Turkish Journal of Analysis and Number Theory, 2016, Vol. 4, No. 3, 74-81
Available online at http://pubs.sciepub.com/tjant/4/3/4

DOI:10.12691/tjant-4-3-4

A Note on the Translated Whitney Numbers and Their *q*-Analogues

Mahid M. Mangontarum[1,*], Omar I. Cauntongan[2], Amerah M. Dibagulun[1]

[1]Department of Mathematics, Mindanao State University-Main Campus, 9700 Marawi City, Philippines
[2]Department of Natural Sciences and Mathematics, Mindanao State University-Maigo School of Arts and Trades, 9206 Maigo, Lanao del Norte, Philippines
*Corresponding author: mmangontarum@yahoo.com, mangontarum.mahid@msumain.edu.ph

Abstract This paper presents natural q-analogues for the translated Whitney numbers. Several combinatorial properties which appear to be q-deformations of those classical ones are obtained. Moreover, we give a combinatorial interpretation of the classical translated Whitney numbers of the first and second kind, and their q-analogues in terms of A-tableaux.

Keywords: *translated Whitney numbers, A-tableaux, q-analogues*

Cite This Article: Mahid M. Mangontarum, Omar I. Cauntongan, and Amerah M. Dibagulun, "A Note on the Translated Whitney Numbers and Their q-Analogues." *Turkish Journal of Analysis and Number Theory*, vol. 4, no. 3 (2016): 74-81. doi: 10.12691/tjant-4-3-4.

1. Introduction

Belbachir and Bousbaa [1] defined the following translated Whitney numbers via combinatorial approach:

- $\begin{bmatrix} n \\ k \end{bmatrix}^{(\alpha)}$:= the number of permutations of n elements with k cycles such that the element of each cycle can mutate in α ways, except the dominant one;
- $\begin{Bmatrix} n \\ k \end{Bmatrix}^{(\alpha)}$:= the number of partitions of the set $\{1,2,3,\ldots,n\}$ into k subsets such that each element of each subset can mutate in α ways, except the dominant one; and
- $\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^{(\alpha)}$:=the number of ways to distribute the set $\{1,2,3,\ldots,n\}$ into k ordered lists such that the elements in each list can mutate with α ways except the dominant one.

These numbers are called translated Whitney numbers of the first, second and third kind, respectively. Mangontarum *et al.* [17], and Mangontarum and Dibagulun [15] established several combinatorial properties related to the translated Whitney numbers of the first and second kinds. For simplicity, the notations

$$\tilde{w}_{(\alpha)}(n,k) := \begin{bmatrix} n \\ k \end{bmatrix}^{(\alpha)} \text{ and } \tilde{W}_{(\alpha)}(n,k) := \begin{Bmatrix} n \\ k \end{Bmatrix}^{(\alpha)}$$

are used. Among the properties of the numbers $\tilde{w}_{(\alpha)}(n,k)$ and $\tilde{W}_{(\alpha)}(n,k)$ are the horizontal generating functions (see [1,17])

$$(x|-\alpha)_n = \sum_{k=0}^{n} \tilde{w}_{(\alpha)}(n,k)x^k \tag{1}$$

and

$$x^n = \sum_{k=0}^{n} \tilde{W}_{(\alpha)}(n,k)(x|\alpha)_k \tag{2}$$

where

$$(x|\alpha)_n = x(x-\alpha)(x-2\alpha) \cdots (x-(n-1)\alpha), (x|\alpha)_0 = 1. \tag{3}$$

Furthermore, Mangontarum and Dibagulun [15] defined the "signed" translated Whitney numbers $w^*_{(\alpha)}(n,k)$ as

$$w^*_{(\alpha)}(n,k) = (-1)^{k-n}\,\tilde{w}_{(\alpha)}(n,k) \tag{4}$$

with the horizontal generating function given by

$$(x|\alpha)_n = \sum_{k=0}^{n} w^*_{(\alpha)}(n,k)x^k. \tag{5}$$

On the other hand, the translated Whitney numbers of the third kind (originally called as translated Whitney-Lah numbers),

$$L_{(\alpha)}(n,k) := \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^{(\alpha)},$$

is known to satisfy the relation (see [1])

$$(x|-\alpha)_n = \sum_{k=0}^{n} L_{(\alpha)}(n,k)(x|\alpha)_k. \tag{6}$$

The above-mentioned numbers are actually particular cases of Hsu and Shiue's [12] generalized Stirling numbers which are defined as coefficients of the inverse relations

$$(t\,|\,\alpha)_n = \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma)(t-\gamma\,|\,\beta)_k \tag{7}$$

and

$$(t\,|\,\beta)_n = \sum_{k=0}^{n} S(n,k;\beta,\alpha,-\gamma)(t+\gamma\,|\,\alpha)_k, \tag{8}$$

for any non-negative integer n and α, β and γ may be real or complex numbers such that $(\alpha,\beta,\gamma) \neq (0,0,0)$. More precisely, the translated Whitney numbers $w^*_{(\alpha)}(n,k)$, $\widetilde{W}_{(\alpha)}(n,k)$ and $L_{(\alpha)}(n,k)$ are given by

$$\begin{aligned} S(n,k;-\alpha,0,0) &= w^*_{(\alpha)}(n,k), \\ S(n,k;0,\alpha,0) &= \tilde{W}_{(\alpha)}(n,k), \\ S(n,k;-\alpha,\alpha,0) &= L_{(\alpha)}(n,k). \end{aligned} \tag{9}$$

Furthermore, the classical Stirling numbers (see [21]) are given by

$$S(n,k;1,0,0) = s(n,k), S(n,k;0,1,0) = S(n,k), \tag{10}$$

where $s(n,k)$ and $S(n,k)$ denote the Stirling numbers of the first and second kind, respectively.

Denoting by $[x]_q$, the q-analogue of an integer x defined by

$$[x]_q = \frac{q^x - 1}{q-1}, \tag{11}$$

and introducing the "q-deformed" generalized factorial given by

$$\begin{aligned} [x|\alpha]_n &= [x]_q [x-\alpha]_q [x-2\alpha]_q \\ & [x-(n-1)\alpha]_q, [x\,|\,\alpha]_0 = 1, \end{aligned} \tag{12}$$

we will define the numbers $w^1_{(\alpha)}[n,k]_q$, $w^2_{(\alpha)}[n,k]_q$ and $w^3_{(\alpha)}[n,k]_q$ as coefficients in the expansions of the resulting relations obtained when the expressions $(x|\alpha)_n$, $(x|-\alpha)_n$ and x are replaced with $[x|\alpha]_n$, $[x|-\alpha]_n$ and $[x]_q$, respectively, in (5), (2) and (6). That is, we have the following defining relations in the form of horizontal generating functions:

$$[x\,|\,\alpha]_n = \sum_{k=0}^{n} w^1_{(\alpha)}[n,k]_q [x]_q^k, \tag{13}$$

$$[x]_q^n = \sum_{k=0}^{n} w^2_{(\alpha)}[n,k]_q [x\,|\,\alpha]_k, \tag{14}$$

and

$$[x\,|-\alpha]_n = \sum_{k=0}^{n} w^3_{(\alpha)}[n,k]_q [x\,|\,\alpha]_k. \tag{15}$$

For clarity, we will refer to the numbers $w^1_{(\alpha)}[n,k]_q$, $w^2_{(\alpha)}[n,k]_q$ and $w^3_{(\alpha)}[n,k]_q$ as the translated q-Whitney numbers of the first, second, and third kind, respectively. By convention, we set

$$w^1_{(\alpha)}[n,k]_q = w^2_{(\alpha)}[n,k]_q = w^3_{(\alpha)}[n,k]_q = 0 \tag{16}$$

when $n < k$ or for $n, k < 0$. Also, we easily observe that when $n = 0$, we have

$$w^1_{(\alpha)}[0,0]_q = w^2_{(\alpha)}[0,0]_q = w^3_{(\alpha)}[0,0]_q = 1. \tag{17}$$

The study of q-analogues of some well-known identities and certain Stirling-type numbers has been the interest of previous authors. Different approaches has been earlier considered by Carlitz [5], Katriel [13], Katriel and Kibler [14], Gould [11], Corcino *et al*. [8], Mansour *et al*. [18], Corcino and Mangontarum [10], Mangontarum and Katriel [16], as well as some of the references therein. Perhaps the reason is due to their various applications in other fields of discipline. For q-analogues of the generalized Stirling numbers, Corcino *at al*. [8] defined the generalized q-Stirling numbers as

$$\begin{aligned} \sigma^1[n,k] &\equiv \sigma^1[n,k;\alpha,\beta,\gamma] \\ &= S\left[n,k;q^\alpha,q^\beta,q^\gamma-1\right](q-1)^{k-n} \end{aligned} \tag{18}$$

and

$$\begin{aligned} \sigma^2[n,k] &\equiv \sigma^2[n,k;\alpha,\beta,\gamma] \\ &= S\left[n,k;q^\beta,q^\alpha,1-q^\gamma\right](q-1)^{k-n}, \end{aligned} \tag{19}$$

where $S[n,k;q^\alpha,q^\beta,q^\gamma-1]$ and $S[n,k;q^\beta,q^\alpha,1-q^\gamma]$ are the exponential-type Stirling numbers [8, Equations (3) and (4)]. The q-analogues $\sigma^1[n,k]$ and $\sigma^2[n,k]$ are known to satisfy the relations

$$\begin{aligned} &[[\beta t]+[\gamma]|[\alpha]]_n \\ &= \sum_{k=0}^{n} \sigma^1[n,k] q^{\beta\binom{k}{2}} \binom{t}{k}_{q^\beta} \prod_{i=1}^{k}[i\beta]_q \end{aligned} \tag{20}$$

and

$$\begin{aligned} &[[\alpha t]-[\gamma]|[\beta]]_n \\ &= \sum_{k=0}^{n} \sigma^2[n,k] q^{\alpha\binom{k}{2}} \binom{t}{k}_{q^\alpha} \prod_{i=1}^{k}[i\alpha]_q \end{aligned} \tag{21}$$

and the explicit formula given by

$$\begin{aligned} \sigma^1[n,k] = &\left(\prod_{i=1}^{k}[i\beta]_q\right)^{-1} \sum_{j=0}^{k} (-1)^{k-j} q^{\beta<k|j>} \\ &\binom{k}{j}_{q^\beta} [[j\beta]+[\gamma]\,|\,[\alpha]]_n, \end{aligned} \tag{22}$$

as reported by Corcino and Mangontarum [9, Lemma 2.2], and Corcino and Barrientos [7, Equation (1.3)], respectively, $< k|j >= \binom{j+1}{2} - kj$ and

$$[[x\beta]+[\gamma]\,|\,[\alpha]]_n = \prod_{l=0}^{n-1}\left([x\beta]_q + [\gamma]_q - [l\alpha]_q\right). \tag{23}$$

Here, the expression $\binom{k}{j}_{q^\beta}$ is the q-binomial coefficients defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}, \tag{24}$$

where

$$[n]_q! = [1]_q[2]_q \cdots [n-1]_q[n]_q, [0]_q! = 1 \tag{25}$$

is the q-falling factorial of n. Other combinatorial properties of $\sigma^1[n,k]$ and $\sigma^2[n,k]$ were mostly established by Corcino and Barrientos [7].

Another special case obtained by suitable assignment of values to the defining relations in (7) and (8) are the r-Whitney numbers of the first and second kind defined by Mező [20] in his attempt to derive a new formula for the Bernoulli polynomials. The r-Whitney numbers of the first and second kind are actually generalizations of the classical Whitney numbers of the first and second kind earlier defined and fully developed by Benoumhani [2,3]. The q-analogues of the r-Whitney numbers of both kinds were recently introduced by Mangontarum and Katriel [16], and are called (q,r) −Whitney numbers of the first and second kind, denoted by $w_{m,r,q}(n,k)$ and $W_{m,r,q}(n,k)$, respectively. The said q-analogues defined via horizontal generating functions given by

$$m^n (a^\dagger)^n a^n = \sum_{k=0}^{n} w_{m,r,q}(n,k)(ma^\dagger a + r)^k \tag{26}$$

and

$$(ma^\dagger a + r)^n = \sum_{k=0}^{n} m^k W_{m,r,q}(n,k)(a^\dagger)^k a^k \tag{27}$$

were developed using the q-Boson operators $a^\dagger$ and a which satisfy the commutation relation

$$[a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = 1. \tag{28}$$

By thoroughly investigating the two pairs of q-analogues presented in (18), (19), (26) and (27), it can be observed that they represent motivations which are different from ours in (13), (14) and (15). This leads us to the conclusion that these distinctly motivated q-analogues are difficult to express in terms of one another.

This study will then focus on establishing some basic combinatorial properties of the translated q-Whitney numbers, as well as to give a combinatorial interpretation in the context of A-tableaux. Most of the full proofs of the results in this paper follow the usual methods employed by previous authors which are quite elementary. Hence, they are left for the readers to explore.

2. Some Combinatorial Properties

2.1. Recurrence Relations

The next theorem contains useful tools in solving for the first few values of the translated q-Whitney numbers$w^1_{(\alpha)}[n,k]_q$, $w^2_{(\alpha)}[n,k]_q$ and $w^3_{(\alpha)}[n,k]_q$.

Theorem 2.1. *The triangular recurrence relations for the translated q-Whitney numbers of the first, second and third kind are given by the following identities*:

$$w^1_{(\alpha)}[n+1,k]_q = q^{-\alpha n} w^1_{(\alpha)}[n,k-1]_q + [-\alpha n]_q w^1_{(\alpha)}[n,k]_q, \tag{29}$$

$$w^2_{(\alpha)}[n+1,k]_q = q^{\alpha(k-1)} w^2_{(\alpha)}[n,k-1]_q + [\alpha k]_q w^2_{(\alpha)}[n,k]_q, \tag{30}$$

and

$$w^3_{(\alpha)}[n+1,k]_q = q^{\alpha(n+k-1)} w^3_{(\alpha)}[n,k-1]_q + [\alpha(n+k)]_q w^3_{(\alpha)}[n,k]_q. \tag{31}$$

Proof. The proofs of these identities follow directly from the definitions in (13), (14) and(15).

Remark 2.2. It is trivial that when $k = n$, we get

$$w^1_{(\alpha)}[n,n]_q = q^{-\alpha\binom{n}{2}}, \quad w^2_{(\alpha)}[n,n]_q = q^{\alpha\binom{n}{2}}, \quad w^3_{(\alpha)}[n,n]_q = q^{2\alpha\binom{n}{2}}. \tag{32}$$

The following corollary can be obtained by successive applications of the recurrence relations in the previous theorem:

Corollary 2.3. *The translated q-Whitney numbers of the first, second and third kind satisfy the vertical recurrence relations given by*

$$w^1_{(\alpha)}[n+1,k+1]_q = \sum_{j=k}^{n} q^{-j\alpha}[-\alpha n|-\alpha]_{n-j} w^1_{(\alpha)}[j,k]_q, \tag{33}$$

$$w^2_{(\alpha)}[n+1,k+1]_q = q^{\alpha k}\sum_{j=k}^{n}[\alpha(k+1)]_q^{n-j} w^2_{(\alpha)}[j,k]_q, \tag{34}$$

and

$$w^3_{(\alpha)}[n+1,k+1]_q = \sum_{j=k}^{n} q^{\alpha(k+j)}[\alpha(n+k+1)|\alpha]_{n-j} w^3_{(\alpha)}[j,k]_q, \tag{35}$$

respectively.

It can be shown that as $q \to 1$, we get

$$\lim_{q\to 1}\sum_{j=k}^{n} q^{-j\alpha}[-\alpha n|-\alpha]_{n-j} w^1_{(\alpha)}[j,k]_q = \sum_{j=k}^{n}(-\alpha)^{n-j} w^*_{(\alpha)}(j,k)(n)_{n-j}, \tag{36}$$

$$\lim_{q\to 1} q^{\alpha k}\sum_{j=k}^{n}\left[\alpha(k+1)\right]_q^{n-j} w_{(\alpha)}^2\left[j,k\right]_q = \sum_{j=k}^{n}(\alpha(k+1))^{n-j}\tilde{W}_{(\alpha)}(n,k), \tag{37}$$

and

$$\lim_{q\to 1}\sum_{j=k}^{n} q^{\alpha(k+j)}\left[\alpha(n+k+1)|\alpha\right]_{n-j} w_{(\alpha)}^3\left[j,k\right]_q = \sum_{j=k}^{n}\alpha^{n-j}L_{(\alpha)}(j,k)(n+k+1)_{n-j}. \tag{38}$$

Remark 2.4. The identities in (36) and (37) are exactly the vertical recurrence relations obtained by Mangontarum and Dibagulun [15], whereas (38) which can be written in the form

$$L_{(\alpha)}(n+1,k+1) = \sum_{j=k}^{n}\alpha^{n-j}L_{(\alpha)}(j,k)(n+k+1)_{n-j}. \tag{39}$$

is a new identity for the translated Whitney-Lah numbers.

Another type of recursion formula is the horizontal recurrence relation. The results in the next corollary can be verified by evaluating the right-hand sides using the triangular recurrence relations.

Corollary 2.5. *The translated q-Whitney numbers of the first kind, second kind and third kind satisfy the horizontal recurrence relations given by*

$$w_{(\alpha)}^1\left[n,k\right]_q = \sum_{j=0}^{n-k}\frac{[-\alpha n]_q^j\, w_{(\alpha)}^1[n+1,k+1+j]_q}{q^{-\alpha n(j+1)}}, \tag{40}$$

$$w_{(\alpha)}^2\left[n,k\right]_q = \sum_{j=0}^{n-k}\frac{(-1)^j\left[\alpha(k+1)|-\alpha\right]_j}{q^{\alpha k(j+1)+\alpha\binom{j+1}{2}}}\cdot w_{(\alpha)}^2[n+1,k+1+j]_q, \tag{41}$$

and

$$w_{(\alpha)}^3\left[n,k\right]_q = \sum_{j=0}^{n-k}\frac{(-1)^j\left[\alpha(k+1)|-\alpha\right]_j}{q^{\alpha(j+1)(n+k)+\alpha\binom{j+1}{2}}}\cdot w_{(\alpha)}^3\left[n+1,k+1+j\right]_q. \tag{42}$$

Taking the limits of these results as q approaches 1 yields

$$w_{(\alpha)}^*(n,k) = \sum_{j=0}^{n-k}(\alpha n)^j w_{(\alpha)}^*(n+1,k+1+j), \tag{43}$$

$$\tilde{W}_{(\alpha)}(n,k) = \sum_{j=0}^{n-k}(-\alpha)^j\tilde{W}_{(\alpha)}(n+1,k+1+j)k+1_j, \tag{44}$$

and

$$L_{(\alpha)}(n,k) = \sum_{j=0}^{n-k}(-\alpha)^j\tilde{L}_{(\alpha)}(n+1,k+1+j)n+k+1_j. \tag{45}$$

Remark 2.6. The vertical recurrence relations in (43) and (44) were due to Mangontarum and Dibagulun [15]. On the other hand, (45) is a new identity for the translated Whitney-Lah numbers.

Before ending this subsection, we note that the vertical and horizontal recurrence relations follow the same pattern illustrated by the well-known *Chu Shih-Chieh's binomial identities* [6].

2.2. Orthogonality and Inverse Relations

The orthogonality and the inverse relations for the classical translated Whitney numbers of the first and second kinds were earlier established by Mangontarum and Dibagulun [15]. In this subsection, we establish analogous properties for the q-analogues $w_{(\alpha)}^1[n,k]_q$ and $w_{(\alpha)}^2[n,k]_q$.

Theorem 2.7. *The orthogonality relations of the translated q-Whitney numbers of the first and second kind are given by*

$$\sum_{k=j}^{n} w_{(\alpha)}^2\left[n,k\right]_q w_{(\alpha)}^1\left[k,j\right]_q = \delta_{jn} \tag{46}$$

and

$$\sum_{k=j}^{n} w_{(\alpha)}^1\left[n,k\right]_q w_{(\alpha)}^2\left[k,j\right]_q = \delta_{jn}, \tag{47}$$

where δ_{jn} is the Kronecker delta defined by

$$\delta_{jn} = \begin{cases}0, if\ j\neq n\\ 1, if\ j=n\end{cases}.$$

Proof. The proof of this theorem is done by combining the defining relations in (13) and (14).

Since $w_{(\alpha)}^1[n,k]_q = w_{(\alpha)}^2[n,k]_q = 0$ when $n<k$, then

$$\sum_{k=0}^{\infty} w_{(\alpha)}^2\left[n,k\right]_q w_{(\alpha)}^1\left[k,j\right]_q = \sum_{k=0}^{\infty} w_{(\alpha)}^1\left[n,k\right]_q w_{(\alpha)}^2\left[k,j\right]_q = \delta_{jn}.$$

Hence, if we define $\mathcal{M}_q^1(\alpha) = \left(w_{(\alpha)}^1[n,k]_q\right)$, and $\mathcal{M}_q^2(\alpha) = \left(w_{(\alpha)}^2[n,k]_q\right)$ to be infinite matrices whose (n,k)-th entries are the translated q-Whitney numbers $w_{(\alpha)}^1[n,k]_q$ and $w_{(\alpha)}^2[n,k]_q$, respectively, then the following corollary is immediately obtained:

Corollary 2.8. *If I is the infinite-dimensional identity matrix, then*

$$\mathcal{M}_q^2(\alpha)\cdot\mathcal{M}_q^1(\alpha) = \mathcal{M}_q^1(\alpha)\cdot\mathcal{M}_q^2 = I. \tag{49}$$

Remark 2.9. It is obvious that $\mathcal{M}_q^1(\alpha)$ and $\mathcal{M}_q^2(\alpha)$ are inverse matrices of each other.

Corollary 2.10. *The inverse relations of the translated q-Whitney numbers of the first and Second kind are given by the following:*

$$f_n = \sum_{k=0}^{n} w^1_{(\alpha)}[n,k]_q g_k \Leftrightarrow g_n = \sum_{k=0}^{n} w^2_{(\alpha)}[n,k]_q f_k, \quad (50)$$

$$f_k = \sum_{n=k}^{\infty} w^1_{(\alpha)}[n,k]_q g_n \Leftrightarrow g_k = \sum_{n=k}^{\infty} w^2_{(\alpha)}[n,k]_q f_n. \quad (51)$$

2.3. Explicit Formulas

If we replace x with αk, then we can rewrite the defining relation in (14) as

$$[\alpha k]_q^n = \sum_{j=0}^{n} w^2_{(\alpha)}[n,j]_q [\alpha k \mid \alpha]_j = \sum_{j=0}^{k} \binom{k}{j}_{q^\alpha} \left\{ \frac{w^2_{(\alpha)}[n,j]_q [\alpha k \mid \alpha]_j}{\binom{k}{j}_{q^\alpha}} \right\}.$$

Using the q-binomial inversion formula given by

$$f_n = \sum_{k=0}^{n} \binom{n}{k}_q g_k \Leftrightarrow g_n = \sum_{k=0}^{n} (-1)^{n-k} q^{\alpha\binom{k-j}{2}} \binom{k}{j}_q f_k, \quad (52)$$

we obtain

$$\frac{w^2_{(\alpha)}[n,j]_q [\alpha k|\alpha]_k}{\binom{k}{k}_{q^\alpha}} = \sum_{k=0}^{n} (-1)^{n-k} q^{\alpha\binom{k-j}{2}} \binom{k}{j}_q [\alpha j]_q^n.$$

Simplifying this expression yields the explicit formula

$$w^2_{(\alpha)}[n,k]_q = \frac{1}{[\alpha k \mid \alpha]_k} \sum_{k=0}^{n} (-1)^{n-k} q^{\alpha\binom{k-j}{2}} \binom{k}{j}_q [\alpha j]_q^n.$$

Multiplying both sides by $\frac{z^n}{[n]_q!}$ and summing over n yields

$$\sum_{n=k}^{\infty} w^2_{(\alpha)}[n,k]_q \frac{z^n}{[n]_q!} = \frac{1}{[k]_{q^\alpha}![\alpha]_q^k} \sum_{k=0}^{n} (-1)^{n-k} q^{\alpha\binom{k-j}{2}} \binom{k}{j}_q e_q\left([\alpha j]_q z\right).$$

where $e_q\left([\alpha j]_q z\right)$ denotes the type 1 q-exponential function defined by

$$e_q(x) = \sum_{i=0}^{\infty} \frac{x^i}{[i]_q!}. \quad (53)$$

Therefore, we have the following theorem:

Theorem 2.11. *The translated q-Whitney numbers of the second kind satisfy the explicit formula*

$$w^2_{(\alpha)}[n,k]_q = \frac{1}{[k]_{q^\alpha}![\alpha]_q^k} \sum_{k=0}^{n} (-1)^{n-k} q^{\alpha\binom{k-j}{2}} \binom{k}{j}_q [\alpha j]_q^n. \quad (54)$$

and the exponential generating function

$$\sum_{n=k}^{\infty} w^2_{(\alpha)}[n,k]_q \frac{z^n}{[n]_q!} = \frac{1}{[k]_{q^\alpha}![\alpha]_q^k} \sum_{k=0}^{n} (-1)^{n-k} q^{\alpha\binom{k-j}{2}} \binom{k}{j}_q e_q\left([\alpha j]_q z\right). \quad (55)$$

Remark 2.12. Taking the limits of (56) and (57) as q approaches 1 gives us

$$\lim_{q\to 1} \frac{1}{[k]_{q^\alpha}![\alpha]_q^k} \sum_{k=0}^{n} (-1)^{n-k} q^{\alpha\binom{k-j}{2}} \binom{k}{j}_q [\alpha j]_q^n = \frac{1}{\alpha^k k!} \sum_{k=0}^{n} (-1)^{n-k} \binom{k}{j} (\alpha j)^n = \tilde{W}_{(\alpha)}(n,k)$$

and

$$\lim_{q\to 1} \frac{1}{[k]_{q^\alpha}![\alpha]_q^k} \sum_{k=0}^{n} (-1)^{n-k} q^{\alpha\binom{k-j}{2}} \binom{k}{j}_q e_q\left([\alpha j]_q z\right) = \frac{1}{k!}\left(\frac{e^{\alpha k}-1}{\alpha}\right)^k = \sum_{n=k}^{\infty} \tilde{W}_{(\alpha)}(n,k) \frac{z^n}{n!}.$$

These identities are the ones obtained by Mangontarum *et al.* [[17], Propositions 2 and 3].

Note that it is difficult to obtain an explicit formula similar to (56) for the translated q-Whitney numbers of the first kind. In the next theorem, we present explicit formulas in symmetric polynomial forms for the translated q-Whitney numbers of both kinds.

Theorem 2.13. *The translated q-Whitney numbers of the first kind satisfy the explicit formula in elementary symmetric polynomial form given by*

$$w^1_{(\alpha)}[n,k]_q = (-1)^{n-k} q^{-\alpha\binom{n}{2}} \sum_{0 \le i_1 < i_2 < \cdots < i_{n-k} \le n-1} \prod_{j=1}^{n-k} \left[\alpha i_j\right]_q \quad (56)$$

and the translated q-Whitney numbers of the second kind satisfy the explicit formula in complete symmetric polynomial form given by

$$w^2_{(\alpha)}[n,k]_q = q^{\alpha\binom{n}{2}} \sum_{0 \le i_1 \le i_2 \le \cdots \le i_{n-k} \le k} \prod_{j=1}^{n-k} \left[\alpha i_j\right]_q. \quad (57)$$

Proof. Since the triangular recurrence relation in (29) can be written in the equivalent form given by

$$w^{1}_{(\alpha)}[n+1,k+1]_q = q^{-\alpha n}\left(w^{1}_{(\alpha)}[n,k-1]_q - [\alpha n]_q\, w^{1}_{(\alpha)}[n,k]_q\right), \tag{58}$$

and taking into consideration the initial values $w^{1}_{(\alpha)}[0,0]_q = 1$ and $w^{1}_{(\alpha)}[n+1,n+1]_q = 1$, we may proceed with the proof of (56) by induction on n. (57) can be deduced similarly.

The following corollary immediately follows from the previous theorem as $q \to 1$:

Corollary 2.14. *The classical translated Whitney numbers have the following explicit formulas*:

$$\overset{*}{w}_{(\alpha)}(n,k) = (-1)^{n-k} \sum_{0\le i_1<i_2<\cdots<i_{n-k}\le n-1} \prod_{j=1}^{n-k} \alpha i_j, \tag{59}$$

$$\tilde{w}_{(\alpha)}(n,k) = \sum_{0\le i_1<i_2<\cdots<i_{n-k}\le n-1} \prod_{j=1}^{n-k} \alpha i_j, \tag{60}$$

$$\tilde{W}_{(\alpha)}(n,k) = \sum_{0\le i_1\le i_2\le\cdots\le i_{n-k}\le k} \prod_{j=1}^{n-k} \alpha i_j. \tag{61}$$

2.4. Translated *q*-Dowling Polynomials and Numbers

The *n*-th translated Dowling polynomial, denoted by $\tilde{D}_{(\alpha)}(n;x)$, is defined as

$$\tilde{D}_{(\alpha)}(n;x) = \sum_{k=0}^{n} \tilde{W}_{(\alpha)}(n,k)\,x^k, \tag{62}$$

While the translated Dowling numbers, denoted by $\tilde{D}_{(\alpha)}(n)$, is defined as

$$\tilde{D}_{(\alpha)}(n) = \tilde{D}_{(\alpha)}(n;1). \tag{63}$$

These polynomials and numbers were defined and developed by Mangontarum *et al.* [17] as common generalizations of the classical Bell polynomials/numbers and the Dowling polynomials/numbers. Naturally, we may define the translated *q*-Dowling polynomials, denoted by $D_\alpha[n;x]_q$, as the *n*-th degree polynomial

$$D_\alpha[n;x]_q = \sum_{k=0}^{n} w^{2}_{(\alpha)}[n,k]_q\, x^k \tag{64}$$

and the translated *q*-Dowling numbers, denoted by $D_\alpha[n]_q$, as the case when $x = 1$. That is,

$$D_\alpha[n]_q = D_\alpha[n;1]_q. \tag{65}$$

Theorem 2.15. *The polynomials $D_\alpha[n;x]_q$ satisfy the exponential generating function given by*

$$\sum_{n=k}^{\infty} D_\alpha[n;x]_q \frac{z^n}{[n]_q!} = \hat{e}_{q^\alpha}\left(-\frac{x}{[\alpha]_q}\right) \sum_{i=0}^{\infty} \frac{e_q([\alpha i]_q z)}{[i]_{q^\alpha}!} \left(\frac{x}{[\alpha]_q}\right)^i \tag{66}$$

and the explicit formula (Dobinski-type formula) given by

$$D_\alpha[n;x]_q = \hat{e}_{q^\alpha}\left(-\frac{x}{[\alpha]_q}\right) \sum_{i=0}^{\infty} \frac{[\alpha i]_q^n}{[i]_{q^\alpha}!} \left(\frac{x}{[\alpha]_q}\right)^i, \tag{67}$$

where

$$\hat{e}_q(x) = \sum_{i=0}^{\infty} q^{\binom{i}{2}} \frac{t^i}{[i]_q!} \tag{68}$$

is the type 2 q-exponential function.

Proof. These identities can be easily verified using the explicit formula in (54).

The following corollary is obtained by setting $x = 1$:

Corollary 2.16. *The translated q-Dowling numbers satisfy the exponential generating function given by*

$$\sum_{n=k}^{\infty} D_\alpha[n]_q \frac{z^n}{[n]_q!} = \hat{e}_{q^\alpha}\left(-\frac{1}{[\alpha]_q}\right) \sum_{i=0}^{\infty} \frac{e_q([\alpha i]_q z)}{[i]_{q^\alpha}!} \left(\frac{1}{[\alpha]_q}\right)^i \tag{69}$$

and the explicit formula given by

$$D_\alpha[n]_q = \hat{e}_{q^\alpha}\left(-\frac{1}{[\alpha]_q}\right) \sum_{i=0}^{\infty} \frac{[\alpha i]_q^n}{[i]_{q^\alpha}!} \left(\frac{1}{[\alpha]_q}\right)^i. \tag{70}$$

Taking the limits as *q* approaches to 1 yields

$$\lim_{q\to 1} \sum_{n=k}^{\infty} D_\alpha[n;x]_q \frac{z^n}{[n]_q!} = \exp\left\{\frac{x}{\alpha}\left(e^{\alpha z}-1\right)\right\} \tag{71}$$

$$\lim_{q\to 1} D_\alpha[n;x]_q = \left(\frac{x}{e}\right)^{\frac{x}{\alpha}} \sum_{i=0}^{\infty} \frac{(i\alpha)^n}{i!}\left(\frac{x}{\alpha}\right)^i. \tag{72}$$

These identities were the ones obtained by Mangontarum *et al.* [[17], Theorems 5 and 7].

3. Combinatorial Interpretations

Before proceeding, we first recall the following definitions (see [19]):

Definition 3.1. A 0-1tableau is a pair $\varphi = (\lambda, f)$, where $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k)$ is a partition of an integer *m* and $f = (f_{ij})_{1\le j\le \lambda_i}$ is a "filling" of the cells of the corresponding Ferrer's diagram of shape λ with 0's and 1's such that exactly one 1 in each column.

Figure 1 below represents the 0-1 tableau$\varphi = (\lambda, f)$, where $\lambda = (9,6,5,3)$ with

$$f_{13} = f_{15} = f_{17} = f_{18} = f_{19} = f_{22} = f_{26} = f_{34} = f_{41} = 1$$

and $f_{ij} = 0$ elsewhere for $1 \le j \le \lambda_i$.

0	0	1	0	1	0	1	1	1
0	1	0	0	0	1			
0	0	0	1	0				
1	0	0						

Figure 1. A 0-1 Tableau φ

Definition 3.2. An A-tableau is a list ϕ of columns c of a Ferrer's diagram of a partition λ (by decreasing order of length) such that the lengths $|c|$ are part of the strictly increasing sequences $A=(a_i)_{i\geq 0}$ of non-negative integers.

Let $\omega: N\to K$ denote a function from the set of non-negative integers N to a ring K(column weights according to length). For an A-tableau Φ with columns of length $|c|\leq h$, we set

$$\omega_A(\Phi)=\prod_{c\in\Phi}\omega(|c|). \tag{73}$$

Also, we denoted by $T^A(x,y)$ the set of all A-tableaux with y columns whose lengths are in the set $\{a_0,a_1,a_2,\dots,a_x\}$, and by $T_d^A(x,y)$ the subset of $T^A(x,y)$ which contains all A-tableau with columns of distinct lengths.

We are now ready to state the next theorem.

Theorem 3.3. *Let $\omega: N\to K$ be a function (N is the set of non-negative integers and K is a ring) defined by $\omega(|c|)=\alpha|c|$, where α is a complex number.*

1. *The translated Whitney numbers of the first kind $\tilde{w}_{(\alpha)}(n,k)$ counts the number of A-tableaux in $T_d^A(n-1,n-k)$ if $|c|$ is the length of column c of an A-tableau in $T_d^A(n-1,n-k)$;*
2. *The translated Whitney numbers of the second kind $\widetilde{W}_{(\alpha)}(n,k)$ counts the number of A-tableaux in $T^A(k,n-k)$ if $|c|$ is the length of column c of an A-tableau in $T^A(k,n-k)$.*

Proof. Let $\Phi\in T_d^A(n-1,n-k)$ and $|c|$ be the length of column c of Φ. By (73), we have

$$\omega_A(\Phi)=\prod_{c\in\Phi}\omega(|c|)=\prod_{j=1}^{n-k}\alpha\,|c_j|\,. \tag{74}$$

Then, by (60),

$$\sum_{\Phi\in T_d^A(n-1,n-k)}\omega_A(\Phi)=\sum_{\Phi\in T_d^A(n-1,n-k)}\prod_{j=1}^{n-k}\alpha\left|c_j\right|=\tilde{w}_{(\alpha)}(n,k). \tag{75}$$

We can also show by similar method that

$$\sum_{\varphi\in T^A(k,n-k)}\omega_A(\varphi)=\sum_{\varphi\in T^A(k,n-k)}\prod_{j=1}^{n-k}\alpha\left|c_j\right|=\tilde{W}_{(\alpha)}(n,k). \tag{76}$$

This completes the proof.

Now, we take the function $\Omega: N\to K$ defined by $\Omega(|c|)=[\alpha|c|]_q$ in place of the function ω. Then it can be showed that

$$\sum_{\Phi\in T_d^A(n-1,n-k)}\Omega_A(\Phi)=\sum_{\Phi\in T_d^A(n-1,n-k)}\prod_{j=1}^{n-k}\left[\alpha\left|c_j\right|\right]_q=(-1)^{n-k}q^{\alpha\binom{n}{2}}w_{(\alpha)}^1[n,k]_q \tag{77}$$

and

$$\sum_{\varphi\in T^A(k,n-k)}\omega_A(\varphi)=\sum_{\varphi\in T^A(k,n-k)}\prod_{j=1}^{n-k}\left[\alpha\left|c_j\right|\right]_q=q^{-\alpha\binom{n}{2}}w_{(\alpha)}^2[n,k]_q. \tag{78}$$

Hence, we propose the following combinatorial interpretations for the q-deformed case:

Proposition 3.4. *Let $\Omega: N\to K$ be a function (N is the set of non-negative integers and K is a ring) defined by $\Omega(|c|)=[\alpha|c|]_q$, where α is a complex number.*

1. *If $|c|$ is the length of column c of an A-tableau in $T_d^A(n-1,n-k)$, then the number of A-tableaux in $T_d^A(n-1,n-k)$ is given by $(-1)^{n-k}q^{\alpha\binom{n}{2}}w_{(\alpha)}^1[n,k]_q$;*
2. *If $|c|$ is the length of column c of an A-tableau in $T^A(k,n-k)$, then the number of A-tableaux in $T^A(k,n-k)$ is given by $q^{-\alpha\binom{n}{2}}w_{(\alpha)}^2[n,k]_q$.*

4. Conclusion

In this paper, we have seen that the translated q-Whitney numbers, being distinctly motivated compared to the works of previous authors, produces combinatorial identities which are generalizations of the known properties of the classical Stirling and Whitney numbers. Furthermore, a type of combinatorial interpretation in terms of the A-tableaux was presented for the classical Whitney numbers of the first and second kinds, and their q-analogues.

Acknowledgement

Mangontarum and Dibagulun would like to thank the System President of the Mindanao State University for issuing the Special Order No. 26 – OP, Series of 2016. Cauntongan's research is also supported by the office of the campus head of the Mindanao State University-Maigo School of Arts and Trades.

References

[1] H. Belbachir and I. Bousbaa, Translated Whitney and r-Whitney numbers: a combinatorial approach, *J. Integer Seq*, 16 (2013), Article 13.8.6.

[2] M. Benoumhani, On Whitney numbers of Dowling lattices, *Discrete Math*., 159 (1996), 13-33.

[3] M. Benoumhani, On some numbers related to the Whitney numbers of Dowling lattices, *Adv. Appl. Math.*, 19 (1997), 106-116.

[4] M. Benoumhani, Log-concavity of Whitney numbers of Dowling lattices, *Adv. Appl. Math*.,22 (1999), 186-189.

[5] L. Carlitz, q-Bernoulli numbers and polynomials, *Duke Math. J.*, 15 (1948), 987-1000.

[6] C. Chen and K. Koh, *Principles and Techniques in Combinatorics*, Singapore, 1992.

[7] R. B. Corcino and C. Barrientos, Some theorems on the q-analogue of the generalized Stirling numbers, *Bull. Malays. Math. Sci. Soc.*, 34 (2011), 487-501.

[8] R. B. Corcino, L. C. Hsu and E. L. Tan, A q-analogue of generalized Stirling numbers, *Fibonacci Quart.*, 44 (2006), 154-167.

[9] R. B. Corcino and M. M. Mangontarum, On Euler and Heine distributions and the generalized q-factorial moments, *Southeast Asian Bull. Math.*, 38 (2014), 627-640.

[10] R. B. Corcino and M. M. Mangontarum, On multiparameter q-noncentral Stirling and Bell numbers, *Ars Combin.*, 118 (2015), 201-220.

[11] H. W. Gould, The q-Stirling numbers of the first and second kinds, *Duke Math. J.*, 28 (1994), 281-289.

[12] L. Hsu and P. J. Shiue, A unified approach to generalized Stirling numbers, *Adv. Appl. Math.*, 20(1998), 366-384.

[13] J. Katriel, Combinatorial aspects of Boson algebra, *Lett. Nuovo Cimento*, 10 (1974), 565-567.

[14] J. Katriel and M. Kibler, Normal ordering for deformed boson operators and operator-valued deformed Stirling numbers, *J. Phys. A: Math. Gen.*, 25 (1992), 2683-2691.

[15] M. M. Mangontarum and A. M. Dibagulun, On the translated Whitney numbers and their combinatorial properties, *British Journal of Applied Science and Technology*, 11 (2015), 1-15.

[16] M. M. Mangontarum and J. Katriel, On q-boson operators and q-analogues of the r-Whitney and r-Dowling numbers, *J. Integer Seq.*, 18 (2015), Article 15.9.8.

[17] M. M. Mangontarum, A. P. M.-Ringia, and N. S. Abdulcarim, The translated Dowling polynomials and numbers, *International Scholarly Research Notices*, 2014, Article ID 678408, 8 pages, (2014).

[18] T. Mansour, M. Schork and M. Shattuck, On a new family of generalized Stirling and Bell numbers, *The Electronic Journal of Combinatorics*, 18 (2011), #P77.

[19] A. de Mèdicis and P. Leroux, Generalized Stirling umbers, convolution formulae and p;q-analogues, *Canad. J. Math.*, 47 (1995), 474-499.

[20] I. Mező, A new formula for the Bernoulli polynomials, *Results Math.*, 58 (2010), 329-335.

[21] J. Stirling, *Methodus Differentialissme Tractus de Summatione et Interpolatione Serierum Infinitarum*, London, 1730.