We note that one can obtain a general formula for $\alpha_{max}(W, P_0)$ for the case of one right half plane at p:

$$\alpha_{max}(W, P_0) = e^{-hp}|W^{-1}(p)n_1^{-1}(p)|$$

where $n_1(s)$ is the Blaschke product of the open right half plane zeros of $P_0(s)$.

We now consider an example of a system with no open right half plane zeros, two right half plane poles, and a pure transmission delay:

$$P_0(s) = \frac{e^{-hs}\hat{P}_1(s)}{(s-1)(s-2)}$$

where $\hat{P}_1(s)$ is invertible in A_2. In this case a routine calculation gives

$$\alpha_{max}(1, P_0) = 0.5e^{-2h}(\sqrt{9e^{2h} + 9 - 14e^h} - 3(e^h - 1)).$$

For $h = 1$,

$$\alpha_{max}(1, P_0) = 0.065.$$

Hence, in this case, the robust stabilization problem for multiplicative perturbation family, section 3 is solvable if and only if $\delta < 0.065$. And the maximal obtainable gain margin for this plant is 2.26 dB! The effect of a right half plane zero would be to reduce α_{max} even further.

4. CONCLUDING REMARKS

In this paper we have given a solution to certain robust stabilization problems for a large class of distributed plants. In particular, our results give concrete necessary and sufficient conditions for robust stabilizability for plants with pure delay and a finite number of right half plane poles and zeros. Our results provide techniques for the assessment of the impact of time-delays on robust stabilizability. Many open problems remain to be solved in this problem area. For example, so far no explicit results have been obtained for the H^∞-weighted sensitivity optimization problem of Zames (1981). Also, the multivariable versions of the problems treated in this paper remain essentially unsolved. Finally, these results need to be extended to systems governed by partial differential equations.

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Robust Controllers for Uncertain Linear Multivariable Systems*

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Necessary and sufficient conditions are derived for a particular controller to achieve robust stabilization in the face of various types of plant uncertainty; in addition, necessary and sufficient conditions are derived for the existence of such a controller.

Key Words—Control system design; feedback control; robust control; mathematical system theory; stability criteria.

Abstract—This paper is addressed to three distinct yet related topics in the design of controllers for imprecisely known linear multivariable systems. In the first part, it is supposed that the plant to be stabilized is subject to additive or multiplicative uncertainties, and necessary and sufficient conditions are derived for the existence of a controller that stabilizes *all* plants within this band of uncertainty. In the second part, in contrast with the first part, it is supposed that the number of unstable poles of the plant to be stabilized is not precisely known. The type of plant uncertainty is the so-called "stable-factor" uncertainty, and necessary and sufficient conditions are given for robust stabilization. In the third part, the model of uncertainty is a ball in the space of rational matrices metrized by the so-called graph metric, and sufficient conditions for robust stabilization are derived.

INTRODUCTION

THIS PAPER is addressed to three distinct yet related problems in the design of stabilizing controllers for imprecisely known linear multivariable systems. To state each of these problems formally, suppose the imprecisely known plant which is to be stabilized is nominally modeled by a rational transfer matrix $P_0(s)$; the "true" plant is not necessarily P_0, but lies within some "domain of uncertainty" containing P_0. The three problems differ in the representation of this domain of uncertainty.

In the first problem, the true plant P is supposed to have the same number of right half-plane (RHP) poles as P_0, though not necessarily at the same locations. In the case of additive uncertainty, P is assumed to satisfy

$$\|P(j\omega) - P_0(j\omega)\| < |r(j\omega)|, \forall\omega, \tag{1.1}$$

where r is a prespecified stable rational function. In the case of multiplicative uncertainty, P is assumed to satisfy

$$P(s) = (I + L(s))P_0(s), \tag{1.2}$$

where

$$\|L(j\omega)\| < |r(j\omega)|\forall\omega. \tag{1.3}$$

Let $A(P_0, r)$ (resp. $M(P_0, r)$) denote the class of all plants P that have the same number of RHP poles as P_0 and satisfy (1.1) [resp. (1.2)]. In Doyle and Stein (1981), Chen and Desoer (1982), necessary and sufficient conditions are presented that a controller must satisfy in order to stabilize all plants in the class $A(P_0, r)$ or $M(P_0, r)$. These papers leave open the question of whether such a controller actually exists. This is the question tackled in Section 3. Specifically, necessary and sufficient conditions are given for the existence of robustly stabilizing controllers in the case of $A(P_0, r)$ and $M(P_0, r)$.

The assumption that the number of unstable poles of the plant to be stabilized is exactly known is rather restrictive. In some applications, such as large flexible spacecraft, the number of unstable poles can and does change as the configuration of the spacecraft is changed. The objective of Section 4 is to put forward a model of plant uncertainty wherein the various plants within the domain of uncertainty need not all have the same number of RHP poles. This model is called stable-factor uncertainty, and can be described as follows: Let P_0 be the nominal plant model, and factor $P_0(s)$ as $N_0(s)[D_0(s)]^{-1}$

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where N_0, D_0 are right-coprime stable rational matrices [see Desoer *et al.* (1980) and Vidyasagar *et al.* (1982)]. The class $S(N_0, D_0, r)$ consists of all plants P that satisfy $P(s) = N(s)[D(s)]^{-1}$ for some stable rational matrices N, D such that

$$\left\| \begin{bmatrix} N - N_0 \\ D - D_0 \end{bmatrix}(s) \right\| < |r(s)| \, \forall s \text{ with } \operatorname{Re} s \geq 0, \quad (1.4)$$

where r is a specified stable rational function. Thus the class $S(N_0, D_0, r)$ depends not just on P_0 and r but on a particular right-coprime factorization of P_0. In Section 4, necessary and sufficient conditions that a controller must satisfy in order to stabilize all plants in the class $S(N_0, D_0, r)$ are derived.

Finally, in Section 5, the model of plant uncertainty is a ball centered at P_0. Thus it is assumed that the true plant is only known to lie within some distance r of the nominal plant P_0, where r is a positive real number, and the distance between P and P_0 is measured using the graph metric defined in Vidyasagar (1984). In this case the domain of uncertainty is

$$\mathbf{B}(P_0, r) = \{P : d(P, P_0) < r\}, \quad (1.5)$$

where d denotes the graph metric distance. This uncertainty model has the advantages that (i) P need not have the same number of RHP poles as P_0, and (ii) the domain of uncertainty $\mathbf{B}(P_0, r)$ depends only on P_0 and r, and not on any particular factorization of P_0. In Section 5, sufficient conditions are given for a controller to stabilize all plants in the class $\mathbf{B}(P_0, r)$. These sufficient conditions are readily extended to the case where both the plant and controller are perturbed.

As a preparation for the principal results, Section 2 contains a precis of known facts. Finally, Section 6 contains some concluding remarks.

2. NOTATION AND PRELIMINARIES

Throughout the paper, $\mathbf{S}$ denotes the set of proper stable rational functions with real coefficients. The symbol $\mathbf{M}(\mathbf{S})$ is a generic symbol denoting the set of all matrices (of whatever order) whose elements belong to $\mathbf{S}$. Thus, $A, B \in \mathbf{M}(\mathbf{S})$ does not imply that A and B have the same order. This notation is very useful because, almost always, the actual orders of the various matrices encountered in the discussion need not be displayed explicitly and can easily be determined should the need arise.

The set $\mathbf{S}$ is a subset of the space H_∞ of analytic functions bounded over the right half-plane. Specifically, H_∞ consists of all complex-valued analytic functions f over the open RHP with the property that

$$\limsup_{\sigma \to 0+} \sup_{\operatorname{Re} s \geq \sigma} |f(s)| < \infty. \quad (2.1)$$

If $f \in H_\infty$, then the domain of definition of f can be extended to include the $j\omega$-axis, and the boundary function $\omega \to f(j\omega)$ is in $L_\infty(-\infty, \infty)$. Moreover, if the norm of $f \in H_\infty$ is defined as

$$\|f\|_\infty = \sup_{\sigma > 0} \sup_{\operatorname{Re} s \geq \sigma} |f(s)|, \quad (2.2)$$

then actually

$$\|f\|_\infty = \operatorname*{ess\,sup}_{\omega} |f(j\omega)|. \quad (2.3)$$

A function $f \in H_\infty$ is symmetric if $f(s) = \overline{f(\bar{s})} \forall s$ in the RHP, where the bar denotes complex conjugation. Note that $\mathbf{S}$ is precisely the set of symmetric rational functions in H_∞. For further basic facts about H_∞, see Duren (1970).

A function $f \in H_\infty$ is inner if $|f(j\omega)| = 1$ for almost all ω. The definition of an outer function is more technical, but a rational function $f \in H_\infty$ is outer if and only if $f(s) \neq 0$ whenever $\operatorname{Re} s > 0$ (however, f can have zeros on the $j\omega$-axis or at infinity). For convenience, let us refer to the $j\omega$-axis plus the point at infinity as the extended $j\omega$-axis. If f is rational, outer, and also does not vanish at any point on the extended $j\omega$-axis, then f is a unit of H_∞, in that the function $s \to 1/f(s)$ also belongs to H_∞. Every rational $f \in H_\infty$ can be factored as $f_i f_o$, where f_i is inner and f_o is outer. In particular, if f is rational and has no zeros on the extended $j\omega$-axis, then its outer factor f_o is a unit of H_∞.

If $F \in \mathbf{M}(H_\infty)$, i.e. if all components of the matrix F are H_∞-functions, then we define

$$\|F\|_\infty = \operatorname*{ess\,sup}_{\omega} \bar{\sigma}(F(j\omega)), \quad (2.4)$$

where $\bar{\sigma}(\cdot)$ denotes the largest singular value of a matrix.

A rational matrix $F \in H_\infty^{m \times n}$ is inner if $F^*(j\omega)F(j\omega) = I \forall \omega$, where * denotes the conjugate transpose, and is outer if $F(s)$ has full row rank at all s in the open RHP. Note that if $F \in H_\infty^{m \times n}$ is inner (resp. outer), then $m \geq n$ (resp. $m \leq n$). Every rational matrix $F \in H_\infty^{m \times n}$ with $m \geq n$ can be factored as $F_i F_o$ where F_i is inner and F_o is square outer. Every rational matrix $F \in H_\infty^{m \times n}$ with $m \leq n$ can be factored as $G_i G_o$ where G_i is square inner and G_o is outer. Finally, if $F \in \mathbf{M}(H_\infty)$ is rational and square, then both types of factorization are possible. If F has full rank at all points on the extended $j\omega$-axis, then its outer factor is a unit matrix in $\mathbf{M}(H_\infty)$, i.e. its inverse also belongs to $\mathbf{M}(H_\infty)$. Note that if F is square inner, then its adjoint matrix F^{adj} and its determinant $|F|$ are also inner. Finally, multiplication (left or right) by an inner matrix preserves norms. Thus, if F, G are inner and $H \in \mathbf{M}(H_\infty)$, then $\|FH\|_\infty = \|H\|_\infty$, $\|HG\|_\infty = \|H\|_\infty$. For further results concerning factorizations, see Sz-Nagy and Foias (1970).

Suppose $\lambda_1, \ldots, \lambda_n$ are distinct points in the open RHP, and $F_1, \ldots, F_n$ are complex matrices, all of the same order, with $\|F_i\| < 1 \forall i$. Then a classical result states that there exists an $F \in \mathbf{M}(H_\infty)$ such that $\|F\|_\infty \leq 1$ and $F(\lambda_i) = F_i \forall i$ if and only if the matrix

$$P = \begin{bmatrix} P_{11} & \ldots & P_{1n} \\ \vdots & \vdots & \vdots \\ P_{n1} & \ldots & P_{nn} \end{bmatrix}, \quad (2.5)$$

where

$$P_{ij} = \frac{I - F_i^* F_j}{\bar{\lambda}_i + \lambda_j}, \quad (2.6)$$

is nonnegative definite. See Walsh (1935) for the scalar case and Delsarte *et al.* (1979) for the matrix case.

Let $\mathbb{R}(s)$ denote the set of rational functions with real coefficients, and suppose a plant has the transfer matrix $P \in \mathbf{M}(\mathbb{R}(s))$. Suppose a feedback controller $C \in \mathbf{M}(\mathbb{R}(s))$ is applied to the plant P, resulting in the closed-loop transfer matrix

$$H(P, C) = \begin{bmatrix} (I + PC)^{-1} & -P(I + CP)^{-1} \\ C(I + PC)^{-1} & (I + CP)^{-1} \end{bmatrix} \quad (2.7)$$

We say that the pair (P, C) is *stable*, or that C *stabilizes* P if $H(P, C) \in \mathbf{M}(\mathbf{S})$. The symbol $S(P)$ denotes the set of all controllers that stabilize P.

Given a plant P, the set $S(P)$ can be explicitly parametrized using the concept of coprime factorizations over $\mathbf{S}$ defined in Vidyasagar (1975) and exploited in Desoer *et al.* (1980); see also Vidyasagar *et al.* (1982). A pair (N, D) is a right-coprime factorization (r.c.f.) of P if

(i) $N, D \in \mathbf{M}(\mathbf{S})$ and $P(s) = N(s)[D(s)]^{-1}$;

(ii) there exist $X, T \in \mathbf{M}(\mathbf{S})$ such that $XN + YD = I$. A left-coprime factorization (l.c.f.) $(\tilde{D}, \tilde{N})$ of P is defined analogously.

Theorem 1. (Desoer *et al.*, 1980; Vidyasagar *et al.*, 1982.) Suppose $P \in \mathbf{M}(\mathbb{R}(s))$, and let (N, D), $(\tilde{D}, \tilde{N})$ be any r.c.f. and any l.c.f. of P. Let $X, Y, \tilde{X}, \tilde{Y}$ be solutions of

$$XN + YD = I, \quad \tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I. \quad (2.8)$$

Then

$$S(P) = \{(Y - R\tilde{N})^{-1}(X + R\tilde{D}) : R \in \mathbf{M}(\mathbf{S}) \text{ and } |Y - R\tilde{N}| \neq 0\} \quad (2.9)$$

$$= \{(\tilde{X} + DR)(\tilde{Y} - NR)^{-1} : R \in \mathbf{M}(\mathbf{S}) \text{ and } |\tilde{Y} - NR| \neq 0\}. \quad (2.10)$$

Moreover, if P is strictly proper, the nonsingularity constraint can be dropped.

3. ADDITIVE AND MULTIPLICATIVE PERTURBATIONS

Suppose $P_0 \in \mathbf{M}(\mathbb{R}(s))$ is a nominal plant, $r \in \mathbf{S}$, and consider the classes $A(P_0, r)$, $M(P_0, r)$ defined in the introduction. Conditions for a controller $C \in S(P_0)$ to stabilize all plants in each of these classes are available in the literature.

Theorem 2. (Doyle and Stein, 1981; Chen and Desoer, 1982). A controller $C \in S(P_0)$ stabilizes all plants in the class $A(P_0, r)$ if and only if

$$\|C(I + P_0C)^{-1}r\|_\infty \leq 1. \quad (3.1)$$

C stabilizes all plants in the class $M(P_0, r)$ if and only if

$$\|P_0C(I + P_0C)^{-1}r\|_\infty \leq 1. \quad (3.2)$$

It is natural to ask whether, given a nominal plant P_0 and a function r, there actually exists a C that satisfies (3.1) or (3.2). The purpose of this section is to answer these questions.

Theorem 3. Suppose a nominal plant P_0, free of extended $j\omega$-axis poles, and a function $r \in \mathbf{S}$, free of zeros on the extended $j\omega$-axis, are specified. Let (N, D), $(\tilde{D}, \tilde{N})$ be any r.c.f. and l.c.f. of P_0, and let $\tilde{X}$, $\tilde{Y} \in \mathbf{M}(\mathbf{S})$ be any particular solutions of the identity $\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I$. Factor D, $\tilde{D}$ in the form

$$D = D_i D_o, \quad \tilde{D} = \tilde{D}_o \tilde{D}_i, \quad (3.3)$$

where D_i, $\tilde{D}_i$ are inner and D_o, $\tilde{D}_o$ are outer, and assume without loss of generality that $|D_i| = |\tilde{D}_i|$. By Kailath (1980, p. 446), $|D|$ and $|\tilde{D}|$ are associates; so are $|D_o|$ and $|\tilde{D}_o|$, since both are units. Hence $|D_i|$ and $|\tilde{D}_i|$ are also associates. Since both are inner, we have $|D_i| = \pm|\tilde{D}_i|$ in any case. Finally, factor r as $r_i r_o$ where r_i is inner and r_o is a unit of $\mathbf{S}$. Under these conditions, there exists a $C \in S(P_0)$ that stabilizes all plants in the class $A(P_0, r)$ if and only if

$$\min_{S \in \mathbf{M}(S)} \|D_i^{\text{adj}} \tilde{X} \tilde{D}_o r_o + \delta S\|_\infty \leq 1, \quad (3.4)$$

where $\delta = |D_i| = |\tilde{D}_i|$.

If δ has only simple RHP zeros (or equivalently, if P_0 has only simple RHP poles), (3.4) can be stated in a more readily testable form.

Corollary 1.1. Suppose δ has only simple zeros, and let $\lambda_1, \ldots, \lambda_n$ denote these zeros. Let

$$F_j = (D_i^{\text{adj}} \tilde{X} \tilde{D}_o r_o)(\lambda_j), \; j = 1, \ldots, n, \text{ and define}$$

$$Q_{jk} = (I - F_j^* F_k)/(\bar{\lambda}_j + \lambda_k), \quad 1 \leq j, k \leq n. \quad (3.5)$$

$$Q = \begin{bmatrix} Q_{11} & \cdots & Q_{1n} \\ \vdots & \vdots & \vdots \\ Q_{n1} & \cdots & Q_{nn} \end{bmatrix}. \tag{3.6}$$

Then there exists a $C \in S(P_0)$ that stabilizes all plants in the class $A(P_0, r)$ if and only if Q is nonnegative definite.

The treatment of multiplicative perturbations is entirely similar to that of additive perturbations.

Theorem 4. Suppose a function $r \in \mathbf{S}$ and a nominal plant $P_0 \in \mathbf{M}(\mathbb{R}(s))$ are specified, and suppose in addition that P_0 has no poles on the extended $j\omega$-axis. Let (N, D), $(\tilde{D}, \tilde{N})$ be any r.c.f. and l.c.f. of P_0, and let X, $Y \in \mathbf{M}(\mathbf{S})$ be any particular solution of the identity $XN + YD = I$. Under these conditions, there exists a $C \in S(P_0)$ that stabilizes all plants in the class $M(P_0, r)$ if and only if

$$\min_{R \in \mathbf{M}(\mathbf{S})} \|N(X + R\tilde{D})\|_\infty \leq 1. \tag{3.7}$$

Suppose in addition that P_0 has at least as many inputs as outputs, and that r has no zeros on the extended $j\omega$-axis. Then the condition (3.7) can be further simplified as follows: factor N, $\tilde{D}$ in the form

$$N = N_i N_o, \qquad \tilde{D} = \tilde{D}_o \tilde{D}_i, \tag{3.8}$$

where N_i, $\tilde{D}_i$ are inner and N_o, $\tilde{D}_o$ are outer. Finally, factor r as $r_i r_o$ where r_i is inner and r_o is a unit of $\mathbf{S}$. Under these conditions, there exists a $C \in S(P_0)$ that stabilizes all plants in the class $M(P_0, r)$ if and only if

$$\min_{R \in \mathbf{M}(\mathbf{S})} \|N_o X r_o \tilde{D}_i^{adj} + \delta R\|_\infty \leq 1, \tag{3.9}$$

where $\delta = |\tilde{D}_i|$.

Proof of Theorem 3. By Theorem 1, the set of all $C \in S(P_0)$ is described by

$$S(P_0) = \{(\tilde{X} + DS)(\tilde{Y} - NS)^{-1} : S \in \mathbf{M}(\mathbf{S}) \text{ and } |\tilde{Y} - NS| \neq 0\}, \tag{3.10}$$

where the nonsingularity constraint is automatically satisfied if P is strictly proper. Moreover, if $C = (\tilde{X} + DS)(\tilde{Y} - NS)^{-1}$, then

$$C(I + P_0 C)^{-1} = (\tilde{X} + DS)\tilde{D}. \tag{3.11}$$

Hence there exists a $C \in S(P_0)$ satisfying (3.1) if and only if there exists an $S \in \mathbf{M}(\mathbf{S})$ such that $\|\tilde{X} + DS)\tilde{D}r\|_\infty \leq 1$, i.e. if and only if

$$\min_{S \in \mathbf{M}(\mathbf{S})} \|\tilde{X} + DS)\tilde{D}r\|_\infty \leq 1. \tag{3.12}$$

A few simple manipulations bring (3.12) into the form (3.4). Note that

$$\begin{aligned} \|\tilde{X}\tilde{D}r + DS\tilde{D}r\|_\infty &= \|\tilde{X}\tilde{D}_o\tilde{D}_i r_o r_i \\ &\quad + D_i D_o S \tilde{D}_o \tilde{D}_i r_o r_i\|_\infty \\ &= \|D_i^{adj}\tilde{X}\tilde{D}_o r_o r_i \delta \\ &\quad + \delta^2 r_i D_o S \tilde{D}_o r_o\|_\infty \end{aligned}$$

after left multiplication by D_i^{adj} and right multiplication by $\tilde{D}_i^{adj}$

$$= \|\delta r_i (D_i^{adj}\tilde{X}\tilde{D}_o r_o + \delta S_1)\|_\infty$$

where $S_1 = D_o S \tilde{D}_o r_o$ is a new free parameter

$$= \|D_i^{adj}\tilde{X}\tilde{D}_o r_o + \delta S_1\|_\infty$$

since δ and r_i are inner. (3.13)

Hence (3.12) and (3.4) are equivalent.

Proof of Corollary 1.1. Let F denote $D_i^{adj}\tilde{X}\tilde{D}_o r_o$, and note that $F \in \mathbf{M}(H_\infty)$. Now, a matrix $G \in \mathbf{M}(H_\infty)$ is of the form $F + \delta S$ for some $S \in \mathbf{M}(H_\infty)$ if and only if $G(\lambda_j) = F(\lambda_j) = F_j \forall j$. Thus (3.4) holds if and only if there exists a matrix $G \in \mathbf{M}(H_\infty)$ with $\|G\|_\infty \leq 1$ such that $G(\lambda_j) = F_j \forall j$. Now apply (2.5) and (2.6).

In (3.4), $\tilde{X}$ is any particular solution of the identity $\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I$. Hence it is nice to know that the test matrix Q is the same no matter which $\tilde{X}$ is used. To see this, let $\tilde{X}_1$, $\tilde{Y}_1$ be another set of matrices in $\mathbf{M}(\mathbf{S})$ satisfying $\tilde{N}\tilde{X}_1 + \tilde{D}\tilde{Y}_1 = I$. Then $\tilde{X}_1 = \tilde{X} + DR$ for some $R \in \mathbf{M}(\mathbf{S})$. Hence

$$\begin{aligned} D_i^{adj}\tilde{X}_1\tilde{D}_o r_o &= D_i^{adj}(\tilde{X} + DR)\tilde{D}_o r_o \\ &= D_i^{adj}\tilde{X}\tilde{D}_o r_o + \delta D_o R \tilde{D}_o \end{aligned} \tag{3.14}$$

since $D = D_i D_o$. Now the second term on the right side of (3.14) vanishes at all zeros of δ. Hence $F_1, \ldots, F_n$ are independent of which particular solution $\tilde{X}$ is used to compute them. Similar remarks apply even in the case where δ has repeated zeros.

If the plant P_0 is scalar, then the expression for F_j (or f_j) in the scalar case is more elegant. In this case, $d_i^{adj} = 1$ (since by convention the adjoint of a 1×1 matrix is 1), and $f_j = (xd_o r_o)(\lambda_j)$. Now, in the scalar case $\delta = \delta_i$; hence it follows from $d = d_i d_o = \delta d_o$ and $nx + dy = 1$ that, at any zero λ_j of δ, we have $n(\lambda_j)x(\lambda_j) = 1$. Hence

$$x(\lambda_j)d_o(\lambda_j) = \frac{d_o(\lambda_j)}{n(\lambda_j)}. \tag{3.15}$$

With the plant p_0, associate the stable plant $q_0 = p_0\delta = n/d_0$. Then

$$f_j = \frac{d_o(\lambda_j)r_o(\lambda_j)}{n(\lambda_j)} = \frac{r_o(\lambda_j)}{q_o(\lambda_j)}. \tag{3.16}$$

The above result was obtained earlier by Kimura (1983) using quite different methods. He also extended these results to the case where both the unperturbed and perturbed plants had a simple pole at $s = 0$.

The treatment of multiplicative perturbations is entirely similar to that of additive perturbations.

Proof of Theorem 4. From Theorem 1, every $C \in S(P_0)$ is of the form $(Y - R\tilde{N})^{-1}(X + R\tilde{D})$ for some $R \in \mathbf{M}(\mathbf{S})$. Moreover,

$$P_0 C(I + P_0 C)^{-1} = P_0(I + CP_0)^{-1}C = N(X + R\tilde{D}). \tag{3.17}$$

Hence (3.2) holds for some $C \in S(P_0)$ if and only if there exists an $R \in \mathbf{M}(\mathbf{S})$ such that (3.7) holds. This proves the first part of the theorem.

To prove the second part, let $\gamma = |N_i|$ and note that γ is inner. Using by now familiar manipulations, we arrive at

$$\begin{aligned} \|NXr + NR\tilde{D}r\|_\infty &= \|N_i N_o X r_o r_i \\ &\quad + N_i N_o R \tilde{D}_o r_o \tilde{D}_i r_i\|_\infty \\ &= \|\gamma r_i (N_o X r_o \tilde{D}_i^{adj} + \delta S)\|_\infty \\ &\quad \text{where } S = N_o R \tilde{D}_o r_o \\ &= \|N_o X r_o \tilde{D}_i^{adj} + \delta S\|_\infty. \end{aligned} \tag{3.18}$$

Now (3.9) follows readily.

If the plant P_0 is scalar, one can again obtain a simple expression for the quantity $N_o X r_o \tilde{D}_i^{adj}$ evaluated at the zeros of δ. Suppose $nx + dy = 1$, and suppose $\delta(\lambda) = 0$ at some point λ in the RHP. Then $d(\lambda) = 0$, $(nx)(\lambda) = 1$, and $(N_o X r_o \tilde{D}_i^{adj})(\lambda) = \dfrac{r_o(\lambda)}{n_i(\lambda)}$.

The foregoing results also lead very naturally to the notion of optimally robust compensators. Suppose a nominal plant P_0 is given, together with a function $f \in \mathbf{S}$, which represents an uncertainty profile. Consider the class $A(P_0, r)$ consisting of all plants satisfying (1.1), where $r = \lambda f$. It is now reasonable to ask: what is the largest value of the parameter λ for which the class $A(P_0, r)$ is robustly stabilizable, and what is a corresponding robustly stabilizing compensator? This problem can be solved very easily using the results derived so far. Factor f as $f_i f_o$ where f_i, f_o are respectively inner and outer, and define

$$\gamma = \min_{S \in \mathbf{M}(\mathbf{S})} \|D_i^{adj}\tilde{X}\tilde{D}_o f_o + \delta S\|_\infty, \tag{3.19}$$

where all symbols are as in Theorem 3. Now, by applying Theorem 3 with $r = \lambda f$, it follows that there exists a single compensator that stabilizes all the plants in the class $A(P_0, r)$ if and only if $\gamma\lambda \leq 1$. Hence the largest value of λ for which there exists a robustly stabilizing compensator is given by $\lambda = 1/\gamma$. Further, if S is any matrix which attains the minimum in (3.19), then the compensator $C = (\tilde{X} + DS)(\tilde{Y} - NS)^{-1}$ is an optimally robust compensator. The case of multiplicative perturbations is entirely similar and is left to the reader.

If the plant P_0 has more rows than columns, then the condition (3.7) for robust stabilizability cannot be simply tested in terms of the nonnegative definiteness of a test matrix of the form (3.6). In this case one has to use the iterative method of Doyle (1983) to test the condition (3.7). If the plant P_0 has at least as many columns as rows, then the simpler condition (3.9) applies, which can be more readily tested.

4. STABLE FACTOR PERTURBATIONS

In the previous section, we studied the case where the perturbed and unperturbed plants had the same number of RHP poles, and neither had poles on the $j\omega$-axis. However, this assumption is not satisfied in certain applications such as large flexible spacecraft. Thus it is desirable to develop a theory that removes this assumption by considering a different class of plant perturbations. The class of stable factor perturbations has this feature. The conditions for the robust stabilizability in the case of stable factor perturbations are given in the next theorem.

Theorem 5. Suppose $P_0 \in \mathbf{M}(\mathbb{R}(s))$. Suppose an r.c.f. (N_0, D_0) of P_0 and a function $r \in \mathbf{S}$ are specified, and define the class $S(N_0, D_0, r)$ as in (1.4). Suppose $C \in S(P_0)$, and select an l.c.f. $(\tilde{D}_c, \tilde{N}_c)$ of C such that $\tilde{D}_c D_0 + \tilde{N}_c N_0 = I$. Then C stabilizes all P in the class $S(N_0, D_0, r)$ if and only if

$$\|[\tilde{D}_c \;\; \tilde{N}_c]r\|_\infty \leq 1. \tag{4.1}$$

Proof. For convenience, define

$$A_0 = \begin{bmatrix} D_0 \\ N_0 \end{bmatrix}, \qquad A = \begin{bmatrix} D \\ N \end{bmatrix}, \quad \tilde{A}_c = [\tilde{D}_c \;\; \tilde{N}_c]. \tag{4.2}$$

"if": Suppose (4.1) holds, and suppose P is an arbitrary plant in the class $S(N_0, D_0, r)$. Then P has an r.c.f. (N, D) such that $\|A(s) - A_0(s)\| \leq |r(s)|$ $\forall s \in C_{+e}$. Now consider the return difference matrix $\tilde{D}_c D + \tilde{N}_c N = \tilde{A}_c A$. Since $\tilde{A}_c A_0 = I$, it follows that $\tilde{A}_c A = I + A_c(A - A_0)$. However, from (4.1), we get $\|[\tilde{A}_c(A - A_0)](s)\| \leq \|\tilde{A}_c(s)\| \, \|[A - A_0](s)\| < 1$ $\forall s \in C_{+e}$. This shows that $|\tilde{A}_c A(s)| \neq 0$ $\forall s \in C_{+e}$, so that $\tilde{A}_c A$ is unimodular. Hence C stabilizes P.

"only if": Suppose (4.1) is false; we will construct a plant $P \in S(N_0, D_0, r)$ that is not stabilized by C. Since (4.1) is false, there exists a ω_0 such that $\|\tilde{A}_c(j\omega_0)\| \, |r(j\omega_0)| > 1$. Select unitary matrices U, V

such that*

$$U\tilde{A}_c(j\omega_0)V = \begin{bmatrix} \sigma_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \sigma_m & 0 \end{bmatrix}, \qquad (4.3)$$

where $\sigma_1 \geq \cdots \geq \sigma_m$ are the singular values of $A_c(j\omega_0)$. Since $A_c(j\omega_0)A(j\omega_0) = I$, it follows that $A(j\omega_0)$ must be of the form

$$A_0(j\omega_0) = V \begin{bmatrix} 1/\sigma_1 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & 1/\sigma_m \\ & T & \end{bmatrix} U \qquad (4.4)$$

where T is some matrix. Now, by assumption, $\sigma_1|r(j\omega_0)| > 1$. Define

$$Q(s) = -V \begin{bmatrix} 1/\sigma_1 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & 0 \\ q & 0 & 0 \end{bmatrix} Ur(s)/r(j\omega_0), \qquad (4.5)$$

where q is a column vector chosen such that (i) $\|Q(j\omega_0)\| < |r(j\omega_0)|$, and (ii) the first column of $A_0(j\omega_0) + Q(j\omega_0)$ is not identically zero.† Let P be the plant ND^{-1} where $[D'\ N']' = A = A_0 + Q$. Then P belongs to the class $S(N_0, D_0, r)$, since $\|Q(s)\|\,|r(s)| < 1\ \forall s \in C_{+e}$. [The matrix Q may not be a real rational matrix, but this is easily fixed; see the discussion following the proof.] However, since

$$A(j\omega_0) = A_0(j\omega_0) + Q(j\omega_0)$$
$$= \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 1/\sigma_2 & & \vdots \\ \vdots & \vdots & \vdots & 1/\sigma_m \\ \times & \times & \times & \times \end{bmatrix} U \qquad (4.6)$$

we see that $|\tilde{A}_c A(j\omega_0)| = 0$, so that $A_c A$ is not unimodular.

The question is: does the singularity of $|\tilde{D}_c D + \tilde{N}_c N|$ at $j\omega_0$ imply that C fails to stabilize P? We can conclude that this is so provided we can show that this singularity does not come about as a result of a nontrivial common right divisor between N and D. To amplify this argument, suppose $N = N_1 B$, $D = D_1 B$, where B is a greatest common right divisor of N and D, and N_1 and D_1 are right-coprime. Then $[\tilde{D}_c D + \tilde{N}_c N| = |\tilde{D}_c D_1 + \tilde{N}_c N_1| \cdot |B|$. Hence, if it can be shown that $|B(j\omega_0)| \neq 0$, then $|\tilde{D}_c D_1 + \tilde{N}_c N_1|$ vanishes at $j\omega_0$, and C does not stabilize P. Thus the proof is complete if it can be shown that $|B(j\omega_0)| \neq 0$. For this it is enough to show that $A_0 + Q = [D'\ N']'$ has full column rank at $j\omega_0$. But this last fact is immediate from (4.6), since the first column of T is nonzero.

Let us digress briefly to consider the possibility that the matrix Q defined in (4.5) may not be a real rational matrix. If $\omega_0 = 0$ or ∞, then $Q(s)$ is clearly real rational. If ω_0 is nonzero and finite, proceed as follows: rewrite (4.5) in the form $Q(s) = r(s)M$ where M is a constant matrix whose definition is self-evident. Now M is a rank one, possibly complex, matrix, and $\|M\| < 1$ since $\sigma_1|r(j\omega_0)| > 1$ by assumption. It is easy to see that the argument in the proof of Theorem 5 is unaffected if the constant matrix M is replaced by a function $T(s)$ so long as (i) $\|T(\cdot)\|_\infty < 1$, and (ii) $T(j\omega_0) = M$. Thus it is shown that, given *any* complex rank one matrix M and any nonzero finite number ω_0, it is possible to construct a real rational function $T(s)$ such that $T(j\omega_0) = M$, and $\|T(\cdot)\|_\infty = \|M\|$. Factor the rank one matrix M in dyadic form as xy', and suppose without loss of generality that ω_0 is greater than zero. Express the vectors x, y in the form

$$x_i = \alpha_i \exp(j\phi_i),\ y_i = \beta_i \exp(j\theta_i), \qquad (4.7)$$

where the α_i, β_i are all real and $\phi_i, \theta_i \in (-\pi, 0]\ \forall i$. The idea is to generate a collection of all-pass functions such that at the frequency ω_0 they have the right values. This is done by defining

$$a_i(s) = \alpha_i \frac{s - \gamma_i}{s + \gamma_i}, \qquad b_i(s) = \beta_i \frac{s - \delta_i}{s + \delta_i}, \qquad (4.8)$$

and adjusting the constants γ_i, δ_i such that $a_i(j\omega_0) = x_i$, $b_i(j\omega_0) = y_i$. The matrix $T(s) = a(s)[b(s)]'$, where a, b are the vectors of the a_i, b_i, has the required properties. As this construction is quite general, it follows that in robustness studies one can as well use complex rational matrices instead of real rational matrices, as the proofs are much more transparent in this case.

There is an interesting anomaly associated with the robustness condition (4.1). Comparing the contents of Doyle and Stein (1981) and Chen and Desoer (1982), we see that it makes very little difference, in the case of additive and multiplicative perturbations, whether the class of perturbations is defined with "<" or "≤". If the class of perturbations is defined with a strict inequality, the robustness condition has a nonstrict inequality, and vice versa. But this is not so in the case of stable factor perturbations. Define the class $\bar{S}(N_0, D_0, r)$ by

* Note that $A_c(s)$ is a "fat" matrix and has full row rank at all $s \in C_{+e}$.

† If the first column of the matrix T in (4.4) is nonzero, simply choose $q = 0$. Otherwise, choose q to be any nonzero vector of sufficiently small norm that $\|Q(j\omega_0)\| < |r(j\omega_0)|$. Since the norm of the top part of $Q(j\omega_0)$ is $1/\sigma < |r(j\omega_0)|$, such a q can always be found.

$$\bar{S}(N_0, D_0, r) =$$
$$\left\{P = ND^{-1} : \left\|\begin{bmatrix} (N - N_0)(s) \\ (D - D_0)(s) \end{bmatrix}\right\| \leq |r(s)|\ \forall s \in C_{+e}\right\}. \qquad (4.9)$$

One might then be tempted to conjecture the following result: $C \in S(P_0)$ stabilizes all P in the class $\bar{S}(N_0, D_0, r)$ if and only if

$$\sup_\omega \|\tilde{A}_c(j\omega)\|\,|r(j\omega)| < 1. \qquad (4.10)$$

But this is false: (4.10) is certainly sufficient for robust stability, but not necessary, as the next example shows.

Example 1. Consider

$$p_0(s) = \frac{4}{s-3} = \frac{n_0(s)}{d_0(s)},$$

where

$$n_0(s) = \frac{4(s+7)}{(s+1)^2}, \qquad d_0(s) = \frac{(s-3)(s+7)}{(s+1)^2}.$$

Let $c(s) = 4/(s+5)$. Then c stabilizes p_0. Moreover, $d_c d_0 + n_c n_0 = 1$, where

$$d_c(s) = \frac{s+5}{s+7}, \qquad n_c(s) = \frac{4}{s+7}.$$

Now consider the class $\bar{S}(n_0, d_0, r)$ with $r = 1$. Since

$$\left\|\begin{bmatrix} d_c(\infty) \\ n_c(\infty) \end{bmatrix}\right\| = \left\|\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\| = 1,$$

(4.10) does not hold. Nevertheless, c stabilizes every p in the class $\bar{S}(n_0, d_0, r)$, which shows that (4.8) is not always necessary for robust stability.

To show this, suppose that $p \in \bar{S}(n_0, d_0, r)$. Thus

$$p(s) = \frac{n(s)}{d(s)} \text{ where } \|[n - n_0\ \ d - d_0]'(s)\|$$
$$\leq 1\ \forall s \in C_{+e}.$$

Let $Q = [q_1\ q_2]'$ denote $[d - d_0\ \ n - n_0]'$. Then there are two cases to consider, namely: (i) $q_1(\infty) + 1 \neq 0$, and (ii) $q_1(\infty) + 1 = 0$. In either case we have $\|[d_c(s)\ n_c(s)]\| < 1$ for all (finite) s in the RHP, so that

$$(d_c d + n_c n)(s) = 1 + [d_c\ n_c]Q|_s \neq 0\ \forall s \in C_+.$$

On the other hand, $[d_c(\infty)\ n_c(\infty)] = [1\ 0]$ so that

$$(d_c d + n_c n)(\infty) = 1 + q_1(\infty).$$

Hence, if $q_1(\infty) \neq 1$, then the return difference $d_c d + n_c n$ has no zeros in C_{+e}, and is thus a unit of **S**. Therefore c stabilizes p.

It only remains to show that c stabilizes p even if $q_1(\infty) = -1$. In this case $q_2(\infty) = 0$ since $\|Q(\infty)\| \leq 1$. Hence $n(\infty) = d(\infty) = 0$, and $\alpha = 1/(s+1)$ is a common divisor of n and d in the ring **S**. Further, since $(d_c d + n_c n)(s) \neq 0$ for all finite s in the RHP, α, its powers and associates are the only possible common divisors of n and d. Now it is claimed that, whatever be q_1, the function $d = d_0 + q_1$ can have only a simple zero at infinity. Let us accept this claim for a moment; then α is a greatest common divisor of n and d, since d/α does not vanish at infinity. Let $n_1 = n/\alpha$, $d_1 = d/\alpha$. Then (n_1, d_1) is a coprime factorization of p. Now

$$(d_c d_1 + n_c n_1)(\infty) = d_1(\infty)$$

$\neq 0$ since $d_1 = d/\alpha$ and d has only a simple zero at infinity. On the other hand, since it has already been established that $(d_c d + n_c n)(s) \neq 0$ for all finite s in the RHP, it follows that

$$(d_c d_1 + n_c n_1)(s) \neq 0\ \forall s \in C_{+e}.$$

This shows that c stabilizes p.

Thus the example is complete if the claim can be established. This is most easily done using the bilinear transformation $z = (s-1)/(s+1)$, which sends the function d_0 into

$$a_0(z) = d_0((1+z)/(1-z)) = (4 - 3z)/(2z - 1)$$

and sends $q_1(s)$ into an associated rational function $t_1(z)$. In this context, the claim is that $a_0 + t_1$ has only a simple zero at $z = 1$ whenever $t_1(1) = -1$ and $\|t_1\|_\infty = 1$. We prove the contrapositive, namely, if g is a rational H_∞-function such that $g(1) = 1$ and $a_0 - g$ has a double zero at $z = 1$, then $\|g\|_{\infty\infty} > 1$. Expand g in a power series around $z = 1$, as

$$g(z) = \sum_{i=0}^{\infty} g_i(z-1)^i.$$

If $a_0 - g$ has (at least) a double zero at $z = 1$, then

$$g(1) = a_0(1) = 1, g'(1) = a_0'(1) = -1.$$

Hence, for $z < 1$ and sufficiently close to 1, we have $g(z) > 1$. Therefore $\|g\| > 1$.

This completes the example.

Theorem 5 provides a necessary and sufficient condition for a $C \in S(P_0)$ to stabilize all plants in the class $S(N_0, D_0, r)$. The issue of whether such a C exists can be formulated in terms of an H_∞-norm minimization problem.

Theorem 6. Suppose a function $r \in$ **S** and a nominal plant $P_0 \in \mathbf{M}(\mathbf{S})$ are specified, together with an r.c.f. (N_0, D_0) and an l.c.f. $(\tilde{D}, \tilde{N})$ of P_0. Let X, $Y \in \mathbf{M}(\mathbf{S})$ satisfy $XN_0 + YD_0 = I$. Then there exists a $C \in S(P_0)$ that stabilizes all plants in the class $S(N_0, D_0, r)$ if and only if

$$\inf_{R \in \mathbf{M}(\mathbf{S})} \|A - RB\| \leq 1, \tag{4.11}$$

where

$$A = [Y \;\; X]r, \qquad B = [-\tilde{N}_0 \;\; \tilde{D}_0]r. \tag{4.12}$$

The proof is obvious and is left to the reader. Note that the matrix B in (4.11) always has more columns than rows. Hence the condition (4.12) can only be verified using the iterative method of Doyle (1983).

5. GRAPH METRIC PERTURBATIONS

In this section, we consider the case where the uncertainty in the plant is characterized by a graph metric perturbation. It turns out that simultaneous perturbations in both the plant and the controller can be very easily handled in this framework.

We begin with a brief description of the graph metric, which is defined in Vidyasagar (1984). An r.c.f. (N, D) of a plant P is said to be normalized if

$$[D'(-s) N'(-s)]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = I, \forall s. \tag{5.1}$$

It can be shown that every P has a normalized r.c.f., which is unique to within right multiplication by an orthogonal matrix. If (N, D) is a normalized r.c.f. of P, then $A = [D' \;\; N']'$ is inner, and as a result multiplication by A is an isometry. Now suppose P_1, P_2 are two plants with normalized r.c.f.s (N_1, D_1), (N_2, D_2), respectively. To define the graph metric distance $d(P_1, P_2)$ between the two plants, let

$$A_i = \begin{bmatrix} D_i \\ N_i \end{bmatrix}, \qquad \text{for } i = 1, 2. \tag{5.2}$$

$$\delta(P_1, P_2) = \inf_{U \in \mathbf{M}(\mathbf{S}),\, \|U\|_\infty \leq 1} \|A_1 - A_2 U\|_\infty, \tag{5.3}$$

$$d(P_1, P_2) = \max\{\delta(P_1, P_2), \delta(P_2, P_1)\}. \tag{5.4}$$

Then d is a metric on $\mathbf{M}(\mathbb{R}(s))$, taking values in the interval $[0, 1]$.

The presentation of the results in this section is made clearer by a bit of notation. Given P, $C \in \mathbf{M}(\mathbb{R}(s))$, define

$$T(P, C) = H(P, C) - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.5}$$

where $H(P, C')$ is defined in (2.7). Using this last equation, one can derive a few other useful representations of $T(P, C)$. For instance.

$$T(P, C') = \begin{bmatrix} -PC(I + PC)^{-1} & -P(I + CP)^{-1} \\ C(I + PC)^{-1} & (I + CP)^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} -P \\ I \end{bmatrix}(I + CP)^{-1}[C \;\; I]. \tag{5.6}$$

Theorem 7. Suppose the pair (P_0, C_0) is stable, and that P_0, C_0 are perturbed to P, C, respectively. Then the pair (P, C') is stable provided

$$d(P, P_0)\|T(P_0, C_0)\|_\infty + d(C, C_0)\|T(C_0, P_0)\|_\infty < 1. \tag{5.7}$$

Corollary 5.1. Under the hypotheses of Theorem 7, the pair (P, C_0) is stable provided $d(P, P_0) < 1/\|T(P_0, C_0)\|_\infty$.

The significance of Theorem 7 and Corollary 5.1 is as follows: the type of plant perturbations studied in these two results is the most unstructured one considered in this paper, in that (i) one is permitted to perturb simultaneously both the plant and the controller, (ii) there is no restriction on the number of RHP poles of the perturbed and unperturbed plant being the same, and (iii) the perturbations are not couched in terms of a particular coprime factorization of the plant. The stability condition (5.7) is interesting in that the effects of the perturbations in the plant and controller enter additively. When only the plant is perturbed, the stability condition given in Corollary 5.1 is reminiscent of the small gain theorem [see e.g. Desoer and Vidyasagar (1975)]. These results also serve to bring out the significance of the concept of the graph metric introduced in Vidyasagar (1984).

The proof requires the following easily proved result.

Lemma 5.1. Suppose (N_p, D_p), (N_c, D_c) are r.c.f.s of P, C, respectively. Then the pair (P, C) is stable if and only if the matrix

$$U = \begin{bmatrix} D_p & -N_c \\ N_p & D_c \end{bmatrix} \tag{5.8}$$

is unimodular.

Proof. Define

$$G = \begin{bmatrix} C & 0 \\ 0 & P \end{bmatrix}, \quad N_g = \begin{bmatrix} N_c & 0 \\ 0 & N_p \end{bmatrix}, \quad D_g = \begin{bmatrix} D_c & 0 \\ 0 & D_p \end{bmatrix}, \tag{5.9}$$

$$F = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \tag{5.10}$$

Then it is easy to verify that (N_g, D_g) is an r.c.f. of G, and that the transfer matrix $H(P, C)$ equals $(I + FG)^{-1}$. Since F is a constant matrix, an r.c.f. of $H(P, C)$ is given by the pair $(D_g, D_g + FN_g)$. Hence $H(P, C) \in \mathbf{M}(\mathbf{S})$ if and only if its "denominator" matrix $D_g + FN_g$ is unimodular. Now routine computation shows that this matrix is precisely U.

Proof of Theorem 7. Let (N_0, D_0), $(\tilde{X}_0, \tilde{Y}_0)$ be normalized r.c.f.s of P_0, C_0, respectively. Then from the stability of the pair (P_0, C_0), the matrix

$$U_0 = \begin{bmatrix} D_0 & -\tilde{X}_0 \\ N_0 & \tilde{Y}_0 \end{bmatrix} \tag{5.11}$$

is unimodular. Let $V_0 = U_0^{-1}$, and partition V_0 as

$$V_0 = \begin{bmatrix} Y_0 & X_0 \\ -\tilde{N}_0 & \tilde{D}_0 \end{bmatrix}. \tag{5.12}$$

Then $(\tilde{D}_0, \tilde{N}_0)$, (Y_0, X_0) are l.c.f.s of P_0, C_0, respectively.

Now select real numbers $\delta_p > d(P, P_0)$, $\delta_c > d(C, C_0)$ such that

$$\delta_p\|T(P_0, C_0)\|_\infty + \delta_c\|T(C_0, P_0)\|_\infty < 1. \tag{5.13}$$

This is possible in view of (5.7). Let (N_p, D_p), (N_c, D_c) be any normalized r.c.f.s of P, C respectively. Then, from the definition of the graph metric, there exist matrices W_p, $W_c \in \mathbf{M}(\mathbf{S})$ with $\|W_p\|_\infty \leq 1$, $\|W_c\|_\infty \leq 1$ such that

$$\left\|\begin{bmatrix} D_0 - D_pW_p \\ N_0 - N_pW_p \end{bmatrix}\right\|_\infty \leq \delta_p, \tag{5.14}$$

$$\left\|\begin{bmatrix} \tilde{Y}_0 - D_cW_c \\ \tilde{X}_0 - N_cW_c \end{bmatrix}\right\|_\infty \leq \delta_c. \tag{5.15}$$

Define F as in (5.11) and observe that F is inner so that multiplication by F is norm-preserving. Thus (5.15) implies that

$$\left\|\begin{bmatrix} -(\tilde{X}_0 - N_cW_c) \\ \tilde{Y}_0 - D_cW_c \end{bmatrix}\right\|_\infty \leq \delta_c. \tag{5.16}$$

Next, define

$$D_g = \begin{bmatrix} D_c & 0 \\ 0 & D_p \end{bmatrix},$$

$$V = \begin{bmatrix} D_pW_p & -N_cW_c \\ N_pW_p & D_cW_c \end{bmatrix}$$
$$= \begin{bmatrix} D_p & -N_c \\ N_p & D_c \end{bmatrix}\begin{bmatrix} W_p & 0 \\ 0 & W_c \end{bmatrix} \tag{5.17}$$

It is now shown that (5.13) implies the unimodularity of V. This will show, *a fortiori*, that W_p and W_c are both unimodular, and that the pair (P, C) is stable, the latter conclusion following from Lemma 5.1. Note that $V - U_0 = [A \;\; B]$ where

$$A = \begin{bmatrix} D_0 - D_pW_p \\ N_0 - N_pW_p \end{bmatrix}, \qquad B = \begin{bmatrix} -(\tilde{X}_0 - N_cW_c) \\ \tilde{Y}_0 - D_cW_c \end{bmatrix} \tag{5.18}$$

Now, if $\|[A \;\; B]U_0^{-1}\|_\infty < 1$, then $\|V - U_0)U_0^{-1}\|_\infty < 1$, which implies that V is unimodular. From (5.12),

$$[A \;\; B]U_0^{-1} = [A \;\; B]V_0 = A[Y_0 \;\; X_0] + B[-\tilde{N}_0 \;\; \tilde{D}_0]. \tag{5.19}$$

$$\|[A \;\; B]U_0^{-1}\|_\infty \leq \|A[Y_0 \;\; X_0]\|_\infty + \|B[-\tilde{N}_0 \;\; \tilde{D}_0]\|_\infty$$
$$\leq \delta_p\|[Y_0 \;\; X_0]\|_\infty + \delta_c\|[-\tilde{N}_0 \;\; \tilde{D}_0]\|_\infty \tag{5.20}$$

where the last step follows from (5.14) and (5.16). The proof is completed by showing that

$$\|[Y_0 \;\; X_0]\|_\infty = \|T(P_0, C_0)\|_\infty,$$
$$\|[-\tilde{N}_0 \;\; \tilde{D}_0]\|_\infty = \|T(C_0, P_0)\|_\infty. \tag{5.21}$$

Then (5.13) and (5.20) will imply the unimodularity of V and the stability of (P, C).

To prove the first part of (5.21), recall that (N_0, D_0) is a normalized r.c.f. of P_0, and (Y_0, X_0) is the corresponding l.c.f. of C_0 such that $Y_0D_0 + X_0N_0 = I$. Hence, from (5.6),

$$T(P_0, C_0) = \begin{bmatrix} -N_0X_0 & -N_0Y_0 \\ D_0X_0 & D_0Y_0 \end{bmatrix}$$
$$= \begin{bmatrix} -N_0 \\ D_0 \end{bmatrix}[X_0 \;\; Y_0]; \tag{5.22}$$

$$\|T(P_0, C_0)\|_\infty = \|[X_0 \;\; Y_0]\|_\infty$$

$$\text{since } \begin{bmatrix} -N_0 \\ D_0 \end{bmatrix} \text{ is an isometry}$$

$$= \|[Y_0 \;\; X_0]\|_\infty. \tag{5.23}$$

The proof of the second half of (5.21) follows essentially by symmetry arguments, after noting that $\|[-\tilde{N}_0 \;\; \tilde{D}_0]\|_\infty = \|[\tilde{N}_0 \;\; \tilde{D}_0]\|_\infty$.

It is not known at present how close the conditions of Theorem 7 and Corollary 5.1 are to being necessary. In particular, it is not known whether the condition $\|T(P_0, C)\|_\infty < r^{-1}$ is necessary for a controller C to stabilize all plants within a distance of r from the plant P_0. Nevertheless, it is reasonable to seek, among all stabilizing controllers for P_0, an "optimally robust" controller C_0 for which $\|T(P_0, C_0)\|$ is as small as possible. The problem of finding such a C_0 can be formulated as an H_∞-norm minimization problem. As C varies over $S(P_0)$, the corresponding $T(P_0, C)$ vary over all matrices of the form

$$T_1(R) = \begin{bmatrix} -N_0(X_0 + R\tilde{D}_0) & -N_0(Y_0 - R\tilde{N}_0) \\ D_0(X_0 + R\tilde{D}_0) & D_0(Y_0 - R\tilde{N}_0) \end{bmatrix} \quad (5.24)$$

where (N_0, D_0), $(\tilde{D}_0, \tilde{N}_0)$ are any r.c.f. and l.c.f. of P_0, X_0, $Y_0 \in \mathbf{M}(\mathbf{S})$ satisfy $X_0N_0 + Y_0D_0 = I$, and $R \in \mathbf{M}(\mathbf{S})$ is a free parameter. Thus, minimizing $\|T(P_0, C)\|_\infty$ over all $C \in S(P_0)$ is equivalent to the unconstrained minimization of $\|T_1(R)\|_\infty$ as R varies over $\mathbf{M}(\mathbf{S})$. Now note that $T_1(R)$ is of the form $U - VRW$, where

$$U = \begin{bmatrix} -N_0 \\ D_0 \end{bmatrix} [X_0 \ \ Y_0], \qquad V = \begin{bmatrix} -N_0 \\ D_0 \end{bmatrix}, \quad (5.25)$$

$$W = [\tilde{D}_0 \ \ -\tilde{N}_0].$$

However, since V has more rows than columns and W has more columns than rows, the quantity

$$\inf_{C \in S(P_0)} \|T(P_0, C)\|_\infty = \inf_{R \in \mathbf{M}(\mathbf{S})} \|U - VRW\|_\infty \quad (5.26)$$

can only be computed iteratively using the method of Doyle (1983).

6. CONCLUSIONS

This paper contains three distinct contributions: (i) in the case of additive and multiplicative perturbations, necessary and sufficient conditions are given for the existence of a robustly stabilizing controller; (ii) in the case of stable-factor perturbations, necessary and sufficient conditions are given for a controller to stabilize all plants within the specified class; (iii) in the case of graph metric perturbations, which also includes simultaneous perturbations in the plant as well as controller, sufficient conditions for robust stabilization are given. In the latter two cases, conditions for the existence of a robustly stabilizing controller are given in the form of the value of a certain H_∞-norm minimization problem being less than one.

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Brief Paper

Fenchel Duality and Smoothness of Solution of the Optimal Routing Problem*

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Key Words—Computer networks; traffic control; convex programming; duality theory; minimum time control.

Abstract—A linear state and control constrained problem arising in optimal routing in communication networks is investigated by Fenchel duality methods. The problem reduces to a dual program having a particularly simple solution.

1. *Introduction*

IN DEALING with state constrained optimal control problems, one encounters discontinuities in the time derivatives of state components whenever some component hits the boundary. These correspond to jumps of multipliers at these time instances. The behaviour of optimal control and state trajectory at these points is practically important and many refining assertions around the necessary conditions of optimality can be obtained, if one restricts the generality of dynamics and constraints (see typically Maurer, 1977). A case in point where smoothness of solution can lead to substantial simplification of on-line algorithms, is the optimal dynamic routing problem in single destination communication networks. As originally attacked by Segall (1977), Moss (1977) and Moss and Segall (1982), this reduces without undue simplifications to a linear dynamics, linear cost, state and control constrained problem, with the state absent from the right hand side of the dynamic equation, state constraints $x(t) \geqq 0$ and convex, time constant control constraints. This model is introduced in Section 2 and has become standard in the communication network literature (Segall, 1977; Moss, 1977; Moss and Segall, 1982; Sarachik and Özgüner, 1982; Hajek and Ogier, 1984; Stassinopoulos and Konstantopoulos, 1985). Segall's original development based on a Kuhn–Tucker theorem (see Section II, Proof of Theorem 1 in Moss and Segall, 1982) leads via an optimal control in feedback form to solutions, where all state components can exhibit discontinuous derivatives of all trajectory components, whenever some component hits its respective boundary. This paper shows (Sections 4–6) that optimal solutions always exist, where this does not happen, so that optimal trajectory components have constant slopes throughout the interval, where these components are nonzero. Hence the problem reduces to a finite dimensional dual program. Furthermore, due to the special geometry of the velocity set, the problem can be solved by a finite algorithm.

The property of constant slopes has been observed (Moss, 1977; Hajek and Ogier, 1984), without a simple proof, free from arguments relevant only to the communication network case. We attack here the corresponding control problem by Fenchel duality methods and illustrate in Section 6 which geometric properties of the velocity set are needed along the way and are thus responsible for the simplicity of the final result. Since our development departs right from the start from the network case and is solely based on the geometry of the velocity set, its scope and applicability are broader. This contrasts with the development in Hajek and Ogier (1984), where the property of the constant trajectory slopes is derived through a network theoretic approach. By restricting in this way the applicability of the method, a finite algorithm of polynomial complexity in the number of nodes is presented (Hajek and Ogier, 1984). Our approach, based on the velocity set is broader, however the resulting finite algorithm is of exponential complexity.

We emphasize that the removal of state constraints, either directly, or by allowing only positive velocities, makes the problem trivial, solvable in a few lines with the results of, say Hermes and Lasalle (1969). We concentrate here on aspects interesting to control theorists and give extensions of importance to communication networks in Stassinopoulos and Konstantopoulos, 1985.

2. *The control problem*

The dynamics are governed by linear equations, with the right hand side independent from the state

$$\dot{x}(t) = Bu(t) \quad (1)$$

with $x(t) \varepsilon R^n$ the state vector with components $x_i(t)$, $u(t) \varepsilon R^m$ the control vector and B a $n \times m$ incidence matrix with elements 1, -1 or 0. We have the following constraints:

$$\text{Control constraints } u \varepsilon U \quad (2)$$

with U the rectangle $U = \{u \varepsilon R^m / 0 \leqq u_j \leqq C_j, j = 1, \ldots, m\}$.

$$\text{State constraints } x(t) \geqq 0 \qquad \forall t. \quad (3)$$

The control problem is to steer to the origin a given initial state $x_0 = x(0)$ lying in the positive orthant $(R^n)^+$, while minimizing either one of the two criteria:

$$\text{Minimum total delay: } J_D = \int_0^\infty \sum_{i=1}^n x_i(t) \mathrm{d}t \quad (4)$$

or

$$\text{Minimum time: } J_T = \int_0^{t^*} \mathrm{d}t \quad (5)$$

where
$$t^* = \min\{t \varepsilon [0, \infty) / x(t) = 0\}. \quad (6)$$

In the following development we will place special interest in the (negative) velocity set

$$V = \{v \varepsilon R^n / v = -Bu, \qquad u \varepsilon U\}. \quad (7)$$

Due to the special form of B and U, V is convex, polyhedral and can be described (Moss, 1977; Stassinopoulos and Konstantopoulos, 1984) by linear inequalities of the form

$$\sum_{i \varepsilon F} v_i \leqq \gamma_F \quad (8)$$

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