 = \int_0^{\infty}e^{-\propto x^{-\beta}}f(x)dx$$

$$= \int_0^{\infty}e^{-\propto x^{-\beta}}\frac{\theta \propto \beta}{e^{\theta}-1}x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}}dx$$

$$= \frac{\theta \propto \beta}{e^{\theta}-1}\int_0^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^r(-\propto r)^s}{r!s!}x^{-\beta s}x^{-\beta-1}e^{-2\propto x^{-\beta}}dx$$

$$= \frac{\theta \propto \beta}{e^{\theta}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^r(-\propto r)^s}{r!s!}\int_0^{\infty}x^{-(\beta+\beta s)-1}e^{-2\propto x^{-\beta}}dx$$

$$= \frac{\theta \propto \beta}{e^{\theta}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^r(-\propto r)^s}{r!s!}\frac{1}{\beta}\frac{\Gamma(s+1)}{(2\propto)^{s+1}}$$

Then,

$$SE = \ln\frac{e^{\theta}-1}{\theta \propto \beta}+\frac{(1-\frac{1}{\beta})}{e^{\theta}-1}\sum_{r=0}^{\infty}\frac{\theta^{r+1}}{(r+1)!}\{\gamma+\ln(\propto(r+1))\}$$
$$+\frac{1}{e^{\theta}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(s+1)\theta^{r+1}(-r)^s}{r!}$$
$$-\frac{1}{2(e^{\theta}-1)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^{r+2}(-\frac{r}{2})^s}{r!} \quad (21)$$

Rènyi entropy of $IWP(\propto, \beta, \theta)$ random variable X can be found as follows,

$$RE = \frac{1}{1-w}\ln\int_0^{\infty}(f(x))^w dx$$

$$= \frac{1}{1-w}\ln\int_0^{\infty}(\frac{\theta \propto \beta}{e^{\theta}-1}x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}})^w dx$$

$$= \frac{1}{1-w}\ln[(\frac{\theta \propto \beta}{e^{\theta}-1})^w\int_0^{\infty}x^{-\beta w-w}e^{-\propto w x^{-\beta}}e^{\theta w e^{-\propto x^{-\beta}}}dx]$$

$$= \frac{1}{1-w}\ln[(\frac{\theta \propto \beta}{e^{\theta}-1})^w\int_0^{\infty}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\left(\begin{matrix}\frac{(\theta w)^u(-\propto u)^s}{u!s!}x^{-\beta s}\\ x^{-\beta w-w}e^{-\propto w x^{-\beta}}\end{matrix}\right)dx]$$

$$= \frac{1}{1-w}\ln\left[\begin{matrix}(\frac{\theta \propto \beta}{e^{\theta}-1})^w\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\theta w)^u(-\propto u)^s}{u!s!}\\ \int_0^{\infty}x^{-(\beta s+\beta w+w-1)-1}e^{-\propto w x^{-\beta}}dx\end{matrix}\right]$$

$$= \frac{1}{1-w}\ln\left[\begin{array}{c}(\frac{\theta \propto \beta}{e^{\theta}-1})^{w}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\theta w)^{u}(-\propto u)^{s}}{u!s!} \\ \frac{1}{\beta}\frac{\Gamma(\frac{\beta s+\beta w+w-1}{\beta})}{(\propto w)^{\frac{\beta s+\beta w+w-1}{\beta}}}\end{array}\right]$$

$$= \frac{1}{1-w}\ln\left[\begin{array}{c}\frac{\beta^{w-1}}{(e^{\theta}-1)^{w}}\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^{u+w}w^{u-s-w-\frac{w-1}{\beta}}(-u)^{s}}{u!s!\propto^{\frac{w-1}{\beta}}} \\ \Gamma(s+w+\frac{w-1}{\beta})\end{array}\right]$$

$$= \frac{1}{1-w}\left\{\begin{array}{l}\ln\left(\frac{\beta^{w-1}}{\left(e^{\theta}-1\right)^{w}}\right)-\frac{w-1}{\beta}\ln(\propto w)-w\ln\left(\frac{\theta}{w}\right) \\ +\ln\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^{u}w^{u-s}(-u)^{s}}{u!s!}\Gamma\left(s+w+\frac{w-1}{\beta}\right)\end{array}\right\} \quad (22)$$

Now, let $f(x)=\frac{\theta\propto\beta}{e^{\theta}-1}\,x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}}$ and $g(x)=\frac{abc}{e^{a}-1}\,x^{-c-1}\,e^{-bx^{-c}}\,e^{ae^{-bx^{-c}}}$

Then, $\frac{f(x)}{g(x)}=\frac{\theta\propto\beta(e^{a}-1)}{abc(e^{\theta}-1)}\,\frac{x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}}}{x^{-c-1}\,e^{-bx^{-c}}\,e^{ae^{-bx^{-c}}}}$,

and

$$\ln\frac{f(x)}{g(x)}=\ln\left(\frac{\theta\propto\beta\left(e^{a}-1\right)}{abc(e^{\theta}-1)}\right)-(\beta+1)\ln(x)$$

$$-\propto x^{-\beta}+\theta e^{-\propto x^{-\beta}}+(c+1)\ln(x)+bx^{-c}-ae^{-bx^{-c}}$$

Then, the Kullback–Leibler divergence can be found as follows,

$$KL=\int_{0}^{\infty}\ln(\frac{f(x)}{g(x)})f(x)dx$$

$$=\ln\left(\frac{\theta\propto\beta\left(e^{a}-1\right)}{abc(e^{\theta}-1)}\right)+(c-\beta)\int_{0}^{\infty}\ln(x)f(x)dx$$

$$-\propto\int_{0}^{\infty}x^{-\beta}f(x)dx+\theta\int_{0}^{\infty}e^{-\propto x^{-\beta}}f(x)dx$$

$$+b\int_{0}^{\infty}x^{-c}f(x)dx-a\int_{0}^{\infty}e^{-bx^{-c}}f(x)dx$$

Since $\int_{0}^{\infty}\ln(x)f(x)dx=\sum_{r=0}^{\infty}\frac{\theta^{r+1}}{\beta\left(e^{\theta}-1\right)(r+1)!}\{\gamma+\ln(\propto(r+1))\}$ and $\int_{0}^{\infty}x^{-\beta}f(x)dx=\frac{\theta\propto\beta}{e^{\theta}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^{r}(-\propto r)^{s}}{r!s!}\frac{1}{\beta}\frac{\Gamma(s+2)}{\propto^{s+2}}$.

and

$$\int_{0}^{\infty}e^{-\propto x^{-\beta}}f(x)dx=\frac{\theta\propto\beta}{e^{\theta}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^{r}(-\propto r)^{s}}{r!s!}\frac{1}{\beta}\frac{\Gamma(s+1)}{(2\propto)^{s+1}}$$

and

$$\int_{0}^{\infty}x^{-c}f(x)dx=\int_{0}^{\infty}x^{-c}\frac{\theta\propto\beta}{e^{\theta}-1}x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}}dx$$

$$=\frac{\theta\propto\beta}{e^{\theta}-1}\int_{0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^{r}(-\propto r)^{s}}{r!s!}x^{-\beta s}x^{-c}x^{-\beta-1}e^{-\propto x^{-\beta}}dx$$

$$=\frac{\theta\propto\beta}{e^{\theta}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^{r}(-\propto r)^{s}}{r!s!}\int_{0}^{\infty}x^{-(\beta s+c+\beta)-1}e^{-\propto x^{-\beta}}dx$$

$$=\frac{\theta\propto\beta}{e^{\theta}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^{r}(-\propto r)^{s}}{r!s!}\frac{1}{\beta}\frac{\Gamma(\frac{\beta s+c+\beta}{\beta})}{(\propto)^{\frac{\beta s+c+\beta}{\beta}}}.$$

To solve

$$\int_{0}^{\infty}e^{-bx^{-c}}f(x)=\int_{0}^{\infty}e^{-bx^{-c}}\frac{\theta\propto\beta}{e^{\theta}-1}x^{-\beta-1}e^{-\propto x^{-\beta}}e^{\theta e^{-\propto x^{-\beta}}}$$

$$=\frac{\theta\propto\beta}{e^{\theta}-1}\int_{0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\left(\begin{array}{c}\frac{\theta^{r}(-\propto r)^{s}(-b)^{u}}{r!s!u!}x^{-\beta s} \\ x^{-cu}x^{-\beta-1}e^{-\propto x^{-\beta}}\end{array}\right)dx$$

$$=\frac{\theta\propto\beta}{e^{\theta}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\frac{\theta^{r}(-\propto r)^{s}(-b)^{u}}{r!s!u!}\int_{0}^{\infty}x^{-(cu+\beta s+\beta)-1}e^{-\propto x^{-\beta}}dx$$

$$=\frac{\theta\propto\beta}{e^{\theta}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\frac{\theta^{r}(-\propto r)^{s}(-b)^{u}}{r!s!u!}\frac{1}{\beta}\frac{\Gamma(\frac{cu+\beta s+\beta}{\beta})}{\propto^{\frac{cu+\beta s+\beta}{\beta}}}$$

Then,

$$\begin{aligned}KL=&\ln(\tfrac{\theta\propto\beta(e^{a}-1)}{abc(e^{\theta}-1)})\\&+\tfrac{(c-\beta)}{\beta(e^{\theta}-1)}\textstyle\sum_{r=0}^{\infty}\frac{\theta^{r+1}}{(r+1)!}\{\gamma+\ln(\propto(r+1))\}\\&-\tfrac{1}{e^{\theta}-1}\textstyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(s+1)\theta^{r+1}(-r)^{s}}{r!}\\&+\tfrac{1}{2(e^{\theta}-1)}\textstyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^{r+2}\left(-\frac{r}{2}\right)^{s}}{r!}\\&+\tfrac{b}{e^{\theta}-1}\textstyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^{r+1}(-r)^{s}}{r!s!}\frac{\Gamma(\frac{c}{\beta}+s+1)}{\propto^{\frac{c}{\beta}}}\\&-\tfrac{a}{e^{\theta}-1}\textstyle\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\frac{\theta^{r+1}(-b)^{u}(-r)^{s}}{r!s!u!}\frac{\Gamma(\frac{cu}{\beta}+s+1)}{\propto^{\frac{cu}{\beta}}}\end{aligned} \quad (23)$$

3.3. Stress-Strength Reliability

Let Y and X be the stress and the strength random variables, independent of each other, follow respectively $IWP(\propto,\beta,\theta)$ and $WP(b,c,a)$, then the Stress-Strength reliability is,

$$R = p(x < y) = \int_0^{\infty} f_x(x) F_y(x)\,dx$$

$$= \int_0^{\infty} \begin{bmatrix} \left(\frac{\theta \propto \beta}{e^{\theta}-1} x^{-\beta-1} e^{-\propto x^{-\beta}} e^{\theta e^{-\propto x^{-\beta}}}\right) \\ \left(\frac{1}{1-e^{a}}\left(1-e^{ae^{-bx^{-c}}}\right)\right) \end{bmatrix} dx$$

$$= \frac{1}{1-e^{a}} - \frac{\theta \propto \beta}{\left(e^{\theta}-1\right)\left(1-e^{a}\right)} \int_0^{\infty} \begin{pmatrix} x^{-\beta-1} e^{-\propto x^{-\beta}} \\ e^{\theta e^{-\propto x^{-\beta}}} e^{ae^{-bx^{-c}}} \end{pmatrix} dx$$

$$= \frac{1}{1-e^{a}} - \frac{\theta \propto \beta}{\left(e^{\theta}-1\right)\left(1-e^{a}\right)} \int_0^{\infty} \begin{pmatrix} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{\theta^{r}(-\propto r)^{s} a^{u}(-bu)^{v}}{r!s!u!v!} \\ x^{-(cv+\beta s+\beta)-1} e^{-\propto x^{-\beta}} \end{pmatrix} dx$$

$$= \frac{1}{1-e^{a}} - \frac{\theta \propto \beta}{\left(e^{\theta}-1\right)\left(1-e^{a}\right)} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \begin{pmatrix} \frac{\theta^{r}(-\propto r)^{s} a^{u}(-bu)^{v}}{r!s!u!v!} \\ \frac{1}{\beta} \frac{\Gamma\left(\frac{cv+\beta s+\beta}{\beta}\right)}{\propto^{\frac{cv+\beta s+\beta}{\beta}}} \end{pmatrix}$$

$$= \frac{1}{1-e^{a}} - \frac{1}{(e^{\theta}-1)(1-e^{a})} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \begin{pmatrix} \frac{\theta^{r+1}(-1)^{s+v} a^{u} r^{s} (bu)^{v}}{r!s!u!v!} \\ \frac{\Gamma\left(\frac{cv}{\beta}+s+1\right)}{\propto^{\frac{cv}{\beta}}} \end{pmatrix}. \quad (24)$$

4. Summary and Conclusions

In view of the great importance of the Statistical lifetime distributions in lifetime data modeling. Recently, various methods of generating new distributions have been proposed in the statistical literature. Among these methods, the compounding of some discrete and important lifetime distributions has been in the foreword of lifetime modeling. So, several families of distributions were derived by compounding some useful lifetime and truncated discrete distributions. In this paper, properties of Weibull-Poisson distribution (WPD) and inverse Weibull-Poisson distribution (IWPD) is derived. We provide forms for characteristic function, *r*th raw moment, mean, variance, Shannon entropy function, Rényi entropy function and Relative entropy function. This paper deals also with the determination of stress-strength reliability R = P[Y < X] when X (strength) and Y (stress) are two independent WPD (IWPD) distributions with different parameters.

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