

# The Discrete Binomial Model for Option Pricing

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## Abstract

This paper introduces the notion of option pricing in the context of financial markets. The discrete time, one-period binomial model is explored and generalized to the multi-period binomial model. The multi-period model is then redeveloped using the sophisticated tools of martingale theory. The paper concludes with a brief extension of the results to continuous time, giving a heuristic derivation of the Black-Scholes equation.

## 1 Introduction

The financial markets heavily utilize *securities*, which are abstract representations of financial value. Securities can be thought of as contracts that have a particular value, and can be traded. Common types of securities include stocks and bonds. A *derivative security* is a contract whose value is derived from the future behavior of another security (called the *underlying asset*), such as a stock. In this paper, we will be focusing on a particular type of derivative security known as an *option*. An option is a contract in which the holder has the right but not the obligation to carry out the terms of the contract. Examples of options include *European options*, where the holder can exercise his right to buy (a *call*) or sell (a *put*) at a specified date, and *American options*, where the holder has the right to buy or sell at any date up to the one specified in the contract. Note that options can be traded at any time before their exercise date.

To enable the purchasing or selling of an option, we would like to be able to determine its value at any point in time. In particular, we would like to know the value at the time the option is created, before the future behavior of the underlying asset is known. Determining an option's value is commonly called *option pricing*. This paper aims to answer the question of option pricing under the simplified framework of the binomial model. We will use a discrete-time setup in order to simplify the mathematics involved; however, the discrete models do capture the fundamental aspects of option pricing in more general continuous time.

We now introduce basic financial terms that will be used throughout this paper:

- A *stock* is a security representing partial ownership of a company. A unit of stock is called a *share*. Stocks are traded in the *stock market*.
- The *money market* consists of risk-free securities, such as bonds, which accrue interest over time.
- In this paper, the *interest rate*  $r > 0$  is defined such that \$1 invested in the money market at time zero will be worth  $\$(1 + r)$  at time one.

- An *arbitrage* is a trading strategy that, beginning with zero wealth, has zero probability of losing money, and has positive probability of making money.
- An investor can *short sell* a stock by borrowing it from the owner and selling it to obtain the proceeds. The investor must repurchase the stock at some point, and return the stock to the owner. If the share price falls after the investor short sells, the investor will make a profit after repurchasing the stock. Mathematically, this is equivalent to purchasing negative shares of stock.
- A *portfolio* is a collection of securities.

In short, a stock is a risky asset whereas assets from the money market are riskless. The stock and money markets form the financial world used in the models discussed below.

## 2 The Binomial Model

The binomial model is based upon a simplification of the financial instruments involved in option pricing, but its implications capture the essential features of more complicated continuous models. We first introduce the one-period binomial model and then discuss the more general multi-period model.

### 2.1 The One-Period Binomial Model

First, the principal assumptions of the one-period model, in which the start of the period is called *time zero* and the end is called *time one*, are:

- A single share of stock can be subdivided for purchasing and selling;
- In each transaction, the price the buyer pays to purchase the stock and the amount the seller receives for selling the stock are the same (i.e. there are no transaction costs or fees);
- The interest rates for borrowing and investing are the same;
- The stock can take only two possible values at time one.

The final condition provides this model with a binomial structure. In practice, these assumptions are far too simplistic, but they provide a good starting point with which to begin.

We consider a single stock with a price per share of  $S_0 > 0$  at time zero. We can imagine the price at time one to be the result of a coin toss, either heads or tails, with probabilities  $p$  and  $q = 1 - p$ , respectively. (Note that  $p$  and  $q$  are not necessarily  $\frac{1}{2}$ .) At time one, the price per share will be either  $S_1(H)$  or  $S_1(T)$ , with probabilities  $p$  and  $q$ .

Let

$$u = \frac{S_1(H)}{S_0}, \quad d = \frac{S_1(T)}{S_0}$$

Assume that both  $d$  and  $u$  are positive, and without loss of generality,  $d < u$ . The situation can then be represented with the following diagram (see p. 1 of [4]):

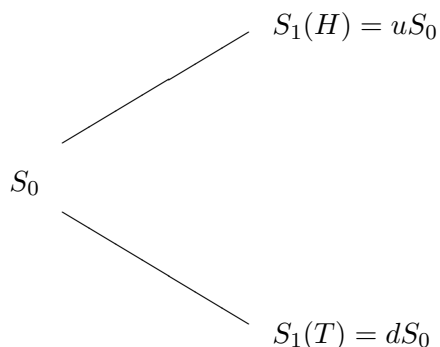


Figure 1: The one-period binomial model for  $t = 0$  and  $t = 1$ .

So, the financial tools available to use in this model consists of the stock described above as well as the money market with interest rate  $r$ .

A key assumption of our model is that it does not allow any arbitrage situations, as the possibility of a riskless profit could lead to contradictory results from the model. Furthermore, any arbitrages in the real world quickly disappear as people take advantage of them. A simple condition on  $d$  and  $u$  will ensure the no-arbitrage requirement.

### Proposition 1

The no-arbitrage assumption implies that  $0 < d < 1 + r < u$ .

### Proof

We have already assumed that  $d > 0$ . Now, assume that  $d \geq 1 + r$ . Then, starting with no wealth at time zero, borrow  $X$  from the money market and use that money to purchase stock. At time one, the debt will be  $(1 + r)X$ . However, if the stock price goes down, the value of the stock will be at least  $(1 + r)X$  since  $d \geq 1 + r$ . Hence, selling the stock will result in enough money to pay off the money market debt. The stock will go up with probability  $p > 0$ , which will lead to a profit since  $u > d$ . Therefore, there is a positive probability of generating a riskless profit which gives an arbitrage, leading to a contradiction. So  $d$  must be less than  $1 + r$ .

Similarly, assume that  $u \leq 1 + r$ . This time, short sell  $X$  of stock at time zero and invest in the money market. At time one, the proceeds from the money market will be  $(1 + r)X$ . The debt from the short selling will have a maximum value of  $uX \leq (1 + r)X$ , so it can be paid off. If the stock value decreases to  $dX$ , a profit will be made. Again there is an arbitrage and so a contradiction. Hence  $u > 1 + r$ .  $\square$

The converse is true as well. However, the proof requires some notation that has not yet been developed, so it will be presented later on in the paper.

We would like to determine the value of an option at time zero. Assume that at time one, a given option pays an amount  $V_1(H)$  if the stock price increases and  $V_1(T)$  if the stock price goes down. The key idea to no-arbitrage option pricing is to create a *replicating portfolio* through the stock and money markets (the stock in the replicating portfolio is the underlying asset of the option). By

constructing a portfolio whose wealth at time one is equal to the value of the option, regardless of heads or tails, we can infer that the value of the option at time zero is simply that of the replicating portfolio. This is a direct result of the no-arbitrage assumption:

### Proposition 2

If two portfolios give the same payoffs at all times, then they must have the same value.

Note that this result applies to all types of portfolios, but for our purposes we apply it to one portfolio consisting of an option and another consisting of single assets in the stock and money markets.

### Proof

For illustrative purposes, consider only a one-period time frame. Assume that there exist two portfolios, one containing Stock 1 and the other containing Stock 2, and both stocks are worth  $V_1$  at time one. At time zero, Stock 1 is worth  $V_0$  and Stock 2 is worth  $V'_0$ , with  $V_0 < V'_0$ . Beginning with no wealth, at time zero, short sell Stock 2, and purchase Stock 1. This leaves us with a total wealth of  $V'_0 - V_0$ . At time one, sell Stock 1, which gives exactly the amount of money needed ( $V_1$ ) to purchase Stock 2. We will then have a net wealth of  $V'_0 - V_0 > 0$ . Therefore, beginning with zero wealth, we are guaranteed to make a profit, giving an arbitrage opportunity (and hence a contradiction). Extending this argument to general portfolios and multiple time-steps gives the result above.  $\square$

The previous discussion has given us the tools to determine the time zero value of an option, and we now follow Chapter 1 of [4] to continue. Assume we have wealth  $X_0$  at time zero, and we purchase  $\Delta_0$  shares of stock. We then have wealth  $X_0 - \Delta_0 S_0$  that is invested in the money market at time zero. At time one, this portfolio will be worth

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \quad (1)$$

$$= (1+r)X_0 + \Delta_0(S_1 - (1+r)S_0). \quad (2)$$

Replication requires that  $X_1(H) = V_1(H)$  and  $X_1(T) = V_1(T)$ , and enforcing these constraints gives a portfolio that replicates the option's value at time one. Rewriting Equation (2) as

$$X_0 + \Delta_0 \left( \frac{1}{1+r} S_1(H) - S_0 \right) = \frac{1}{1+r} V_1(H) \quad (3)$$

$$X_0 + \Delta_0 \left( \frac{1}{1+r} S_1(T) - S_0 \right) = \frac{1}{1+r} V_1(T) \quad (4)$$

to incorporate the unknown result of the coin toss, we have a system of two linear equations with two unknowns,  $X_0$  and  $\Delta_0$ . One might be tempted to solve this simple system using linear algebraic techniques, but it is more informative to proceed as follows:

Multiply the first equation by a number denoted  $\tilde{p}$  and the second by  $\tilde{q} = 1 - \tilde{p}$ . Adding them gives

$$X_0 + \Delta_0 \left( \frac{1}{1+r} [\tilde{p} S_1(H) + \tilde{q} S_1(T)] - S_0 \right) = \frac{1}{1+r} [\tilde{p} V_1(H) + \tilde{q} V_1(T)]. \quad (5)$$

To eliminate the term involving  $\Delta_0$ , pick  $\tilde{p}$  so that

$$S_0 = \frac{1}{1+r} [\tilde{p}S_1(H) + \tilde{q}S_1(T)]. \quad (6)$$

This leads to the equation

$$X_0 = \frac{1}{1+r} [\tilde{p}V_1(H) + \tilde{q}V_1(T)]. \quad (7)$$

Substituting in  $S_1(H) = uS_0$  and  $S_1(T) = dS_0$ , we also see that

$$S_0 = \frac{1}{1+r} [\tilde{p}uS_0 + (1-\tilde{p})dS_0] = \frac{S_0}{1+r} [(u-d)\tilde{p} + d]. \quad (8)$$

Some rearrangement implies

$$\tilde{p} = \frac{1+r-d}{u-d}, \quad \tilde{q} = \frac{u-1-r}{u-d}. \quad (9)$$

Solving for  $\Delta_0$  gives us the *delta-hedging formula*:

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}. \quad (10)$$

Therefore, starting with wealth  $X_0$  and buying  $\Delta_0$  shares of stock at time zero implies that if at time one the coin is heads, the portfolio will be worth  $V_1(H)$ , and if the coin comes up tails, the portfolio will be worth  $V_1(T)$ . Hence, according to the above discussion, the option should be priced as

$$V_0 = \frac{1}{1+r} [\tilde{p}V_1(H) + \tilde{q}V_1(T)] \quad (11)$$

at time zero.

Similar arguments to those used in the proof of Proposition 1 indicate that the value  $V_0$  given by Equation (11) is the only non-arbitrage value. This uniqueness result can be also be obtained by recasting the pricing problem in terms of matrices. Equations (3) and (4) then become

$$\underbrace{\begin{bmatrix} 1 & \frac{1}{1+r}S_1(H) - S_0 \\ 1 & \frac{1}{1+r}S_1(T) - S_0 \end{bmatrix}}_{Ax = b} \begin{bmatrix} X_0 \\ \Delta_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{1+r}V_1(H) \\ \frac{1}{1+r}V_1(T) \end{bmatrix} \quad (12)$$

This matrix equation will have a unique solution if  $\det(A) \neq 0$ .

$$\det(A) = \left( \frac{1}{1+r}S_1(T) - S_0 \right) - \left( \frac{1}{1+r}S_1(H) - S_0 \right) \quad (13)$$

$$= \frac{1}{1+r} [S_1(T) - S_1(H)] \quad (14)$$

Our assumption that  $d < 1 + r < u$  implies that  $S_1(T) - S_1(H)$  is strictly negative, so we indeed have a unique solution to the option pricing problem.

We finish with the proof of the converse of Proposition 1:

**Proposition 1 (converse)**

If  $0 < d < 1 + r < u$ , then there is no arbitrage.

**Proof**

Assume that  $0 < d < 1 + r < u$ , and that both heads and tails have positive probability of occurring. Then, if  $X_0 = 0$ , Equation (2) implies that

$$\begin{aligned} X_1(H) &= \Delta_0 S_0(u - (1 + r)) > 0 \\ X_1(T) &= \Delta_0 S_0(d - (1 + r)) < 0 \end{aligned}$$

where  $uS_0$  and  $dS_0$  are substituted in for  $S_1(H)$  and  $S_1(T)$  respectively. Then  $X_1$  is strictly positive with positive probability (if the coin toss is a head), but it is also strictly negative with positive probability (if the coin is a tail). This result is true for all values of  $\Delta_0$ , and hence there cannot be any arbitrage opportunities.  $\square$

**Example**

We illustrate the ideas above with an example. Consider a European call option over one time period, where the holder has the right but not the obligation to purchase one share of stock at time one. The price paid for the stock, called the *strike price*  $K$ , is specified in the contract. In this example, assume that  $S_0 = 4$ ,  $S_1(T) = 2$ ,  $S_1(H) = 8$ , and that  $K = 5$ . Thus  $d = \frac{1}{2}$  and  $u = 2$ . The situation is summarized below in Figure 2. Also let  $r = \frac{1}{4}$ .

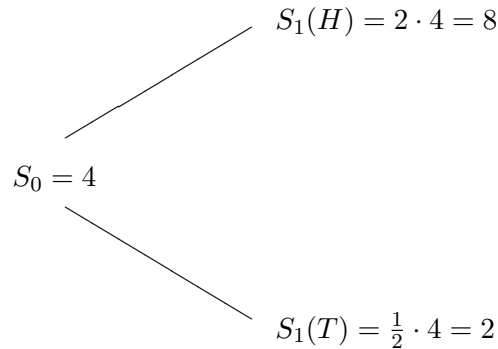


Figure 2: The one-period binomial model for the Example.

If the share price decreases to  $S_1(T)$ , the holder will choose not to exercise the option, and so its value will be worth 0 at time one. On the other hand, if the share price increases to  $S_1(H)$ , the holder will exercise the option at time one, realizing a profit of  $S_1(H) - K = 3$ . Hence at time one, the option is worth  $\max(S_1 - K, 0)$ , which depends on the result of the coin toss. Rewriting this in terms of the notation above, we have  $V_1(H) = 3$  and  $V_1(T) = 0$ . We calculate  $\tilde{p}$  and  $\tilde{q}$  using Equation (9):

$$\tilde{p} = \frac{1 + \frac{1}{4} - \frac{1}{2}}{1 - \frac{1}{4}} = \frac{1}{2}, \quad \tilde{q} = \frac{2 - 1 - \frac{1}{4}}{2 - \frac{1}{2}} = \frac{1}{2}.$$

Equation (7) gives the necessary initial wealth required to replicate the option as

$$X_0 = \frac{1}{1 + \frac{1}{4}} \left[ \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 0 \right] = \frac{6}{5}$$

and Equation (10) says that the number of shares of stock to be purchased in the replicating portfolio is

$$\Delta_0 = \frac{3 - 0}{8 - 2} = \frac{1}{2}.$$

Since  $V_0$ , the value of the option at time zero, is equal to  $X_0$ , the value of the replicating portfolio at time zero, we conclude that the value of the option at time zero is 1.2. A quick calculation verifies Equations (3) and (4).  $\square$

We now explain why Equations (3) and (4) were solved by introducing the variables  $\tilde{p}$  and  $\tilde{q}$ . Note that, due to the no-arbitrage assumption, both  $\tilde{p}$  and  $\tilde{q}$  are positive, and  $\tilde{p} + \tilde{q} = 1$ . Hence,  $\tilde{p}$  and  $\tilde{q}$  can be interpreted as the probabilities of the coin being heads or tails, but they do not necessarily equal the actual probabilities of the coin toss,  $p$  and  $q$ . We say that  $\tilde{p}$  and  $\tilde{q}$  are the *risk-neutral probabilities* of the option pricing problem.

Recall that Equation (6) says that  $S_0 = \frac{1}{1+r} [\tilde{p}S_1(H) + \tilde{q}S_1(T)]$ . When we multiply both sides by  $1+r$ , this equation indicates that, if the actual probabilities governing the stock were the risk-neutral probabilities, the average rate of growth of the stock would be equal to the rate of growth of an investment in the money market. This cannot be true in the real world, since investors would not take on the risk of investing in the stock market if they could obtain the same average (but riskless) growth in the money market; instead,

$$S_0 < \frac{1}{1+r} [pS_1(H) + qS_1(T)]. \quad (15)$$

As a result, the risk-neutral probabilities make the average rate of growth of any portfolio consisting of assets in the stock and money markets appear to equal the rate of growth of assets in the money market alone. Hence Equation 7 gives the correct value of  $X_0$ , ensuring that the replicating portfolio has the value of  $V_1$  at time one.

A key observation is that the time zero value of an option, given by Equation (11), is independent of the actual probabilities  $p$  and  $q$ , which is somewhat surprising!. However, this result is to be expected because we have constructed a pricing method that works for all possible stock paths. Only the potential prices the stock can take, controlled by  $d$  and  $u$ , influence the value of the option. We say that Equation (11) is the *risk-neutral pricing formula* for the one-period binomial model. In the continuous-time models, it can be shown that the value of an option depends on the volatility of stock prices, but not on their average rates of growth. Hence, we can conclude that this simple binomial model does capture some of the essential features of the continuous models.

## 2.2 The Multi-Period Binomial Model

We now move from the one-period model to a multi-period model. In the previous section, we assumed that, given an initial price of  $S_0$ , the price of a stock could increase by a factor of  $u$  or decrease by a factor of  $d$  at time one. Now assume that at time two, the stock price can again increase or decrease by the multiplicative factors  $u$  and  $d$ , respectively. Then at time two, the possible stock prices are:

$$\begin{aligned} S_2(HH) &= uS_1(H) = u^2S_0, & S_2(HT) &= dS_1(H) = duS_0 \\ S_2(TH) &= uS_1(T) = udS_0, & S_2(TT) &= dS_1(T) = d^2S_0 \end{aligned}$$

Continuing this pattern for multiple time steps gives a binomial tree of stock prices. We still assume the first three conditions given at the beginning of Section 2. First, by generalizing Equation (2) to multiple time steps, we can deduce the following equation, called the *wealth equation*:

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n) \quad (16)$$

The following theorem states the multi-period results analogous to Equations (10) and (11):

**Theorem 1: Replication in the multi-period binomial**

Consider an  $N$ -period binomial option pricing model, with  $0 < d < 1+r < u$ , and with

$$\tilde{p} = \frac{1+r-d}{u-d}, \quad \tilde{q} = \frac{u-1-r}{u-d}.$$

Let  $V_N$  be an option, depending on the first  $N$  coin tosses  $\omega_1\omega_2\cdots\omega_N$ , which is to be exercised at time  $N$ . Define recursively backward in time the sequence of random variables  $V_{N-1}, V_{N-2}, \dots, V_0$ , by

$$V_n(\omega_1\omega_2\cdots\omega_N) = \frac{1}{1+r} [\tilde{p}V_{n+1}(\omega_1\omega_2\cdots\omega_nH) + \tilde{q}V_{n+1}(\omega_1\omega_2\cdots\omega_nT)]. \quad (17)$$

Then each  $V_n$  depends on the first  $n$  tosses  $\omega_1\omega_2\cdots\omega_n$ , where  $n$  ranges between  $N-1$  and 0. Now, define

$$\Delta_n(\omega_1\omega_2\cdots\omega_n) = \frac{V_{n+1}(\omega_1\omega_2\cdots\omega_nH) - V_{n+1}(\omega_1\omega_2\cdots\omega_nT)}{S_{n+1}(\omega_1\omega_2\cdots\omega_nH) - S_{n+1}(\omega_1\omega_2\cdots\omega_nT)} \quad (18)$$

where again  $n$  ranges between 0 and  $N-1$ . If we set  $X_0 = V_0$  and define recursively forward in time the portfolio values  $X_1, X_2, \dots, X_N$  by the wealth equation (16), then we will have

$$X_N(\omega_1\omega_2\cdots\omega_N) = V_N(\omega_1\omega_2\cdots\omega_N) \quad (19)$$

for all coin toss sequences  $\omega_1\cdots\omega_N$ .

For  $n = 1, 2, \dots, N$ , the random variable  $V_n(\omega_1\omega_2\cdots\omega_n)$  is defined to be the *value of the option at time  $n$*  if the outcomes of the first  $n$  tosses are  $\omega_1\omega_2\cdots\omega_n$ . The *value of the option at time zero* is defined to be  $V_0$ .



The number of shares of stock that should be held in the portfolio at time  $n$  is given by  $\Delta_n(\omega_1\omega_2\cdots\omega_N)$ . We say that the portfolio process  $\Delta_0, \Delta_1, \dots, \Delta_{N-1}$  is *adapted* since  $\Delta_n$  depends on the first  $n$  coin tosses.

**Proof**

See pp. 13-14 of [4]. □

The recursive relationship found Equation (17) can be motivated as follows: first think of the  $N$ -period model as a one-period model from time  $N - 1$  to time  $N$ . Then apply the formula for the time zero option value found in the one-period model, Equation (11), to this reduced  $N$ -period model, giving

$$V_{N-1}(\omega_1\omega_2\cdots\omega_{N-1}) = \frac{1}{1+r} [\tilde{p}V_N(\omega_1\omega_2\cdots\omega_{N-1}H) + \tilde{q}V_N(\omega_1\omega_2\cdots\omega_{N-1}T)] \quad (20)$$

Now that we know the value of the option at time  $N$ , reduce the problem to an  $(N - 1)$ -period model. Consider just time  $N - 1$ , and again apply Equation (11). Continue the recursion until the time zero option value is known. This algorithm indicates that the multi-period binomial model is simply a recursive version of the one-period model.

We say that the multi-period binomial model is *self-financing* since the portfolio at time  $n + 1$  can be financed entirely from the wealth in the portfolio at time  $n$ ; no money needs to be added to the portfolio from outside resources to carry out the replication of the option.

As one would expect given the results of the one-period model, the no-arbitrage value of an option under the multi-period binomial model does not depend on the actual probabilities  $p$  and  $q$ . Also, as in the one-period framework, every option can be replicated by a portfolio consisting of the underlying stock and a money market asset. We say that the multi-period binomial model is a *complete market*. The next section illustrates an *incomplete market*.

### 3 The Trinomial Model

Consider a one-period model where the stock price can also take an intermediate price between  $dS_0$  and  $uS_0$  at time one; call this value  $mS_0$ . Hence,  $d < m < u$ , and it is not necessary that  $m = 1$ . We introduce a new random variable  $M$  to describe the situation that the stock takes price  $mS_0$  at time one (call this possibility “edge”). The situation is portrayed in Figure 3.

The same analysis as in Section 2.1 leads to the matrix equation

$$\begin{bmatrix} 1 & \frac{1}{1+r}S_1(H) - S_0 \\ 1 & \frac{1}{1+r}S_1(M) - S_0 \\ 1 & \frac{1}{1+r}S_1(T) - S_0 \end{bmatrix} \begin{bmatrix} X_0 \\ \Delta_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{1+r}V_1(H) \\ \frac{1}{1+r}V_1(M) \\ \frac{1}{1+r}V_1(T) \end{bmatrix} \quad (21)$$

This is a system of three linear equations and two unknowns, so in general it will not have a solution. Hence the technique of determining the value of an option via a replicating portfolio does not work in the trinomial model. We can however find upper and lower bounds on the option value. For

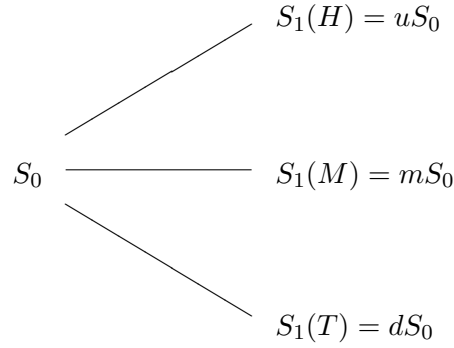


Figure 3: The one-period trinomial model for  $t = 0$  and  $t = 1$ .

example, consider a European call option with strike price  $K$  such at  $mS_0 < K < uS_0$ . At time one, the option will be worth  $uS_0 - K$  if heads occurs, and 0 if the coin is edge or tails. It can be shown (see pp. 17-18 of [1]) that an upper bound for the time zero value of the option is

$$V_0^+ = \frac{(1 + r - d)}{u - d} \cdot \frac{(uS_0 - K)}{1 + r}$$

If  $m \geq 1 + r$ , the lower bound on the value is  $V_0^- = 0$ , and if  $m < 1 + r$ , it is

$$V_0^- = \frac{(1 + r - m)}{u - m} \cdot \frac{(uS_0 - K)}{1 + r}$$

There is no unique time zero no-arbitrage value, but these bounds provide some information about the value of the option.

## 4 Probability Theory Applied to Option Pricing

### 4.1 Basic Terms and Definitions

The above sections have shown how to price an option under the assumptions of the binomial model. However, these techniques are somewhat limited in scope, and they are not particularly elegant! We now provide a brief introduction to some basic probability theory, which will be used to reformulate some of the previously mentioned results into a more cohesive theory. First, some definitions:

#### Definition 1

A *finite probability space* consists of a *sample space*  $\Omega$  and a *probability measure*  $\mathbb{P}$ . The sample space  $\Omega$  is a nonempty finite set and the probability measure  $\mathbb{P}$  is a function that assigns to each element  $\omega$  of  $\Omega$  a number in  $[0, 1]$  so that

$$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1.$$

An *event* is a subset of  $\Omega$ , and the probability of an event  $A$  is defined to be

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega). \quad \square$$

**Definition 2**

$(\Omega, \mathbb{P})$  is a finite probability space. A *random variable* is a real-valued function defined on  $\Omega$ .  $\square$

**Definition 3**

Let  $X$  be a random variable defined on a finite probability space  $(\Omega, \mathbb{P})$ . The *expectation* of  $X$  is defined to be

$$\mathbb{E}X = \sum_{\omega \in \Omega} X(\omega) \mathbb{P}(\omega). \quad \square$$

**Definition 4**

Let  $n$  satisfy  $1 \leq n \leq N$ , and let  $\omega_1 \cdots \omega_n$  be given and, for the moment, fixed. There are  $2^{N-n}$  possible continuations  $\omega_{n+1} \cdots \omega_N$  of the sequence of fixed  $\omega_1 \cdots \omega_n$ . Denote by  $\#H(\omega_{n+1} \cdots \omega_N)$  the number of heads in the continuation  $\omega_{n+1} \cdots \omega_N$  and by  $\#T(\omega_{n+1} \cdots \omega_N)$  the number of tails. We define

$$\mathbb{E}_n[X](\omega_1 \cdots \omega_n) = \sum_{\omega_{n+1} \cdots \omega_N} p^{\#H(\omega_{n+1} \cdots \omega_N)} q^{\#T(\omega_{n+1} \cdots \omega_N)} X(\omega_1 \cdots \omega_n \omega_{n+1} \cdots \omega_N) \quad (22)$$

to be the *conditional expectation* of  $X$  based on the information at time  $n$ .

Furthermore, the conditional expectation of  $X$  given no information is defined by

$$\mathbb{E}_0[X] = \mathbb{E}X \quad (23)$$

and the conditional expectation of  $X$  given the information of all  $N$  coin tosses is defined by

$$\mathbb{E}_N[X] = X. \quad (24)$$

$\square$

From now on, the symbol  $\mathbb{P}$  denotes the probability measure associated with the actual probabilities  $p$  and  $q$ . The probability measure associated with the risk-neutral probabilities  $\tilde{p}$  and  $\tilde{q}$  will be represented by  $\tilde{\mathbb{P}}$ . Then the expectation using the risk-neutral probability measure  $\tilde{\mathbb{P}}$  is

$$\tilde{\mathbb{E}}X = \sum_{\omega \in \Omega} X(\omega) \tilde{\mathbb{P}}(\omega)$$

and  $\tilde{\mathbb{E}}_n[X]$  is the conditional expectation of  $X$  based on the information at time  $n$  under the risk-neutral probabilities.

The properties of conditional expectations will be very useful in proving some later results, so we state them below.

**Theorem 2**

Let  $N$  be a positive integer, and let  $X$  and  $Y$  be random variables depending on the first  $N$  coin tosses. Let  $0 \leq n \leq N$  be given. Then the following properties hold:

- **Linearity of conditional expectations:** For all constants  $c_1$  and  $c_2$ , we have

$$\mathbb{E}_n[c_1X + c_2Y] = c_1\mathbb{E}_n[X] + c_2\mathbb{E}_n[Y].$$

- **Taking out what is known:** If  $X$  only depends on the first  $n$  coin tosses, then

$$\mathbb{E}_n[XY] = X \cdot \mathbb{E}_n[Y].$$

- **Iterated conditioning:** If  $0 \leq n \leq m \leq N$ , then

$$\mathbb{E}_n[\mathbb{E}_m[X]] = \mathbb{E}_n[X].$$

In particular,  $\mathbb{E}[\mathbb{E}_m[X]] = \mathbb{E}X$ .

- **Independence:** If  $X$  depends only on tosses  $n + 1$  through  $N$ , then

$$\mathbb{E}_n[X] = \mathbb{E}X.$$

### Proof

See p. 177 of [4]. □

## 4.2 Martingale Theory Applied to Option Pricing

First, for notation's sake, we assume that for a sequence of random variables  $A_0, A_1, \dots$ , the expression  $A_n(\omega_1 \cdots \omega_n)$  can be denoted by  $A_n$ . Also, we can shorten  $A_{n+1}(\omega_1 \cdots \omega_n H)$  to  $A_{n+1}(H)$ , and similarly for  $T$ . Now, recall from Equation (9) that

$$\tilde{p} = \frac{1 + r - d}{u - d}, \quad \tilde{q} = \frac{u - 1 - r}{u - d}.$$

Then it follows immediately that

$$\frac{\tilde{p}u + \tilde{q}d}{1 + r} = 1. \tag{25}$$

Multiplying both sides by  $S_n$  and using the fact that  $S_{n+1}(H) = uS_n$  and  $S_{n+1}(T) = dS_n$ , then

$$S_n(\omega_1 \cdots \omega_n) = \frac{1}{1 + r} [\tilde{p}S_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}S_{n+1}(\omega_1 \cdots \omega_n T)]. \tag{26}$$

We can rewrite Equation (26) using the notation of Definition 4, giving

$$S_n = \frac{1}{1 + r} \tilde{\mathbb{E}}_n[S_{n+1}]. \tag{27}$$

Dividing Equation (27) by  $(1 + r)^n$  and applying Theorem 2 gives

$$\frac{S_n}{(1 + r)^n} = \tilde{\mathbb{E}}_n \left[ \frac{S_{n+1}}{(1 + r)^{n+1}} \right]. \tag{28}$$

We call the quantity  $\frac{S_n}{(1+r)^n}$  the *discounted* stock price, as it is multiplied by  $\frac{1}{(1+r)^n} < 1$ . Since \$1 at time zero will be worth  $\$(1+r)^n$  at time  $n$ , we can interpret the discounted price as the time zero worth of the price of the stock at time  $n$ . (This is an illustration of the depreciation of money over time). Equation (28) indicates that the risk-neutral probabilities are chosen so that the best estimate of the discounted stock price at time  $n+1$ , under the risk-neutral probabilities and based on the information at time  $n$ , is the discounted stock price at time  $n$ . We say that this process is a *martingale*. The formal definition of a martingale is as follows:

**Definition 5**

Consider the binomial option pricing model. Let  $M_0, M_1, \dots, M_N$  be a sequence of random variables, with each  $M_n$  depending only on the first  $n$  coin tosses (and  $M_0$  constant). Such a sequence of random variables is called an *adapted stochastic process*.

- If  $M_n = \mathbb{E}_n[M_{n+1}]$ ,  $n = 0, 1, \dots, N-1$ , this process is a *martingale*.
- If  $M_n \leq \mathbb{E}_n[M_{n+1}]$ ,  $n = 0, 1, \dots, N-1$ , this process is a *submartingale*.
- If  $M_n \geq \mathbb{E}_n[M_{n+1}]$ ,  $n = 0, 1, \dots, N-1$ , this process is a *supermartingale*. □

The following is a useful property of martingales:

**Proposition 3**

The expectation of a martingale is constant over time: Assume that  $M_0, M_1, \dots, M_N$  is a martingale. Then

$$M_0 = \mathbb{E}M_n, \quad n = 0, 1, \dots, N.$$

**Proof**

If  $M_n = \mathbb{E}_n[M_{n+1}]$ ,  $n = 0, 1, \dots, N-1$ , then

$$\begin{aligned} \mathbb{E}M_n &= \mathbb{E}[\mathbb{E}_n[M_{n+1}]] \\ &= \mathbb{E}M_{n+1}. \end{aligned} \quad (\text{by properties of conditional expectation})$$

Hence,  $\mathbb{E}M_0 = \mathbb{E}M_1 = \dots = \mathbb{E}M_{N-1} = \mathbb{E}M_N$ . Since  $M_0$  is constant,  $\mathbb{E}M_0 = M_0$ , and so we get the result. □

This next theorem formalizes the discovery made in Equation (28).

**Theorem 3**

Consider the general binomial model with  $0 < d < 1+r < u$ . Let the risk-neutral probabilities be given by

$$\tilde{p} = \frac{1+r-d}{u-d}, \quad \tilde{q} = \frac{u-1-r}{u-d}.$$

Then, under the risk-neutral measure, the discounted stock price is a martingale.

**Proof**

$$\begin{aligned}
\tilde{\mathbb{E}}_n \left[ \frac{S_{n+1}}{(1+r)^{n+1}} \right] &= \tilde{\mathbb{E}}_n \left[ \frac{S_n}{(1+r)^{n+1}} \cdot \frac{S_{n+1}}{S_n} \right] \\
&= \frac{S_n}{(1+r)^n} \tilde{\mathbb{E}}_n \left[ \frac{1}{1+r} \cdot \frac{S_{n+1}}{S_n} \right] && \text{(taking out what is known)} \\
&= \frac{S_n}{(1+r)^n} \cdot \frac{1}{1+r} \tilde{\mathbb{E}} \left[ \frac{S_{n+1}}{S_n} \right] && \text{(independence)} \\
&= \frac{S_n}{(1+r)^n} \frac{\tilde{p}u + \tilde{q}d}{1+r} \\
&= \frac{S_n}{(1+r)^n}.
\end{aligned}$$

□

Previously, we discussed how recasting the option pricing problem in terms of the risk-neutral probabilities implied that the average rate of growth of any portfolio consisting of assets in the stock and money markets equals the rate of growth of a money market account. Hence, the average rate of growth of an investor's wealth will be equal to the interest rate, and so the wealth process is also a martingale. This result is formalized in the theorem below.

**Theorem 4**

Consider the binomial model with  $N$  periods. Let  $\Delta_0, \Delta_1, \dots, \Delta_{N-1}$  be an adapted portfolio process from Theorem 1, let  $X_0$  be a real number, and let the wealth process  $X_1, \dots, X_N$  be generated recursively by

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n), n = 0, 1, \dots, N-1.$$

Then the discounted wealth process  $\frac{X_n}{(1+r)^n}, n = 0, 1, \dots, N$ , is a martingale under the risk-neutral measure.

**Proof**

The proof is very similar to the proof of Theorem 3. □

**Corollary**

Under the conditions of Theorem 4,

$$\tilde{\mathbb{E}} \left[ \frac{X_n}{(1+r)^n} \right] = X_0, \quad n = 0, 1, \dots, N. \quad (29)$$

**Proof**

This is an application of Proposition 3. □

Theorem 4 and its corollary can be used to provide a more general proof of the converse of Proposition 1.

**Proposition 4**

There can be no arbitrage in the binomial model.

### Proof

Proceed by contradiction, and assume that there is an arbitrage. Then, beginning with  $X_0 = 0$ , there is a portfolio process  $\Delta_1, \dots, \Delta_N$  such that its corresponding wealth process  $X_1, \dots, X_N$  satisfies  $X_N \geq 0$  for all coin toss sequences  $\omega_1, \dots, \omega_N$ , and furthermore  $X_N > 0$  for at least one coin toss sequence. But then  $\tilde{\mathbb{E}}X_0 = 0$  and  $\tilde{\mathbb{E}}\frac{X_N}{(1+r)^N} > 0$ , contradicting the corollary.  $\square$

Theorem 4 also leads to a sophisticated, more general analogue of Equation (17), which gave the value of an option at time  $n$ ,  $0 < n < N - 1$ .

### Theorem: Risk-neutral pricing formula

Consider an  $N$ -period binomial asset-pricing model with  $0 < d < 1 + r < u$  and with risk-neutral probability measure  $\tilde{\mathbb{P}}$ . Let  $V_N$  be a random variable (an option being exercised at time  $N$ ) depending on the coin tosses. Then, for  $n$  between 0 and  $N$ , the value of the option at time  $n$  is given by the *risk-neutral pricing formula*

$$V_n = \tilde{\mathbb{E}}_n \left[ \frac{V_N}{(1+r)^{N-n}} \right]. \quad (30)$$

Furthermore, the discounted value of the option is a martingale under  $\tilde{\mathbb{P}}$ , i.e.

$$\frac{V_n}{(1+r)^n} = \tilde{\mathbb{E}}_n \left[ \frac{V_{n+1}}{(1+r)^{n+1}} \right], n = 0, 1, \dots, N-1. \quad (31)$$

### Proof

This proof proceeds in several parts.

- First, we show that if  $M_0, M_1, \dots, M_N$  and  $M'_0, M'_1, \dots, M'_N$  are martingales under the risk-neutral probability measure  $\tilde{\mathbb{P}}$ , and  $M_N = M'_N$  for every possible sequence of coin tosses, then  $M_n = M'_n$  for  $0 \leq n \leq N$ .

Assume that  $M_N = M'_N$ . By the martingale property,  $M_{N-1} = \tilde{\mathbb{E}}_{N-1}[M_N]$ . As  $M_N = M'_N$ , then  $\tilde{\mathbb{E}}_{N-1}[M_N] = \tilde{\mathbb{E}}_{N-1}[M'_N] = M'_{N-1}$  since  $M'_0, \dots, M'_N$  is a martingale, and hence  $M_{N-1} = M'_{N-1}$ . Proceeding by induction gives that  $M_n = M'_n$  for all  $0 \leq n \leq N$ .

- Next, let  $V_N$  be the payoff at time  $N$  of an option, and define  $V_{N-1}, V_{N-2}, \dots, V_0$  by Equation (17). Then,

$$\begin{aligned} \tilde{\mathbb{E}}_n \left[ \frac{V_{n+1}}{(1+r)^{n+1}} \right] &= \frac{1}{(1+r)^{n+1}} \tilde{\mathbb{E}}_n[V_{n+1}] \\ &= \frac{1}{(1+r)^{n+1}} [\tilde{p}V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \cdots \omega_n T)] \\ &= \frac{1}{(1+r)^{n+1}} \cdot (1+r)V_n \quad (\text{by Equation (17)}) \\ &= \frac{V_n}{(1+r)^n}. \end{aligned}$$

Hence  $V_0, \frac{V_1}{1+r}, \dots, \frac{V_N}{(1+r)^N}$  is a martingale under  $\tilde{\mathbb{P}}$ .

- Now, define

$$V'_n = \tilde{\mathbb{E}}_n \left[ \frac{V_N}{(1+r)^{N-n}} \right], \quad n = 0, \dots, N-1.$$

Then

$$\begin{aligned} \tilde{\mathbb{E}}_n \left[ \frac{V'_{n+1}}{(1+r)^{n+1}} \right] &= \frac{1}{(1+r)^{n+1}} \tilde{\mathbb{E}}_n[V'_{n+1}] \\ &= \frac{1}{(1+r)^{n+1}} \tilde{\mathbb{E}}_n \left[ \tilde{\mathbb{E}}_{n+1} \left[ \frac{V_N}{(1+r)^{N-(n+1)}} \right] \right] \\ &= \frac{1}{(1+r)^{n+1}} \tilde{\mathbb{E}}_n \left[ \frac{V_N}{(1+r)^{N-(n+1)}} \right] \\ &= \frac{1}{(1+r)^{n+1}} \tilde{\mathbb{E}}_n \left[ \frac{V_N}{(1+r)^{N-n}} \right] \\ &= \frac{V'_n}{(1+r)^n}. \end{aligned}$$

So,  $V'_0, \frac{V'_1}{1+r}, \dots, \frac{V'_N}{(1+r)^N}$  is also a martingale under  $\tilde{\mathbb{P}}$ .

- Now note that that  $V'_N = \tilde{\mathbb{E}}_N[V_N] = V_N$  by Equation (24). Hence from the first part of the proof, we conclude that  $V_n = V'_n$  for all  $n$ .

Therefore, the recursive algorithm for calculating  $V_n$  given by Equation (17) gives the same option value at time  $n$  as the risk-neutral pricing formula in Equation (30), and the discounted option value is a martingale under  $\tilde{\mathbb{P}}$ .  $\square$

We have now answered the question of how to price an option under the assumptions of the binomial model. To conclude this section, we mention a few results tying together the ideas of arbitrage, replicating portfolios that give unique option values, and martingales.

### 4.3 Further Results

#### Definition 6

Two probability measures  $\mathbb{P}$  and  $\mathbb{Q}$  for a process are called *equivalent* if they agree on which sets of paths have zero probability and which sets of paths have positive probability.  $\square$

#### Definition 7

Given a process and a probability measure  $\mathbb{P}$ , a probability measure  $\mathbb{Q}$  is an *equivalent martingale measure* if  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$  and the process is a martingale under the measure  $\mathbb{Q}$ .  $\square$

Hence we can say that the risk-neutral probability measure  $\tilde{\mathbb{P}}$  is an equivalent probability measure to the measure of the actual probabilities,  $\mathbb{P}$ , since the risk-neutral probabilities simply rescale the probability on the paths but do not alter which paths have zero or positive probability.

This definition leads to an important result about arbitrage, which we present without proof:



### The First Fundamental Theorem of Option Pricing

If we can find a risk-neutral equivalent probability measure in a model, then there is no arbitrage in the model.  $\square$

Finally, here is a theorem which gives credence to the idea of determining the value of an option via a replicating portfolio. The proof comes from p. 4.18 in [6].

### Martingale Representation Theorem

If  $A_0, A_1, \dots, A_N$  is a martingale with respect to a probability measure  $\mathbb{P}$ , then for any other martingale  $B_0, B_1, \dots, B_N$  with respect to  $\mathbb{P}$ , there is an adapted portfolio process  $\Delta_0, \Delta_1, \dots, \Delta_{N-1}$  such that

$$B_n = B_0 + \sum_{k=1}^n \Delta_{k-1}(A_k - A_{k-1}) \quad (32)$$

for  $n = 0, \dots, N$ .

### Proof

Since  $\{A_n\}$  is a martingale, then

$$A_n = \mathbb{E}_n[A_{n+1}] = pA_{n+1}(H) + qA_{n+1}(T)$$

and hence

$$p(A_{n+1}(H) - A_n) = -q(A_{n+1}(T) - A_n). \quad (33)$$

Also, as  $\{B_n\}$  is a martingale,

$$p(B_{n+1}(H) - B_n) = -q(B_{n+1}(T) - B_n). \quad (34)$$

Dividing Equation (34) by Equation (33) gives

$$\frac{B_{n+1}(H) - B_n}{A_{n+1}(H) - A_n} = \frac{B_{n+1}(T) - B_n}{A_{n+1}(T) - A_n} := \Delta_n. \quad (35)$$

Hence we have defined  $\Delta_n$  to be either of the two fractions above, depending on the result of the coin toss, and so  $\Delta_n$  is independent of the coin toss. Therefore, we conclude that

$$\Delta_n = \frac{B_{n+1} - B_n}{A_{n+1} - A_n}. \quad (36)$$

Multiplying both sides by  $A_{n+1} - A_n$  implies that  $B_{n+1} - B_n = \Delta_n(A_{n+1} - A_n)$ , and hence by recursion,

$$B_n = B_0 + \sum_{k=1}^n \Delta_{k-1}(A_k - A_{k-1})$$

as required.  $\square$

This result indicates that any martingale under a particular probability measure can be replicated at each time step by another martingale under the same probability measure, using the adapted

portfolio process  $\Delta_0, \dots, \Delta_{N-1}$ . This is a general result not related to option pricing theory, so let us see what happens when we apply it to our problem of option pricing.

We have previously shown that the discounted stock price  $S_0, \frac{S_1}{(1+r)}, \dots, \frac{S_N}{(1+r)^N}$  and the discounted option value  $V_0, \frac{V_1}{(1+r)}, \dots, \frac{V_N}{(1+r)^N}$  are martingales under  $\tilde{\mathbb{P}}$ . Hence, applying the martingale representation theorem,

$$\frac{V_n}{(1+r)^n} = V_0 + \sum_{k=1}^n \Delta_{k-1} \left[ \frac{S_k}{(1+r)^k} - \frac{S_{k-1}}{(1+r)^{k-1}} \right]$$

and so

$$\frac{V_{n+1}}{(1+r)^{n+1}} - \frac{V_n}{(1+r)^n} = \Delta_n \left[ \frac{S_{n+1}}{(1+r)^{n+1}} - \frac{S_n}{(1+r)^n} \right].$$

Multiplying by  $(1+r)^{n+1}$  implies that

$$V_{n+1} - (1+r)V_n = \Delta_n [S_{n+1} - (1+r)S_n]$$

and hence

$$\Delta_n = \frac{V_{n+1} - (1+r)V_n}{S_{n+1} - (1+r)S_n}. \quad (37)$$

Now, note that  $(1+r)V_n = \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)$  by the martingale property of the discounted option value, and similarly  $(1+r)S_n = \tilde{p}S_{n+1}(H) + \tilde{q}S_{n+1}(T)$ . Then, Equation (37) becomes

$$\Delta_n = \frac{V_{n+1} - \tilde{p}V_{n+1}(H) - \tilde{q}V_{n+1}(T)}{S_{n+1} - \tilde{p}S_{n+1}(H) - \tilde{q}S_{n+1}(T)} = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} \quad (38)$$

regardless of whether a head or tail occurs at the  $n+1$  coin toss. But this equation is the same equation for the portfolio process  $\Delta_0, \dots, \Delta_{N-1}$ , Equation (18), derived in Section 2.2. This is another confirmation that the martingale view for option pricing gives the same results derived in the first half of the paper, and in a simpler and more sophisticated fashion.

As a point of interest, recasting the trinomial model in terms of martingales gives two equivalent martingale measures. Taking any convex combination of these measures will give another equivalent martingale measure, and so there are an infinite number of equivalent martingale measures and hence no unique option value. This analysis also leads to the same bounds for the option value, as expected.

## 5 Overture to Continuous Models

The results and conclusions about option pricing have been, up till now, based on discrete-time models with a period of length one. It is illuminating to investigate the behavior of the binomial model as the time step decreases from one to zero, and the non-rigorous discussion that follows (see pp. 41-43 in [2]) will culminate in a derivation of the famous Black-Scholes equation for the value of a European call option.

Consider a multi-period binomial model with the length of time between periods denoted by  $\delta t$ , which is a more general period spacing. The interest rate is adapted to be more appropriate for continuous time, and hence \$1 at time  $n\delta t$  will be worth  $\$e^{r\delta t}$  at time  $(n+1)\delta t$ . Also, the stock price at time  $n\delta t$  is now assumed to satisfy

$$S_{(n+1)\delta t} = \begin{cases} S_{(n+1)\delta t}(H) = S_{n\delta t} \exp(\mu\delta t + \sigma\sqrt{\delta t}) \\ S_{(n+1)\delta t}(T) = S_{n\delta t} \exp(\mu\delta t - \sigma\sqrt{\delta t}) \end{cases}$$

where  $\sigma$  is the noisiness and  $\mu$  is the stock growth rate. The actual probabilities  $p$  and  $q$  are set equal to  $\frac{1}{2}$ . Now fix a time  $t$ , which gives  $n = t/\delta t$  as the number of periods to reach time  $t$ . It can be shown that the stock price at time  $t$  is

$$S_t = S_0 \exp \left( \mu t + \sigma \sqrt{t} \left( \frac{2X_n - n}{\sqrt{n}} \right) \right).$$

where  $X_n$  is the total number of coin tosses that were heads.  $X_n$  has a binomial distribution with mean  $np = n/2$  and variance  $np(1-p) = n/4$ . Hence the random variable  $(2X_n - n)/\sqrt{n}$  has a mean of zero and a variance of one. An application of the central limit theorem indicates that as  $n \rightarrow \infty$ , the distribution of the random variable  $(2X_n - n)/\sqrt{n}$  tends to a normal distribution with zero mean and variance one. Hence, as  $\delta t$  decreases in size and  $n$  increases, the distribution of  $S_t$  tends to a log-normal distribution. This result has been derived in the context of the actual probability measure  $\mathbb{P}$ .

Now consider the risk-neutral probability measure  $\tilde{\mathbb{P}}$ . It can be shown that

$$\tilde{p} \approx \frac{1}{2} \left( 1 - \sqrt{\delta t} \left( \frac{\mu + \frac{1}{2}\sigma^2 - r}{\sigma} \right) \right).$$

$X_n$  remains binomially distributed, but its mean is  $nq$  and its variance is  $nq(1-q)$ . Hence  $(2X_n - n)/\sqrt{n}$  has mean  $-\sqrt{t}(\mu + \frac{1}{2}\sigma^2 - r)/\sigma$  and a variance that approaches one asymptotically. Again, the central limit theorem says that this distribution tends towards a normal distribution with the same mean and variance one. Therefore, the marginal distribution of  $S_t$  under  $\tilde{\mathbb{P}}$  is log-normal, i.e.

$$S_t = S_0 \exp(\sigma\sqrt{t}Z + (r - \frac{1}{2}\sigma^2)t)$$

where  $Z$  is a standard normal random variable under  $\tilde{\mathbb{P}}$ .

We apply these results to a European call option to be exercised at time  $T$  with strike price  $K$ . The value of the option at time  $T$  is  $\max(S_T - K, 0)$ , and its value at time zero is

$$V_0 = \tilde{\mathbb{E}} \left[ \max(S_0 \exp(\sigma\sqrt{T}Z - \frac{1}{2}\sigma^2 T) - K \exp(-rT), 0) \right].$$

This can be evaluated to give

$$V_0 = S_0 \Phi \left( \frac{\log \frac{S_0}{K} + (r + \frac{1}{2}\sigma^2 T)}{\sigma\sqrt{T}} \right) - K e^{-rT} \Phi \left( \frac{\log \frac{S_0}{K} + (r + \frac{1}{2}\sigma^2 T)}{\sigma\sqrt{T}} \right) \quad (39)$$

where  $\Phi$  is the normal distribution function, i.e.  $\Phi(x)$  is the probability that a standard normal random variable is less than or equal to  $x$ . Equation (39) is the Black-Scholes equation for the value of a European call option at time zero. Note that the option value does not depend on  $\mu$ , the stock growth rate, which is analogous to the fact that the option value in the one-period model did not depend on the actual probabilities governing the stock price movement.

## 6 Conclusions

We have seen that the problem of how to price derivative securities, and in particular, options, can be approached by using both algebraic and probabilistic techniques and working in discrete time. The assumption of no-arbitrage plays a central role in determining the option value at all times, as well as the idea of a replicating portfolio. Surprisingly, the value of an option at any time does *not* depend on the probability that the stock price increases or decreases, but only on the values the stock price can take. Our brief foray into continuous models reveals the analogous result: the time zero option value does not depend on the stock growth rate.

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