Mathematics Learning Centre

The University of Sydney

Introduction to Differential Calculus

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Contents

1 Introduction

In day to day life we are often interested in the extent to which a change in one quantity affects a change in another related quantity. This is called a *rate of change*. For example, if you own a motor car you might be interested in how much a change in the amount of fuel used affects how far you have travelled. This rate of change is called *fuel consumption*. If your car has high fuel consumption then a large change in the amount of fuel in your tank is accompanied by a small change in the distance you have travelled. Sprinters are interested in how a change in time is related to a change in their position. This rate of change is called *velocity*. Other rates of change may not have special names like fuel consumption or velocity, but are nonetheless important. For example, an agronomist might be interested in the extent to which a change in the amount of fertiliser used on a particular crop affects the yield of the crop. Economists want to know how a change in the price of a product affects the demand for that product.

Differential calculus is about describing in a precise fashion the ways in which related quantities change.

To proceed with this booklet you will need to be familiar with the concept of the *slope* (also called the *gradient*) of a straight line. You may need to revise this concept before continuing.

1.1 An example of a rate of change: velocity

1.1.1 Constant velocity

Figure 1 shows the graph of part of a motorist's journey along a straight road. The vertical axis represents the distance of the motorist from some fixed reference point on the road, which could for example be the motorist's home. Time is represented along the horizontal axis and is measured from some convenient instant (for example the instant an observer starts a stopwatch).

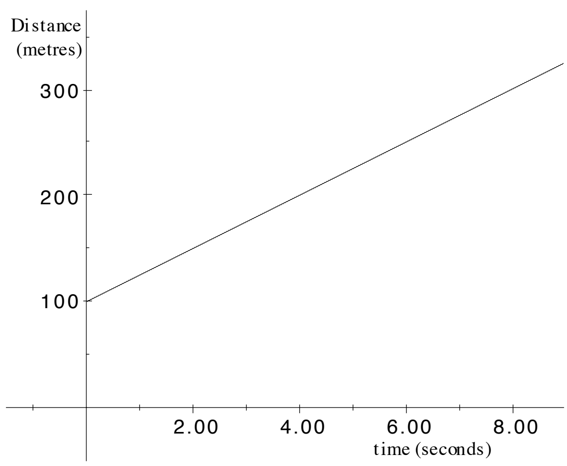

Figure 1: Distance versus time graph for a motorist's journey.

Exercise 1.1

How far is the motorist in Figure 1 away from home at time $t = 0$ and at time $t = 6$?

Exercise 1.2

How far does the motorist travel in the first two seconds (ie from time $t = 0$ to time $t = 2$)? How far does the motorist travel in the two second interval from time $t = 3$ to $t = 5$? How far do you think the motorist would travel in any two second interval of time?

The shape of the graph in Figure 1 tells us something special about the type of motion that the motorist is undergoing. *The fact that the graph is a straight line tells us that the motorist is travelling at a constant velocity.*

- At a constant velocity equal increments in time result in equal changes in distance.
- For a straight line graph equal increments in the horizontal direction result in the same change in the vertical direction.

In Exercise 1.2 for example, you should have found that in the first two seconds the motorist travels 50 metres and that the motorist also travels 50 metres in the two seconds between time $t = 3$ and $t = 5$.

Because the graph is a straight line we know that the motorist is travelling at a constant velocity. What is this velocity? How can we calculate it from the graph? Well, in this situation, velocity is calculated by dividing distance travelled by the time taken to travel that distance. At time $t = 6$ the motorist was 250 metres from home and at time $t = 2$ the motorist was 150 metres away from home. The distance travelled over the four second interval from time $t = 2$ to $t = 6$ was

$$\text{distance travelled} = 250 - 150 = 100$$

and the time taken was

$$\text{time taken} = 6 - 2 = 4$$

and so the velocity of the motorist is

$$\text{velocity} = \frac{\text{distance travelled}}{\text{time taken}} = \frac{250 - 150}{6 - 2} = \frac{100}{4} = 25 \text{ metres per second.}$$

But this is exactly how we would calculate the slope of the line in Figure 1. Take a look at Figure 2 where the above calculation of velocity is shown diagramatically.

The slope of a line is calculated by vertical rise divided by horizontal run and if we were to use the two points $(2, 150)$ and $(6, 250)$ to calculate the slope we would get

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{250 - 150}{6 - 2} = 25.$$

To summarise:

The fact that the car is travelling at a constant velocity is reflected in the fact that the distance-time graph is a straight line. The velocity of the car is given by the slope of this line.

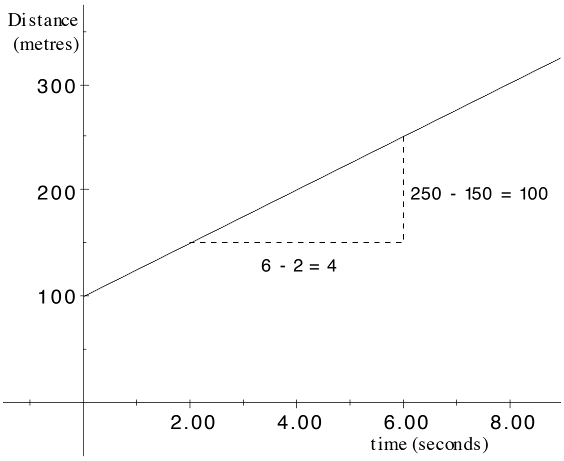

Figure 2: Calculation of the velocity of the motorist is the same as the calculation of the slope of the distance - time graph.

1.1.2 Non-constant velocity

Figure 3 shows the graph of a different motorist's journey along a *straight* road. This graph is not a straight line. The motorist is not travelling at a constant velocity.

Exercise 1.3

How far does the motorist travel in the two seconds from time $t = 60$ to time $t = 62$?

How far does the motorist travel in the two second interval from time $t = 62$ to $t = 64$?

Since the motorist travels at different velocities at different times, when we talk about the velocity of the motorist in Figure 3 we need to specify the particular time that we mean. Nevertheless we would still like somehow to interpret the velocity of the motorist as the slope of the graph, even though the graph is curved and not a straight line.

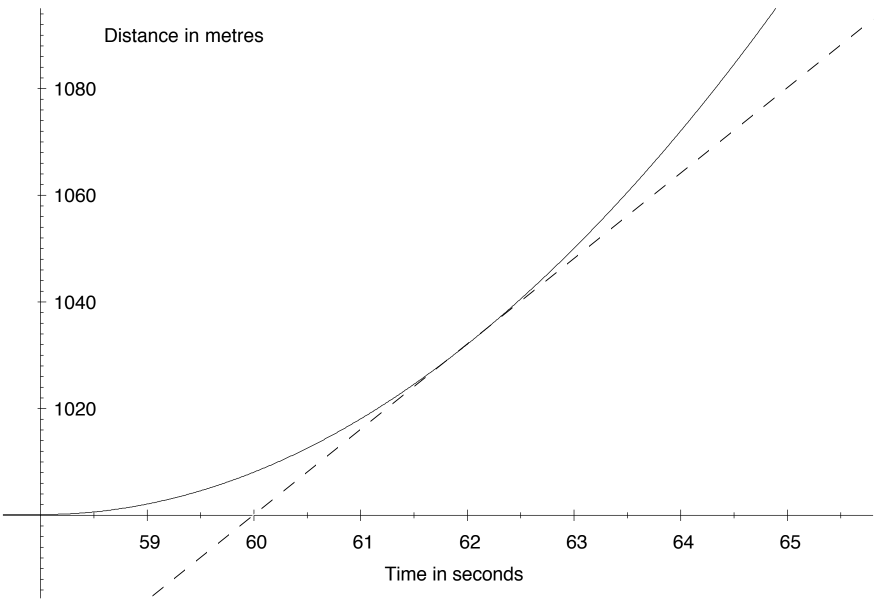

Figure 3: Position versus time graph for a motorist's journey.

What do we mean by the slope of a curve? Suppose for example that we are interested in the velocity of the motorist in Figure 3 at time $t = 62$. In Figure 3 we have drawn in a dashed line. Notice that this line just grazes the curve at the point on the curve where $t = 62$. The dashed line is in fact the *tangent* to the curve at that point. We will talk more about tangents to curves in Section 2. For now you can think of the dashed line like this: if you were going to draw a straight line through this point on the curve, and if you wanted that straight line to look as much like the curve near that point as it possibly could, this is the line that you would draw. This solves our problem about interpreting the slope of the curve at this point on the curve.

The slope of the curve at the point on the curve where $t = 62$ *is the slope of the tangent to the curve at that point: that is the slope of the dashed line in Figure 3.*

The velocity of the motorist at time $t = 62$ is the slope of the dashed line in that figure. Of course if we were interested in the velocity of the motorist at time $t = 64$ then we would draw the tangent to the curve at the point on the curve where $t = 64$ and we would get a different slope. At different points on the curve we get different tangents having different slopes. At different times the motorist is travelling at different velocities.

1.2 Other rates of change

The situation above described a car moving in one direction along a straight road away from a fixed point. Here, the word *velocity* describes how the distance changes with time. Velocity is a *rate of change.* For these type of problems, the velocity corresponds to the rate of change of distance with respect to time. Motion in general may not always be in one direction or in a straight line. In this case we need to use more complex techniques.

Velocity is by no means the only rate of change that we might be interested in. Figure 4 shows a graph representing the yield a farmer gets from a crop depending on the amount of fertiliser that the farmer uses.

The shape of this graph makes good sense. If no fertiliser is used then there is still some crop yield (50 tonnes to be precise). As more fertiliser is used the crop yield increases,

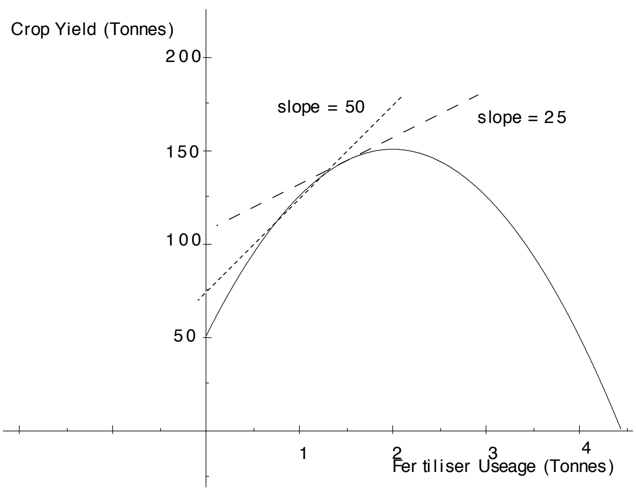

Figure 4: Crop yield versus fertiliser useage for a hypothetical crop.

as you would expect. Note though that at a certain point putting on more fertiliser does not improve the yield of the crop, but in fact decreases it. The soil is becoming poisoned by too much fertiliser. Eventually the use of too much fertiliser causes the crop to die altogether and no yield is obtained.

On the graph the tangents to the curve corresponding to fertiliser usage of 1 tonne (the dotted line) and of 1.5 tonnes (the dashed line) are drawn. The slope of these tangents give the rate of change of crop yield with respect to fertiliser usage.

The slope of the dotted tangent is 50. This means that if fertiliser usage is increased from 1 tonne by a very small amount then the crop yield will increase by 50 times that small change. For example an increase in fertiliser usage from 1 tonne (1000 kg) to 1005 kg will increase the crop yield by approximately $50 \times 5 = 250$ kg. If we are using 1 tonne of fertiliser then the rate of change of crop yield with respect to fertiliser useage is quite high. On the other hand the slope of the dashed tangent is 25. The same increase (by 5 kg) in fertiliser useage from 1500 kg (1.5 tonnes) to 1505 kg will increase the crop yield by about $25 \times 5 = 125$ kg.

2 What is the derivative?

If you are not completely comfortable with the concept of a function and its graph then you need to familiarise yourself with it before continuing. The booklet *Functions* published by the Mathematics Learning Centre may help you.

In Section 1 we learnt that differential calculus is about finding the rates of change of related quantities. We also found that a rate of change can be thought of as the slope of a tangent to a graph of a function. Therefore we can also say that:

Differential calculus is about finding the slope of a tangent to the graph of a function, or equivalently, differential calculus is about finding the rate of change of one quantity with respect to another quantity.

If we are going to go to all this trouble to find out about the slope of a tangent to a graph, we had better have a good idea of just what a tangent is.

2.1 Tangents

Look at the curve and straight line in Figure 5.

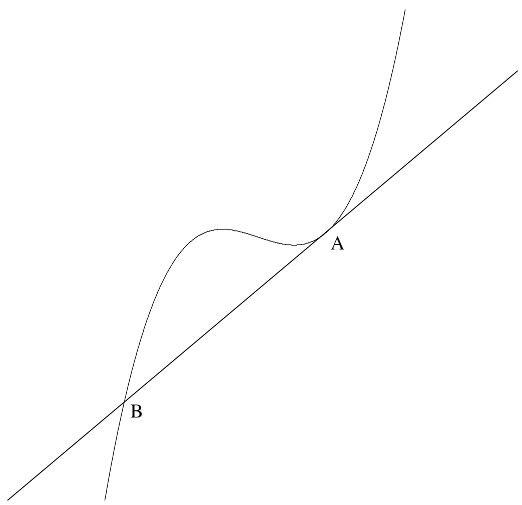

Figure 5: The line is tangent to the curve at point A but not at point B.

Imagine taking a very powerful magnifying glass and looking very closely at this figure near the point A. Figure 6 shows two views of this curve at successively greater magnifications. The closer we look at the curve near the point A the straighter the curve appears to be. The more we zoom in the more the curve begins to look like the straight line. This straight line is called the *tangent to the curve at the point* A. If we want to draw a straight line that most resembles the curve near the point A, the tangent line is the one that we would draw. It is pretty clear from Figure 5 that no matter how closely we look at the curve near the point B the curve is never going to look like the straight line we have drawn in here. That line is tangent to the curve at A but not at B. The curve does have a tangent at B, but it is not shown on Figure 5.

Note that it is not necessarily true that the tangent line only cuts the curve at one point or that curve lies entirely on one side of the line. These properties hold for some special curves like circles, but not for all curves, and certainly not for the one in Figure 5.

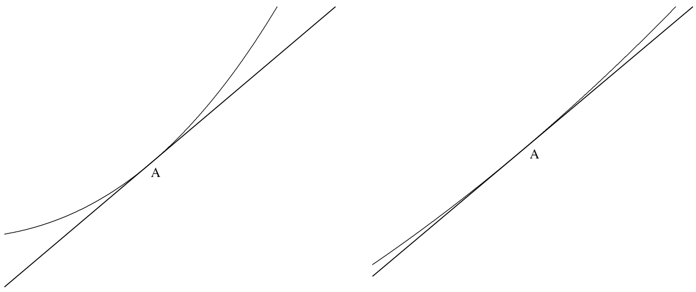

Figure 6: Two close up views of the curve in Figure 5 near the point A. The closer we look near the point A the more the curve looks like the tangent.

2.2 The derivative: the slope of a tangent to a graph

Terminology The slope of the tangent at the point $(x, f(x))$ on the graph of f is called the *derivative of f at x* and is written $f'(x)$.

Look at the graph of the function $y = f(x)$ in Figure 7. Three different tangent lines have

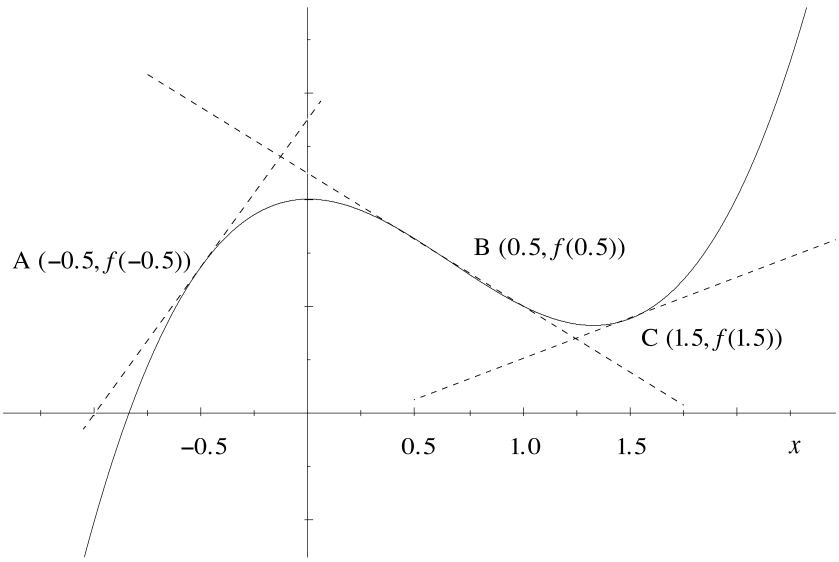

Figure 7: Tangent lines to the graph of $f(x)$ drawn at three different points on the graph.

been drawn on the graph, at A, B and C, corresponding to three different values of the independent variable, $x = -0.5$, $x = 0.5$ and $x = 1.5$.

If we were to make careful measurements of the slopes of the three tangents shown we would find that $f'(-0.5) \approx 2.75$, $f'(0.5) \approx -1.25$ and $f'(1.5) \approx 0.75$. Here the symbol $\approx$ means 'is approximately'. We can only say approximately here because there is no way that we can make completely accurate measurements from a graph, and no way even to draw a completely accurate graph. However this graphical approach to finding the approximate derivative is often very useful, and in some situations may be the only technique that we have.

At different points on the graph we get different tangents having different slopes. The slope of the tangent to the graph depends on where on the graph we draw the tangent. Because we can specify a point on the graph by just giving its x coordinate (the other coordinate is then $f(x)$), we can say that *the slope of the tangent to the graph of a function depends on the value of the independent variable* x, or the value of $f'(x)$ depends on x. In other words, f' is a *function* of x.

Terminology The function f' is called the *derivative* of f.

Terminology The process of finding the derivative is called *differentiation.*

The derivative of a a function f is another function, called f', which tells us about the slopes of tangents to the graph of f. Because there are several different ways of writing functions, there are several different ways of writing the derivative of a function. Most of the ways that are commonly used are expressed in the following table.

Function	**Derivative**
$f(x)$	$f'(x)$ or $\frac{df(x)}{dx}$
f	f' or $\frac{df}{dx}$
y	y' or $\frac{dy}{dx}$
$y(x)$	$y'(x)$ or $\frac{dy(x)}{dx}$

Exercise 2.1 (You will find this exercise easier to do if you use graph paper.)

Draw a careful graph of the function $f(x) = x^2$. Draw the tangents at the points $x = 1$, $x = 0$ and $x = -0.5$. Find the slopes of these lines by picking two points on them and using the formula

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$

These slopes are the (approximate) values of $f'(1)$, $f'(0)$ and $f'(-0.5)$ respectively.

Exercise 2.2

Repeat Exercise 2.1 with the function $f(x) = x^3$.

3 How do we find derivatives (in practice)?

Differential calculus is a procedure for finding the exact derivative directly from the formula of the function, without having to use graphical methods. In practise we use a few rules that tell us how to find the derivative of almost any function that we are likely to encounter. In this section we will introduce these rules to you, show you what they mean and how to use them.

Warning! To follow the rest of these notes you will need feel comfortable manipulating expressions containing indices. If you find that you need to revise this topic you may find the Mathematics Learning Centre publication *Exponents and Logarithms* helpful.

3.1 Derivatives of constant functions and powers

Perhaps the simplest functions in mathematics are the constant functions and the functions of the form x^n.

Rule 1 If k is a constant then $\frac{d}{dx}k = 0$.

Rule 2 If n is any number then $\frac{d}{dx}x^n = nx^{n-1}$.

Rule 1 at least makes sense. The graph of a constant function is a horizontal line and a horizontal line has slope zero. The derivative measures the slope of the tangent, and so the derivative is zero.

How you approach Rule 2 is up to you. You certainly need to know it and be able to use it. However we have given no justification for why Rule 2 works! In fact in these notes we will give little justification for any of the rules of differentiation that are presented. We will show you how to apply these rules and what you can do with them, but we will not make any attempt to prove any of them.

Examples If $f(x) = x^7$ then $f'(x) = 7x^6$.

If $y = x^{-0.5}$ then $\frac{dy}{dx} = -0.5x^{-1.5}$.

$\frac{d}{dx}x^{-3} = -3x^{-4}$.

If $g(x) = 3.2$ then $g'(x) = 0$.

If $f(t) = t^{\frac{1}{2}}$ then $f'(t) = \frac{1}{2}t^{-\frac{1}{2}}$.

If $h(u) = -13.29$ then $h'(u) = 0$.

In the examples above we have used Rules 1 and 2 to calculate the derivatives of many simple functions. However we must not lose sight of what it is that we are calculating here. The derivative gives the slope of the tangent to the graph of the function.

For example, if $f(x) = x^2$ then $f'(x) = 2x$. To find the slope of the tangent to the graph of x^2 at $x = 1$ we substitute $x = 1$ into the derivative. The slope is $f'(1) = 2 \times 1 = 2$. Similarly the slope of the tangent to the graph of x^2 at $x = -0.5$ is found by substituting $x = -0.5$ into the derivative. The slope is $f'(-0.5) = 2 \times -0.5 = -1$. This is illustrated in Figure 8.

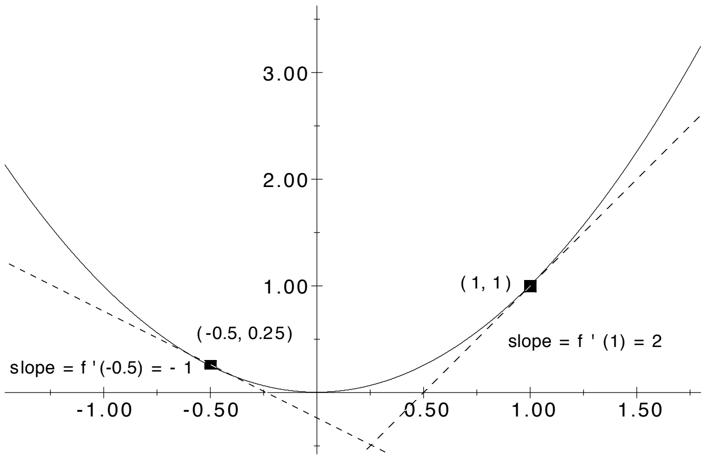

Figure 8: Slopes of tangents to the graph of $y = x^2$.

Example

Find the slope of the tangent to the graph of the function $g(t) = t^4$ at the point on the graph where $t = -2$.

Solution

The derivative is $g'(t) = 4t^3$, and so the slope of the tangent line at $t = -2$ is $g'(-2) = 4 \times (-2)^3 = -32$.

Example

Find the equation of the line tangent to the graph of $y = f(x) = x^{\frac{1}{2}}$ at the point $x = 4$.

Solution

$f(4) = 4^{\frac{1}{2}} = \sqrt{4} = 2$, so the coordinates of the point on the graph are $(4, 2)$. The derivative is

$$f'(x) = \frac{x^{-\frac{1}{2}}}{2} = \frac{1}{2\sqrt{x}}$$

and so the slope of the tangent line at $x = 4$ is $f'(4) = \frac{1}{4}$. We therefore know the slope of the line and we know one point through which the line passes.

Any non vertical line has equation of the form $y = mx + b$ where m is the slope and b the vertical intercept.

In this case the slope is $\frac{1}{4}$, so $m = \frac{1}{4}$, and the equation is $y = \frac{x}{4} + b$. Because the line passes through the point $(4, 2)$ we know that $y = 2$ when $x = 4$.

Substituting we get $2 = \frac{4}{4} + b$, so that $b = 1$. The equation is therefore $y = \frac{x}{4} + 1$.

Notice that in the examples above the independent variable is not always called x. We have also used u and t, and in fact we can and will use many different letters for the independent variable. Notice also that we might not stick to the symbol f to stand for function. Many other symbols are used. Some of the common ones are g and h. Throughout this booklet we will use a variety of symbols for functions and variables to get you used to the fact that our choice of symbols makes no difference to the ideas that we are introducing. On the other hand, we can make life easier for ourselves if we make sensible choices of symbols. For example if we were discussing the revenue obtained by a manufacturer who sells articles for a certain price it might be sensible for us to choose the symbol p to mean price, and r to mean revenue, and to write $r(p)$ to express the fact that the revenue is a function of the price. In this way the symbols we have chosen remind us of their meaning, much more than if we had chosen x to represent price and f to represent revenue and written $f(x)$. On the other hand, because the symbol d has a special use in calculus, to express the derivative $\frac{df(x)}{dx}$, we almost never use d for any other purpose. For this reason you will often see the letter s used to represent di**S**placement.

We now know how to differentiate any function that is a power of the variable. Examples are functions like x^3 and $t^{-1.3}$. You will come across functions that do not at first appear to be a power of the variable, but can be rewritten in this form. One of the simplest examples is the function

$$f(t) = \sqrt{t},$$

which can also be written in the form

$$f(t) = t^{\frac{1}{2}}.$$

The derivative is then

$$f'(t) = \frac{t^{-\frac{1}{2}}}{2} = \frac{1}{2\sqrt{t}}.$$

Similarly, if

$$h(s) = \frac{1}{s} = s^{-1}$$

then

$$h'(s) = -s^{-2} = -\frac{1}{s^2}.$$

Examples

If $f(x) = \dfrac{1}{\sqrt[3]{x}} = x^{-\frac{1}{3}}$ then $f'(x) = -\dfrac{1}{3}x^{-\frac{4}{3}}$.

If $y = \dfrac{1}{x\sqrt{x}} = x^{-\frac{3}{2}}$ then $\dfrac{dy}{dx} = -\dfrac{3}{2}x^{-\frac{5}{2}}$.

Exercises 3.1

Differentiate the following functions:

a. $f(x) = x^4$ **b.** $y = x^{-7}$ **c.** $f(u) = u^{2.3}$

d. $f(t) = t^{-\frac{1}{3}}$ **e.** $f(t) = t^{\frac{22}{7}}$ **f.** $g(z) = z^{-\frac{3}{2}}$

g. $y = t^{-3.8}$ **h.** $z = x^{\frac{3}{7}}$

Exercise 3.2

Express the following as powers and then differentiate:

a. $\dfrac{1}{x^2}$ **b.** $t\sqrt{t}$ **c.** $\sqrt[3]{x}$

d. $\dfrac{1}{x^2\sqrt{x}}$ **e.** $\dfrac{1}{x\sqrt[4]{x}}$ **f.** $\dfrac{s^3\sqrt{s}}{\sqrt[3]{s}}$

g. $\dfrac{1}{u^3}$ **h.** $\dfrac{t}{t^2\sqrt{t}}$ **i.** $x^{\frac{1}{2}}\dfrac{\sqrt{x}}{x}$

Exercise 3.3

Find the equation of the line tangent to the graph of $y = \sqrt[3]{x}$ when $x = 8$.

3.2 Adding, subtracting, and multiplying by a constant

So far we know how to differentiate powers of the independent variable. Many of the functions that you will encounter are made up in simple ways from powers. For example, a function like $3x^2$ is just a constant multiple of x^2. However neither Rule 1 nor Rule 2 tell us how to differentiate $3x^2$. Nor do they tell us how to differentiate something like $x^2 + x^3$ or $x^2 - x^3$.

Rules 3 and 4 specify how to differentiate combinations of functions that are formed by multiplying by constants, or by adding or subtracting functions.

Rule 3 If $f(x) = cg(x)$, where c is a constant, then $f'(x) = cg'(x)$.

Rule 4 If $f(x) = g(x) \pm h(x)$ then $f'(x) = g'(x) \pm h'(x)$.

Examples If $f(x) = 3x^2$ then $f'(x) = 3 \times \dfrac{d}{dx}x^2 = 6x$.

If $g(t) = 3t^2 + 2t^{-2}$ then $g'(t) = \dfrac{d}{dt}3t^2 + \dfrac{d}{dt}2t^{-2} = 6t - 4t^{-3}$.

If $y = \frac{3}{\sqrt{x}} - 2x\sqrt[3]{x} = 3x^{-\frac{1}{2}} - 2x^{\frac{4}{3}}$ then $\frac{dy}{dx} = -\frac{3}{2}x^{-\frac{3}{2}} - \frac{8}{3}x^{\frac{1}{3}}$.

If $y = -0.3x^{-0.4}$ then $\frac{dy}{dx} = 0.12x^{-1.4}$.

$\frac{d}{dx}2x^{0.3} = 0.6x^{-0.7}$.

Warning! Although Rule 4 tells us that $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$, *the same is not true for multiplication or division.* To differentiate $f(x) \times g(x)$ or $f(x) \div g(x)$ we cannot simply find $f'(x)$ and $g'(x)$ and multiply or divide them. *Be careful of this!* The methods of differentiating products of functions or quotients of functions are discussed in Sections 3.3 and 3.4.

Exercise 3.4

Differentiate the following functions:

a. $f(x) = 5x^2 - 2\sqrt{x}$ **b.** $y = 2x^{-7} + \frac{3}{x^2}$ **c.** $f(t) = 2.5t^{2.3} + \frac{t}{\sqrt{t}}$

d. $h(z) = z^{-\frac{1}{3}} + 5z$ e. $f(u) = u^{\frac{5}{3}} - 3u^{-7}$ **f.** $g(z) = 8z^{-2} - \frac{5}{z}$

g. $y = 5t^{-8} + \frac{t}{\sqrt{t}}$ **h.** $z = 4x^{\frac{1}{7}} + 2x^{-\frac{1}{2}}$

3.3 The product rule

Another way of combining functions to make new functions is by multiplying them together, or in other words by forming *products*. The product rule tells us how to differentiate functions like this.

Rule 5 (The product rule) If $f(x) = u(x)v(x)$ then

$$f'(x) = u(x)v'(x) + u'(x)v(x).$$

Examples

If $y = (x + 2)(x^2 + 3)$ then $y' = (x + 2)2x + 1(x^2 + 3)$.

If $f(x) = \sqrt{x}(x^3 - 3x^2 + 7)$ then $f'(x) = \sqrt{x}(3x^2 - 6x) + \frac{1}{2}x^{-\frac{1}{2}}(x^3 - 3x^2 + 7)$.

If $z = (t^2 + 3)(\sqrt{t} + t^3)$ then $\frac{dz}{dt} = (t^2 + 3)(\frac{1}{2}t^{-\frac{1}{2}} + 3t^2) + 2t(\sqrt{t} + t^3)$.

Exercise 3.5

Use the product rule to differentiate the functions below:

a. $f(x) = (4x^3 + 2)(1 - 3x)$

b. $g(x) = (x^2 + x + 2)(x^2 + 1)$

c. $h(x) = (3x^3 - 2x^2 + 8x - 5)(x^2 - 2x + 4)$

d. $f(s) = (1 - \frac{1}{2}s^2)(3s + 5)$

e. $g(t) = (\sqrt{t} + \frac{1}{t})(2t - 1)$

f. $h(y) = (2 - \sqrt{y} + y^2)(1 - 3y^2)$

Exercise 3.6

If $r = (t + \frac{1}{t})(t^2 - 2t + 1)$, find the rate of change of r with respect to t when $t = 2$.

Exercise 3.7

Find the slope of the tangent to the curve $y = (x^2 - 2x + 1)(3x^3 - 5x^2 + 2)$ at $x = 2$.

3.4 The Quotient Rule

This rule allows us to differentiate functions which are formed by dividing one function by another, ie by forming *quotients* of functions. An example is such as

$$f(x) = \frac{2x + 3}{3x - 5}.$$

Rule 6 (The quotient rule)

$$\begin{aligned} f(x) &= \frac{u(x)}{v(x)} \\ f'(x) &= \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2} \\ &= \frac{vu' - uv'}{v^2}. \end{aligned}$$

Warning! Because of the minus sign in the numerator (ie in the top line) it is important to get the terms in the numerator in the correct order. This is often a source of mistakes, so be careful. Decide on your own way of remembering the correct order of the terms.

Examples

If $y = \dfrac{2x^2+3x}{x^3+1}$, then $\dfrac{dy}{dx} = \dfrac{(x^3+1)(4x+3)-(2x^2+3x)3x^2}{(x^3+1)^2}$.

If $g(t) = \dfrac{t^2+3t+1}{\sqrt{t}+1}$ then $g'(t) = \dfrac{(\sqrt{t}+1)(2t+3)-(t^2+3t+1)(\frac{1}{2}t^{-\frac{1}{2}})}{(\sqrt{t}+1)^2}$.

Exercise 3.8

Use the Quotient Rule to find derivatives for the following functions:

a. $f(x) = \dfrac{x-1}{x+1}$ b. $g(x) = \dfrac{2x+3}{3x-2}$

c. $h(x) = \dfrac{x^2+2}{x^2+5}$ d. $f(t) = \dfrac{2t}{1+2t^2}$

e. $f(s) = \dfrac{1+\sqrt{s}}{1-\sqrt{s}}$ f. $h(x) = \dfrac{x^2-1}{x^3+4}$

g. $f(u) = \dfrac{u^3+u-4}{3u^4+5}$ h. $g(t) = \dfrac{t(t+6)}{t^2+3t+1}$

3.5 The composite function rule (also known as the chain rule)

Have a look at the function $f(x) = (x^2+1)^{17}$. We can think of this function as being the result of combining two functions. If $g(x) = x^2+1$ and $h(t) = t^{17}$ then the result of substituting $g(x)$ into the function h is

$$h(g(x)) = (g(x))^{17} = (x^2+1)^{17}.$$

Another way of representing this would be with a diagram like

$$x \overset{g}{\longmapsto} x^2+1 \overset{h}{\longmapsto} (x^2+1)^{17}.$$

We start off with x. The function g takes x to x^2+1, and the function h then takes x^2+1 to $(x^2+1)^{17}$. Combining two (or more) functions like this is called *composing* the functions, and the resulting function is called a *composite function.* For a more detailed discussion of composite functions you might wish to refer to the Mathematics Learning Centre booklet *Functions.*

Using the rules that we have introduced so far, the only way to differentiate the function $f(x) = (x^2+1)^{17}$ would involve expanding the expression and then differentiating. If the function was $(x^2+1)^2 = (x^2+1)(x^2+1)$ then it would not take too long to expand these two sets of brackets. But to expand the seventeen sets of brackets involved in the function $f(x) = (x^2+1)^{17}$ (or even to expand using the binomial theorem) would take a long time. The composite function rule shows us a quicker way.

Rule 7 (The composite function rule (also known as the chain rule))

If $f(x) = h(g(x))$ then $f'(x) = h'(g(x)) \times g'(x)$.

In words: differentiate the 'outside' function, and then multiply by the derivative of the 'inside' function.

To apply this to $f(x) = (x^2 + 1)^{17}$, the outside function is $h(\cdot) = (\cdot)^{17}$ and its derivative is $17(\cdot)^{16}$. The inside function is $g(x) = x^2 + 1$ which has derivative $2x$. The composite function rule tells us that $f'(x) = 17(x^2 + 1)^{16} \times 2x$.

As another example let us differentiate the function $1/(z^3 + 4z^2 - 3z - 3)^6$. This can be rewritten as $(z^3 + 4z^2 - 3z - 3)^{-6}$. The outside function is $(\cdot)^{-6}$ which has derivative $-6(\cdot)^{-7}$. The inside function is $z^3 + 4z^2 - 3z - 3$ with derivative $3z^2 + 8z - 3$. The chain rule says that

$$\frac{d}{dz}(z^3 + 4z^2 - 3z - 3)^{-6} = -6(z^3 + 4z^2 - 3z - 3)^{-7} \times (3z^2 + 8z - 3).$$

There is another way of writing down, and hence remembering, the composite function rule.

Rule 7 (The composite function rule (alternative formulation))

If y is a function of u and u is a function of x then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

This makes the rule very easy to remember. The expressions $\frac{dy}{du}$ and $\frac{du}{dx}$ are not really fractions but rather they stand for the derivative of a function with respect to a variable. However for the purposes of remembering the chain rule we can think of them as fractions, so that the du cancels from the top and the bottom, leaving just $\frac{dy}{dx}$.

To use this formulation of the rule in the examples above, to differentiate $y = (x^2 + 1)^{17}$ put $u = x^2 + 1$, so that $y = u^{17}$. The alternative formulation of the chain rules says that

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 17u^{16} \times 2x \\ &= 17(x^2 + 1)^{16} \times 2x. \end{aligned}$$

which is the same result as before. Again, if $y = (z^3 + 4z^2 - 3z - 3)^{-6}$

then set $u = z^3 + 4z^2 - 3z - 3$ so that $y = u^{-6}$ and

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= -6u^{-7} \times (3z^2 + 8z - 3). \end{aligned}$$

You select the formulation of the chain rule that you find easiest to use. They are equivalent.

Example

Differentiate $(3x^2 - 5)^3$.

Solution

The first step is always to **recognise** that we are dealing with a composite function and then to split up the composite function into its components. In this case the outside function is $(\cdot)^3$ which has derivative $3(\cdot)^2$, and the inside function is $3x^2 - 5$ which has derivative $6x$, and so by the composite function rule,

$$\frac{d(3x^2-5)^3}{dx} = 3(3x^2-5)^2 \times 6x = 18x(3x^2-5)^2.$$

Alternatively we could first let $u = 3x^2 - 5$ and then $y = u^3$. So

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 \times 6x = 18x(3x^2-5)^2.$$

Example

Find $\frac{dy}{dx}$ if $y = \sqrt{x^2+1}$.

Solution

The outside function is $\sqrt{\cdot} = (\cdot)^{\frac{1}{2}}$ which has derivative $\frac{1}{2}(\cdot)^{-\frac{1}{2}}$, and the inside function is $x^2 + 1$ so that

$$y' = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x.$$

Alternatively, if $u = x^2 + 1$, we have $y = \sqrt{u} = u^{\frac{1}{2}}$. So

$$\frac{dy}{dx} = \frac{1}{2}\,u^{-\frac{1}{2}} \times 2x = \frac{1}{2}\,(x^2+1)^{-\frac{1}{2}} \times 2x.$$

Exercise 3.9

Differentiate the following functions using the composite function rule.

a. $(2x+3)^2$ **b.** $(x^2+2x+1)^{12}$ **c.** $(3-x)^{21}$

d. $(x^3-1)^5$ **e.** $f(t) = \sqrt{t^2-5t+7}$ **f.** $g(z) = \frac{1}{\sqrt{2-z^4}}$

g. $y = (t^3 - \sqrt{t})^{-3.8}$ **h.** $z = (x + \frac{1}{x})^{\frac{3}{7}}$

Exercise 3.10

Differentiate the functions below. You will need to use both the composite function rule and the product or quotient rule.

a. $(x+2)(x+3)^2$ **b.** $(2x-1)^2(x+3)^3$ **c.** $x\sqrt{(1-x)}$

d. $x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}$ **e.** $\dfrac{x}{\sqrt{1-x^2}}$

3.6 Derivatives of exponential and logarithmic functions

If you are not familiar with exponential and logarithmic functions you may wish to consult the booklet *Exponents and Logarithms* which is available from the Mathematics Learning Centre.

You may have seen that there are two notations popularly used for natural logarithms, $\log_e$ and ln. These are just two different ways of writing exactly the same thing, so that $\log_e x \equiv \ln x$. In this booklet we will use both these notations.

The basic results are:

$$\begin{aligned} \frac{d}{dx}e^x &= e^x \\ \frac{d}{dx}(\log_e x) &= \frac{1}{x}. \end{aligned}$$

We can use these results and the rules that we have learnt already to differentiate functions which involve exponentials or logarithms.

Example

Differentiate $\log_e(x^2+3x+1)$.

Solution

We solve this by using the chain rule and our knowledge of the derivative of $\log_e x$.

$$\begin{aligned} \frac{d}{dx}\log_e(x^2+3x+1) &= \frac{d}{dx}(\log_e u) \qquad (\text{where } u = x^2+3x+1) \\ &= \frac{d}{du}(\log_e u) \times \frac{du}{dx} \qquad (\text{by the chain rule}) \\ &= \frac{1}{u} \times \frac{du}{dx} \\ &= \frac{1}{x^2+3x+1} \times \frac{d}{dx}(x^2+3x+1) \\ &= \frac{1}{x^2+3x+1} \times (2x+3) \\ &= \frac{2x+3}{x^2+3x+1}. \end{aligned}$$

Example
Find $\frac{d}{dx}(e^{3x^2})$.

Solution
This is an application of the chain rule together with our knowledge of the derivative of e^x.

$$\begin{aligned}\frac{d}{dx}(e^{3x^2}) &= \frac{de^u}{dx} \qquad \text{where } u = 3x^2 \\ &= \frac{de^u}{du} \times \frac{du}{dx} \qquad \text{by the chain rule} \\ &= e^u \times \frac{du}{dx} \\ &= e^{3x^2} \times \frac{d}{dx}(3x^2) \\ &= 6xe^{3x^2}.\end{aligned}$$

Example
Find $\frac{d}{dx}(e^{x^3+2x})$.

Solution
Again, we use our knowledge of the derivative of e^x together with the chain rule.

$$\begin{aligned}\frac{d}{dx}(e^{x^3+2x}) &= \frac{de^u}{dx} \qquad (\text{where } u = x^3 + 2x) \\ &= e^u \times \frac{du}{dx} \qquad (\text{by the chain rule}) \\ &= e^{x^3+2x} \times \frac{d}{dx}(x^3 + 2x) \\ &= (3x^2 + 2) \times e^{x^3+2x}.\end{aligned}$$

Example
Differentiate $\ln(2x^3 + 5x^2 - 3)$.

Solution
We solve this by using the chain rule and our knowledge of the derivative of $\ln x$.

$$\begin{aligned}\frac{d}{dx}\ln(2x^3 + 5x^2 - 3) &= \frac{d\ln u}{dx} \qquad (\text{where } u = (2x^3 + 5x^2 - 3) \\ &= \frac{d\ln u}{du} \times \frac{du}{dx} \qquad (\text{by the chain rule}) \\ &= \frac{1}{u} \times \frac{du}{dx} \\ &= \frac{1}{2x^3 + 5x^2 - 3} \times \frac{d}{dx}(2x^3 + 5x^2 - 3) \\ &= \frac{1}{2x^3 + 5x^2 - 3} \times (6x^2 + 10x) \\ &= \frac{6x^2 + 10x}{2x^3 + 5x^2 - 3}.\end{aligned}$$

There are two shortcuts to differentiating functions involving exponents and logarithms. The four examples above gave

$$\begin{aligned}
\frac{d}{dx}(\log_e(x^2+3x+1)) &= \frac{2x+3}{x^2+3x+1} \\
\frac{d}{dx}(e^{3x^2}) &= 6xe^{3x^2} \\
\frac{d}{dx}(e^{x^3+2x}) &= (3x^2+2)e^{3x^2} \\
\frac{d}{dx}(\log_e(2x^3+5x^2-3)) &= \frac{6x^2+10x}{2x^3+5x^2-3}.
\end{aligned}$$

These examples suggest the general rules

$$\begin{aligned}
\frac{d}{dx}(e^{f(x)}) &= f'(x)e^{f(x)} \\
\frac{d}{dx}(\ln f(x)) &= \frac{f'(x)}{f(x)}.
\end{aligned}$$

These rules arise from the chain rule and the fact that $\frac{de^x}{dx} = e^x$ and $\frac{d\ln x}{dx} = \frac{1}{x}$. They can speed up the process of differentiation but it is not necessary that you remember them. If you forget, just use the chain rule as in the examples above.

Exercise 3.11

Differentiate the following functions.

a. $f(x) = \ln(2x^3)$ **b.** $f(x) = e^{x^7}$ **c.** $f(x) = \ln(11x^7)$

d. $f(x) = e^{x^2+x^3}$ **e.** $f(x) = \log_e(7x^{-2})$ **f.** $f(x) = e^{-x}$

g. $f(x) = \ln(e^x + x^3)$ **h.** $f(x) = \ln(e^x x^3)$ **i.** $f(x) = \ln\left(\frac{x^2+1}{x^3-x}\right)$

3.7 Derivatives of trigonometric functions

To understand this section properly you will need to know about trigonometric functions. The Mathematics Learning Centre booklet *Introduction to Trigonometric Functions* may be of use to you.

There are only two basic rules for differentiating trigonometric functions:

$$\begin{aligned} \frac{d}{dx}\sin x &= \cos x \\ \frac{d}{dx}\cos x &= -\sin x. \end{aligned}$$

For differentiating all trigonometric functions these are the only two things that we need to remember.

Of course all the rules that we have already learnt still work with the trigonometric functions. Thus we can use the product, quotient and chain rules to differentiate functions that are combinations of the trigonometric functions.

For example, $\tan x = \frac{\sin x}{\cos x}$ and so we can use the quotient rule to calculate the derivative.

$$\begin{aligned} f(x) &= \tan x = \frac{\sin x}{\cos x}, \\ f'(x) &= \frac{\cos x.(\cos x) - \sin x.(-\sin x)}{(\cos x)^2} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos x} = \frac{1}{\cos^2 x} \qquad (\text{since } \cos^2 x + \sin^2 x = 1) \\ &= \sec^2 x \end{aligned}$$

Note also that

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$$

so it is also true that

$$\frac{d}{dx}\tan x = \sec^2 x = 1 + \tan^2 x.$$

Example

Differentiate $f(x) = \sin^2 x$.

Solution

$f(x) = \sin^2 x$ is just another way of writing $f(x) = (\sin x)^2$. This is a composite function, with the outside function being $(\cdot)^2$ and the inside function being $\sin x$.

By the chain rule, $f'(x) = 2(\sin x)^1 \times \cos x = 2\sin x \cos x$. Alternatively using the other method and setting $u = \sin x$ we get $f(x) = u^2$ and

$$\frac{df(x)}{dx} = \frac{df(x)}{du} \times \frac{du}{dx} = 2u \times \frac{du}{dx} = 2\sin x \cos x.$$

Example

Differentiate $g(z) = \cos(3z^2 + 2z + 1)$.

Solution

Again you should recognise this as a composite function, with the outside function being $\cos(\cdot)$ and the inside function being $3z^2 + 2z + 1$. By the chain rule $g'(z) = -\sin(3z^2 + 2z + 1) \times (6z + 2) = -(6z + 2)\sin(3z^2 + 2z + 1)$.

Example

Differentiate $f(t) = \frac{e^t}{\sin t}$.

Solution

By the quotient rule

$$f'(t) = \frac{e^t \sin t - e^t \cos t}{\sin^2 t} = \frac{e^t(\sin t - \cos t)}{\sin^2 t}.$$

Example

Use the quotient rule or the composite function rule to find the derivatives of $\cot x$, $\sec x$, and cosec x.

Solution

These functions are defined as follows:

$$\begin{aligned} \cot x &= \frac{\cos x}{\sin x} \\ \sec x &= \frac{1}{\cos x} \\ \csc x &= \frac{1}{\sin x}. \end{aligned}$$

By the quotient rule

$$\frac{d\cot x}{dx} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x}.$$

Using the composite function rule

$$\frac{d\sec x}{dx} = \frac{d(\cos x)^{-1}}{dx} = -(\cos x)^{-2} \times (-\sin x) = \frac{\sin x}{\cos^2 x}.$$

$$\frac{d\csc x}{dx} = \frac{d(\sin x)^{-1}}{dx} = -(\sin x)^{-2} \times \cos x = -\frac{\cos x}{\sin^2 x}.$$

Exercise 3.12

Differentiate the following:

a. $\cos 3x$ **b.** $\sin(4x+5)$ **c.** $\sin^3 x$ **d.** $\sin x \cos x$ **e.** $x^2 \sin x$

f. $\cos(x^2+1)$ **g.** $\dfrac{\sin x}{x}$ **h.** $\sin\dfrac{1}{x}$ **i.** $\tan(\sqrt{x})$ **j.** $\dfrac{1}{x}\sin\dfrac{1}{x}$

4 What is differential calculus used for?

4.1 Introduction

The development of mathematics stands as one of the most important achievements of humanity, and the development of the calculus, both the differential calculus and integral calculus is one of most important achievements in mathematics. The practical applications of differential calculus are so wide ranging that it would be impossible to mention them all here. Suffice to say that differential calculus is an indispensable tool in *every* branch of science and engineering.

In elementary mathematics there are two main applications of differential calculus. One is to help in sketching curves, and the other is in optimisation problems. For a treatment of the uses of calculus in curve sketching see the Mathematics Learning Centre publication *Curve Sketching*. In this section we will give a brief introduction to how differential calculus is used in optimisation problems.

4.2 Optimisation problems

There are many practical situations in which we would like to make a quantity as small as we possibly can or as large as we possibly can. For example, a manufacturer of bicycles trying to decide how much to charge for a model of bicycle would think that if he charges too little for the bicycles then he will probably sell a lot of bicycles but that he won't make much profit because the price is too low, and that if he charges too much for the bicycle then he won't make much profit because not many people will buy his bicycles. The manufacturer would like to find just the right price to charge to *maximise* his profit. Similarly a farmer might realise that if she uses too little fertiliser on her crops then her yield will be very low, and if she uses too much fertiliser then she will poison the soil and her yield will be low. The farmer might like to know just how much fertiliser to use to maximise the crop yield. A manufacturer of sheet metal cans that are meant to hold one litre of liquid might like to know just what shape to make the can so that the amount of sheet metal that is used is a *minimum*. These are all examples of optimisation problems.

If we were to draw a graph of the profit versus price for the bicycle manufacturer mentioned above then finding the maximum profit is equivalent to finding the highest point on the graph. Similarly a minimisation problem may be thought of geometrically as finding the lowest point on the graph of a funcion.

4.2.1 Stationary points - the idea behind optimisation

As a thought experiment, let us imagine that a person wearing a blindfold is walking along a road, and that the road has a hill on it. Let us imagine also that the blindfolded person is searching for the highest point on the road. How would this person be able to decide when they were at the top of the hill? Well, while they were walking uphill the person would know that this wasn't the top of the hill - because they are still going up! And of course while they are walking downhill they would know that they are not at the top of the hill because they are going down. In other words, while they are on a sloping bit of the road the blindfolded person would know that this is not the top of the hill.

Right at the top of the hill there would be a little bit of level road. The slope at the top of the hill would be zero! Without even being able to see the road, the blindfolded person would know that they could not possibly be at the top of the hill unless they were standing on level ground. The same idea would apply if the road had a valley in it, and the person was searching for the lowest piece of road. Right at the lowest point of the valley the slope of the road would be zero.

So if we are searching for the highest (or lowest) point on a road, of all the possible places we only have to consider those places where the road has slope zero. This is the idea behind using calculus for optimisation. If we are searching for the highest or lowest points on the graph of a function we have to look for those places where the graph has slope zero. These points are called *stationary points.*

Definition For a function $y = f(x)$ the points on the graph where the graph has zero slope are called *stationary points*. In other words stationary points are where $f'(x) = 0$.

To find the stationary points of a function we differentiate, set the derivative equal to zero and solve the equation.

Example Find the stationary points of the function $f(x) = 2x^3 + 3x^2 - 12x + 17$.

Solution $f'(x) = 6x^2 + 6x - 12$. Setting $f'(x) = 0$ and solving we obtain

$$\begin{aligned} 6x^2 + 6x - 12 &= 0 \\ x^2 + x - 2 &= 0 \\ (x-1)(x+2) &= 0 \\ x &= 1, -2. \end{aligned}$$

This gives us the values of x for which the function f is stationary. The corresponding values of the function are found by substituting 1 and -2 into the function.

They are $f(1) = 2 \times 1^3 + 3 \times 1^2 - 12 \times 1 + 17 = 10$ and

$f(-2) = 2 \times (-2)^3 + 3 \times (-2)^2 - 12 \times (-2) + 17 = 37$. The stationary points are therefore $(1, 10)$ and $(-2, 37)$.

Example Find the stationary points of the function $g(t) = e^{t^2}$.

Solution Differentiating and setting the derivative equal to zero we obtain the equation $g'(t) = 2te^{t^2} = 0$. Since e^{t^2} is never zero, the only solution to this equation is where $2t = 0$, ie $t = 0$. Substituting into the formula for g we obtain the function value $g(0) = e^{0^2} = 1$. Thus the stationary point is $(0, 1)$.

4.2.2 Types of stationary points

In our thought experiment above we mentioned two types of stationary points: one was the top of the hill and the other was the bottom of the valley. The top of the hill is called a local maximum, and the bottom of the valley is called a local minimum. The word 'local' conveys the fact that at the top of the hill the blindfolded person is not necessarily at the highest point in the world, but merely at the highest point in the local vicinity. Sometimes you will see local maxima and local minima called *relative* maxima and *relative*

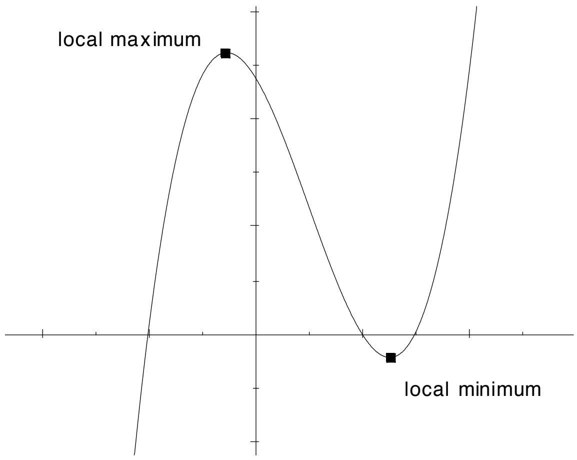

Figure 9: Graph of a function showing a local maximum and a local minimum.

minima. Figure 9 shows a function with a local maximum and a local minimum. Note that at each of these points the slope of the curve is zero.

Local maxima and local minima are not the only types of stationary points. There is a third kind. Figure 10 shows a stationary point that is neither a local maximum nor a local minimum. This type of stationary point is called a *stationary point of inflection.* Don't worry about why it is given this name. That is beyond the scope of this booklet. You just need to be aware of the fact that stationary points exist that are neither local maxima nor local minima.

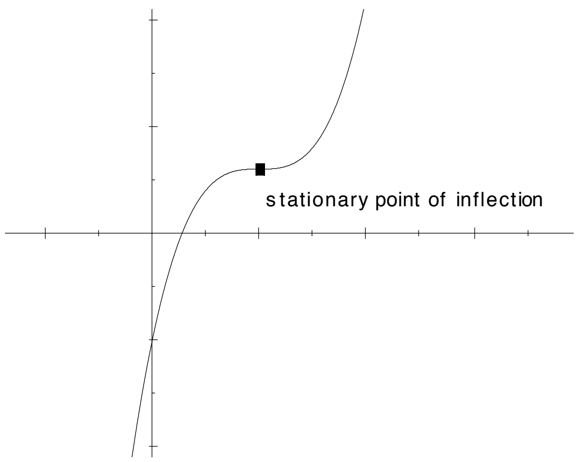

Figure 10: Graph of a function showing a stationary point of inflection.

Let us now return to the first of the examples in the previous section. We found that the function $f(x) = 2x^3 + 3x^2 - 12x + 17$ had stationary points at $(1, 10)$ and $(-2, 37)$. What type of stationary points are they? At the moment you probably have no idea just what the graph of $f(x) = 2x^3 + 3x^2 - 12x + 17$ looks like. How can you tell what type of stationary points these are? If you could see the graph you would be able to tell what types of stationary points they were, but it takes a lot of work to draw the graph of a function. What we need is a way of testing a stationary point that will tell us whether we have found a local maximium, a local minimum or neither (in other words a stationary point of inflection) without drawing the graph. There are several ways of doing this, but in this booklet we will look at only one of them. This is called the *first derivative test.*

Really we are pretty much in the shoes of the blindfolded person now. We can't see the whole graph, so how can we tell what type of stationary point we have got? Imagine the blindfolded person standing on a piece of level ground, and wanting to know whether this was the top of a hill (a local maximum), the bottom of a valley (a local minimum) or neither (a stationary point of inflection). One thing the person could do is take a step backwards from the level spot. Which way does the ground slope here? And then take a step forwards from the level spot. Which way does the ground slope here? If the person took a step backward and found that the ground in front of them sloped up, then returned to the original position and took a step forward and the ground sloped down, then the level spot must have been the top of the hill. On the other hand if the person took a step backward and the ground sloped down, and a step forward and the ground sloped up, then the level spot must have been the bottom of a valley. You should be able to figure out what the blindfolded person would find for a stationary point of inflection. This idea is the basis of the first derivative test.

The first derivative test

If x_0 is a stationary point of the function f, so that $f'(x_0) = 0$ then to find out the nature of the stationary point check the sign (ie positive or negative) of f' just either side of x_0. If $f' < 0$ to the left of x_0 (ie for $x < x_0$) and $f' > 0$ to the right of x_0 (ie for $x > x_0$) then x_0 is a local minimum. If $f' > 0$ to the left of x_0 and $f' < 0$ to the right of x_0 then x_0 is a local maximum. Otherwise x_0 is a stationary point of inflection. Have a look at Figures 11 and 12.

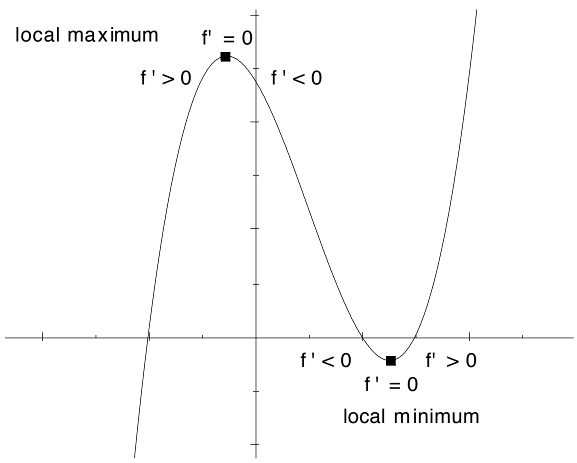

Figure 11: First derivative test for local minima and local maxima.

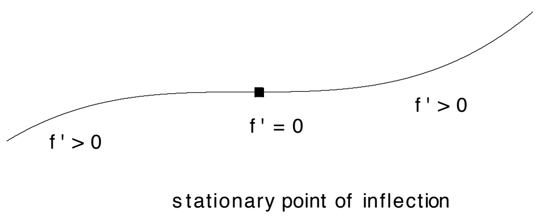

Figure 12: First derivative test for point of inflection.

4.2.3 Optimisation

Okay, now we are in a position to be able to do some optimisation problems. To maximise a function $f(x)$ in a certain region of the x values we are looking for the greatest value that $f(x)$ can possibly take for x in the region that we are interested in. This may or may not be at a stationary point. Figure 13 illustrates this. In this figure, we are looking for the maximum and minimum of the function in the region $2 \le x \le 7$. In this region there are two stationary points, one a local maximum and one a local minimum. However notice that the maximum value of the function does not occur at the local maximum, but at the endpoint of the region, ie where $x = 7$. This point is not at the top of the hill, so it is not a stationary point, but it is still the maximum value of the function for $2 \le x \le 7$ because we are ignoring any x which is bigger than 7. On the other hand, in this case the minimum value of this function for $2 \le x \le 7$ is found at a stationary point. Now we are in a position to tell you exactly how to find the maximum or minimum of a function.

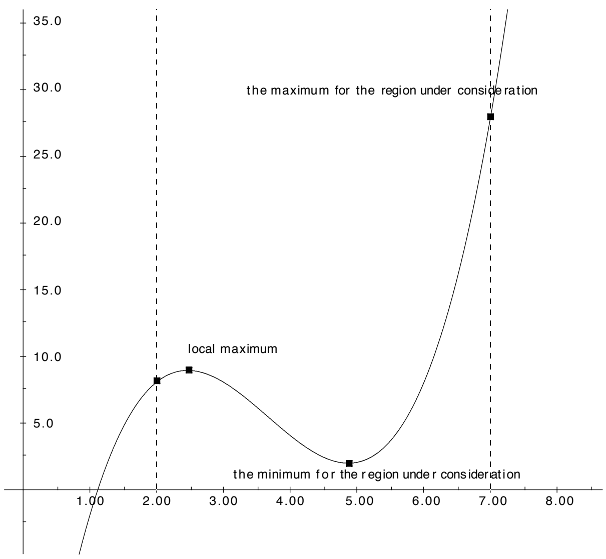

Figure 13: The maximum is found at the endpoint of the region under consideration, and not at a stationary point. The minimum is found inside the region under consideration at a stationary point

The location of maxima and minima

A function $f(x)$ may or may not have a maximum or minimum value in a particular region of x values. However, if they do exist the maximum and the minimum values must occur at one of three places:

1. At the endpoints (if they exist) of the region under consideration.
2. Inside the region at a stationary point.
3. Inside the region at a point where the derivative does not exist.

Notes

1. It is easy to find an example of a function which has no maximum or minimum in a particular region. For example the function $f(x) = x$ has neither a maximum nor a

minimum value for $-\infty < x < \infty$. Its graph simply keeps increasing as the values of x increase. Referring to Point 1 above, if for example the region under consideration was $-\infty < x < \infty$ then this region has no endpoints. As another example, the region $x \geq 1$ has only one endpoint, $x = 1$.
2. A note about Point 3 above: in this booklet we will not treat points where the derivative does not exist. However you should be aware that there may be such points, and that the maximum or minimum may be found at one. For more information consult a more comprehensive calculus text.

Now that we know exactly where the maxima or minima can occur, we can give a procedure for finding them.

Procedure for finding the maximum or minimum values of a function.

1. Find the endpoints of the region under consideration (if there are any).
2. Find all the stationary points in the region.
3. Find all points in the region where the derivative does not exist.
4. Substitute each of these into the function and see which gives the greatest (or smallest) function value.

Example Find the minimum value and the maximum value of the function $f(x) = x^2e^x$ for $-4 \leq x \leq 1$.

Solution We will follow the procedure outlined above. The endpoints are -4 and 1. Differentiating we obtain $f'(x) = x^2e^x + 2xe^x = x(x+2)e^x$. Setting $f'(x) = 0$ and solving we get stationary points at $x = 0$ and $x = -2$. There are no points where the derivative does not exist. Therefore the maximum and minimum values will be found at one of the points $x = -4, -2, 0, 1$. Substituting we obtain $f(-4) \approx 0.29$, $f(-2) \approx 0.54$, $f(0) = 0$ and $f(1) = e \approx 2.7$. therefore the maximum value occurs at $x = 1$ and is equal to e, and the minimum value occurs at $x = 0$ and is 0.

Example Find the maximum and minimum values of the function $g(t) = \frac{1}{3}t^3 - t + 2$ for $0 \leq t \leq 3$.

Solution The endpoints are $t = 0$ and $t = 3$. Differentiating and equating to zero we get $g'(t) = t^2 - 1 = (t-1)(t+1) = 0$ so the stationary points are at $t = -1, 1$. Since -1 is not in the region, the possible locations of the maximum and the minimum are $t = 0, 1, 3$. Substituting into g we obtain $g(0) = 2$, $g(1) = \frac{4}{3}$ and $g(3) = 8$. The maximum is therefore $g(3) = 8$ and the minimum is $g(1) = \frac{4}{3}$.

Example A farmer is to make a rectangular paddock. The farmer has 100 metres of fencing and wants to make the rectangle that will enclose the greatest area. What dimensions should the rectangle be?

Solution There are many rectangular paddocks that can be made with 100 metres of fencing. If we call one side of the rectangle x, then because the perimeter is 100, the other side of the rectangle is $50 - x$. The area of the paddock is then $A(x) = x(50 - x)$. We must maximise the function $A(x)$ for $0 \leq x \leq 50$ (since the sides of the rectangle cannot have negative length). Now $\frac{dA}{dx} = 50 - 2x$ which is zero when $x = 25$. Thus $x = 25$ is the only stationary point and the maximum is found at one of the points $x = 0, 25, 50$. Substituting these values into $A(x)$ we find that the maximum occurs when $x = 25$. The rectangular paddock with the maximum area is a square.

Exercise 4.1 Find the maximum and the minimum of the function $f(x) = x^4 - 2x^2$ for $-1 \leq x \leq 2$

Exercise 4.2 Maximise the function $g(t) = te^{-t^2}$ for $-2 < t < 2$.

Exercise 4.3 Find the minimum value of $h(u) = 2u^3 + 3u^2 - 12u + 5$ in the region $-3 \leq u \leq 2$.

Exercise 4.4 A farmer wishes to make a rectangular chicken run using an existing wall as one side. He has 16 metres of wire nettting. Find the dimensions of the run which will give the maximum area. What is this area?

5 The clever idea behind differential calculus (also known as differentiation from first principles)

In this section we will have a look at the idea behind differential calculus. While it is important that you at least see this idea once, in practice you calculate the derivative of a function using the procedures explained in Section 3. These procedures work because of the clever idea that we are going to describe now, but in practice we just use them without keeping in mind the whole time where they came from.

Figure 14 shows a portion of the graph of the function $f(x) = x^2$.

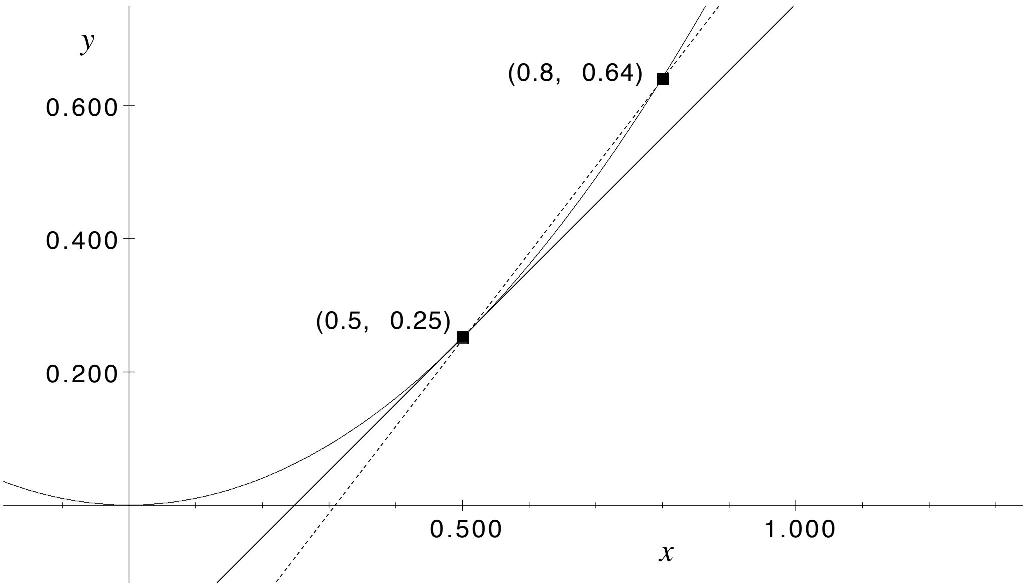

Figure 14: Graph of $y = x^2$. The solid line is the tangent to the graph at $x = 0.5$, and the dotted line is an approximate tangent line drawn through the points $(0.5, 0.25)$ and $(0.8, 0.64)$ which both lie on the graph of the function.

The tangent to the graph at the point $(x, y) = (0.5, (0.5)^2)$ is represented by the solid line. We are going to find the exact slope of this tangent.

To work out the slope of a line we need to know two points on the line. If we know the points (x_1, y_1) and (x_2, y_2) on the line then the rise between these two points is $y_2 - y_1$, and the run between them is $x_2 - x_1$, and so the slope of the line is given by

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

We cannot use this formula directly to work out the slope of the tangent, because we only know the exact location of *one* point on the tangent line, the point $(0.5, 0.25)$. If we were to pick another point from the diagram that looks like it is on the line, then we would be back to using the approximate graphical methods from Section 2. If we want to get the exact answer, we must use another way. This is the clever idea behind differential calculus.

We look at another line which which has slope *nearly* equal to the slope of the tangent, and on which we do know two points. In Figure 14 we have drawn such a line (the dotted line) going through the points $(0.5, (0.5)^2)$, which is the actual point on the curve where we are trying to find the tangent, and $(0.8, (0.8)^2)$ which is another point on the curve. This second point is not far from the point $(0.5, (0.5)^2)$, and so the slope of the line joining them is not too different from the slope of the tangent at $(0.5, (0.5)^2)$. Because we know two points on the dotted line, we can work out its slope. It is

$$\begin{aligned}\text{slope} &= \frac{(0.8)^2 - (0.5)^2}{0.8 - 0.5} \\ &= 1.3\end{aligned}$$

The slope of the tangent is therefore about (but certainly not exactly) 1.3.

To get a better approximation we might try taking the second point closer to the point $(0.5, (0.5)^2)$. In Figure 15 we have done this.

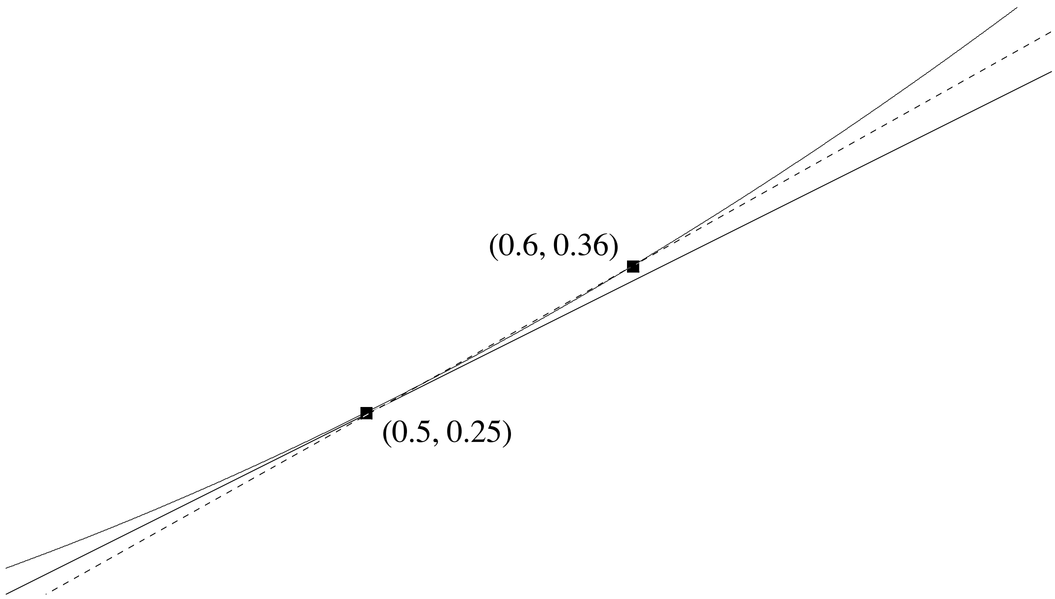

Figure 15: Graph of $y = x^2$. The solid line is the tangent to the graph at $x = 0.5$, and the dotted line is an approximate tangent line drawn through the points $(0.5, 0.25)$ and $(0.6, 0.36)$ which both lie on the graph of the function.

Here the second point is just 0.1 units to the right of $(0.5, (0.5)^2)$. The slope of the dotted line in this figure is

$$\begin{aligned}\text{slope} &= \frac{(0.6)^2 - (0.5)^2}{0.6 - 0.5} \\ &= 1.1\end{aligned}$$

The exact slope of the tangent is closer to 1.1 than it is to 1.3, though we still don't know its precise value.

If we wanted an even better approximation then we could choose the second point to be even closer to $(0.5, (0.5)^2)$. For example we could try the second point to be $(0.49, (0.49)^2)$. Notice that this point is to the *left* of $(0.5, (0.5)^2)$, whereas previously we had chosen points to the right of the point. This is not important. What is important is that it is just 0.01

units to the left of this point. The line joining these two points is very close to the actual tangent, and the slope of this line is

$$\begin{aligned} \text{slope} &= \frac{(0.49)^2 - (0.5)^2}{0.49 - 0.5} \\ &= 0.99 \end{aligned}$$

It seems that the closer the second point gets to the point $(0.5, (0.5)^2)$ the closer the slope of the line joining the two points gets to 1. We might guess that the slope of the tangent to the curve at the point $(0.5, (0.5)^2)$ must be 1. We can be sure of this with the following calculation.

Suppose that the second point is just h units to the right (or left if $h < 0$) of $x = 1$. We can think of h as being a very small number. For example we have used $h = 0.3$, $h = 0.1$, and $h = -0.01$ in our examples above. The coordinates of the second point will be $(0.5 + h, (0.5 + h)^2)$. We can work out the slope of the line joining the points $(0.5, (0.5)^2)$ and $(0.5 + h, (0.5 + h)^2)$ in the same way that we did above. It is

$$\begin{aligned} \text{Slope} &= \frac{(0.5 + h)^2 - 0.5^2}{(0.5 + h) - 0.5} \\ &= \frac{(0.5 + h)^2 - 0.5^2}{h} \\ &= \frac{(.25 + h + h^2) - .25}{h} \\ &= \frac{h + h^2}{h} \\ &= 1 + h \end{aligned}$$

Moving the second point closer and closer to the first is the same as making h closer and closer to zero. But the slope is $1 + h$ so the closer that h gets to zero the closer the slope gets to 1. The slope of the tangent is therefore *exactly* 1. This puts the matter beyond doubt. We are no longer relying on approximations or guesses. We have shown that the slope of the tangent to the graph of $y = x^2$ at the point $(0.5, 0.25)$ is exactly 1. In symbols,

$$\left.\frac{d}{dx}x^2\right|_{x=0.5} = 1.$$

With the same method we could have found the slope of the tangent to the curve when $x = 1$, or $x = -0.37$, or indeed at any value of the independent variable x.

Exercise 5.1 **(a)** Using the same technique as above, find the slope of the tangent to the graph of x^2 at $x = 2$. Check that this agrees with the answer that you would have obtained using the results of Section 3.

(b) Using the idea introduced above, find the slope of the tangent to the graph of x^3 at $x = 1$. Check your answer by using the techniques from Section 3.

We don't really have to specify any particular value of x, but can leave it as unknown.

$$\begin{aligned}\text{Slope of line through } (x, x^2) \text{ and } (x+h, (x+h)^2) &= \frac{(x+h)^2 - x^2}{x+h-x} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= 2x + h\end{aligned}$$

and as $h \to 0$ the line through (x, x^2) and $(x + h, (x + h)^2)$ gets closer to the tangent at x and the slope of this line gets closer and closer to $2x$. The slope of the tangent is therefore exactly $2x$. This works no matter what the value of x is. For example the slope of the tangent at $x = -0.37$ is $2 \times (-0.37) = -0.74$. We can say that the derivative of the function x^2 is $2x$. In symbols,

$$\frac{d}{dx}x^2 = 2x.$$

We have chosen the function $f(x) = x^2$ for this example, because it is perhaps the simplest function that gets across the idea. The same method works for any function, though the resulting algebra will often be more difficult.

Exercise 5.2 **(a)** Using the ideas of this section, find the derivative of x^3.

(b) Use the same ideas to find the derivative of $x^3 + 2x$.

By now you probably have little doubt that the derivative of x^2 is $2x$, and that the derivative of x^3 is $3x^2$. Hopefully you are now willing to believe that the derivative of x^n is nx^{n-1}, no matter what value n has. It would not be too difficult for us to prove this fact, but the proof is beyond the scope of this booklet. However what you have seen in this section is the basic idea that underlies all of differential calculus, and all of the rules and techniques of Section 3 come from it.

6 Solutions to exercises

Exercise 1.1

From the graph, at time $t = 0$ the motorist is 100 metres from home, and at time $t = 6$ the motorist is 250 metres from home.

Exercise 1.2

At time $t = 0$ the motorist is 100 metres from home and at time $t = 2$ the motorist is 150 metres from home, so in the first 2 seconds the motorist has travelled $150 - 100 = 50$ metres. At time $t = 3$ the motorist is 175 metres from home and at time $t = 5$ the motorist is 225 metres from home so in the time from $t = 3$ to $t = 5$ the motorist has travelled $225 - 175 = 50$ metres.

Exercise 1.3

A time $t = 60$ the motorist is 1008 metres from home and at time $t = 62$ the motorist is 1032 metres from home so in the 2 second interval from time $t = 60$ to time $t = 62$ the motorist travelled $1032 - 1008 = 24$ metres. A time $t = 64$ the motorist is 1072 metres from home so in the 2 second interval from time $t = 62$ to time $t = 64$ the motorist has travelled $1072 - 1032 = 40$ metres.

Exercise 2.1

Refer to Figure 16. We have used the indicated points on the lines to calculate the slopes. You may have chosed different points, but your answers should be close to those here. Remember this is only an approximate way of finding the slopes, so you shouldn't consider yourself wrong if you don't get exactly the same answers as here.

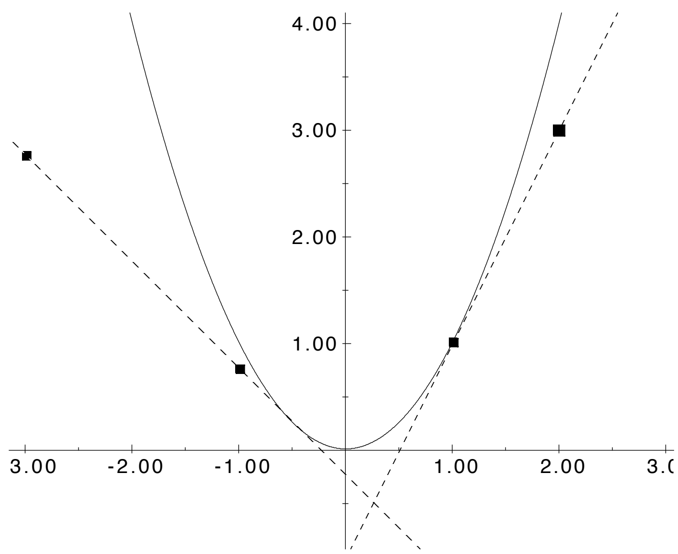

Figure 16: Tangents to graph of $f(x) = x^2$.

Slope of tangent to $f(x) = x^2$ at $x = 1$ is

$$f'(1) \approx \frac{3-1}{2-1} = 2.$$

The tangent at $x = 0$ is the x-axis, which has slope 0, so $f'(0) = 0$.

Slope of tangent to $f(x) = x^2$ at $x = -0.5$ is

$$f'(-0.5) \approx \frac{2.75 - 0.75}{-3 - (-1)} = -1.$$

Exercise 2.2

Refer to Figure 17.

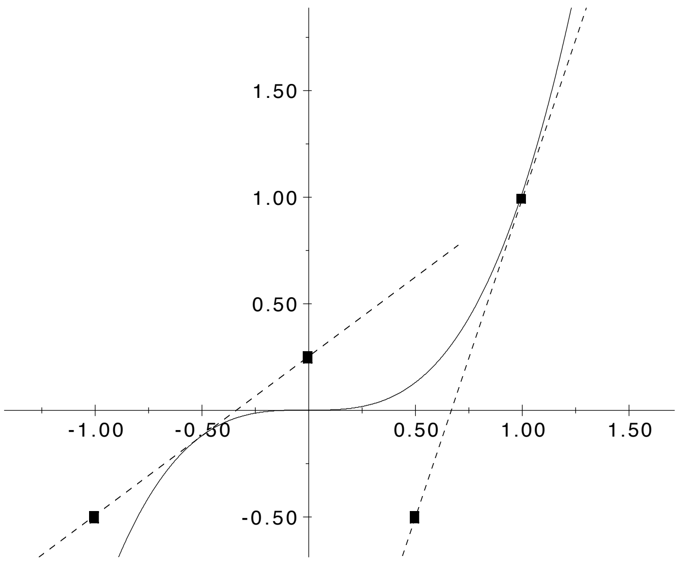

Figure 17: Tangents to graph of $f(x) = x^3$.

Slope of tangent to $f(x)$ at $x = 1$ is

$$f'(1) \approx \frac{1 - (-0.5)}{1 - 0.5} = 3.$$

As in Exercise 2.1, the tangent at $x = 0$ is the x-axis which has slope 0, so $f'(0) = 0$.

Slope of tangent to $f(x)$ at $x = -0.5$ is

$$f'(-0.5) \approx \frac{0.25 - (-0.5)}{0 - (-1)} = 0.75.$$

Exercise 3.1

(a) $f'(x) = 4x^3$ **(b)** $\frac{dy}{dx} = -7x^{-8}$ **(c)** $f'(u) = 2.3u^{1.3}$ **(d)** $f'(t) = -\frac{1}{3}t^{-\frac{4}{3}}$

(e) $f'(t) = \frac{22}{7}t^{\frac{15}{7}}$ **(f)** $g'(z) = -\frac{3}{2}z^{-\frac{5}{2}}$ **(g)** $\frac{dy}{dt} = -3.8t^{-4.8}$ **(h)** $\frac{dz}{dx} = \frac{3}{7}x^{-\frac{4}{7}}$

Exercise 3.2

(a) $\dfrac{1}{x^2} = x^{-2}$ so $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right) = -2x^{-3}$

(b) $t\sqrt{t} = t^{\frac{3}{2}}$ so $\dfrac{d}{dt}\left(t\sqrt{t}\right) = \dfrac{3}{2}t^{\frac{1}{2}}$

(c) $\dfrac{d}{dx}\sqrt[3]{x} = \dfrac{d}{dx}x^{\frac{1}{3}} = \dfrac{1}{3}x^{-\frac{2}{3}}$

(d) $\dfrac{d}{dx}\left(\dfrac{1}{x^2\sqrt{x}}\right) = \dfrac{d}{dx}x^{-\frac{5}{2}} = -\dfrac{5}{2}x^{-\frac{7}{2}}$

(e) $\dfrac{d}{dx}\left(\dfrac{1}{x\sqrt[4]{x}}\right) = -\dfrac{d}{dx}x^{5/4} = \dfrac{-5}{4}x^{-\frac{9}{4}}$

(f) $\dfrac{d}{dx}\left(\dfrac{s^3\sqrt{s}}{\sqrt[3]{s}}\right) = \dfrac{d}{ds}s^{\frac{19}{6}} = \dfrac{19}{6}s^{\frac{13}{6}}$

(g) $\dfrac{d}{du}\left(\dfrac{1}{u^3}\right) = \dfrac{du^{-3}}{du} = -3u^{-4}$

(h) $\dfrac{d}{dt}\left(\dfrac{t}{t^2\sqrt{t}}\right) = \dfrac{d}{dt}t^{-\frac{3}{2}} = -\dfrac{3}{2}t^{-\frac{5}{2}}$

(i) $\dfrac{d}{dx}\left(x^{\frac{1}{2}}\dfrac{\sqrt{x}}{x}\right) = \dfrac{d}{dx}1 = 0$

Exercise 3.3

When $x = 8$ we have $y = \sqrt[3]{8} = 2$ so the point $(8, 2)$ is on the line. Now $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$ and so $\frac{dy}{dx} = \frac{1}{12}$ when $x = 8$. The tangent therefore has equation

$$y = \frac{1}{12}x + b.$$

Substituting $x = 8$ and $y = 2$ into this equation we obtain

$$2 = \frac{1}{12}8 + b$$

so that $b = \frac{4}{3}$. The equation is therefore $y = \frac{x}{12} + \frac{4}{3}$.

Exercise 3.4

(a) $f'(x) = 10x - x^{-\frac{1}{2}}$ **(b)** $\frac{dy}{dx} = -14x^{-8} - 6x^{-3}$ **(c)** $f'(t) = 5.75t^{1.3} + \frac{1}{2}t^{\frac{-1}{2}}$

(d) $h'(z) = -\frac{1}{3}z^{-\frac{4}{3}} + 5$ **(e)** $f'(u) = \frac{5}{3}u^{\frac{2}{3}} + 21u^{-8}$ **(f)** $g'(z) = -16z^{-3} + 5z^{-2}$

(g) $\frac{dy}{dt} = -40t^{-9} + \frac{1}{2}t^{-\frac{1}{2}}$ **(h)** $\frac{dz}{dx} = \frac{4}{7}x^{-\frac{6}{7}} - x^{-\frac{3}{2}}$

Exercise 3.5

(a) $f'(x) = 12x^2(1-3x) - 3(4x^3+2)$

(b) $g'(x) = (2x+1)(x^2+1) + (x^2+x+2)2x$

(c) $h'(x) = (9x^2-4x+8)(x^2-2x+4) + (3x^3-2x^2+8x-5)(2x-2)$

(d) $f'(s) = -s(3s+5) + 3(1-\frac{s^2}{2})$

(e) $g'(t) = (\frac{t^{-\frac{1}{2}}}{2} - t^{-2})(2t-1) + 2(\sqrt{t}+\frac{1}{t})$

(f) $h'(y) = (-\frac{y^{-\frac{1}{2}}}{2} + 2y)(1-3y^2) - 6y(2-\sqrt{y}+y^2)$

Exercise 3.6

The rate of change of r with respect to t is

$$\frac{dr}{dt} = (1-t^{-2})(t^2-2t+1) + (t+\frac{1}{t})(2t-2).$$

Substituting $t=2$ we obtain $(1-\frac{1}{4})(4-4+1) + (2+\frac{1}{2})(4-2) = \frac{23}{4}$.

Exercise 3.7

The gradient of the tangent is given by

$$\frac{dy}{dx} = (2x-2)(3x^3-5x^2+2) + (x^2-2x+1)(9x^2-10x).$$

Substituting $x=2$ we obtain 28.

Exercise 3.8

(a) $f'(x) = \dfrac{(x+1)-(x-1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$

(b) $g'(x) = \dfrac{(3x-2)2-(2x+3)3}{(3x-2)^2} = \dfrac{-13}{(3x-2)^2}$

(c) $h'(x) = \dfrac{(x^2+5)2x-(x^2+2)2x}{(x^2+5)^2} = \dfrac{6x}{(x^2+5)^2}$

(d) $f'(t) = \dfrac{(1+2t^2)2-8t^2}{(1+2t^2)^2} = \dfrac{2-4t^2}{(1+2t^2)^2}$

(e) $f'(s) = \dfrac{(1-\sqrt{s})\frac{1}{2}s^{-\frac{1}{2}} + (1+\sqrt{s})\frac{1}{2}s^{-1/2}}{(1-\sqrt{s})^2} = \dfrac{s^{-\frac{1}{2}}}{(1-\sqrt{s})^2}$

(f) $h'(x) = \dfrac{(x^3+4)2x-(x^2-1)3x^2}{(x^3+4)^2} = \dfrac{-x^4+3x^2+8x}{(x^3+4)^2}$

(g) $f'(u) = \dfrac{(3u^4+5)(3u^2+1)-(u^3+u-4)12u^3}{(3u^4+5)^2} = \dfrac{-3u^6-9u^4+48u^3+15u^2+5}{(3u^4+5)^2}$

(h) $g'(t) = \dfrac{(t^2+3t+1)(2t+6)-(t^2+6t)(2t+3)}{(t^2+3t+1)^2} = \dfrac{-3t^2+2t+6}{(t^2+3t+1)^2}$

Exercise 3.9

(a) $\dfrac{d}{dx}\left((2x+3)^2\right) = 8x + 12$

(b) $\dfrac{d}{dx}\left((x^2+2x+1)^{12}\right) = 12(x^2+2x+1)^{11}(2x+2)$

(c) $\dfrac{d}{dx}\left((3-x)^{21}\right) = -21(3-x)^{20}$

(d) $\dfrac{d}{dx}\left((x^3-1)^5\right) = 5(x^3-1)^4 3x^2 = 15x^2(x^3-1)^4$

(e) $\dfrac{d}{dt}\sqrt{t^2-5t+7} = \dfrac{d}{dt}(t^2-5t+7)^{\frac{1}{2}} = \dfrac{1}{2}(t^2-5t+7)^{-\frac{1}{2}}(2t-5)$

(f) $\dfrac{d}{dz}\left(\dfrac{1}{\sqrt{2-z^4}}\right) = \dfrac{d}{dz}\left((2-z^4)^{-\frac{1}{2}}\right) = 2z^3(2-z^4)^{-\frac{3}{2}}$

(g) $\dfrac{d}{dt}\left((t^3-\sqrt{t})^{-3.8}\right) = -3.8(t^3-\sqrt{t})^{-4.8}(3t^2 - \dfrac{1}{2\sqrt{t}})$

(h) $\dfrac{d}{dx}\left((x+\dfrac{1}{x})^{\frac{3}{7}}\right) = \dfrac{3}{7}(x+\dfrac{1}{x})^{-\frac{4}{7}}(1-\dfrac{1}{x^2})$

Exercise 3.10

(a) $\dfrac{d}{dx}\left((x+2)(x+3)^2\right) = (x+3)^2 + 2(x+2)(x+3)$

(b) $\dfrac{d}{dx}\left((2x-1)^2(x+3)^3\right) = 4(2x-1)(x+3)^3 + 3(2x-1)^2(x+3)^2$

(c) $\dfrac{d}{dx}\left(x\sqrt{1-x}\right) = \sqrt{1-x} - \dfrac{x}{2\sqrt{1-x}}$

(d) $\dfrac{d}{dx}\left(x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}\right) = \dfrac{1}{3}x^{-\frac{2}{3}}(1-x)^{\frac{2}{3}} - \dfrac{2}{3}x^{\frac{1}{3}}(1-x)^{-\frac{1}{3}}$

(e) $\dfrac{d}{dx}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \dfrac{\sqrt{1-x^2} + x^2(1-x^2)^{-\frac{1}{2}}}{1-x^2}$

Exercise 3.11

(a) $f'(x) = \dfrac{6x^2}{2x^3} = \dfrac{3}{x}$

Alternatively write $f(x) = \ln 2 + 3\ln x$ so that $f'(x) = 3\dfrac{1}{x}$.

(b) $f'(x) = 7x^6 e^{x^7}$

(c) $f'(x) = \frac{7}{x}$

(d) $f'(x) = (2x+3x^2)e^{x^2+x^3}$

(e) Write $f(x) = \log_e 7 - 2\log_e x$ so that $f'(x) = -\frac{2}{x}$.

(f) $f'(x) = -e^{-x}$

(g) $f'(x) = \dfrac{e^x + 3x^2}{e^x + x^3}$

(h) Write $f(x) = \ln e^x + 3\ln x$ so that $f'(x) = 1 + \dfrac{3}{x}$.

(i) Write $f(x) = \ln(x^2+1) - \ln(x^3 - x)$ so that $f'(x) = \dfrac{2x}{x^2+1} - \dfrac{3x^2-1}{x^3-x}$.

Exercise 3.12

(a) $\dfrac{d}{dx}\cos 3x = -3\sin 3x$

(b) $\dfrac{d}{dx}\sin(4x+5) = 4\cos(4x+5)$

(c) $\dfrac{d}{dx}\sin^3 x = 3\sin^2 x\cos x$

(d) $\dfrac{d}{dx}\sin x\cos x = \cos^2 x - \sin^2 x$

(e) $\dfrac{d}{dx}x^2\sin x = 2x\sin x + x^2\cos x$

(f) $\dfrac{d}{dx}\cos(x^2+1) = -2x\sin(x^2+1)$

(g) $\dfrac{d}{dx}\left(\dfrac{\sin x}{x}\right) = \dfrac{x\cos x - \sin x}{x^2}$

(h) $\dfrac{d}{dx}\sin\dfrac{1}{x} = -\dfrac{1}{x^2}\cos\dfrac{1}{x}$

(i) $\dfrac{d}{dx}\tan\sqrt{x} = \dfrac{1}{2\sqrt{x}}\sec^2\sqrt{x}$

(j) $\dfrac{d}{dx}\left(\dfrac{1}{x}\sin\dfrac{1}{x}\right) = -\dfrac{1}{x^2}\sin\dfrac{1}{x} - \dfrac{1}{x^3}\cos\dfrac{1}{x}$

Exercise 4.1

$f'(x) = 4x^3 - 4x$ so $f'(x) = 0$ at $x = 0, \pm 1$ and the maxima and minima must occur at the points $x = -1, 0, 1, 2$. Substituting these values into $f(x)$ we find that the maximum occurs at $x = 2$ and the minimum occurs at $x = -1$ and at $x = 1$.

Exercise 4.2

$g'(t) = (1-2t^2)e^{-t^2}$. Setting this equal to zero and solving we find that the stationary points are at $t = \pm\frac{1}{\sqrt{2}}$ and the maximum must occur at one of the points $t = -2, \pm\frac{1}{\sqrt{2}}, 2$. Substituting into $g(t)$ we find that the maximum value occurs at $t = \frac{1}{\sqrt{2}}$.

Exercise 4.3

$h'(u) = 6u^2 + 6u - 12 = 6(u^2 + u - 2)$. The stationary points are at $u = -2, 1$ and the minimum value occurs at one of the points $u = -3, -2, 1, 2$. Substituting into $h(u)$ we find that the minimum occurs at $u = 1$.

Exercise 4.4

If we let the side of the run that is opposite the existing wall have length x, then the other side of the run has length $8 - \frac{x}{2}$.

The area of the run is $A(x) = x(8 - \frac{x}{2})$ and we must maximise this function in the region $0 \le x \le 16$. Differentiating gives $A'(x) = 8 - x$ so the only stationary point is at $x = 8$. The maximum occurs at one of $x = 0, 8, 16$. Substituting, we see that the maximum occurs when $x = 8$, giving an area of 32 square metres.

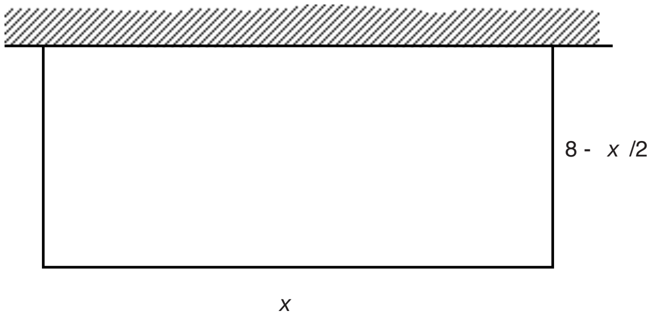

Figure 18: A chicken run built against the side of an existing wall, with 16 metres of netting.

Exercise 5.1

b.

$$\begin{aligned}\text{Slope} &= \frac{(1+h)^3 - 1^3}{(1+h) - 1} \\ &= \frac{1 + 3h + 3h^2 + h^3 - 1}{h} \\ &= \frac{3h + 3h^2 + h^3}{h} \\ &= 3 + 3h + h^2.\end{aligned}$$

So, the slope of the tangent to the graph of x^3 at $x = 1$ is 3.

Exercise 5.2

b. Slope of the line through $(x, x^3 + 2x)$ and $((x+h), (x+h)^3 + 2(x+h))$ is

$$\begin{aligned}\text{Slope} &= \frac{((x+h)^3 + 2(x+h)) - (x^3 + 2x)}{(x+h) - x} \\ &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - x^3 - 2x}{h} \\ &= \frac{3x^2h + 3xh^2 + h^3 + 2h}{h} \\ &= 3x^2 + 3xh + h^2 + 2.\end{aligned}$$

As $h \to 0$ the slope of this line $\to 3x^2 + 2$.

So, the slope of the tangent to the curve is exactly $3x^2 + 2$.