Approximation Properties of Some Modified Summation-Integral Type Operator

Ekta Pandey, R.K.Mishra, S.P.Pandey

Abstract- ***In the present paper, we introduce some Stancu type generalization of Szász-Mirakyan-Baskakov type operators. We estimate the moments for these operators using the hypergeometric series, which can be related to Laguerre polynomials. We estimate point wise convergence, asymptotic expansion and error estimate in terms of higher order modulus of continuity of function in simultaneous approximation for these generalized operators. We use the technique of linear approximating method viz. Steklov mean.***

Keywords. ***linear positive operators, simultaneous approximation, Lebesgue function, modulus of continuity, Steklov mean, hypergeometric series, Laguerre polynomial .***

I. Introduction

In the theory of approximation we have several integral modification of the Szász operators, which include Szász-Durrmeyer operator, Szász-Kantorovich operators, Szász-Beta operators, and Szász-Mirakyan operators. The approximation properties of several linear positive operators have been discussed in the past few decades. Some of the important results have been given in the recent books by Aral-Gupta-Agarwal [1] and Gupta-Agarwal [3]. Gupta and Srivastava [6] proposed a family of linear positive operators, by combining the Szász-Mirakyan and Baskakov basis functions to approximate Lebesgue integrable functions on $[0,\infty)$ as follows:

$$M_n(f,x)=(n-1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}q_{n,k}(t)f(t)dt\,,\ x\in[0,\infty),\qquad(1.1)$$

where

$$p_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!},q_{n,k}(t)=\frac{(n+k-1)!}{k!(n-1)!}\frac{t^k}{(1+t)^{n+k}}.\qquad(1.2)$$

We give a more interesting way to represent these operators in terms of confluent hyper-geometric function, defined as

$${}_1F_1(a;b;x)=\frac{(a)_k\,x^k}{(b)_k\,k!},$$

where $(a)_k$ is called Pochhammer symbol and is defined as $(a)_k=a(a+1)(a+2)....(a+k-1)$ and $(1)_k=k!$. Then $M_n(f,x)$ can be written as:

$$M_n(f,x)=(n-1)\int_0^{\infty}\frac{e^{-nx}f(t)}{(1+t)^n}\ {}_1F_1(n;1;\frac{nxt}{1+t})dt.\qquad 1.3)$$

This is an alternate form of the operator (1.1) in terms of confluent hypergeometric series.

In the year 1972 D. D. Stancu [12] introduced a very general class of positive linear operator generalizing the well known operators as those of Bernstein, Baskakov and Favard Szász operator. In recent years a great work has been done on Stancu type operators by several authors [13], [8], [9]. Motivated by the recent work on Stancu type operators due to Gupta-Yadav [9] and Gupta-Tachev [7], here we introduce the Stancu type generalization of the Szász-Mirakyan-Baskakov type operator defined by (1.1) based on two parameters α,β such that $0\le\alpha\le\beta$ as follows:

$$M_{n,\alpha,\beta}(f,x)=(n-1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}q_{n,k}(t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt,\quad x\in[0,\infty),\qquad(1.4)$$

where the Baskakov and Szász-Mirakyan operators are defined in (1.2). Also, in terms of hypergeometric series, one can write the operators as follows:

$$M_n(f,x)=(n-1)\int_0^{\infty}\frac{e^{-nx}f\left(\frac{nt+\alpha}{n+\beta}\right)}{(1+t)^n}\ {}_1F_1\left(n;1;\frac{nxt}{1+t}\right)dt.\qquad(1.5)$$

Remark 1.1 *If we put* $\alpha=0$ *and* $\beta=0$ ***then the operator (1.3) will reduce to the operator defined in (1.1).***

In the present paper we estimate point wise convergence, asymptotic expansion and error estimate in terms of higher order modulus of continuity of function in simultaneous approximation for these generalized operators. We use the technique of linear approximating method viz. Steklov mean.To establish the main results we need the following auxiliary results.

II. Auxiliary Results

Lemma 1. [10] Let $r\in\mathbb{N}\cup 0$, if the r-th order moment is defined as

$$U_{n,r}(x)=\sum_{k=0}^{\infty}p_{n,k}(x)\left(\frac{k}{n}-x\right)^r,$$

then there exists a recurrence relation

$$nU_{n,r+1}(x)=x[U'_{n,r}(x)+rU_{n,r-1}(x)].$$

Consequently

1. $U_{n,r}(x)$ is a polynomial in x of degree $\le$ r.
2. $U_{n,r}(x)=O(n^{-[(r+1)/2]})$ as $n\to\infty$.

Manuscript Received on March 2015.

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Lemma 2. [13] Let the central moment be defined as

$$T_{n,r}(x) = M_{n,\alpha,\beta}((t-x)^r, x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} q_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^r dt,$$

then $T_{n,0}(x)=1$, $T_{n,1}(x)=\frac{\alpha-\beta x}{n+\beta}+\frac{n(1+2x)}{(n-2)(n+\beta)}$ and for $n>1$ we have the following recurrence relation:

$$(n+\beta)(n-r-2)T_{n,r+1} = nxT_{n,r}^{'}(x)+rT_{n,r-1}(x) + \{n(r+1)+2(r+1)((n+\beta)x-\alpha)+n(\alpha-\beta x)\}T_{n,r}(x)$$

$$+\left\{nr\left(x-\frac{\alpha}{n+\beta}\right)+r(n+\beta)\left(x-\frac{\alpha}{n+\beta}\right)^2\right\}T_{n,r-1}(x).$$

From the recurrence relation, it can easily be verified that,

$$T_{n,r}(x) = O(n^{-[(m+1)/2]}) \text{ as } n\to\infty \text{ for all } x\in[0,\infty).$$

Lemma 3. $M_{n,\alpha,\beta}(t^r, x)$ is a polynomial in x of degree exactly r, $\forall r\in N^0$, we have

$$M_{n,\alpha,\beta}(t^r,x) = \frac{n^{2r}(n-r-2)!}{(n+\beta)^r(n-2)!}x^r + \frac{n^{2r-2}r(n-r-2)!}{(n+\beta)^r(n-2)!}\{nr+\alpha(n-r-1)\}x^{r-1}$$

$$+\frac{n^{2r-4}r(r-1)(n-r-2)!}{(n+\beta)^r(n-2)!}\left\{\frac{n^2}{2}r(r-1)+\alpha(r-1)n(n-r-1)+\frac{\alpha^2}{2}(n-r)(n-r-1)\right\}x^{r-2}$$

$$+O(n^{-2}).$$

Proof. Using Pochhammer symbol and confluent hypergeometric function, defined in section 1, we can write as

$$M_n(t^r,x) = e^{-nx}\frac{(n-r-2)!r!}{(n-2)!}\,{}_1F_1(r+1;1;nx). \qquad (2.1)$$

Also by Kummer's transformation ${}_1F_1(a;b;x) = e^x\,{}_1F_1(b-a;b;-x)$, we can write

$$M_n(t^r,x) = \frac{(n-r-2)!r!}{(n-2)!}\,{}_1F_1(-r;1;-nx).$$

Further the confluent hypergeometric function is related with generalized Laguerre polynomial with the relation

$$L_n^m(x) = \frac{(m+n)!}{m!n!}\,{}_1F_1(-n;m+1;x),$$

therefore we can write $M_n(t^r,x)$ in terms of Laguerre polynomial as follows,

$$M_n(t^r,x) = \frac{(n-r-2)!r!}{(n-2)!}L_r(-nx),$$

where $L_k(x) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{x^j}{j!}$. So we can write

$$M_n(t^r,x) = \frac{(n-r-2)!r!}{(n-2)!}\sum_{j=0}^{r}\binom{r}{j}\frac{(nx)^j}{j!}. \qquad (2.2)$$

Further using binomial theorem, the relation between the operator (1.3) and (1.4) can be defined as follows

$$M_{n,\alpha,\beta}(t^r,x) = \sum_{j=0}^{r}\binom{r}{j}\frac{\alpha^{r-j}n^j}{(n+\beta)^r}M_n(t^j,x).$$

Now by using (2.2) one can obtain the required result.

Lemma 4. [6] There exist the polynomial $\phi_{i,j,r}(x)$ independent of n and k such that

$$x^r\frac{d^r}{dx^r}[e^{-nx}(nx)^k] = \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i(k-nx)^j\phi_{i,j,r}(x)[e^{-nx}(nx)^k].$$

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Lemma 5. Let f be r times differentiable on $[0,\infty)$ such that $f^{(r-1)} = O(t^{\alpha})$, for some $\alpha > 0$ as $t \to \infty$ then for r= 1,2,3.... and $n > \alpha + r$, we have

$$M^{(r)}_{n,\alpha,\beta}(f,x) = \frac{n^{2r}(n-r-1)!}{(n-2)!(n+\beta)^r}\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} q_{n-r,k+r}(t) f^{(r)}(\frac{nt+\alpha}{n+\beta})dt.$$

Proof. Using Leibnitz theorem

$$M^{(r)}_{n,\alpha,\beta}(f,x) = (n-1)\sum_{i=0}^{r}\sum_{k=i}^{\infty}\binom{r}{i}\frac{(-1)^{r-i} n^r e^{-nx}(nx)^{k-i}}{(k-i)!}\int_0^{\infty} q_{n,k}(t) f(\frac{nt+\alpha}{n+\beta})dt$$

$$= (n-1)\sum_{k=0}^{\infty}\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i} n^r p_{n,k}(x)\int_0^{\infty} q_{n,k+i}(t) f(\frac{nt+\alpha}{n+\beta})dt$$

$$= (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty}(-1)^r\sum_{i=0}^{r}\binom{r}{i}(-1)^i n^r q_{n,k+i}(t) f(\frac{nt+\alpha}{n+\beta})dt.$$

By Leibnitz theorem

$$q^{(r)}_{n-r,k+r}(t) = \frac{(n-1)!}{(n-r-1)!}\sum_{i=0}^{r}(-1)^i\binom{r}{i} q_{n,k+i}(t),$$

we obtain after simple computation

$$M^{(r)}_{n,\alpha,\beta}(f,x) = \frac{n^r(n-r-1)!}{(n-2)!}\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty}(-1)^r q^{(r)}_{n-r,k+r}(t) f(\frac{nt+\alpha}{n+\beta})dt.$$

Further integrating by part r times, we get the required result.

III. Main Results

In this section we give the direct results, we establish here point wise approximation, asymptotic formula and an error estimation in simultaneous approximation. We need the following definitions to prove the results:

Definition 1. The m th order modulus of continuity for a function continuous on $[a,b]$ is defined as

$$\omega_m(f,\delta,[a,b]) = sup\{|\Delta_h^m f(x)|:|h|\le\delta; x, x+h \in [a,b]\}.$$

Definition2. Let

$C_\gamma[0,\infty) = \{f \in C[0,\infty) : |f(t)| \le Mt^{\gamma}, for some \gamma > 0\}$,

the norm $\|.\|$ on $C_\gamma[0,\infty)$ is defined by

$\|f\|_\gamma = \sup_{0\le t\le\infty}|f(t)|t^{-\gamma}$,

then for $0 < a < a_1 < b_1 < b < \infty$, for sufficiently small $\eta > 0$ the Steklov mean $f_{\eta,2}$ of 2^{nd} order corresponding to $f \in C_\gamma[a,b]$ and $t \in [a,b]$ is defined as follows:

$$f_{\eta,2}(t) = \eta^{-2}\int_{-\eta/2}^{\eta/2}\int_{-\eta/2}^{\eta/2}(f(t) - \Delta_h^2 f(t))dt_1 dt_2,$$

where $h = (t_1 + t_2)/2$ and Δ_h^2 is the second order forward difference operator with step length h. For $f \in C[a,b]$, $f_{\eta,2}$ satisfy the following properties:

(1) $f_{\eta,2}$ has continuous derivatives up to order 2 over $[a_1,b_1]$,

(2) $\|f_{\eta,2}\|_{C[a_1,b_1]} \le C\omega_r(f,\eta,[a,b])$,

(3) $\|f - f_{\eta,2}\|_{C[a_1,b_1]} \le C\omega_2(f,\eta,[a,b])$,

(4) $\|f_{\eta,2}\|_{C[a_1,b_1]} \le C\eta^{-2}\|f\|_{C[a,b]}$,

(5) $\|f_{\eta,2}\|_{C[a_1,b_1]} \le C\|f\|_\gamma$,

where by C, we mean certain constants not same at each occurrence and are independent of f and η.

Theorem 1. *(Point wise convergence)* Let α, β be the two parameters satisfying the conditions $0 \le \alpha \le \beta$. If $r \in N \in, f \in C_\gamma[0,\infty)$ for some $\gamma > 0$ and $f^{(r)}$ exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty}\left[M^{(r)}_{n,\alpha,\beta}(f,w)\right]_{w=x} = f^{(r)}(x).$$

Proof . By Taylor's expansion of f

$$f(t) = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$

$$\left[\frac{d^r}{dw^r}M_{n,\alpha,\beta}(f,w)\right]_{w=x} = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\left[M^{(r)}_{n,\alpha,\beta}((t-x)^i,w)\right]_{w=x}$$

$$+\left[M^{(r)}_{n,\alpha,\beta}(\varepsilon(t,x)(t-x)^r,w)\right]_{w=x}$$

$$=: I_1 + I_2.$$

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Now from Lemma 3 and Lemma 5

$$I_1 = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \left[M_{n,\alpha,\beta}^{(r)}(t^j, w)\right]_{w=x}$$

$$= \frac{f^{(r)}(x)}{r!} \frac{n^{2r}(n+r-1)!r!}{(n+\beta)^r(n-2)!}$$

$$\to f^{(r)}(x) \ \ as \ \ n \to \infty.$$

Now estimate I_2 by using Lemma 4, we obtain

$$I_2 = (n-1) \sum_{\substack{2i+j\le r \\ i,j\ge 0}} \frac{n^i}{x^r} \phi_{i,j,r}(x) \sum_{k=0}^{\infty} \frac{(k-nx)^j}{k!} e^{-nx}(nx)^k$$

$$\times \int_0^{\infty} q_{n,k}(t)\varepsilon(t,x)\left(\frac{nt+\alpha}{n+\beta} - x\right)^r dt$$

$$= (n-1) \sum_{\substack{2i+j\le r \\ i,j\ge 0}} \frac{n^i}{x^r} \phi_{i,j,r}(x) \sum_{k=0}^{\infty} \frac{(k-nx)^j}{k!} e^{-nx}(nx)^k$$

$$\int_0^{\infty} q_{n,k}(t)\varepsilon(t,x)\left(\frac{nt+\alpha}{n+\beta} - x\right)^r dt.$$

This gives

$$|I_2| \le (n-1) \sum_{\substack{2i+j\le r \\ i,j\ge 0}} \frac{n^i}{x^r} |\phi_{i,j,r}(x)| \sum_{k=0}^{\infty} |(k-nx)^j| \, p_{n,k}(x)$$

$$\times \int_0^{\infty} q_{n,k}(t) |\varepsilon(t,x)| \left|\frac{nt+\alpha}{n+\beta} - x\right|^r dt.$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$, for a given $\varepsilon > 0$ there exist $\delta > 0$ such that $|\varepsilon(t,x)|$ whenever $|t-x| < \delta$, further if λ is any integer $\ge max\{\gamma, r\}$ then we find a constant $K > 0$ such that $|\varepsilon(t,x)| \left|\frac{nt+\alpha}{n+\beta} - x\right|^r \le K \left|\frac{nt+\alpha}{n+\beta} - x\right|^\gamma$. Thus for some $c_1 > 0$ we can write

$$I_2 = (n-1)C_1 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} n^i \sum_{k=0}^{\infty} |(k-nx)^j| \, p_{n,k}(x) \left\{ \int_{|t-x|<\delta} q_{n,k}(t) |\varepsilon(t,x)| \left|\frac{nt+\alpha}{n+\beta} - x\right|^r dt \right.$$

$$\left. + \int_{|t-x|\ge\delta} K q_{n,k}(t) \left|\frac{nt+\alpha}{n+\beta} - x\right|^\gamma dt \right\}$$

$$=: I_3 + I_4,$$

where

$$C_1 = \sup_{\substack{2i+j\le r \\ i,j>0}} \frac{|\phi_{i,j,r}(x)|}{x^r},$$

and K is independent of C_1. Next, using Schwarz inequality for the integration and summation, we obtain

$$I_3 \le (n-1)\varepsilon C_1 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} n^i \sum_{k=0}^{\infty} |(k-nx)^j| \, p_{n,k}(x) \left(\int_0^{\infty} q_{n,k}(t)dt\right)^{1/2}$$

$$\times \left(\int_0^{\infty} q_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^{2r} dt\right)^{1/2}$$

$$= \varepsilon C_1 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} n^i \left(\sum_{k=0}^{\infty} p_{n,k}(x)(k-nx)^{2j}\right)^{1/2}$$

$$\times \left(\sum_{k=0}^{\infty} (n-1) p_{n,k}(x) \int_0^{\infty} q_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^{2r} dt\right)^{1/2},$$

as $\int_0^{\infty} q_{n,k}(t)dt = \frac{1}{n-1}$. Now making use of Lemma 1 we get

$$\sum_{k=0}^{\infty} p_{n,k}(x)(k-nx)^{2j} = n^{2j}\left[\sum_{k=0}^{\infty} p_{n,k}(x)\left(\frac{k}{n} - x\right)^{2j}\right] = n^{2j}\left[O(n^{-j})\right] = O(n^j).$$

Also, by Lemma 2

$$(n-1)\sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} q_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^{2r} dt = O(n^{-r}).$$

Thus

$$I_3 \le \varepsilon C_1 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} n^i O(n^{j/2}) O(n^{-r/2}) = \varepsilon O(1).$$

Similarly using Schwarz inequality

$$I_4 = O(1).$$

Thus due to arbitrariness of ε, it follows that $I_2 = O(1)$. Combining the estimates of I_1 and I_2, we obtain the desired result.

Theorem 2. *(Asymptotic expansion)* Let $f \in C_\gamma[0,\infty)$ be bounded on every finite sub-interval of $[0,\infty)$ admitting the derivative of order $(r+2)$ at a fixed

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$x\in(0,\infty)$. Let $f(t)=O(t^{\gamma})$ as $t\to\infty$ for some $\gamma>0$, then we have

$$\lim_{n\to\infty} n\left(\left(M^{(r)}_{n,\alpha,\beta}(f,w)\right)_{w=x}-f^{(r)}(x)\right)=r\{\frac{r+3}{2}-\beta\}f^{(r)}(x)$$
$$+\{(r+1)+(r+2)x+\alpha-\beta x\}f^{(r+1)}(x)$$
$$+\{x^2+2(r+2+\alpha)x\}f^{(r+2)}(x).$$

Proof. By Taylor's expansion

$$f(t)=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{(i)!}(t-x)^i+\varepsilon(t,x)(t-x)^{(r+2)},$$

where $\varepsilon(t,x)\to x$ as $t\to x$ and $\varepsilon(t,x)=o((t-x)^{\delta})$ as $t\to\infty$ for some $\delta>0$.

Using Lemma 4, we can write

$$n\left(\left(M^{(r)}_{n,\alpha,\beta}(f,w)\right)_{w=x}-f^{(r)}(x)\right)=n\left(\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{(i)!}\left[M^{(r)}_{n,\alpha,\beta}((t-x)^i,w)\right]_{w=x}-f^{(r)}(x)\right)$$

$$+n(M^{(r)}_{n,\alpha,\beta}(\varepsilon(t,x)(t-x)^{(r+2)},w))_{w=x}$$
$$=:I_1+I_2.$$

By Lemma 2 and 3, we have

$$I_1=nf^{(r)}(x)\{\frac{n^{2r}(n-r-2)!}{(n+\beta)^r(n-2)!}-1\}$$

$$+n\frac{f^{(r+1)}(x)}{(r+1)!}\{(r+1)\frac{(-x)n^{2r}(n-r-2)!r!}{(n+\beta)^r(n-2)!}$$
$$+\frac{n^{2r+2}(n-r-3)!}{(n+\beta)^{r+1}(n-2)!}(r+1)!x$$

$$+\frac{n^{2r}(r+1)(n-r-3)!}{(n+\beta)^{r+1}(n-2)!}[n(r+1)+\alpha(n-r-2)]r!\}$$

$$+n\frac{f^{(r+2)}(x)}{(r+2)!}\{\frac{n^{2r+4}(n-r-4)!(r+2)!}{(n+\beta)^{r+2}(n-2)!}\frac{x^2}{2}$$

$$+\frac{n^{2r+2}(r+2)(n-r-4)!}{(n+\beta)^{r+2}(n-2)!}[n(r+2)+\alpha(n-r-3)](r+1)!x$$

$$+\frac{n^{2r}(r+2)(r+1)(n-r-4)!}{(n+\beta)^{r+2}(n-2)!}[\frac{n^2}{2}(r+2)(r+1)$$

$$+\alpha(r+1)n(n-r-3)+\frac{\alpha^2}{2}(n-r-2)(n-r-3)]r!$$

$$+(r+2)(-x)(\frac{n^{2r+2}(n-r-3)!(r+1)!x}{(n+\beta)^{r+1}(n-2)!}$$

$$+\frac{n^{2r}(r+1)(n-r-3)!}{(n+\beta)^{r+1}(n-2)!}[n(r+1)+\alpha(n-r-2)]r!)$$

$$+\frac{(r+2)(r+1)}{2}x^2\frac{n^{2r}(n-r-2)!}{(n+\beta)^r(n-2)!}r!\}.$$

Now taking limit $n\to\infty$ and using induction hypothesis coefficient of $f^{(r)}(x), f^{(r+1)}(x)$ and $f^{(r+2)}(x)$ in the above expression are respectively $r(\frac{r+3}{2}-\beta)$, $\{(r+1)+(r+2)x+\alpha-\beta x\}$ and $\{x(x+2(r+2+\alpha))\}$. Hence in order to prove the theorem it is sufficient to show that $x^r I_2\to 0$ as $n\to\infty$, which follows on proceeding along the lines in the estimation of I_2 as done in Theorem 1.

Theorem 3 *(Error estimation)* Let $f\in C_{\gamma}[0,\infty)$ for some $\gamma>0$ and $0<a<a_1<b_1<b<\infty$. Then for n sufficiently large, we have

$$\| M^{(r)}_{n,\alpha,\beta}(f,.)-f^{(r)}\|_{C[a_1,b_1]}\le C_1\omega_2(f^{(r)},n^{-1/2},[a,b])+C_2n^{-1}\|f\|_{\gamma},$$

where $C_1=C_1(r)$ and $C_2=C_2(r,f)$.

Proof. By linearity property

$$\left\|M^{(r)}_{n,\alpha,\beta}(f,.)-f^{(r)}\right\|_{C[a_1,b_1]}\le\left\|M^{(r)}_{n,\alpha,\beta}((f-f_{\eta,2}),.)\right\|_{C[a_1,b_1]}$$
$$+\left\|M^{(r)}_{n,\alpha,\beta}(f_{\eta,2},.)-f^{(r)}_{\eta,2})\right\|_{C[a_1,b_1]}$$
$$+\left\|f^{(r)}-f^{(r)}_{\eta,2})\right\|_{C[a_1,b_1]}$$

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$$=: T_1 + T_2 + T_3,$$

since $f_{\eta,2}^{(r)} = (f^{(r)})_{\eta,2}$, hence by property (3) of Steklov mean, we get

$$T_3 \le C_1 \omega_2(f^{(r)}, \eta, [a,b]).$$

Next, using Theorem 2, we obtain

$$T_2 \le C_2 n^{-1} \sum_{i=r}^{2+r} \left\| f_{\eta,2}^{(i)} \right\|_{C[a,b]}$$

$$\le C_4 n^{-1} \left\{ \left\| f_{\eta,2} \right\|_{C[a,b]} + \left\| f_{\eta,2}^{(2+r)} \right\|_{C[a,b]} \right\},$$

now by properties (2) and (4) of Steklov mean, we obtain

$$T_2 \le C_4 n^{-1} \left\{ \| f \|_\gamma + \eta^{-2} \omega_2(f^{(r)}, \eta, [a,b]) \right\}.$$

Finally, we estimate T_1 choosing a,b satisfying the condition $0 < a < a^* < a_1 < b_1 < b^* < b < \infty$. Also let $\chi(t)$ denotes the characteristic function in the interval $[a^*, b^*]$, then

$$T_1 \le \left\| M_{n,\alpha,\beta}^{(r)}(\chi(t)(f(t) - f_{\eta,2}(t)),.) \right\|_{C[a_1,b_1]}$$

$$+ \left\| M_{n,\alpha,\beta}^{(r)}((1-\chi(t))(f(t) - f_{\eta,2}(t)),.) \right\|_{C[a_1,b_1]}$$

$$=: T_4 + T_5.$$

Now by Lemma 5

$$M_{n,\alpha,\beta}^{(r)}(\chi(t)(f(t) - f_{\eta,2}(t)),x) = \frac{n^{2r}(n-r-1)!}{(n-2)!(n+\beta)^r} \sum_{k=0}^{\infty} p_{n,k}(x)$$

$$\times \int_0^\infty q_{n-r,k+r}(t)\chi(t)\{ f^{(r)}(\frac{nt+\alpha}{n+\beta}) - f_{\eta,2}^{(r)}(\frac{nt+\alpha}{n+\beta})\} dt.$$

Hence

$$T_4 \le C_5 \left\| f^{(r)} - f_{\eta,2}^{(r)} \right\|_{C[a^*,b^*]}.$$

Now for $x \in [a_1, b_1]$ and $t \in [0,\infty) \setminus [a^*, b^*]$, we choose $\delta_1 > 0$ satisfying $\left| \frac{nt+\alpha}{n+\beta} \right| \ge \delta_1 > 0$. By Lemma 4 and Schwarz inequality, we have

$$\left| \frac{d^r}{dw^r} M_{n,\alpha,\beta}((1-\chi(t))(f(t) - f_{\eta,2}(t)), w) \right|_{w=x} \le (n-1) \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i \frac{|\phi_{i,j,r}(x)|}{x^r} \sum_{k=0}^{\infty} p_{n,k}(x) |k - nx|^j$$

$$\times \int_0^\infty q_{n,k}(t)(1-\chi(t)) \left| f\left(\frac{nt+\alpha}{n+\beta}\right) - f\eta,2\left(\frac{nt+\alpha}{n+\beta}\right) \right| dt$$

$$\le C_6 \| f \|_\gamma (n-1) \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i \sum_{k=0}^{\infty} p_{n,k}(x) |k - nx|^j$$

$$\times \int_{|t-x| \ge \delta_1} q_{n,k}(t)\, dt$$

$$\le C_6 \delta_1^{-2l} \| f \|_\gamma (n-1) \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i \sum_{k=0}^{\infty} p_{n,k}(x) |k - nx|^j$$

$$\times \left\{ \int_{k=0}^\infty q_{n,k}(t) dt \right\}^{1/2} \left\{ \int_0^\infty q_{n,k}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{4l} dt \right\}^{1/2}$$

$$\le C_6 \delta_1^{-2l} \| f \|_\gamma \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) |k - nx|^{2j} \right\}^{1/2}$$

$$\times \left\{ (n-1)(\sum_{k=0}^{\infty} p_{n,k}(x) \int_0^\infty q_{n,k}(t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{4l} dt) \right\}^{1/2}.$$

Hence by Lemma 1 and lemma 2 we have

$$T_5 \le C_7 \| f \|_\gamma \sum n^{(i+\frac{j}{2}-l)} \le C_7 n^{(-v)} \| f \|_\gamma,$$

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where $v=(l-\frac{r}{2})$. Now choose $\eta>0$ such that $v>1$. Then $T_5\leq C_7 n^{-1}\| f\|_\gamma$. Therefore by property (3) of Steklov mean, we obtain

$$T_1 \leq C_8 \left\| f^{(r)} - f_{\eta,2}^{(r)} \right\|_{C[a^*,b^*]} + C_7 n^{-1} \| f \|_\gamma$$

$$\leq C_9 \omega_2(f^{(r)},\eta,[a,b]) + C_7 n^{-1} \| f \|_\gamma .$$

Hence with $\eta = n^{-1/2}$, the theorem follows.

Acknowledgments

The authors are thankful to the referee for making valuable suggestions leading better presentation of the paper.

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