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A 3D Brans-Dicke Theory Model

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Abstract The evidence of an accelerated universe and the gap of 70 percent in the total energy, collected by WMAP, are problems that cannot be solved by standard models based on General relativity. Therefore the search for an alternative theory that solves the open questions in General Relativity became an important research field in the past few years. A particular alternative is the Brans-Dicke theory, which has been broadly studied as concerned to k-essence type fields in 4D. However, this theory is almost unexplored in the context of the dimensional reduction in 3D. In the present work, we study the Brans-Dicke Theory in a dimensional reduction context. In order to do this, we consider the Brans-Dicke theory in the vacuum, and in the presence of matter fields.

Keywords: *cosmology, dimensional reduction, general relativity, lower dimensional gravitation*

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1. Introduction

The evidence of an accelerated expansion of the universe, observed by WMAP [3], opened the discussion of the general relativity eventual limits. There are many options for alternative theories of gravity, and among them we can cite scalar-tensor theories like supergravity, Kaluza-Klein theories, dual string theories, M-Theory, etc [1,2,4,5,14]. One particular kind of scalar-tensor theory to describe an accelerate expansion of the universe, called Brans-Dicke theory [6,7], was proposed in the early sixties . This theory uses the principle of Mach and the hypothesis of Dirac [8], considering an eventual variation in time of the Newton's gravitational constant, thus ensuring the universality of free fall (equivalence principle) [20, 22, 23].

Most of the works which have been published in this theory up to now consider four flat dimensions, and some of them have tried to associate the scalar field of the Brans-Dicke theory as quintessence field [9], $\Lambda C\ DM$ models [19] and as a type of K-essence field [10,18]. Others have tried to find a solution for the observed accelerated expansion using a dimensional reduction of the 5*D* Brans-Dicke theory without matter [11,12].

Concerning three dimensions, a broad study has been done in gravitational theories since the publication of [13], motivated by the fact that 3*D* theories avoid some complications found in higher dimensions [15,16]. However, there are not as many results about 3D scalartensor theories, and it would be interesting to find some results in this subject, more specifically in 3*D* Brans-Dicke theory. For instance, we can see some problems like the association of the scalar field of the Brans-Dicke theory to K-essence fields which models the dark energy, as done in [10]. The goal in this paper is to construct and then analyze the behavior of a 3*D* Brans-Dicke model. For this we consider a Brans-Dicke scalar field which depends only on the cosmic time.

2. The 3D Brans-Dicke Theory

The action of Brans-Dicke theory in 3*D* will be described by [5]

$$S=\frac{1}{16}\int d^3x\sqrt{\left|g^{(3)}\right|}\left[\phi R^{(3)}-\frac{\omega}{\phi}g^{(3)}_{ab}\nabla^a\phi\nabla^b\phi\right]+\int d^3x\sqrt{\left|g^{(3)}\right|}\mathcal{L}^{(3)}_M, \tag{1}$$

where R is the scalar curvature associated to the 3*D* metric $g^{(3)}_{ab}$, ϕ is the scalar FIeld of the Brans-Dicke theory, ω is a theory dimensionless parameter, and $\mathcal{L}_M$ represents the lagrangian of the matter. The matter FIelds do not depend on the scalar field ϕ (this is necessary in order to preserve the weak equivalence principle [5]). Note that, in 3*D*, the field ϕ has dimensions of inverse length.

The equations for the gravitational field derived from (1), can be written in the following form

$$G_{ab}=\frac{8\pi}{\phi}T^{(3)}_{ab}+\frac{\omega}{\phi^2}\left[\nabla_a\phi\left(\nabla_b\phi\right)-\frac{1}{2}g^{(3)}_{ab}\left(\nabla_c\phi\right)\left(\nabla^c\phi\right)\right]+\frac{1}{\phi}\left(\nabla_a\nabla_b\phi-g^{(3)}_{ab}\nabla^2\phi\right), \tag{2}$$

where $\nabla^2=\nabla^a\nabla_a$, and $T^{(3)}_{ab}$ is the energy momentum tensor associated with the matter fields for 3*D*. From the equation (1) it is possible to find that the field equation for the scalar field ϕ is given by

$$\nabla^c\nabla_c\phi=\frac{8\pi}{\omega+2}T^{(3)}, \tag{3}$$

where $T^{(3)} = g^{ab}T_{ab}$ is the trace of energy momentum tensor. In the context of a perfect fluid approximation, it can be seen from the second term in the equation (2) as the energy momentum tensor associated with the BD scalar field ϕ, and write

$$8\pi T_{ab}^{BD} = \frac{\omega}{\phi^2}\left[(\nabla_a\phi)(\nabla_b\phi) - \frac{1}{2}g_{ab}(\nabla_c\phi)(\nabla^c\phi)\right] + \frac{1}{\phi}\left(\nabla_a\nabla_b\phi - g_{ab}\nabla^2\phi\right). \quad (4)$$

We consider a perfect fluid approximation for the matter fields. Then the energy momentum tensor associated with matter field will be given by

$$T_{ab} = g_{ab}p + (\rho + p)u_a u_b, \quad (5)$$

where p and ρ represent the pressure and energy density associated with the matter-fluid and u_a is the velocity vector. The pressure and energy density are related by the state equation $p = \sigma\rho$, where σ is a parameter of proportionality. Comparing this relation with equation (3) we see that the BD scalar field and this matter-fluid are related by

$$\nabla^c\nabla_c\phi = \frac{8\pi}{\omega + 2}(2\rho - p). \quad (6)$$

As the BD scalar field ϕ has non-minimal coupling with gravity, it is impossible to write an expression similar to the equation (5), relating the associated pressure p_{BD} and energy density ρ_{BD} for the field ϕ, these quantities being identified as the spatial and temporal components of the energy momentum tensor associated with the BD scalar field, given by the equation (4). However, they are related by the equation of state

$$p_{BD} = \sigma_{BD}\rho_{BD}.$$

3. 3D Brans-Dicke Cosmology in the Absence of Matter

Let the spacetime be described by the 3D FRLW metric, that is given by [5]

$$ds^2 = g_{ab}dx^a dx^b = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\varphi^2\right], \quad (7)$$

where $a(t)$ is a dimensionless cosmic scale factor, and k is a parameter which can assume only one of three values: −1 (open universe), 0 (at universe) and +1 (closed universe). The observations done by WMAP, show that for $4D$ standard model the universe is about at. Then following this tendency, we will consider only the at case for the $3D$ model. Replacing (7) and $\phi = \phi(t)$ in (2); we find that the pressure p_{BD} and the energy density ρ_{BD} associated with the BD scalar field are given by

$$\rho_{BD} = \frac{1}{8\pi}\left[\frac{\omega}{2}\left(\frac{\dot\phi}{\phi}\right)^2 - 2\frac{\dot a}{a}\frac{\dot\phi}{\phi}\right], \quad (8)$$

$$p_{BD} = \frac{1}{8\pi}\left[\frac{\omega}{2}\left(\frac{\dot\phi}{\phi}\right)^2 - 2\frac{\dot a}{a}\frac{\dot\phi}{\phi} + \frac{\ddot\phi}{\phi}\right]. \quad (9)$$

If the BD scalar field is assumed as a perfect fluid, then the equations (8) and (9) are complemented by the equation of state [5,24,25]

$$\dot\rho_{BD} + 2\frac{\dot a}{a}(\rho_{BD} + p_{BD}) = 0. \quad (10)$$

Considering the at metric from (7) in the equations (2) and (6) we find

$$\left(\frac{\dot a}{a}\right)^2 = 8\pi\rho_{BD}, \ddot\phi + 2\frac{\dot a}{a}\dot\phi = 0. \quad (11)$$

In order to solve this system of coupled equations, we consider the ansatz

$$\phi(t) = \frac{1}{G_0}a^n(t), a(t) = a_0\left(1 + \frac{\beta}{\sqrt{G_0}}t\right)^{\alpha}, \quad (12)$$

where G_0 denotes the present value for the Newton's gravitational constant and β is a dimensionless parameter. Using the equation (12) in (11) we get a condition that relates the α to n parameters, namely

$$\alpha = \frac{1}{n+2}. \quad (13)$$

Now, substitution of the ansatz in (10) results

$$\omega = -1,$$

that implies in two possibilities for the parameter n

$$n = 0 \ \ or \ \ n = -2.$$

The only acceptable possibility for the n parameter is $n = 0$, resulting in $\alpha = \frac{1}{2}$, because the value $n = -2$ results in a divergent parameter α, without a physical meaning. For this value of α, the Brans-Dicke scalar field is given by $\phi(t) = \frac{1}{G_0}$, and the scale factor, has the form

$$a(t) = a_0\left(1 + \frac{\beta}{\sqrt{G_0}}t\right)^{\frac{1}{2}}. \quad (14)$$

Thus, when we consider ϕ depending only on time, the $3D$ Brans-Dicke model reduces exactly to the results of the $3D$ general relativity model. Therefore for a massless universe, the $3D$ Brans-Dicke theory does not have any K-essence behavior.

4. 3D Brans-Dicke Cosmology in the Presence of Matter

In this section we address the cosmology problem derived from the $3D$ Brans-Dicke theory in the presence of matter for a at background. As can be seen in Section II, the matter will be modeled like a perfect fluid with associated energy density ρ and pressure p. Substituting the FRLW metric (7) in the field equations (2), we see that the resulting Friedmann equations, in this case, are given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{\phi}\rho + 8\pi\rho_{BD}, \tag{15}$$

$$\ddot{\phi}+2\left(\frac{\dot{a}}{a}\right)\dot{\phi}=\frac{8\pi}{(\omega+2)}(\rho\text{-}2p). \tag{16}$$

The energy density ρ and the pressure p are related by

$$p = \sigma\rho.$$

Thus the equation (16) can be rewritten as

$$\ddot{\phi}+2\left(\frac{\dot{a}}{a}\right)\dot{\phi}=\frac{8\pi(1-2\sigma)}{(\omega+2)}\rho. \tag{17}$$

If we require the conservation of the energy momentum tensor, then we see that the equations of state for the matter fields and for the BD scalar field ϕ are given by

$$\dot{\rho}+2\left(\frac{\dot{a}}{a}\right)(1+\sigma)\rho = 0, \tag{18}$$

$$\dot{\rho}_{BD} + 2\left(\frac{\dot{a}}{a}\right)(\rho_{BD}+p_{BD}) = \frac{\dot{\phi}}{\phi^2}\rho. \tag{19}$$

The non-minimal coupling between the BD scalar field ϕ and gravity in the equation (1) amounts to a mixed term in the equation of state for ϕ involving the matter density ρ, which makes its solution a non-trivial one. Also, we see that, since the BD scalar field ϕ is massless and does not interact with matter, the equation (18) preserves the weak equivalence principle. And, by integrating equation (18), we obtain that the energy density ρ is related to the scale factor $a(t)$ by means of

$$\rho = \left(\frac{a(t)}{a(0)}\right)^{-2(1+\sigma)}.$$

As in the previous section, now we propose the ansatz

$$\phi(t) = \frac{1}{G_0}(1+\chi t)^{\gamma}, a(t) = a_0(1+\chi t)^{\beta}, \tag{20}$$

as solutions for the equations (15) and (17), where χ is a constant. Replacing this ansatz in the coupled field equations (15) and (17) gives us the results

$$\beta = \frac{\omega+2}{2\omega+4\sigma+2}, \gamma = \pm\beta\sqrt{\frac{2(1-2\sigma)}{2(\omega+2)+\omega(1-2\sigma)}}. \tag{21}$$

Unlike the scalar-vaccum configuration, now it is not possible to determine the parameter ω; since the state equation (19) is coupled. The equation (21) shows that there are two possibilities for γ, but for small values of ω the positive value of γ allows a negative value for the energy density, which is forbidden by the strong energy condition [17]. Therefore the negative choice is the only possibility for γ.

If we replace the ansatz (20) in the general equations (8) and (9) we find that the equations for the energy density and pressure associated with the BD scalar field are given by

$$\rho_{BD} = \frac{1}{8\pi}\gamma\left(\frac{\omega\gamma}{2}-2\beta\right)\frac{\chi^2}{(1+\chi t)^2}, \tag{22}$$

$$p_{BD} = \frac{1}{8\pi}\left[2\beta\gamma+\frac{\omega\gamma^2}{2}+\gamma(\gamma-1)\right]\frac{\chi^2}{(1+\chi t)^2}, \tag{23}$$

and the Brans-Dicke energy density and pressure are related by

$$p_{BD} = \sigma_{BD}\rho_{BD}.$$

If the strong energy condition [17] is considered, we find that the deceleration parameter for the *3D* model constructed with the Brans-Dicke theory, is given by

$$q = \frac{\Omega(1-\sigma)}{\phi(t)} + \Omega_{BD}(1-\sigma_{BD}), \tag{24}$$

where

$$\Omega = \frac{8\pi}{H(t)^2}\rho \;\; and \;\; \Omega_{BD} = \frac{8\pi}{H(t)^2}\rho_{BD},$$

and H is the Hubble parameter. We observe that the first term in the right side is related to the matter fields, while the second term is related to the BD scalar field.

The behavior of the *3D* Brans-Dicke model as a Kessence theory, depends on the possible values for the dimensionless parameter ω of theory, parameter σ_{BD} that relates the Brans-Dicke energy density and pressure, and the parameter σ that relates the energy density and pressure of matter fields. For this last parameter, there are two possibilities: $\sigma = 0$ that describes a Matter Dominate Era (**MDE**); $\sigma = \frac{1}{2}$ Radiation Dominate Era (**RDE**).

For the **RDE**, the 3*D* Brans Dicke theory recovers the cosmological standard model results. For the **MDE** the behavior of the parameter σ_{BD} is described by Figure 1.

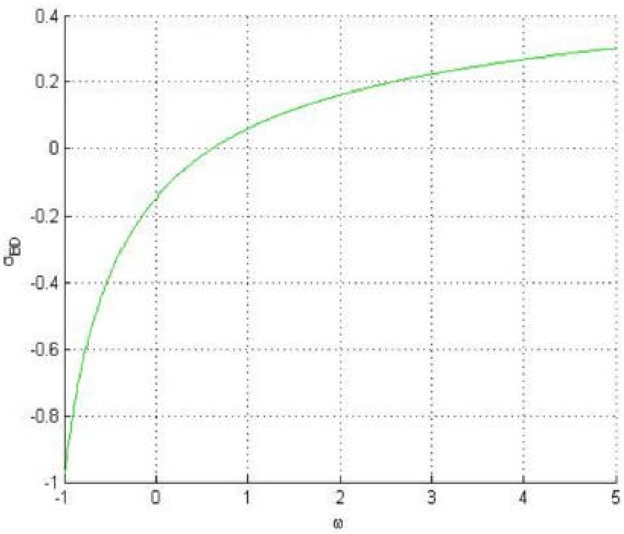

Figure 1. σ_{BD} parameter in terms of dimensionless parameter of theory ω

Analyzing Figure 1, we found that σ_{BD} has negative values for $-1.32 < \omega < 0.61$. However, the Brans-Dicke Theory is inconsistent for small values of the dimensionless parameter ω in a solar system level, this consistence is restored only for larger values of ω [5]. For large values of $\omega\left(\omega \geq 50.000\right)$, σ_{BD} does not assume positive values, which is inconsistent with the description of an accelerated expansion of a 3D universe.

A comparison between the scale factors of the 3*D* and 4*D* Brans-Dicke models is shown in Figure 2. As expected, the models have a different behavior as presented in [10]. Figure 2 shows that there is a value of time that the two models converge to the same result, diverging again shortly thereafter.

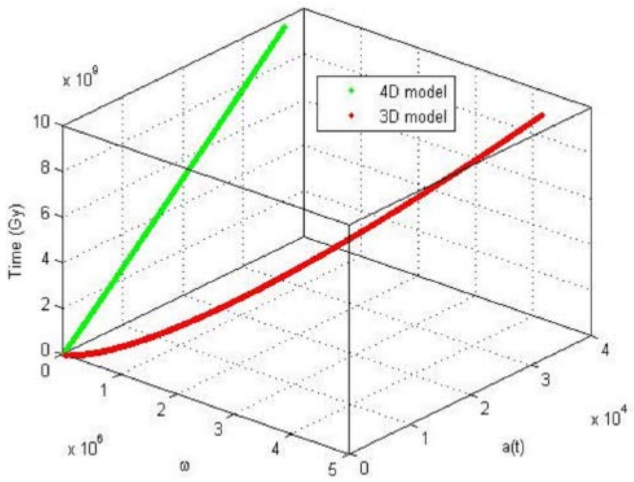

Figure 2. Comparison between scale factor *a*(*t*) behavior for vacuum of 3*D* (red) and 4*D* (green) models

For the MDE, the energy density of Brans-Dicke scalar field is non null. This is expected because the Brans-Dicke scalar field is generated by matter fields (3). The analysis of Figure 1 shows that the Brans-Dicke model can only describe a K-essence field $\left(\sigma_{BD} < 0\right)$ for some values of the dimensionless parameter $\omega\left(-1.32 < \omega < 0.61\right)$. However for the consistency of the theory, only larger values of ω are allowed in a solar system level $\left(\omega \geq 50.000\right)$ [5,21]; and for $\omega > 0.61$, the 3*D* theory no describe a K-essence model $\left(\sigma_{BD} > 0\right)$. In the RDE the 3*D* and 4*D* models has the same behaviour. The 3*D* Brans-Dicke model restores the General Relativity results when $\omega \to \infty$.

Table 1. 3*D* and 4*D* Brans-Dicke models

Era model	3D BD	3D BD
vacuum	$a(t) \sim t^{\frac{1}{2}}$	$a(t) \sim t^{m} m > 1$
RDE	$a(t) \sim t^{\frac{1}{2}}$	$a(t) \sim t^{\frac{1}{2}}$
MDE	$a(t) \sim t^{\frac{\omega+2}{2(\omega+1)}}$	$a(t) \sim t^{\frac{2(\omega+1)}{\omega+4}}$
BD	*DNE*	$a(t) \sim t$

Table 1 shows a comparation between the scale factor behavior of 3*D* and 4*D* Brans-Dicke models for the different evolution eras. An interesting result is that there is no difference between vacuum Era and **RDE** in the 3*D* Brans-Dicke model. Therefore, the 3*D* model has only two distinct evolution Eras, while the 4*D* Brans-Dicke model has four distinct Eras (see Table 1) [10]. This comparison shows that for the scalar tensor theories like Brans-Dicke theory, the choice of spatial dimension is a very important factor for the theory behavior.

5. Conclusions

In the absence of matter fields (vacuum), the energy density of the Brans-Dicke scalar field vanishes, i.e. $\rho_{BD} = 0$, and the 3*D* Brans-Dicke theory recovers the 3*D* General Relativity results. This result is expected since the Brans-Dicke scalar field is generated by matter fields (3). However the 4*D* Brans-Dicke model does not reduce to the 4*D* General Relativity model when the matter fields vanish [10], and a new Era of scalar field emerges in the 4*D* Brans-Dicke model. This new Era in the 4*D* model is due to K-essence behavior of Brans-Dicke scalar field in 4D model, which does not happen in the 3*D* model.

In the presence of matter fields, the energy density of the 3*D* scalar fields is non-null. However, even in the presence of matter fields, the 3*D* Brans-Dick model does not have a K-essence behavior like found in the 4*D* model [10]. Another important information is that 3*D* Brans-Dicke model has only two distinct evolution Eras, the vacuum and RDE Eras are the same and the MDE Era. In this sense we conclude that the difference of scale factor behavior between 3*D* and 4*D* Brans-Dicke models is mainly due to the dimensional factor of the field equations.

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