VECTOR BUNDLES AND GROMOV–HAUSDORFF DISTANCE

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Abstract. We show how to make precise the vague idea that for compact metric spaces that are close together for Gromov–Hausdorff distance, suitable vector bundles on one metric space will have counterpart vector bundles on the other. Our approach employs the Lipschitz constants of projection-valued functions that determine vector bundles. We develop some computational techniques, and we illustrate our ideas with simple specific examples involving vector bundles on the circle, the two-torus, the two-sphere, and finite metric spaces. Our topic is motivated by statements concerning "monopole bundles" over matrix algebras in the literature of theoretical high-energy physics.

Introduction

The purpose of this paper is to make precise the vague idea that if two compact metric spaces are close together then there should be a relationship between the vector bundles on the two spaces. I was led to examine this idea by statements in the theoretical high-energy physics literature [62, 2, 21, 60, 3, 63, 8, 21] stating, for example, that for matrix algebras "close" to the two-sphere certain "vector bundles" for the matrix algebras are *the* "monopole bundles" corresponding to the ordinary monopole bundles over the two-sphere. I was able to make sense of the statement that matrix algebras are close to the two-sphere [48] by introducing [47] a definition of quantum metric spaces and showing how to view the matrix algebras as such, and then by defining a quantum Gromov–Hausdorff distance, which supplied a distance between the matrix algebras and the two-sphere. But I did not find in the literature any discussion for ordinary metric spaces of a relationship between vector bundles and ordinary Gromov–Hausdorff distance that

2000 *Mathematics Subject Classification.* Primary 53C23; Secondary 46L85, 55R50.

Key words and phrases. vector bundles, Gromov–Hausdorff distance, Lipschitz, projections, monopole bundle.

The research reported here was supported in part by National Science Foundation Grant DMS-0500501.

I could then adapt to the quantum setting. The purpose of this paper is to develop such a relationship for ordinary metric spaces. I believe I have also found a path to such a relationship for quantum spaces, but it appears to be substantially more complicated and indirect. (See [51], where I have laid more foundation for establishing that relationship, motivated by the results of the present paper.) Thus I feel that it is worthwhile to first explain separately this relationship just for ordinary metric spaces. That is the aim of the present paper.

In order to see the issues involved, let us consider a relatively simple situation. Consider, for example, the two-sphere S^2 of radius 1 with its usual metric. For some small $\varepsilon > 0$ let F be a finite subset of S^2 which is ε-dense in S^2, that is, every point of S^2 is within distance ε of a point of F. (Matrix algebras are often considered to be the algebras of "functions" on "*quantum* finite sets".) Put on F the metric from S^2. Any vector bundle over S^2 restricts to a vector bundle over F. But all vector bundles over F are trivial. So, for example, the various inequivalent line bundles over S^2 will restrict to equivalent line bundles over F. At the topological level there seems to be little more that one can say about this. It is hard to see how one can say that one line bundle over F corresponds to the monopole bundle on S^2, while a different one corresponds to the trivial line bundle on S^2.

But vague intuition suggests that the restriction to F of a non-trivial line bundle over S^2 should somehow twist more rapidly than the restriction to F of a trivial bundle over S^2. We want to make this intuition precise. We need metrics in order to make sense of "more rapidly". To see how to use metrics, let us drop back to the simpler example of the circle and its simplest non-trivial vector bundle, the Möbius-strip bundle. If you imagine explaining to a friend what this bundle is, using only your hand, you will probably move your hand around in a circle, but twisting your hand as it moves. It makes sense to ask for the rate of twisting of your hand with respect to arc-length along the circle. To formulate this idea mathematically it seems necessary to view your hand as indicating a one-dimensional subspace twisting in the normal bundle to the circle within $\mathbb{R}^3$. This one-dimensional subspace can be specified by the orthogonal projection onto it. More generally, it is well-known that any vector bundle over a compact base space can be described up to equivalence (in many ways) by means of a continuous projection-valued function on the base space. Given a metric on the base space, we can consider the Lipschitz constant, $L(p)$, of a projection-valued function p. We will see that this provides a quite effective measure of how rapidly the bundle determined by p

is twisting. In fact, I have not seen any better way to quantitatively discuss how rapidly a vector bundle twists.

We will find that for a given compact base metric space and a given constant K the set of projection-valued functions of given rank and size can have several path components (even for a finite metric space). These components reflect different amounts and types of twisting. We will find that for examples such as an ε-dense $F \subset S^2$ and a p defined on F, and for more general Gromov–Hausdorff contexts, the constant K can be chosen in terms of ε so that connected components of the set of p's with $L(p) < K$ can only come from corresponding projection-valued functions q on S^2 which give isomorphic vector bundles. That is, we have a uniqueness theorem (Theorems 4.5 and 4.7) in this context. We also have an existence theorem, which states in quantitative terms that if $\varepsilon L(p)$ is small enough, then there will exist a corresponding projection-valued function q on the big space whose restriction is p, such that $L(q)$ too is appropriately small. We also have (and need) homotopy versions of these theorems (Theorems 6.2 and 6.5). To give an indication of the nature of our results, we state the following imprecise version of Theorem 6.5.

Theorem 0.1 (Imprecise version of Theorem 6.5)**.** *Let X and Y be compact metric spaces, and let ρ be a metric on their disjoint union that restricts to the given metrics on X and Y, and for which the Hausdorff distance between them is less than ϵ. Let p_0 and p_1 be functions on X whose values are projections in $M_n(\mathbb{R})$, so that they determine vector bundles on X. There are positive constants $r_1,\ r_2,\ r_3$, depending on ϵ and n, such that the following holds. Assume that there is a continuous path p_t of projection-valued functions on X going from p_0 to p_1 such that $L_X(p_t) < r_1$ for all t, where L_X denotes Lipschitz constant for the metric on X. Then there exist functions q_0 and q_1 on Y whose values are projections in $M_n(\mathbb{R})$ such that $L^\rho(p_j \oplus q_j) < r_2$ for $j = 0, 1$. Furthermore, for any such q_0 and q_1 there is a continuous path q_t of projection-valued functions on Y going from q_0 to q_1 such that $L_Y(q_t) < r_3$ for all t. In particular, the vector bundles determined by q_0 and q_1 are isomorphic. It is appropriate to view these vector bundles on Y as corresponding to the vector bundles determined by p_0 and p_1.*

The precise version of Theorem 6.5 gives, among other things, formulas for $r_1,\ r_2,\ r_3$.

When the metric spaces are sufficiently "nice", standard facts in comparison geometry can be used to relate vector bundles on ones that are close together. For example, the contents of section 3 of [38] show that if ρ is a metric on the disjoint union $X \dot\cup Y$ of compact spaces such that X is ϵ-close to Y, and if Y is suitably locally geometrically $(n-1)$-connected where n is the covering dimension of X, then there is a map f from X to Y such that $\rho(x, f(x))$ is suitably small for all $x \in X$; furthermore any two such f's are homotopic if X is an absolute neighborhood retract. Such an f can be used to pull back vector bundles from Y to X. However, the techniques that we will use here do not require conditions of finite dimension, local connectivity, and ANR on the compact metric spaces that we consider. In particular, we very much want to permit Y to be finite, so not connected. (Full matrix algebras are viewed as the algebras of functions on nonexistent "quantum finite sets".) We also need to have good control of the Lipschitz constants of projection-valued functions for the vector bundles, and it is not clear to me to what extent techniques such as those in [38] can be used to get quantitative estimates on Lipschitz constants that are as strong as those that we obtain. Perhaps a combination of the techniques could in favorable situations give stronger estimates than those obtained here. (It is interesting to recall that Vietoris first defined homology groups in the context of compact metric spaces, using the metric in an essential way. See [40] and its references. His methods seem to be in the spirit of those used in the present paper, but I have not seen a specific relationship.)

Let us mention some related ideas. If one looks back at the high-energy physics literature which discusses "vector bundles" on matrix algebras that converge to bundles on the two-sphere or other spaces (see references above), no definition of "convergence" is given, but what is noted is that the formulas for the Chern classes of the "vector bundles" on the matrix algebras appear to converge to the Chern classes of the limit bundle. However, given a convergent sequence of any kind, one can always change any one given term of the sequence without affecting the convergence of the sequence. Thus it does not seem possible to use this approach to justify asserting that a given "vector bundle" on a given matrix algebra is *the* monopole bundle. As another approach, if one has a sequence of ordinary compact metric spaces which converges to a given compact metric space, and if one has vector bundles over all of these spaces, one can try to put a compatible metric on the disjoint union of the metric spaces so that the sequence converges to the limit for Hausdorff distance, in such a way that one can combine the vector bundles to form a (continuous) vector bundle over this disjoint union. If one can do this, then one can say that the sequence of vector bundles converges to the bundle on the limit space. This approach is taken in [22, 23]. But again it does not seem possible to use this approach to justify saying that a given bundle on one given space of the sequence is *the* bundle on that space which corresponds to the bundle on the limit space.

The first two sections of this paper develop the basic properties of the Lipschitz seminorms that we will need, while the third section relates these seminorms to projections. Section 4 contains our general uniqueness theorems for extending vector bundles, while the next two sections develop our existence theorems for extending vector bundles, and discuss their consequences. The next nine sections examine simple examples that illustrate our general theory. These examples involve vector bundles on the circle, the two-torus and the two-sphere (including monopole bundles), as well as on finite metric spaces. We develop techniques for actually finding projections corresponding to given vector bundles, and for calculating the Lipschitz constants of these projections. In Section 10 we indicate how Chern classes can be used to obtain lower bounds for the Lipschitz constants. Sections 14 and 15 discuss vector bundles on compact homogeneous spaces in preparation for discussing monopole bundles in Section 16. Our treatment is far from exhaustive, even for the simple spaces we consider. The purpose of the examples we discuss is only to provide "proof of concept". There is much more to be explored (and we say nothing here about non-compact spaces).

The audience I have had in mind when writing this paper consists of geometers and topologists. For their convenience I have selectively included discussion of certain known facts from analysis that we need, rather than just giving references to the literature.

I developed an early part of the material reported here during a two-month visit at the Institut Mittag-Leffler and a one-month visit at the Institut des Hautes Etudes Scientifique in the Fall of 2003. I am very appreciative of the quite stimulating and enjoyable conditions provided by both institutes.

Contents

1. The setting

Let (X, ρ_X) and (Y, ρ_Y) be two compact metric spaces. Suppose that the Gromov–Hausdorff distance [20, 5, 55] between them is $< \varepsilon$. Then, by definition, there exist a compact metric space (Z, ρ) and isometric inclusions of X and Y into Z such that, when X and Y are viewed as subsets of Z, their Hausdorff distance is $< \varepsilon$. Our aim is to show that such an inclusion provides a correspondence between certain vector bundles over X and certain vector bundles over Y. Since we can always cut Z down to $X \cup Y$, the issues essentially come down to relating vector bundles on X to those on Z. Thus most of our technical development will take place in the setting of a given compact metric space (Z, ρ) and a closed subset X of Z, with the metric on X being the restriction of ρ to X. To say that the Hausdorff distance from X to Z is $\leq \varepsilon$ is the same as saying that X is ε-dense in Z, that is, every element of Z is within distance ε of some point of X. We will usually use this terminology "ε-dense".

We are interested in both real and complex vector bundles. But we will see that the bookkeeping for complex vector bundles is slightly more complicated than that for real vector bundles. For this reason our general theoretical discussion, here and later, will be phrased primarily for complex bundles, and we will usually only discuss real vector bundles in those places where it is not entirely clear how our discussion for complex vector bundles can be adapted to the case of real vector bundles. We will let $C(X)$ denote the algebra of continuous complex-valued functions on a compact space X, and we will let $C_{\mathbb{R}}(X)$ denote the corresponding algebra of real-valued functions. We equip both algebras with the supremum norm.

We can only expect to deal with vector bundles up to the usual bundle isomorphism. It is well known [1, 6, 11, 27, 53, 66] that, up to isomorphism, every complex vector bundle on X corresponds (in many ways) to a continuous function, p, on X whose values are projection operators. Specifically, p has values in an $n \times n$-matrix algebra $M_n(\mathbb{C})$ for some n; for any $x \in X$ the fiber at x of the vector bundle corresponding to p will be the subspace $p_x(\mathbb{C}^n)$ of $\mathbb{C}^n$. Each p_x is idempotent ($(p_x)^2 = p_x$), and we can, and will, require that p_x is self-adjoint ($(p_x)^* = p_x$). (This corresponds to choosing a Hermitian metric on the vector bundle.) We can equally well view p as an element of the algebra $M_n(A)$ where $A = C(X)$. Then we simply have $p^2 = p = p^*$. If X is connected then the rank of p must be constant. In this case p can be viewed as having values in a Grassman manifold [24], since one way to define a Grassman manifold is as the space of all projections of a given rank on a given vector space.

How then will the metric control vector bundles on X? We will use ρ to define a corresponding Lipschitz seminorm, $L = L^\rho$, on $M_n(A)$, and then use $L(p)$ for control. To define L we must first say that on $M_n(\mathbb{C})$ we use the usual operator norm obtained by viewing elements of $M_n(\mathbb{C})$ as operators on the inner-product space $\mathbb{C}^n$. We then view $M_n(A)$ as consisting of continuous functions on X with values in $M_n(\mathbb{C})$, and define the norm on $M_n(A)$ by $\|a\| = \|a\|_\infty = \sup\{\|a(x)\| : x \in X\}$. In terms of the norm on $M_n(\mathbb{C})$ we define $L = L^\rho$ on $M_n(A)$ by

$$L^\rho(a) = \sup\{\|a(x) - a(y)\|/\rho(x,y) : x, y \in X,\ x \neq y\}$$

for $a \in M_n(A)$. We note that L can easily be discontinuous for the operator norm, and we can easily have $L(a) = +\infty$. Of course, viewing A as $M_1(A)$, we have L defined on A, and then the definition of L given above becomes the traditional definition of the Lipschitz seminorm. Let $\mathcal{A} = \{f \in A : L(f) < \infty\}$, the $*$-subalgebra of "Lipschitz functions". It is well-known [65] and easily seen that $\mathcal{A}$ is a dense $*$-subalgebra of A, where $*$ refers to complex conjugation. It is also easily seen that for any $a \in M_n(A)$ we have $L(a) < \infty$ if and only if each entry of the matrix a is in $\mathcal{A}$, that is, $a \in M_n(\mathcal{A})$. Of course, $M_n(\mathcal{A})$ is a dense $*$-subalgebra of $M_n(A)$. Also, one should notice that for $a \in M_n(A)$ we have $L(a) = 0$ if and only if a is a constant function with value in $M_n(\mathbb{C})$. (Thus $L(a) = 0$ for more elements a than just the scalar multiples of the identity element of $M_n(A)$.) We will see later (Proposition 3.1) that every vector bundle is given by a p in $M_n(\mathcal{A})$, not just in $M_n(A)$. Then $L(p) < \infty$. We will control a vector bundle by the $L(p)$'s for the possible p's that represent it. In fact, we will see through examples why it is reasonable that projections p be the main focus of our attention.

In the situation in which $X \subseteq Z$, let $B = C(Z)$, and let $\pi : B \to A$ be the surjective homomorphism consisting of restricting functions on

Z to X. We will let π also denote the corresponding $*$-homomorphism from $M_n(B)$ onto $M_n(A)$. We will let L denote also the Lipschitz seminorms on B and $M_n(B)$, though we may write L_B (or L_A for A) when this may be helpful. We let $\mathcal{B}$ denote the dense $*$-subalgebra of B consisting of the functions on Z for which L is finite.

If q is a projection in $M_n(\mathcal{B})$, then $p = \pi(q)$ will be a projection in $M_n(\mathcal{A})$, and the vector bundle associated to p is easily seen to be the restriction to X of the bundle on Z associated with q. Thus an important part of our task will be to show that under suitable conditions in terms of $L(p)$ and the Hausdorff distance between X and Z, we can prove that given a projection $p \in M_n(\mathcal{A})$ there exists a projection $q \in M_n(\mathcal{B})$ such that $\pi(q) = p$, and such that $L(q)$ satisfies estimates that imply that q is unique up to homotopy. (We will see that Lipschitz-controlled homotopy is the appropriate equivalence in our setting.) We will also deal with the situation in which we have two projections p_1 and p_2 in $M_n(\mathcal{B})$ which are homotopic. (Homotopic projections give isomorphic vector bundles — see 1.7 and 4.2 of [6].) For all of this we first need to show that L has strong properties.

2. Properties of the Lipschitz seminorm L

For our present purposes it will be crucial that L^ρ satisfies certain technical properties.

Definition 2.1. Let L be a seminorm, possibly discontinuous and possibly taking value $+\infty$, on a normed unital algebra A (for which $\|ab\| \leq \|a\|\|b\|$ and $\|1\| = 1$). We will say that L satisfies the *Leibniz* property (or "is Leibniz") if for every $a, b \in A$ we have

$$L(ab) \leq L(a)\|b\| + \|a\|L(b).$$

If, in addition, whenever a^{-1} exists in A we have

$$L(a^{-1}) \leq L(a)\|a^{-1}\|^2,$$

then we will say that L is *strongly Leibniz*. If A has an involution and if $L(a^*) = L(a)$ for all $a \in A$, we say that a is a *$*$-seminorm*.

We will say that L is *lower-semicontinuous* if for one $r \in \mathbb{R}$ with $r > 0$ (hence for every $r > 0$) the set $\{a \in A : L(a) \leq r\}$ is a norm-closed subset of A. We say that L is *closed* if for one $r \in \mathbb{R}$ with $r > 0$ (hence for every $r > 0$) the set $\{a \in A : L(a) \leq r\}$ is closed in the completion, $\bar{A}$, of A. We will say that L is *semi-finite* if $\{a \in A : L(a) < \infty\}$ is dense in A.

It is evident that if L is closed then it is lower-semicontinuous. Every lower-semicontinuous L extends to a closed seminorm (see proposition 4.4 of [45]). Let $\mathcal{A} = \{a \in A : L(a) < \infty\}$. We can always define a new norm, $\|\cdot\|_L$, on $\mathcal{A}$ by

$$\|a\|_L = \|a\| + L(a).$$

If L is closed, then $\mathcal{A}$ is complete for $\|\cdot\|_L$, as can be seen by adapting the proof of proposition 1.6.2 of [65]. If L is Leibniz, then $\|\cdot\|_L$ is a normed-algebra norm, i.e., $\|ab\|_L \leq \|a\|_L\|b\|_L$. Thus if L is both closed and Leibniz, then $\mathcal{A}$ is a Banach algebra for $\|\cdot\|_L$. I do not know of an example of a finite Leibniz seminorm which is not strongly Leibniz. (See the comments after definition 1.2 of [51].)

Proposition 2.2. *For a compact metric space (X, ρ), let $A = C(X)$ or $C_{\mathbb{R}}(X)$, and let $L = L^\rho$ on $M_n(A)$ be defined as in the previous section. Then L on $M_n(A)$ is a closed semi-finite strongly-Leibniz $*$-seminorm.*

Proof. That L is a $*$-seminorm is trivial to verify. The semi-finiteness was indicated in the previous section. That L on A is Leibniz is well-known and easily shown by the same device as is used to show that a first derivative has the related Leibniz property. For L on $M_n(A)$ the proof is the same except that one must keep terms in the proper order, respecting the non-commutativity of $M_n(A)$.

That L is strongly Leibniz seems not so well-known, so we include the proof here. Let $a \in M_n(A)$ and assume that a is invertible. If $L(a) = \infty$ the desired inequality is trivially true. So assume that $L(a) < \infty$. For $x, y \in X$ with $x \neq y$ we have

$$a(x)^{-1} - a(y)^{-1} = a(x)^{-1}(a(y) - a(x))a(y)^{-1},$$

so that

$$\|a(x)^{-1} - a(y)^{-1}\|/\rho(x, y) \leq \|a^{-1}\|^2\|a(x) - a(y)\|/\rho(x, y) \leq \|a^{-1}\|^2 L(a).$$

Upon taking the supremum over x and y we obtain the desired inequality. In particular, $a^{-1} \in M_n(\mathcal{A})$.

Finally, for any fixed x and y with $x \neq y$ clearly the function $a \mapsto \|a(x) - a(y)\|/\rho(x, y)$ is norm-continuous. But L is the supremum of these continuous functions as x and y range over X. Thus L is lower-semicontinuous on $M_n(A)$ (where L may take value $+\infty$). Since $M_n(A)$ is complete for its norm, it follows that L is closed. ☐

There are some further properties of L and $M_n(\mathcal{A})$ that will be essential to us in producing projections controlled by L. For any unital algebra A over $\mathbb{C}$ and any $a \in A$, the spectrum, $\sigma(a)$, of a is defined by

$$\sigma(a) = \{\lambda \in \mathbb{C} : (\lambda - a)^{-1} \text{ does not exist in } A\}$$

(where we systematically write λ instead of $\lambda 1_A$). We make the analogous definition for algebras over $\mathbb{R}$. Note that for strongly Leibniz seminorms the "strongly" part implies that for $a \in \mathcal{A}$ its spectrum in $\mathcal{A}$ coincides with its spectrum in A.

For $f \in C(X)$ its spectrum coincides with the range of f. Let $A = C(X)$ and let $a \in M_n(A)$. Then a^{-1} exists in $M_n(A)$ exactly if $a(x)^{-1}$ exists in $M_n(\mathbb{C})$ for all x. From this one sees that $\sigma(a) = \bigcup_{x \in X} \sigma(a(x))$. Basic Banach-algebra arguments [26, 54] show that $\sigma(a)$ is a closed bounded subset of $\mathbb{C}$.

Let $f \in A = C(X)$, and let φ be a holomorphic function defined on an open neighborhood of $\sigma(f)$, that is, on the range of f. Then the composition $\varphi \circ f$ is well-defined and in A. We write it as $\varphi(f)$. But we also need to define $\varphi(a)$ for $a \in M_n(A)$, where now we ask that φ be holomorphic in a neighborhood of $\sigma(a)$. It is not immediately clear how to do this, but a basic Banach-algebra technique (see 3.3 of [26], or [54]) does this by means of the Cauchy integral formula, and this technique is called the "holomorphic-function (or symbolic) calculus". In the standard way used for ordinary Cauchy integrals, we let γ be a collection of piecewise-smooth oriented closed curves in the domain of φ that surrounds $\sigma(a)$ but does not meet $\sigma(a)$, such that φ on $\sigma(a)$ is represented by its Cauchy integral using γ. Then $z \mapsto (z-a)^{-1}$ will, on the range of γ, be a well-defined and continuous function with values in $M_n(A)$. Thus we can define $\varphi(a)$ by

$$\varphi(a) = \frac{1}{2\pi i} \int_\gamma \varphi(z)(z-a)^{-1} dz.$$

For a fixed neighborhood of $\sigma(a)$ the mapping $\varphi \mapsto \varphi(a)$ is a unital homomorphism from the algebra of holomorphic functions on this neighborhood of $\sigma(a)$ into $M_n(A)$ [26, 54].

Proposition 2.3. *With notation as above, if $a \in M_n(\mathcal{A})$ then $\varphi(a) \in M_n(\mathcal{A})$. In fact,*

$$L(\varphi(a)) \leq \left(\frac{1}{2\pi} \int_\gamma |\varphi(z)| d|z| \right) (M_\gamma(a))^2 L(a)$$

where $M_\gamma(a) = \max\{\|(z-a)^{-1}\| : z \in \mathrm{range}(\gamma)\}$.

Proof. Because L is lower-semicontinuous, it can be brought within the integral defining $\varphi(a)$, with the evident inequality. (Think of approximating the integral by Riemann sums.) Because L is strongly Leibniz, this gives

$$L(\varphi(a)) \leq \frac{1}{2\pi} \int_\gamma |\varphi(z)| \; \|(z-a)^{-1}\|^2 \; L(a) \; d|z|.$$

On using the definition of $M_\gamma(a)$ we obtain the desired inequality. $\square$

The commonly used terminology for the fact that if $a \in M_n(\mathcal{A})$ then $\varphi(a) \in M_n(\mathcal{A})$ is that "$M_n(\mathcal{A})$ is closed under the holomorphic-function calculus of $M_n(A)$". See, e.g., section 3.8 of [18]. We remark that Schweitzer has shown [56] that if A is any unital C^*-algebra and if $\mathcal{A}$ is a unital $*$-subalgebra which is closed under the holomorphic-function calculus of A, then $M_n(\mathcal{A})$ is closed under the holomorphic-function calculus of $M_n(A)$. Thus our Proposition 2.3 is a special case of Schweitzer's theorem, but it is good to see the above simple direct proof for our special case.

The proof of the above proposition depends strongly on working over $\mathbb{C}$. But we will to some extent be able to apply it when working over $\mathbb{R}$, in the following way. (See also [17].) Notice that $M_n(C_\mathbb{R}(X))$ is a $*$-subring of $M_n(C(X))$ which is closed under multiplication by scalars in $\mathbb{R}$. We will accordingly speak of "real C^*-subrings", etc. For a real $*$-subring A of a C^*-algebra B we will say that A is closed under the holomorphic-function calculus for B if for every $a \in A$ with $a = a^*$ (so that $\sigma_B(a) \subset \mathbb{R}$) and for every $\mathbb{C}$-valued function φ holomorphic in a neighborhood of $\sigma_B(a)$ and taking real values on $\sigma_B(a)$ we have $\varphi(a) \in A$, where $\varphi(a)$ is initially defined as before to be an element of B.

Proposition 2.4. *Let (X, ρ) be a compact metric space, let $A = C_\mathbb{R}(X)$, let L and $\mathcal{A}$ be as just above for A, and let $B = C(X)$. Then $M_n(\mathcal{A})$ is closed under the holomorphic-function calculus for $M_n(B)$, for every n. If $a \in M_n(\mathcal{A})$ with $a = a^*$, and if φ is holomorphic in an open neighborhood of $\sigma_B(a)$ and $\varphi(\sigma_B(a)) \subset \mathbb{R}$, then the estimate of Proposition 2.3 for $L(\varphi(a))$ holds here also.*

Proof. Let $a \in M_n(\mathcal{A})$ with $a = a^*$, and let φ be a $\mathbb{C}$-valued function holomorphic in a neighborhood of $\sigma_B(a)$ with $\varphi(\sigma_B(a)) \subset \mathbb{R}$. The closed unital $*$-subalgebra over $\mathbb{C}$ generated by a in $M_n(C(X))$ is naturally isomorphic to $C(\sigma(a))$ by one version of the spectral theorem, i.e. the "continuous-function calculus" (see 1.2.4 of [26], or page 8 of [12], or VI.6.3 of [13]), with a represented by the function $\hat{a}$ given by $\hat{a}(r) = r$ for $r \in \sigma(a)$. (Here $\sigma(a)$ is the spectrum of a as an element of $M_n(C(X))$.) Then for any continuous $\mathbb{C}$-valued function ψ on $\sigma(a)$ we can form $\psi \circ \hat{a}$, and this will correspond to an element of $M_n(C(X))$, which we denote by $\psi(a)$. If $\psi(\sigma(a)) \subset \mathbb{R}$, then by approximating ψ on $\sigma(a)$ uniformly by polynomials over $\mathbb{R}$ it is clear that $\psi(a) \in M_n(C_\mathbb{R}(X))$. But if θ is a polynomial over $\mathbb{C}$, then one can check that $\theta(a)$ obtained by substituting a into θ coincides with defining $\theta(a)$ via the holomorphic-function calculus, and that if θ is over $\mathbb{R}$ then $\theta(a) \in M_n(A)$ either way. From this one can check that $\varphi(a)$ defined via $\varphi \circ \hat{a}$ coincides with its definition via the holomorphic-function calculus, and $\varphi(a) \in M_n(C_{\mathbb{R}}(X))$ since $\varphi(\sigma(a)) \subset \mathbb{R}$. But from Proposition 2.3 we see that $L(\varphi(a)) < \infty$ so that $\varphi(a) \in M_n(\mathcal{A})$. The inequality for $L(\varphi(a))$ then follows from Proposition 2.3. $\square$

Most of the examples that we discuss later involve manifolds, and for manifolds it is useful to be able to apply calculus to our considerations. One tool relating calculus to our Lipschitz context, which we will find useful later, is given by the following proposition.

Proposition 2.5. *Let A be a unital Banach algebra and suppose that L is a lower-semicontinuous seminorm on A. Let α be a (strongly continuous) action of a Lie group G on A, and assume that this action is by isometries for L, that is, $L(\alpha_x(a)) = L(a)$ for all $a \in A$ and $x \in G$. Let A^∞ denote the algebra of smooth elements of A for the action α. (It is closed under the holomorphic-function calculus of A — see proposition 3.45 of* [18]*). Then for any $a \in A$ and any $\varepsilon > 0$ there is a $b \in A^\infty$ such that $\|b\| \leq \|a\|$, $L(b) \leq L(a)$, and $\|b - a\| < \varepsilon$.*

Proof. We use the usual smoothing argument (see, e.g., section 0.2 of [61]). Let $f \in C_c^\infty(G)$ and assume further that $f \geq 0$ and $\int_G f(x)dx = 1$ for left Haar measure on G. For a given $a \in A$ set

$$b = \int_G f(x)\alpha_x(a)dx.$$

Standard simple arguments show that $b \in A^\infty$, that $\|b\| \leq \|a\|$, and that $\|b - a\| < \varepsilon$ if f is supported sufficiently closely to the identity element of G. But by the lower semicontinuity of L we have

$$\begin{aligned} L(b) &= L\left(\int_G f(x)\alpha_x(a)dx\right) \\ &\leq \int_G f(x)L(\alpha_x(a))dx = \int_G f(x)L(a)dx = L(a). \end{aligned}$$

$\square$

3. Projections and Lipschitz seminorms

To control a vector bundle we will use bounds on $L(p)$ for projections representing the vector bundle, since we take $L(p)$ as a measure of how rapidly a vector bundle twists. For this to work well we first need to know that we can always find representing projections p such that $L(p) < \infty$. After showing this, we will establish a related fact for homotopies between projections. A well-known fact that is important for all we do here is that if two projections are homotopic then the corresponding vector bundles are isomorphic. See, for example, section 6 of chapter 1 of [53], especially corollary 1.6.12.

The fact that we can find p's with $L(p) < \infty$ is a special case of a general well-known [27] fact about any dense $*$-subalgebra closed under the holomorphic-function calculus in any unital C^*-algebra (see, e.g., section 3.8 of [18]). We sketch the proof here in several steps, both for the reader's convenience and because we will need similar arguments later. The normed $*$-algebras $M_n(C(Z))$ are examples of C^*-algebras, and whenever we write "C^*-algebra" readers can just think of $M_n(C(Z))$ if they prefer.

Proposition 3.1. *Let A be a unital C^*-algebra, and let $\mathcal{A}$ be a dense $*$-subalgebra closed under the holomorphic-function calculus in A. Let p be a projection in A. Then for any $\delta > 0$ there is a projection p_1 in $\mathcal{A}$ such that $\|p - p_1\| < \delta$. If $\delta < 1$ then p_1 is homotopic to p through projections in A. The same conclusions hold if A is a real C^*-subring of a C^*-algebra, as in the setting of Proposition 2.4.*

Proof. For the first part we can assume that $\delta < 1/2$. Since $\mathcal{A}$ is dense in A and $p = p^*$ we can find $a \in \mathcal{A}$ such that $a^* = a$ and $\|p - a\| < \delta < 1/2$. We will use the next step several times again, so we state it as:

Lemma 3.2. *If p is a projection in A and if $a \in A$ with $a = a^*$ and $\|p - a\| < \delta$, then*

$$\sigma(a) \subset [-\delta, \delta] \cup [1 - \delta, 1 + \delta].$$

Proof. (e.g., lemma 2.2.3 of [52]) Note that $\sigma(p) \subseteq \{0, 1\}$. Let $\lambda \in \mathbb{R}$ with $\lambda \neq 0, 1$ so that $\lambda - p$ is invertible. Then

$$1 \;-\; (\lambda - p)^{-1}(\lambda - a) \;=\; (\lambda - p)^{-1}(a - p).$$

Thus if $\|(\lambda - p)^{-1}(a - p)\| < 1$ then $(\lambda - p)^{-1}(\lambda - a)$ is invertible (by Neumann series, i.e., geometric series, converging in A), and so $\lambda - a$ is left invertible. But

$$\|(\lambda - p)^{-1}(a \;-\; p)\| \leq \|(\lambda - p)^{-1}\|\|a \;-\; p\| < \|(\lambda - p)^{-1}\|\delta,$$

and $\|(\lambda-p)^{-1}\| = \max\{|\lambda|^{-1}, |1-\lambda|^{-1}\}$. Thus if $\delta < |\lambda|$ and $\delta < |1-\lambda|$ then $\lambda - a$ is left invertible. A small variation of this argument shows that $\lambda - a$ is also right invertible. Thus $\lambda \notin \sigma(a)$. This yields the desired result, since $\sigma(a) \subset \mathbb{R}$. ☐

We continue sketching the proof of Proposition 3.1. Let χ be defined on $\mathbb{C}$ by $\chi(z) = 0$ if $\mathrm{Re}(z) \leq 1/2$, while $\chi(z) = 1$ if $\mathrm{Re}(z) > 1/2$. Since

$\delta < 1/2 < 1-\delta$, we see from Lemma 3.2 that χ is holomorphic in an open neighborhood of $\sigma(a)$, and thus $\chi(a)$ is defined and is in $\mathcal{A}$. Since $\varphi \mapsto \varphi(a)$ is a homomorphism, $p_1 = \chi(a)$ is a projection in $\mathcal{A}$. By considering the continuous-function calculus [26, 12, 13] it is easily seen that $\|p_1 - a\| < \delta$, and so $\|p - p_1\| < 2\delta$. Replacing δ by $\delta/2$ everywhere above, we obtain the desired result.

The proof that p and p_1 are homotopic is a simpler version of the proof of our next proposition, below.

On looking at Proposition 2.4 and its proof one can see easily how to adapt the above discussion to treat a real C^*-subring of a C^*-algebra. □

When we apply Proposition 3.1 to $M_n(C(X))$ or $M_n(C_{\mathbb{R}}(X))$ we can interpret it as saying that every vector bundle over X has a Lipschitz structure with respect to ρ.

Proposition 3.3. *Let A and $\mathcal{A}$ be as in Proposition 3.1, and let L be a lower-semicontinuous strongly-Leibniz $*$-seminorm on $\mathcal{A}$. Let p_0 and p_1 be two projections in A. Suppose that $\|p_0 - p_1\| \leq \delta < 1$, so that there is a norm-continuous path, $t \mapsto p_t$, of projections in A going from p_0 to p_1 [6, 52]. If p_0 and p_1 are in $\mathcal{A}$ then we can arrange that $p_t \in \mathcal{A}$ and that*

$$L(p_t) \leq (1-\delta)^{-1} \max\{L(p_0), L(p_1)\}$$

for every t. The same conclusions hold if A is a real C^*-subring of a C^*-algebra.

Proof. We begin the proof in a standard way (e.g. proposition 2.2.4 of[52]) by setting $a_t = (1-t)p_0 + tp_1$ for $t \in [0,1]$. If $0 \leq t \leq 1/2$ then $\|a_t - p_0\| \leq \delta/2$, while if $1/2 \leq t \leq 1$ then $\|a_t - p_1\| \leq \delta/2$. From Lemma 3.2 and the evident facts that a_t is positive and $\|a_t\| \leq 1$, it follows that $\sigma(a_t) \subseteq [0, \delta/2] \cup [1-\delta/2, 1]$. Much as in the proof of Proposition 3.1, define χ on $\mathbb{C}$ by setting $\chi(z) = 0$ if $\mathrm{Re}(z) \leq (1+\delta)/4$, and $\chi(z) = 1$ if $\mathrm{Re}(z) > (1+\delta)/4$. Since $\delta < 1$ we see that χ is holomorphic in an open neighborhood of $\sigma(a_t)$ for each t.

Instead of using the curve around $[1-\delta/2, 1]$ that we used in our earlier versions of this paper, we use the family, $\{\gamma_s\}$, of curves used by Hanfeng Li in his proof of proposition 3.1 of [35]. This gives substantially improved estimates compared to those in our earlier versions. For the reader's convenience we include the details from his proof here. For $s > 0$ we let γ_s be the oriented curve that traces counter-clockwise at unit speed the boundary of the rectangle with vertices $(1/2) - si$, $1 + s - si$, $1 + s + si$, $(1/2) + si$. Notice that because of where we have chosen the line of discontinuity for χ, the curves γ_s lie in the domain where χ is holomorphic. We would also usually choose a curve around $[0, \delta/2]$, but since $\chi = 0$ near there, this is unnecessary. For any given t we now set

$$p_t = \chi(a_t) = \frac{1}{2\pi i}\int_{\gamma_s} \chi(z)(z - a_t)^{-1} dz.$$

This integral is independent of s for the usual reasons for holomorphic functions. Then as in the proof of Proposition 3.1 we see that p_t is a projection, and that each p_t is in $\mathcal{A}$ if p_0 and p_1 are. It is not difficult to see then that $t \mapsto p_t$ is a continuous path from p_0 to p_1.

We now estimate $L(p_t)$. For the same reasons as given in the proof of Proposition 2.3 we have

$$L(p_t) \le \frac{L(a_t)}{2\pi}\left(\int_{\gamma_s} d|z| \ \|(z - a_t)^{-1}\|^2\right) .$$

Let γ_s^1 be the segment of γ_s given by $\gamma_s^1(r) = (1/2) + r - si$ for $0 \le r \le (1/2) + s$. For z on γ_s^1, because a_t is self-adjoint, we have $\|(z - a_t)^{-1}\| \le s^{-1}$. Thus

$$\int_{\gamma_s^1} d|z| \ \|(z - a_t)^{-1}\|^2 \ \le \ \int_0^{(1/2)+s} s^{-2} \ dr \ = \ s^{-2}((1/2) + s) \ .$$

This same estimate applies to the segment γ_s^3 of γ_s going from $(1/2) + s + si$ to $(1/2) + si$. Notice that these estimates show that as s goes to $+\infty$ these integrals go to 0. Now let γ_s^2 be the segment of γ_s given by $\gamma_s^2(r) = 1 + s + ri$ for $-s \le r \le s$. For z on γ_s^2 we again have the estimate $\|(z - a_t)^{-1}\| \le s^{-1}$. Thus

$$\int_{\gamma_s^2} d|z| \ \|(z - a_t)^{-1}\|^2 \ \le \ \int_{-s}^{s} s^{-2} \ dr \ = 2s^{-1} \ .$$

Notice again that this integral goes to 0 as s goes to $+\infty$. Finally, let γ_s^4 be the segment of γ_s given by $\gamma_s^4(r) = (1/2) - ri$ for $-s \le r \le s$. For z on γ_s^4 we have the estimate $\|(z - a_t)^{-1}\|^2 \le ((1/2) - \delta/2)^2 + r^2)^{-1}$. Thus

$$\begin{aligned}\int_{\gamma_s^4} d|z| \ \|(z - a_t)^{-1}\|^2 \ &\le \ \int_{-\infty}^{\infty} ((1/2) - \delta/2)^2 + r^2)^{-1} \ dr \\ &= \ \pi((1/2) - \delta/2)^{-1} = 2\pi(1 - \delta)^{-1}.\end{aligned}$$

Taking the sum over the 4 segments and letting s go to $+\infty$, we obtain

$$L(p_t) \le L(a_t)(1 - \delta)^{-1}.$$

Since clearly $L(a_t) \le \max\{L(p_0), L(p_1)\}$, we obtain the desired estimate.

It is easy to see how to adapt the above argument to the case of a real C^*-subring of a C^*-algebra. $\square$

By using the same techniques, or by applying directly proposition 3.1 of [35], we can obtain the following continuation of Proposition 2.5:

Proposition 3.4. *Let A be a unital C^*-algebra and let L be a closed strongly-Leibniz $*$-seminorm on A. Let $\mathcal{A}$ be its subalgebra of Lipschitz elements. Let α be an action of a Lie group G on A by isometries for L, and let A^∞ be the subalgebra of smooth elements for α. Then for any projection p in $\mathcal{A}$ and any $\delta > 0$ there is a projection p_1 in A^∞ such that $\|p - p_1\| < \delta$ and $L(p_1) \leq (1-2\delta)^{-1}L(p)$. In particular, if $\delta < 1/2$ then p and p_1 are homotopic through projections in $\mathcal{A}$. The same conclusion holds if A is a real C^*-subring of a C^*-algebra.*

Proof. According to Proposition 2.5, given a projection $p \in \mathcal{A}$ and a $\delta > 0$ we can find a $b \in A^\infty$ such that $\|b\| \leq \|p\| = 1$, $L(b) \leq L(p)$, and $\|p - b\| < \delta$. In fact, examination of the proof of Proposition 2.5 shows that we can also assume that $b^* = b$, and even more that $b \geq 0$, since $p \geq 0$. Thus $\sigma(b) \subseteq [0, 1]$. Then from Lemma 3.2 we conclude that $\sigma(b) \subseteq [0, \delta] \cup [1-\delta, 1]$. We can assume that $\delta < 1/2$, so that these intervals are disjoint. Then from proposition 3.1 of [35] we obtain $L(p_1) \leq (1-2\delta)^{-1}L(p)$, as desired. $\square$

This proposition is related to the main theorem of [35], which answers a question that I asked in an earlier version of this paper. We state Li's theorem here, since we will use it later.

Theorem 3.5. *Let M be a closed connected compact Riemannian manifold, equipped with its usual metric from its Riemannian metric, and let A be a real C^*-subring of a C^*-algebra. For any projection $p \in C(M, A)$ and any $\epsilon > 0$ there exists a projection $q \in C^\infty(M, A)$ such that $\|p - q\|_\infty < \epsilon$ and $L(q) \leq L(p) + \epsilon$*

4. The uniqueness of extended vector bundles

Let (Z, ρ) be a compact metric space, and let X be a closed non-empty subset of Z. As before, we equip X with the restriction to it of ρ, and we let $A = C(X)$ and $B = C(Z)$. Let $L = L^\rho$ be defined as earlier on A and B, and also on $M_n(A)$ and $M_n(B)$, and let $\mathcal{A}$ and $\mathcal{B}$ denote the dense subalgebras of Lipschitz functions. We let $\pi : M_n(B) \to M_n(A)$ denote the surjective $*$-homomorphism consisting of restricting functions from Z to X. It is easily seen that for $b \in M_n(B)$ we have $L(\pi(b)) \leq L(b)$, so that $\pi(M_n(\mathcal{B})) \subseteq M_n(\mathcal{A})$. We say that a projection $q \in M_n(B)$ extends a projection $p \in M_n(A)$ if $\pi(q) = p$. This corresponds exactly to extension for the corresponding vector bundles.

As mentioned earlier, if two projections, q_1 and q_2, in $M_n(B)$ are homotopic through projections in $M_n(B)$, then the corresponding vector bundles are isomorphic. Thus we seek conditions such that if p is a projection in $M_n(A)$ and if q_0 and q_1 are projections in $M_n(B)$ such that $\pi(q_0) = p = \pi(q_1)$, then q_0 and q_1 are homotopic. Simple examples show that this need not hold without further conditions. As indicated in Section 1, our results will depend on the Hausdorff distance between X and Z, and we will express this by supposing that X is ε-dense in Z.

Key Lemma 4.1. *Suppose that X is ε-dense in Z. Then for any $b \in M_n(B)$ we have*

$$\|b\| \leq \|\pi(b)\| + \varepsilon L(b).$$

Proof. Let $z \in Z$ be given. Then there is an $x \in X$ such that $\rho(z, x) \leq \varepsilon$. Thus

$$\|b(z)\| \leq \|b(x)\| + \|b(z) - b(x)\| \leq \|\pi(b)\| + \varepsilon L(b).$$

□

When we use the word "path" in discussing homotopies we will almost always mean a continuous function whose domain is the interval $[0, 1]$.

Theorem 4.2. *Suppose that X is ε-dense in Z. Let p_0 and p_1 be projections in $M_n(\mathcal{A})$, and let q_0 and q_1 be projections in $M_n(\mathcal{B})$ such that $\pi(q_0) = p_0$ and $\pi(q_1) = p_1$. Set*

$$\delta = \|p_0 - p_1\| + \varepsilon(L(q_0) + L(q_1)).$$

If $\delta < 1$, then there is a path, $t \mapsto q_t$, through projections in $M_n(\mathcal{B})$, from q_0 to q_1, such that

$$L(q_t) \leq (1 - \delta)^{-1} \max\{L(q_0), L(q_1)\}$$

for all $t \in [0, 1]$. The same conclusion holds if $A = C_{\mathbb{R}}(X)$, etc.

Proof. From Key Lemma 4.1 we see that

$$\begin{aligned} \|q_0 - q_1\| &\leq \|\pi(q_0 - q_1)\| + \varepsilon L(q_0 - q_1) \\ &\leq \|p_0 - p_1\| + \varepsilon(L(q_0) + L(q_1)) = \delta. \end{aligned}$$

Assume now that $\delta < 1$. Then according to Proposition 3.3 applied to q_0 and q_1 for $\mathcal{A} = \mathcal{B}$, there is a path $t \to q_t$ from q_0 to q_1 with the stated properties. □

Under the conditions of the above theorem, $t \mapsto \pi(q_t)$ will be a continuous path from p_0 to p_1 through projections in $M_n(\mathcal{A})$. Thus the vector bundles corresponding to p_0 and p_1 will be isomorphic, as will their lifts corresponding to q_0 and q_1.

Notice that the bound on $L(q_t)$ stated in the above theorem is independent of n. This is in contrast to the existence results that we will obtain in Section 6. If $p_0 = p_1$ above then we can obtain some additional information:

Proposition 4.3. *Let $p \in M_n(\mathcal{A})$, and let q_0 and q_1 be projections in $M_n(\mathcal{B})$ such that $\pi(q_0) = p = \pi(q_1)$. If $\varepsilon L(q_0) < 1/2$ and $\varepsilon L(q_1) < 1/2$, then there is a path, $t \to q_t$, through projections in $M_n(\mathcal{B})$, from q_0 to q_1, such that $\pi(q_t) = p$ and*

$$L(q_t) \leq (1-\delta)^{-1} \max\{L(q_0), L(q_1)\}$$

for all t, where $\delta = \varepsilon(L(q_0) + L(q_1))$. The same conclusion holds when $A = C_{\mathbb{R}}(X)$, etc.

Proof. The proof follows the same lines as the proof of Theorem 4.2, except that now we must show that $\pi(q_t) = p$ for all t. For this, much as in the proof of Proposition 3.3 we set $a_t = (1-t)q_0 + tq_1$. It is clear that $\pi(a_t) = p$ for all t. From the proof of Proposition 3.3 we have $q_t = \frac{1}{2\pi i}\int_\gamma (z-a_t)^{-1}dz$, and it follows easily that $\pi(q_t) = p$ for all t. $\square$

We can combine the above results to obtain some information that does not depend on p_0 and p_1 being close together:

Corollary 4.4. *Let p_0 and p_1 be projections in $M_n(\mathcal{A})$, and let q_0 and q_1 be projections in $M_n(\mathcal{B})$ such that $\pi(q_0) = p_0$ and $\pi(q_1) = p_1$. Let N be a constant such that $L(q_j) \leq N$ for $j = 0, 1$. Assume further that there is a path p from p_0 to p_1 such that for each t there is a projection $\tilde{q}_t$ in $M_n(\mathcal{B})$ such that $\pi(\tilde{q}_t) = p_t$ and $L(\tilde{q}_t) \leq N$. Assume that $2\varepsilon N < 1$, and pick δ such that $2\varepsilon N < \delta < 1$. Then there is a continuous path $t \mapsto q_t$ of projections in $M_n(\mathcal{B})$ going from q_0 to q_1 such that*

$$L(q_t) \leq (1-\delta)^{-1}N$$

for each t. (But we may not have $\pi(q_t) = p_t$ for all t.) The same conclusion holds when $A = C_{\mathbb{R}}(X)$, etc.

Proof. Pick a finite increasing sequence $\{t_i\}_{i=0}^k$, of points in $[0,1]$ such that $t_0 = 0$, $t_k = 1$, and $\|p_{t_{i+1}} - p_{t_i}\| \leq \delta - 2\varepsilon N$ for each i for $0 \leq i \leq k-1$. For each such i pick a projection q'_i in $M_n(\mathcal{B})$ such that $\pi(q'_i) = p_{t_i}$ and $L(q'_i) \leq N$, with $q'_0 = q_0$ and $q'_k = q_1$. Then, according to Key Lemma 4.1, for each such i we have

$$\|q'_{i+1} - q'_i\| \leq \|p_{t_{i+1}} - p_{t_i}\| + \varepsilon(L(q'_{i+1}) + L(q'_i)) \leq \delta - 2\varepsilon N + 2\varepsilon N = \delta.$$

According to Theorem 4.2, for each i there is a continuous path of projections in $M_n(\mathcal{B})$, going from q'_i to q'_{i+1}, all of whose Lipschitz norms are no greater than $(1-\delta)^{-1}N$. We can then concatenate these paths in the usual way to obtain the desired path $t \to q_t$. (Notice that the function $t \mapsto L(q_t)$ need not be continuous, and we need not have $\pi(q_t) = p_t$.) □

Let us now see what consequences the above uniqueness results have for metric spaces that are close together. In doing this it seems simplest to notice that in defining Gromov–Hausdorff distance between two compact metric spaces it is sufficient to consider their *disjoint* isometric embeddings into other metric spaces. Since we can always then cut down to their union, it suffices to take $Z = X\dot{\cup}Y$, where this denotes the disjoint union. Thus it suffices to consider metrics ρ on $X\dot{\cup}Y$ whose restrictions to X and Y are their given metrics ρ_X and ρ_Y. We will write $\mathrm{dist}_H^\rho(X,Y) < \varepsilon$ to signify that the Hausdorff distance between X and Y in $Z = X\dot{\cup}Y$ for ρ is less than ε.

From $Z = X\dot{\cup}Y$ we have $C(Z) = C(X) \oplus C(Y)$ as $*$-Banach algebras. As before let $A = C(X)$, $B = C(Z)$, etc., and now also let $D = C(Y)$, with subalgebra of Lipschitz elements $\mathcal{D}$. A projection in $M_n(B)$ will now be of the form $p \oplus q$ where p and q are projections in $M_n(A)$ and $M_n(D)$ respectively. Roughly speaking, our idea is that p and q will correspond if $L(p \oplus q)$ is relatively small. Notice that which projections then correspond to each other will strongly depend on ρ. We will only consider that projections correspond (for a given ρ) if there is some uniqueness to the correspondence. The following immediate consequences of Proposition 4.3 and Theorem 4.2 give appropriate expression for this uniqueness. These consequences also hold when working over $\mathbb{R}$.

Theorem 4.5. *Let A, D, etc., be as just above, with* $\mathrm{dist}_H^\rho(X,Y) < \varepsilon$.

a) *Let $p \in M_n(\mathcal{A})$ and $q \in M_n(\mathcal{D})$ be projections, and suppose that $\varepsilon L(p \oplus q) < 1/2$. If q_1 is any other projection in $M_n(\mathcal{D})$ such that $\varepsilon L(p \oplus q_1) < 1/2$ then there is a path $t \mapsto q_t$ through projections in $M_n(\mathcal{D})$, going from q to q_1, such that*

$$L(p \oplus q_t) \leq (1-\delta)^{-1} \max\{L(p \oplus q), L(p \oplus q_1)\}$$

for all t, where $\delta = \varepsilon(L(p \oplus q) + L(p \oplus q_1))$. If instead there is a $p_1 \in M_n(\mathcal{A})$ such that $\varepsilon L(p_1 \oplus q) < 1/2$ then there is a corresponding path from p to p_1 with corresponding bound for $L(p_t \oplus q)$.

b) *Let p_0 and p_1 be projections in $M_n(\mathcal{A})$ and let q_0 and q_1 be projections in $M_n(\mathcal{D})$. Set*

$$\delta = \|p_0 - p_1\| + \varepsilon(L(p_0 \oplus q_0) + L(p_1 \oplus q_1)).$$

If $\delta < 1$ then there are continuous paths $t \mapsto p_t$ and $t \mapsto q_t$ from p_0 to p_1 and q_0 to q_1, respectively, such that

$$L(p_t \oplus q_t) \leq (1-\delta)^{-1} \max\{L(p_0 \oplus q_0), L(p_1 \oplus q_1)\}$$

for all t.

We remark that a more symmetric way of stating part b) above is to define δ by

$$\delta = \max\{\|p_0 - p_1\|, \|q_0 - q_1\|\} + \varepsilon(L(p_0 \oplus q_0), L(p_1 \oplus q_1)).$$

Let us now examine the consequences of Corollary 4.4. This is best phrased in terms of:

Notation 4.6. For any n let $\mathcal{P}_n(X)$ denote the set of projections in $M_n(\mathcal{A})$. For any $r \in \mathbb{R}^+$ let

$$\mathcal{P}_n^r(X) = \{p \in \mathcal{P}_n(X) : L_A(p) < r\},$$

and similarly for Y and Z.

Now $\mathcal{P}_n^r(X)$ may have many path components. We will see examples shortly. As suggested in the introduction, it may be appropriate to view these different path components as representing inequivalent vector bundles, notably if X is a finite set. Let Π be one of these path components. Let Φ_X denote the evident restriction map from $\mathcal{P}_n(Z)$ to $\mathcal{P}_n(X)$ (for $Z = X \dot{\cup} Y$). For a given $s \in \mathbb{R}^+$ with $s \geq r$ it may be that $\Phi_X(\mathcal{P}_n^s(Z)) \cap \Pi$ is non-empty. This is an existence question, which we deal with in the next sections. But at this point, from Corollary 4.4 we can conclude that:

Theorem 4.7. *Let notation be as above, with $\mathrm{dist}_H^\rho(X,Y) < \varepsilon$, and let $r \in \mathbb{R}^+$ with $\varepsilon r < 1/2$. Let Π be a path component of $\mathcal{P}_n^r(X)$. Let $s \in \mathbb{R}^+$ with $s \geq r$ and $\varepsilon s < 1/2$. Let p_0, $p_1 \in \Pi$ and suppose that there are q_0 and q_1 in $\mathcal{P}_n^s(Y)$ with $L(p_j \oplus q_j) \leq s$ for $j = 0, 1$. Assume, even more, that there is a path $\tilde{p}$ in Π connecting p_0 and p_1 that lies in $\Phi_X(\mathcal{P}_n^s(Z))$. Then for any δ with $2\varepsilon s < \delta < 1$ there exist a path p in $\mathcal{P}_n(X)$ going from p_0 to p_1 and a path q in $\mathcal{P}_n(Y)$ going from q_0 to q_1 such that $L(p_t \oplus q_t) < (1-\delta)^{-1}s$ for each t. The situation is symmetric between X and Y, so the roles of X and Y can be interchanged in the above statement.*

Thus, in the situation described in the theorem, if Π represents some particular class of bundles on X, such as "monopole" bundles on a sphere, then the projections $q \in \mathcal{P}_n^s(Y)$ paired with ones in Π by the requirement that $L(p \oplus q) < s$ will be homotopic, and in particular will determine isomorphic bundles on Y. We emphasize that the above pairing of projections depends strongly on ρ, and not just on the Gromov-Hausdorff distance between X and Y. This reflects the fact that Gromov–Hausdorff distance is only a metric on *isometry classes* of compact metric spaces.

Notice that the homotopies obtained above between q_0 and q_1 need not lie in $\mathcal{P}_n^s(Y)$. We can only conclude that they lie in $\mathcal{P}_n^{s'}$ where $s' = (1-\delta)^{-1}s$. But at least we can say that s' approaches s as ε, and so δ, goes to 0.

In Theorem 6.4 and Corollary 6.7 we will deal with the existence of actual lifts of homotopics between p_0 and p_1.

5. Extending Lipschitz functions

To obtain the *existence* of extensions to Z of vector bundles on X in a manner controlled by the metric, we need to extend projection-valued functions on X to projection-valued functions on Z with control of the Lipschitz norm. We approach this by first extending projection-valued functions on X just to general functions on Z with values in $M_n^s(\mathbb{C})$, the space of self-adjoint matrices. We treat this problem in this section, and then in the next section we see how to modify the extended functions so as to be projection-valued.

We must also treat here homotopy versions of this extension problem. For this purpose we let T denote a compact space which will be the parameter space for the homotopics, so that eventually T will be an interval in $\mathbb{R}$. We will consider functions F on $X \times T$, and for any $t \in T$ we let F_t denote the function $x \mapsto F(x,t)$. When F has values in a Banach space we can then consider $L(F_t)$ for each t, as defined earlier. For a discrete set Γ we let $\ell^\infty_{\mathbb{R}}(\Gamma)$ denote the Banach space of all bounded real-valued functions on Γ with the supremum norm. The following proposition is a homotopy version of a well-known fact which appears as proposition 2.2 of [49].

Proposition 5.1. *Let (Z,ρ) be a compact metric space, and let X be a closed subset of Z. Let Γ be a set (discrete, and possibly uncountable). Let F be a continuous function from $X \times T$ to $\ell^\infty_{\mathbb{R}}(\Gamma)$. Suppose that there is a constant, N, such that $L(F_t) \le N$ for all $t \in T$. Then there is a continuous extension, G, of F to $Z \times T$ such that $L(G_t) \le N$ for all t, and $\|G\|_\infty = \|F\|_\infty$.*

Proof. For each $\gamma \in \Gamma$ and $x \in X$ define H^x_γ on $Z \times T$ by

$$H^x_\gamma(z,t) = F(x,t)(\gamma) - N\rho(z,x).$$

Note that $H^x_\gamma(z,t) \le \|F\|_\infty$. Much as in the standard proof of the non-homotopy version (as in theorem 2.1 of [49]), define $H : Z \times T \to \ell^\infty_{\mathbb{R}}(\Gamma)$ by

$$H(z,t)(\gamma) = \sup\{H^x_\gamma(z,t) : x \in X\}.$$

Clearly H is well-defined and $H \le \|F\|_\infty$ as functions. It is easily seen that for each γ and t we have $L(z \mapsto H(z,t)(\gamma)) \le N$, and that H is an extension of F. It follows easily that $L(H_t) \le N$ for all $t \in T$.

Next, we must show that H is continuous on $Z \times T$. Let $(z_0, t_0) \in Z \times T$, and let $\varepsilon > 0$ be given. Because F is continuous and $Z \times T$ is compact, a little compactness argument shows that there is a neighborhood, $\mathcal{N}$, of t_0 such that for every $x \in X$ and $t \in \mathcal{N}$ we have $\|F(x,t) - F(x,t_0)\|_\infty < \varepsilon/2$, and so $|F(x,t)(\gamma) - F(x,t_0)(\gamma)| < \varepsilon/2$ for each $\gamma \in \Gamma$. Let $\mathcal{B}$ be the ball about z_0 of radius $\varepsilon/(2N)$ in Z. Then for any $(z,t) \in \mathcal{B} \times \mathcal{N}$, any $x \in X$, and any $\gamma \in \Gamma$ we have

$$\begin{aligned} |H^x_\gamma(z,t) \quad &- \quad H^x_\gamma(z_0,t_0)| \\ &\le \quad |F(x,t)(\gamma) - F(x,t_0)(\gamma)| + N|\rho(z,x) - \rho(z_0,x)| < \varepsilon. \end{aligned}$$

A simple argument very similar to the proof of proposition 1.5.5 of [65] then shows that for any $(z,t) \in \mathcal{B} \times \mathcal{N}$ and $\gamma \in \Gamma$ we have

$$|H(z,t)(\gamma) - H(z_0,t_0)(\gamma)| < \varepsilon.$$

It follows that $\|H(z,t) - H(z_0,t_0)\| \le \varepsilon$. Thus H is continuous.

Finally, view $-\|F\|_\infty$ as a constant function on $Z \times T$, and set $G = H \vee (-\|F\|_\infty)$, where $\vee$ means "maximum". Then G has the desired properties. $\square$

Now let V be a finite-dimensional real Banach space (such as $M^s_n(\mathbb{C})$). By definition, the projection constant, $\mathcal{PC}(V)$, of V is the smallest constant c such that whenever V is isometrically embedded into a Banach space W there is a projection P from W onto V such that $\|P\| \le c$. (Such a smallest constant exists — see, e.g., proposition 1.4 of [49].)

Proposition 5.2. *Let (Z,ρ), X and T be as in Proposition* 5.1*, and let V be a finite-dimensional real Banach space. Let F be a continuous function from $X \times T$ into V for which there is a constant, N, such that $L(F_t) \le N$ for all $t \in T$. Then there is a continuous extension, G, of F to $Z \times T$ such that $L(G_t) \le N(\mathcal{PC}(V))$ for all $t \in T$ and $\|G\|_\infty \le \|F\|_\infty(\mathcal{PC}(V))$.*

Proof. We can isometrically embed V into $\ell^\infty_{\mathbb{R}}(\Gamma)$ for some discrete set Γ. (For example, take Γ to be the unit ball of the dual space V' with the discrete topology.) We can then view F as a function from $X \times T$ into $\ell^\infty_{\mathbb{R}}(\Gamma)$, and apply Proposition 5.1 to find a continuous extension $\tilde{G}$ of $Z \times T$ into $\ell^\infty_{\mathbb{R}}(\Gamma)$ such that $L(\tilde{G}_t) \leq N$ for all t and $\|\tilde{G}\|_\infty = \|F\|_\infty$. Let P be a projection from $\ell^\infty_{\mathbb{R}}(\Gamma)$ onto V such that $\|P\| \leq \mathcal{PC}(V)$. Then the function $G = P \circ \tilde{G}$ has the desired properties. □

For a given Banach space V it is usually not easy to determine the precise value of $\mathcal{PC}(V)$. However, our projection-valued functions can be viewed to have values in the Banach space $M^s_n(\mathbb{C})$ of self-adjoint matrices, and in theorem 7.2 of [49] we find that

$$\mathcal{PC}(M^s_n(\mathbb{C})) = 2n\left(\frac{n}{n+1}\right)^{n-1} - 1. \tag{5.3}$$

It is also noticed there that when the right-hand side is written as $n\omega(n)$, then $\omega(n)$ converges to $2e^{-1}$ as $n \to \infty$. In theorem 1.5 of [49] we also find that $\mathcal{PC}(V) = \mathcal{LE}(V)$ where $\mathcal{LE}(V)$ is the Lipschitz extension constant of V, that is, the smallest constant c such that every V-valued function f on a subset X of a metric space Z can be extended to a function g on Z such that $L(g) \leq cL(f)$. Thus in Proposition 5.2 above the constant $\mathcal{PC}(V)$ is the smallest constant c for which we can always find a G for which $L(G) \leq cL(F)$. (After [49] was published I learned that the statement of theorem 1.5 of [49] essentially already appears as proposition 5.1 of [39].)

However, one can ask whether the constant in the inequality $\|G\|_\infty \leq \|F\|_\infty(\mathcal{PC}(V))$ of Proposition 5.2 can be improved. In some cases, such as when V is a Hilbert space, $\mathcal{PC}(V)$ can be replaced by 1 in this inequality. But I have not been able to determine whether this constant can be improved for the case of $V = M^s_n(\mathbb{C})$. However, in proposition 8.3 of [49] it is shown by means of radial retractions that for any V this constant can be replaced by 1 if the Lipschitz inequality is weakened to $L(g) \leq L(f)(2\mathcal{PC}(V))$. This applies equally well to the homotopy situation we consider here. We will use this fact in the rest of our paper, since to have $\|\cdot\|_\infty$ preserved under extensions will simplify our bookkeeping. If eventually situations are found where this has undesirable effects, the extra bookkeeping can be done easily. For the rest of this paper we will use:

Notation 5.4. For each positive integer n we let λ_n denote the smallest constant c such that, with notation as above, for any Z, X, T, and for any continuous $F : X \times T \to M^s_n(\mathbb{C})$ for which there is a constant N

such that $L(F_t) \leq N$ for all t, there is a continuous extension, G, of F to $Z \times T$ such that

$$L(G_t) \leq \lambda_n N \text{ for all } t \quad \text{ and } \quad \|G\|_\infty = \|F\|_\infty.$$

Thus if we let λ_n^* be given by the formula in (5.3), then we have

$$\lambda_n^* \leq \lambda_n \leq 2\lambda_n^*.$$

Suppose we are instead working over $\mathbb{R}$. Given a function $F : X \times T \to M_n^s(\mathbb{R})$, we can view it as taking values in $M_n^s(\mathbb{C})$, and then find an extension, $\tilde{G}$, of it to $Z \times T$ which satisfies the estimates of Notation 5.4. But the map which for each matrix in $M_n^s(\mathbb{C})$ replaces all of its entries by their real part is norm-decreasing and $\mathbb{R}$-linear. Let G be the composition of this map with $\tilde{G}$. We thus obtain:

Proposition 5.5. *Let F be a continuous function from $X \times T$ to $M_n^s(\mathbb{R})$ for which there is a constant N such that $L(F_t) \leq N$ for each t. Then there is a continuous extension, $G : Z \times T \to M_n^s(\mathbb{R})$, of F such that $L(G_t) \leq \lambda_n N$ for each t, and $\|G\|_\infty = \|F\|_\infty$.*

One can probably replace λ_n by a smaller constant by using the techniques of section 7 of [49] to compute $\mathcal{PC}(M_n^s(\mathbb{R}))$.

There is another aspect of extensions which might seem relevant, namely that one can define certain classes of compact metric spaces for which the constant for extending Lipschitz functions into any Banach space is smaller than for arbitrary compact metric spaces. For example, in [25] it is shown that there is a universal constant C such that if Z is a compact subset of some d-dimensional Banach space, with metric from the Banach space, then for every closed subset X of Z and for every Banach space V, every function f from X to V can be extended to a function g from Z to V such that $L(q) \leq CdL(f)$. Notice that this inequality is independent of the dimension of V, unlike our results above. Even more, from part 5 of theorem 5.1 of [34] we see that if Z is in fact a compact subset of a d-dimensional Hilbert space, then the above inequality can be improved to $L(q) \leq C(d^{1/2})L(f)$. See also [7].

But these results do not seem to be useful to us here for the following reason. Ultimately we want to consider two compact metric spaces (X, ρ_X) and (Y, ρ_Y), and then, in order to use their Gromov–Hausdorff distance we must consider all isometric embeddings of them into compact metric spaces (Z, ρ). If we were to restrict X and Y to be subsets of a d-dimensional Banach space, I see no reason why we could, for example, restrict (Z, ρ_Z) to also be (isometrically) a subset of a d-dimensional Banach space. But this type of question would be interesting to explore.

6. Extending vector bundles

We extend vector bundles in a controlled way by extending their projections. As before, let π denote the restriction map from $B = C(Z)$ onto $A = C(X)$, and let p be a projection in $M_n(\mathcal{A})$. In terms of Notation 5.4 we can find $b \in M_n^s(B)$ such that $\pi(b) = p$, $L(b) \leq \lambda_n L(p)$ and $\|b\| = \|p\| = 1$. (The fact that $\|b\| = 1$ is one of the places where our definition of λ_n simplifies the bookkeeping.) Suppose now that X is ε-dense in Z. Then from Key Lemma 4.1 we see that

$$\begin{aligned} \|b^2 - b\| &\leq \|\pi(b^2 - b)\| + \varepsilon L(b^2 - b) \\ &\leq \|p^2 - p\| + \varepsilon(2\|b\|L(b) + L(b)) = \varepsilon 3L(b). \end{aligned}$$

Note the crucial use made here of the Leibniz property of L. Set $\delta = \varepsilon 3L(b)$, so that $\|b^2 - b\| \leq \delta$. Much as in Lemma 3.2 we have:

Lemma 6.1. *Let b be an element in a unital C^*-algebra such that $b^* = b$ and $\|b^2 - b\| \leq \delta$. Then*

$$\sigma(b) \subseteq [-2\delta, 2\delta] \cup [1 - 2\delta, 1 + 2\delta].$$

Proof. Let $\lambda \in \sigma(b)$. The polynomial $(x^2 - x) - (\lambda^2 - \lambda)$ has value 0 at $x = \lambda$, and so factor as $(x - \lambda)p(x)$ for some polynomial p. On substituting b for x in this factorization, one sees easily that $\lambda^2 - \lambda \in \sigma(b^2 - b)$. Thus $|\lambda^2 - \lambda| \leq \delta$, that is, $|\lambda||\lambda - 1| \leq \delta$. If $|\lambda| \geq 1/2$, then $(1/2)|\lambda - 1| \leq |\lambda||\lambda - 1| \leq \delta$, so that $|\lambda - 1| \leq 2\delta$ and $\lambda \in [1-2\delta, 1+2\delta]$. If $|\lambda| \leq 1/2$ then $|\lambda - 1| \geq 1/2$, so that $|\lambda|(1/2) \leq |\lambda||\lambda - 1| \leq \delta$. Thus $|\lambda| \leq 2\delta$ so that $\lambda \in [-2\delta, 2\delta]$. □

Suppose now that $\delta < 1/4$, so that the two intervals $[-2\delta, 2\delta]$ and $[1 - 2\delta, 1 + 2\delta]$ are disjoint. Then we are in almost the same situation as in the proof of Proposition 3.3, but with slightly different conditions on the spectrum. We now present the result that we seek, but in the greater generality involving homotopies, much as in Notation 5.4. In particular, let λ_n be as in Notation 5.4, let $A = C(X)$, etc., and let T be a compact space which serves as a parameter space.

Theorem 6.2. *Let $p : T \to M_n(A)$ be a continuous function such that p_t is a projection for each t. Assume that there is a constant, N, such that $L(p_t) \leq N$ for all t. Suppose that X is ε-dense in Z. If $\varepsilon\lambda_n N < 1/12$, then there exists a continuous function $q : T \to M_n(B)$ such that q_t is a projection, $\pi(q_t) = p_t$, and*

$$L(q_t) < \lambda_n N(1 - 12\varepsilon\lambda_n N)^{-1},$$

for each $t \in T$. If, instead, $A = C_{\mathbb{R}}(X)$, etc., one has the same conclusions.

Proof. We can view p as a continuous function on $X \times T$. Then according to the definition of λ_n in Notation 5.4 we can find a continuous function $G : T \to M_n(B)$ such that for each $t \in T$ we have $G_t^* = G_t$, $\pi(G_t) = p_t$, $\|G_t\| = \|p_t\| = 1$ and $L(G_t) \leq \lambda_n L(p_t) \leq \lambda_n N$. From the discussion given before Lemma 6.1 we see that for any $t \in T$ we have $\|G_t^2 - G_t\| \leq \varepsilon 3 L(G_t) \leq \varepsilon 3 \lambda_n N$. Let $\delta = \varepsilon 3 \lambda_n N$. Then according to Lemma 6.1 and the comment immediately after it, if $\delta < 1/4$ then $\sigma(G_t)$ is contained in the union of the disjoint intervals $[-2\delta, 2\delta]$ and $[1-2\delta, 1+2\delta]$. So we now assume that $\delta < 1/4$. Much as in the proof of Proposition 3.3, let χ be defined on $\mathbb{C}$ by setting $\chi(z) = 0$ if $\mathrm{Re}(z) \leq \delta + \frac{1}{4}$, and $\chi(z) = 1$ otherwise. For any nice curve γ around $[1-2\delta, 1+2\delta]$ that lies in the domain where χ is holomorphic we now set, for each $t \in T$,

$$q_t = \frac{1}{2\pi i} \int_\gamma \chi(z)(z - G_t)^{-1} dz.$$

Then as in the proof of Proposition 3.1 we see that q_t is a projection in $M_n(\mathcal{B})$. From the facts that $\pi(G_t) = p_t$ and that p_t is a projection it is easily verified that $\pi(q_t) = p_t$.

We need to estimate $L(q_t)$. Instead of using the curve that we used in earlier versions of this paper we can now directly apply proposition 3.1 of [35] much as we did in the proof of Proposition 3.3. This gives

$$L(q_t) \leq L(G_t)(1 - 4\delta)^{-1} \leq \lambda_n N (1 - 12\varepsilon\lambda_n N)^{-1}.$$

Finally, we must show that q is continuous in t. Given $s, t \in T$ we have, by a familiar maneuver (basically the "resolvant equation" [28]),

$$\begin{aligned} q_t - q_s &= \frac{1}{2\pi i} \int_\gamma \chi(z)((z - G_t)^{-1} - (z - G_s)^{-1}) dz \\ &= \frac{1}{2\pi i} \int_\gamma (z - G_t)^{-1}((z - G_s) - (z - G_t))(z - G_s)^{-1} dz \\ &= \frac{1}{2\pi i} \int_\gamma (z - G_t)^{-1}(G_t - G_s)(z - G_s)^{-1} dz. \end{aligned}$$

Thus

$$\|q_t - q_s\| \leq \|G_t - G_s\| \frac{1}{2\pi} \int_\gamma \|(z - G_t)^{-1}\| \|(z - G_s)^{-1}\| d|z| \leq K \|G_t - G_s\|$$

for a suitable constant K obtained by the kind of estimates used above to bound $L(q_t)$. Since G is continuous, it follows that q is also.

On looking at Proposition 2.4 and its proof it is easy to see how to adapt the above proof to the case in which $A = C_{\mathbb{R}}(X)$, etc. □

Corollary 6.3. *Let p be a projection in $M_n(A)$. Assume that X is ε-dense in Z. If $\varepsilon\lambda_n L(p) < 1/12$, then there exists a projection $q \in M_n(B)$ such that $\pi(q) = p$ and*

$$L(q) < \lambda_n L(p)(1 - 12\varepsilon\lambda_n L(p))^{-1}.$$

We remark that the role of the Cauchy integrals used in this section can be viewed as follows: The set of elements $b \in M_n^s(A)$ such that $\|b^2 - b\| < 1/4$ is a neighborhood of the set of projections in $M_n^s(A)$, and the Cauchy integrals give a retraction from this neighborhood onto the set of projections in such a way that one keeps control of the Lipschitz constants.

It would be interesting to know whether different techniques, perhaps not involving at all the Lipschitz extension properties of functions into $M_n^s(\mathbb{C})$ as used in Section 5, but rather maybe working directly just with functions into the space of projections (or into a Grassman manifold if X is connected), for example by using in part the methods of [38], could yield extensions with smaller increase in the Lipschitz constants than is obtained in the above theorem and corollary.

Notice that Corollary 6.3 gives a criterion for extending a vector bundle from X to Z which is quite independent of how complicated the topologies of X and Z are. All that is required is that there be a metric on Z and a projection $p \in M_n(C(X))$ representing the bundle such that X is ε-dense in Z and $\varepsilon\lambda_n L(p) < 1/12$. There is no requirement that any obstructions from algebraic topology vanish, or that spaces have finite dimension or be locally geometrically n-connected as seems to be needed in [38].

We now combine Theorem 6.2 with the uniqueness given by Proposition 4.3 to treat the case of a path from p_0 to p_1 for which we already have lifts of p_0 and p_1.

Theorem 6.4. *Let $p : [0, 1] \to M_n(A)$ be a path of projections, and let q_0 and q_1 be projections in $M_n(B)$ such that $\pi(q_0) = p_0$ and $\pi(q_1) = p_1$. Suppose that there is a constant, N, such that $L(p_t) \le N$ for all t. Set $N' = \max\{L(q_0), L(q_1)\}$. Assume that X is ε-dense in Z. If $\varepsilon\lambda_n N < 1/14$ and $\varepsilon N' < 1/2$ then there exists a path, q, of projections from q_0 to q_1 such that $\pi(q_t)$ is in the range of the path p for each t (though $\pi(q)$ may have a different parametrization) and*

$$L(q_t) \le ((1/2) - \varepsilon N')^{-1} \max\{7\lambda_n N, N'\}.$$

The same conclusion holds if $A = C_{\mathbb{R}}(X)$, etc.

Proof. From Theorem 6.2 we see that there is a path $\tilde{q}$ of projections in $M_n(B)$ such that $\pi(\tilde{q}_t) = p_t$ and

$$L(\tilde{q}_t) \le \lambda_n N(1 - 12\varepsilon\lambda_n N)^{-1} \le \lambda_n N(1 - (6/7))^{-1} = 7\lambda_n N$$

for all $t \in [0, 1]$. In particular, for all t

$$\varepsilon L(\tilde{q}_t) \le 7\varepsilon\lambda_n N < 7/14 = 1/2.$$

Since $\pi(\tilde{q}_0) = p_0 = \pi(q_0)$ and $\varepsilon L(q_0) \le \varepsilon N' < 1/2$, we can apply Proposition 4.3 to obtain a path of projections, $t \mapsto q_t^0$, joining q_0 to $\tilde{q}_0$ and such that for each t we have $\pi(q_t^0) = p_0$ and

$$\begin{aligned} L(q_t^0) &\le (1 - \varepsilon(L(q_0) + L(\tilde{q}_0)))^{-1} \max\{L(q_0), L(\tilde{q}_0)\} \\ &\le (1 - \varepsilon L(q_0) - 1/2)^{-1} \max\{L(q_0), 7\lambda_n N\} \\ &\le ((1/2) - \varepsilon N')^{-1} \max\{N', 7\lambda_n N\}. \end{aligned}$$

In the same way there is a path of projections, $t \mapsto q_t^1$, connecting $\tilde{q}_1$ to q_1 with corresponding bound on $L(q_t^1)$. We concatenate the three paths q^0, $\tilde{q}$ and q^1 to obtain a path, q, of projections connecting q_0 to q_1 such that each $\pi(q_t)$ is in the range of the path p.

Since $((1/2) - \varepsilon N')^{-1} > 1$ and $L(\tilde{q}_t) < 7\lambda_n N$, we see that the bound given above for $L(q_t^0)$ and $L(q_t^1)$ is also a bound for $L(\tilde{q}_t)$, and thus for $L(q_t)$ for all t. □

Let us now see what consequences the above existence results have for metric spaces that are close together. As we did near the end of Section 4, we let $Z = X \dot{\cup} Y$, with ρ on Z restricting to the given metrics on X and Y. We also let $D = C(Y)$ as before, so that $B = A \oplus D$. We use the notation $\mathcal{P}^r(X)$, etc., introduced in Notation 4.6. By using Corollary 6.3 to satisfy the hypothesis concerning $\Phi_X(\mathcal{P}_n^s(Z))$ in Theorem 4.7 we obtain:

Theorem 6.5. *Let $r \in \mathbb{R}^+$ be given. Let ε be small enough that $\varepsilon\lambda_n r < 1/14$. Set $s = \lambda_n r(1 - 12\varepsilon\lambda_n r)^{-1}$, so that $\varepsilon s < 1/2$. Finally, assume that $\mathrm{dist}_H^\rho(X, Y) < \varepsilon$. Let p_0 and p_1 be projections in $\mathcal{P}_n^r(X)$ which lie in the same path component of $\mathcal{P}_n^r(X)$. By Corollary 6.3 there exist q_0 and $q_1 \in \mathcal{P}^s(Y)$ such that $L(p_j \oplus q_j) < s$ for $j = 0, 1$. For any such q_0 and q_1 and for any δ with $2\varepsilon s < \delta < 1$ there exist a path p in $\mathcal{P}_n(X)$ going from p_0 to p_1 and a path q in $\mathcal{P}_n(Y)$ going from q_0 to q_1 such that*

$$L(p_t \oplus q_t) < (1 - \delta)^{-1} s$$

for all t. In particular, the vector bundles determined by q_0 and q_1 are isomorphic.

The important comments made in the two paragraphs following Theorem 4.7 apply equally well to Theorem 6.5.

Suppose that we have a specific homotopy in $\mathcal{P}_n(X)$ and we want a homotopy in $\mathcal{P}_n(Y)$ which corresponds to it, but we do not have specific endpoints in $\mathcal{P}_n(Y)$ that we require be joined by the homotopy. Then we can apply Theorem 6.2 to obtain:

Theorem 6.6. *Let $r \in \mathbb{R}^+$ be given. Let ε be small enough that $\varepsilon\lambda_n r < 1/12$. Assume that $\mathrm{dist}_H^\rho(X,Y) < \varepsilon$. Set $s = \lambda_n r(1-12\varepsilon\lambda_n r)^{-1}$. Then for any path p in $\mathcal{P}_n^r(X)$ there exists a path q in $\mathcal{P}_n^s(Y)$ such that $L(p_t \oplus q_t) < s$ for every t.*

Suppose finally that we have a specific homotopy p in $\mathcal{P}_n(X)$ and specific q_0 and q_1 in $\mathcal{P}_n(Y)$ which correspond to p_0 and p_1 for ρ, and we want a corresponding homotopy in $\mathcal{P}_n(Y)$ joining q_0 and q_1. We can apply Theorem 6.4 to immediately obtain:

Corollary 6.7. *Let $r \in \mathbb{R}^+$ be given and let ε be small enough that $\varepsilon\lambda_n r < 1/14$. Let p be a path in $\mathcal{P}_n^r(X)$. Let q_0 and q_1 be projections in $M_n(\mathcal{D})$, and set $N = \max\{L(p_0 \oplus q_0), L(p_1 \oplus q_1)\}$. Assume further that ε is small enough that $\varepsilon N < 1/2$. Finally, assume that $\mathrm{dist}_H^\rho(X,Y) < \varepsilon$. Then there exists a path q of projections in $M_n(\mathcal{D})$ going from q_0 to q_1, and a reparametrization $\tilde{p}$ of the path p, still with domain $[0,1]$, such that*

$$L(\tilde{p}_t \oplus q_t) \leq ((1/2) - \varepsilon N)^{-1} \max\{7\lambda_n r, N\}$$

for all t.

Of course, vector bundles can be represented by projections of different sizes. In particular, if $p \in \mathcal{P}_n(X)$, then for $m > n$ the projection $\tilde{p} = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathcal{P}_m(X)$, for the 0's of appropriate size, will represent the same bundle as p, and we will have $L(\tilde{p}) = L(p)$. But because λ_n grows with n, I have so far not seen anything really useful to say about how projections of different sizes should be related within our context of Gromov–Hausdorff distance.

7. Projective modules and frames

We now make some preparations for our discussion of specific examples. Naturally-arising vector bundles are not often presented by means of projections, and there is usually no canonical choice of a projection for them. We recall in this section some elementary tools for obtaining projections corresponding to vector bundles.

As before, we set $A = C(X)$ for X a compact space. (With evident modifications, everything in this section works just as well for $A =$

$C_{\mathbb{R}}(X)$.) Most of the discussion in this section applies without change to a general unital C^*-algebra A, and so we will in some places write it in that generality, but the reader can take A to be $C(X)$ with no disadvantage for reading the next sections. Let Ξ be an A-module. We use right-module notation, both because it eases the bookkeeping somewhat, and also in view of the generalizations that we will consider elsewhere in which A is non-commutative, for which most writers use right modules. By an A-valued inner product on Ξ (for example, a Riemannian or Hermitian metric on Ξ according to whether we work over $\mathbb{R}$ on $\mathbb{C}$) we mean [52, 66] a sesquilinear form $\langle \cdot, \cdot \rangle_A$ on Ξ with values in A such that for $\xi, \eta \in \Xi$ and $a \in A$ we have

1) $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$,
2) $(\langle \xi, \eta \rangle_A)^* = \langle \eta, \xi \rangle_A$ (with $*$ = complex conjugation),
3) $\langle \xi, \xi \rangle_A \geq 0$, with $= 0$ only if $\xi = 0$.

For naturally-arising vector bundles there is often a natural choice of $C(X)$-valued inner product, even when there is no natural choice of projection. We will see this in the examples in the later sections.

On A^n as a right A-module we have the standard inner product defined by

$$\langle (a_j), (b_k) \rangle_A = \sum a_j^* b_j.$$

If $\Xi = pA^n$ for some projection $p \in M_n(A)$, then the restriction to Ξ of the inner product on A^n will be an inner product on Ξ. Thus every (finitely generated) projective A-module (that is, a summand of A^n for some n) has an inner product. If we set $\eta_j = pe_j$ for each j, where $\{e_j\}$ is the "standard basis" for A^n, then $\{\eta_j\}$ is a "standard module frame" for Ξ. We recall [14, 15, 36] the general definition, valid for modules over any unital C^*-algebra (over $\mathbb{C}$ or $\mathbb{R}$).

Definition 7.1. Let A be a unital C^*-algebra and let Ξ be a right A-module. Let Ξ be equipped with an A-valued inner-product, $\langle \cdot, \cdot \rangle_A$. By a (finite) *standard module frame* for Ξ (with respect to the inner-product) we mean a finite family $\{\eta_j\}$ of elements of Ξ such that for any $\xi \in \Xi$ the reconstruction formula

$$\xi = \sum \eta_j \langle \eta_j, \xi \rangle_A$$

is valid.

The relationships that we need between standard module frames and the projections corresponding to projective modules are given (see, for example, scattered places in [41, 14, 15]) by:

Proposition 7.2. *Let Ξ be a right module over a unital C^*-algebra, A, and suppose that Ξ is equipped with an A-valued inner-product. If Ξ has* *a standard module frame,* $\{\eta_j\}_{j=1}^n$*, then* Ξ *is a projective* A*-module. In fact,* $\Xi \cong pA^n$ *isometrically, where* p *is the projection in* $M_n(A)$ *defined by* $p_{jk} = \langle \eta_j, \eta_k \rangle_A$*. Furthermore,* Ξ *is self-dual for its inner product, in the sense that for any* $\varphi \in \mathrm{Hom}_A(\Xi, A_A)$ *there is a (unique)* $\zeta_\varphi \in \Xi$ *such that* $\varphi(\xi) = \langle \zeta_\varphi, \xi \rangle_A$ *for all* $\xi \in \Xi$*.*

Proof. Let $\{\eta_j\}_{j=1}^n$ be a standard module frame for Ξ. Define $\Phi : \Xi \to A^n$ by

$$(\Phi\xi)_j = \langle \eta_j, \xi \rangle_A.$$

Clearly Φ is an A-module homomorphism. From the reconstruction formula in the definition of a standard module frame it is clear that Φ is injective. Clearly $p^* = p$. Furthermore

$$\begin{aligned}(p^2)_{ik} &= \sum_j p_{ij}p_{jk} = \sum_j \langle \eta_i, \eta_j \rangle_A \langle \eta_j, \eta_k \rangle_A \\ &= \Big\langle \eta_i, \ \sum \eta_j \langle \eta_j, \eta_k \rangle_A \Big\rangle_A = \langle \eta_i, \eta_k \rangle_A = p_{ik}.\end{aligned}$$

Thus $p^2 = p$, and so p is a projection. Now for every $\xi \in \Xi$ we have

$$\begin{aligned}(p(\Phi\xi))_j &= \sum \langle \eta_j, \eta_k \rangle_A (\Phi\xi)_k = \sum \langle \eta_j, \eta_k \rangle_A \langle \eta_k, \xi \rangle_A \\ &= \Big\langle \eta_j, \sum \eta_k \langle \eta_k, \xi \rangle_A \Big\rangle_A = \langle \eta_j, \xi \rangle_A = (\Phi\xi)_j.\end{aligned}$$

Thus the range of p contains the range of Φ. On the other hand if v is an element of A^n in the range of p, so that $v = pv$, then, since $v_j \in A$,

$$v_j = \sum \langle \eta_j, \eta_k \rangle_A v_k = \Big\langle \eta_j, \sum \eta_k v_k \Big\rangle_A.$$

Thus if we set $\xi = \sum \eta_k v_k$, then $v = \Phi\xi$. Hence p is exactly the projection onto the range of Φ. It follows that Ξ is a projective module.

We now show that Φ is isometric. For $\xi, \zeta \in \Xi$ we have

$$\begin{aligned}\langle \Phi\xi, \Phi\zeta \rangle_A &= \sum \langle \eta_j, \xi \rangle_A^* \langle \eta_j, \zeta \rangle_A \\ &= \Big\langle \xi, \sum \eta_j \langle \eta_j, \zeta \rangle_A \Big\rangle_A = \langle \xi, \zeta \rangle_A.\end{aligned}$$

Finally, we show that Ξ is self-dual for its inner product. Let $\varphi \in \mathrm{Hom}_A(\Xi, A_A)$, where A_A means that A is viewed as a right module over itself. Then for any $\xi \in \Xi$ we have

$$\varphi(\xi) = \varphi\Big(\sum \eta_j \langle \eta_j, \xi \rangle_A\Big) = \sum \varphi(\eta_j) \langle \eta_j, \xi \rangle_A = \Big\langle \sum \eta_j (\varphi(\eta_j))^*, \xi \Big\rangle_A.$$

Thus $\zeta_\varphi = \sum \eta_j (\varphi(\eta_j))^*$ is the desired element of Ξ representing φ. $\square$

Suppose that we have a projective module Ξ that is already equipped with an A-valued inner product $\langle \cdot, \cdot \rangle_A$ for which it is self-dual, and suppose that Φ is an isomorphism from Ξ to pA^n for some projection p. Let $\langle \cdot, \cdot \rangle'_A$ denote the pull-back to Ξ of the restriction to pA^n of the standard inner product on A^n. From the self-duality of the inner products it is easily seen that there is an $S \in \mathrm{End}_A(\Xi)$ which is invertible and positive (for either inner product) such that

$$\langle \xi, \eta \rangle_A = \langle S\xi, S\eta \rangle'_A$$

for all $\xi, \eta \in \Xi$. Define $\Phi' : \Xi \to pA^n$ by $\Phi'(\xi) = \Phi(S\xi)$. Then for $\xi, \eta \in \Xi$ we have

$$\langle \Phi'\xi, \Phi'\eta \rangle_A = \langle \Phi(S\xi), \Phi(S\eta) \rangle_A = \langle S\xi, S\eta \rangle'_A = \langle \xi, \eta \rangle_A.$$

Thus for a given self-dual inner product on Ξ, and for any projection p representing Ξ we can assume that our isomorphism $\Phi : \Xi \to pA^n$ preserves the inner products (i.e., is "isometric"). Then on setting $\eta_j = \Phi^{-1}(pe_j)$ for each j we obtain a standard module frame for Ξ such that

$$p_{jk} = \langle \eta_j, \eta_k \rangle_A$$

for all j, k. We thus obtain:

Proposition 7.3. *Let Ξ be a projective A-module equipped with a fixed A-valued inner product for which it is self-dual. Every projection p such that $\Xi \cong pA^n$ for some n is of the form*

$$p_{jk} = \langle \eta_j, \eta_k \rangle_A$$

for some standard module frame $\{\eta_j\}$ for Ξ.

Thus, in the presence of a metric ρ on X, to calculate $L(p)$ for various projections p representing Ξ it suffices to consider standard module frames and their corresponding projections.

8. The Möbius strip

In this section and the next we show that the simplest non-trivial vector bundle, the Möbius-strip bundle, already provides interesting examples that illustrate our general theory. This requires working over $\mathbb{R}$.

Let $\mathbb{T}$ denote the circle, viewed either as $\mathbb{R}/\mathbb{Z}$, or as $I = [0, 1]$ with endpoints identified. Let $A = C_{\mathbb{R}}(\mathbb{T})$, which we will usually view as consisting of functions on $\mathbb{R}$ periodic of period 1. As before, we equip the free A-module A^n with its "standard" inner-product, defined by

$$\langle v, w \rangle_A(r) = \sum v_j(r)w_j(r)$$

for $v, w \in A^n$. We take as our metric ρ the metric coming from the absolute-value on $\mathbb{R}$. Thus

$$\rho(r, s) = \min\{|r - s - n| : n \in \mathbb{Z}\}.$$

We let L denote the corresponding Lipschitz seminorm on A.

Notation 8.1. The Möbius-strip A-module Ξ consists of the $\mathbb{R}$-valued continuous functions ξ on $\mathbb{R}$ which satisfy the condition that for any $r \in \mathbb{R}$

$$\xi(r - 1) = -\xi(r).$$

The action of A on Ξ is by pointwise multiplication of functions. We define an A-valued inner-product (Riemannian metric) on Ξ by

$$\langle \xi, \eta \rangle_A(r) = \xi(r)\eta(r).$$

If Ξ were a free A-module then it would contain an element ξ such that $\langle \xi, \xi \rangle_A$ is nowhere 0, which is easily seen not to happen. So we seek standard module frames for Ξ. Suppose that $\{\eta_j\}_{j=1}^n$ is a standard module frame for Ξ, and define a function $u : \mathbb{R} \to \mathbb{R}^n$ by $u(r) = (\eta_j(r))_{j=1}^n$. It is easily seen that $\mathrm{End}_A(\Xi)$ can be identified with A itself (essentially because Ξ comes from a line bundle). The reconstruction formula for $\{\eta_j\}$ then implies that $\|u(r)\| = 1$ for all r, where the norm here is the Euclidean norm on $\mathbb{R}^n$. Because $\eta_j \in \Xi$ for each j, we also have $u(r - 1) = -u(r)$ for each r. It is easily seen that conversely, if u is a continuous function from $\mathbb{R}$ to $\mathbb{R}^n$ such that $\|u(r)\| = 1$ and $u(r-1) = -u(r)$ for each r, then the component functions of u form a standard module frame for Ξ. For a standard module frame $\{\eta_j\}$ and its u, the corresponding projection p has as entries $p_{jk}(r) = \eta_j(r)\eta_k(r)$ at r. From this we easily see that $p(r)$ is just the rank-1 projection onto $u(r)$, which we like to denote by $\langle u(r), u(r) \rangle_0$. Briefly, $p = \langle u, u \rangle_0$. (Notice that $p(r - 1) = p(r)$.)

For now and later we need the undoubtedly well-known:

Proposition 8.2. *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, and let $v, w \in \mathcal{H}$ with $\|v\| = 1 = \|w\|$, and with corresponding rank-1 projections $\langle v, v \rangle_0$ and $\langle w, w \rangle_0$. Then*

$$\|\langle v, v \rangle_0 - \langle w, w \rangle_0\| = (1 - |\langle v, w \rangle_{\mathcal{H}}|^2)^{1/2} \leq \|v - w\|.$$

If $\mathcal{H}$ is over $\mathbb{R}$, the middle term is equal to $|\sin \theta|$ where θ is the angle between v and w.

Proof. If $w = av$ with $a \in \mathbb{C}$ and $|a| = 1$ then the left-hand side is 0. If v and w are linearly independent, let $\{e_1, e_2\}$ be an orthonormal basis for the subspace spanned by v and w, with $e_1 = v$. Let $w = ae_1 + be_2$

for scalars a and b, so that $|a|^2 + |b|^2 = 1$. Let $T = \langle v, v\rangle_0 - \langle w, w\rangle_0$. Then the matrix for T for the basis $\{e_1, e_2\}$ is

$$\begin{pmatrix} 1 - a\bar{a} & -a\bar{b} \\ -\bar{a}b & -b\bar{b} \end{pmatrix} = \begin{pmatrix} b\bar{b} & -a\bar{b} \\ -\bar{a}b & -b\bar{b} \end{pmatrix}.$$

Its trace is 0 and its determinant is $-|b|^2$. Thus its norm is $|b| = (1 - |\langle v, w\rangle_{\mathcal{H}}|^2)^{1/2}$, while

$$\|v - w\|^2 = |1 - a|^2 + |b|^2 \geq |b|^2,$$

giving the desired inequality. □

Because for our examples most of our spaces will be manifolds and we will use the letters X, Y for vector fields, we will at times denote our metric space by M.

Corollary 8.3. *Let (M, ρ) be a metric space and let $\mathcal{H}$ be a Hilbert space (over $\mathbb{R}$ or $\mathbb{C}$). Let u be a function from M to $\mathcal{H}$ with $\|u(m)\| = 1$ for all $m \in M$. Define a function p by $p(m) = \langle u(m), u(m)\rangle_0$ for all $m \in M$. Then $L(p) \leq L(u)$.*

Proof. For $m, n \in M$ we have from Proposition 8.2 $\|p(m) - p(n)\| \leq \|u(m) - u(n)\|$. Now divide by $\rho(m, n)$. □

Since $\mathbb{T}$ is a manifold, it is helpful to use calculus. Because we have chosen a metric that is invariant under "rotation" of $\mathbb{T}$, we can apply Proposition 2.5. Specifically, we will apply that proposition to functions on $\mathbb{R}$ which are periodic of period two, and so to the components of a function $u : \mathbb{R} \to \mathbb{R}^n$ that satisfies $\|u(r)\| = 1$ and $u(r - 1) = -u(r)$ for each $r \in \mathbb{R}$. We conclude that for any $\varepsilon > 0$ such a function can be approximated by a function $h : \mathbb{R} \to \mathbb{R}^n$ which is infinitely differentiable, in such a way that $\|u - h\|_\infty < \varepsilon$ and $L(h) \leq L(u)$. Furthermore, the smoothing argument in the proof of Proposition 2.5 can easily be seen to give $h(r - 1) = -h(r)$. But we need to obtain a smooth unit-vector-valued function in order to obtain a projection. From the relation to u we see that $\|h(r)\| \geq 1 - \varepsilon$ for all r. Define g by $g(r) = (1 - 2\varepsilon)^{-1}h(r)$. Then $\|g(r)\| > 1$ for all r, and $L(g) \leq (1 - 2\varepsilon)^{-1}L(u)$. Also, $\|h - g\| \leq 2\varepsilon(1 - 2\varepsilon)^{-1}\|h\| \leq 2\varepsilon(1 + \varepsilon)(1 - 2\varepsilon)^{-1}$, with corresponding estimate for $\|u - g\|$. Finally, let v be the composition of g with the radial retraction from $\mathbb{R}^m$ onto its unit ball. The radial retraction for a Hilbert space has Lipschitz constant 1, and is smooth at points strictly outside the unit ball. It follows that v is smooth, that $L(v) \leq (1 - 2\varepsilon)^{-1}L(u)$, and that v can be as close to u as desired by making ε small enough. Let $q = \langle v, v\rangle_0$. Then q is smooth, $L(q) \leq$

$(1-2\varepsilon)^{-1}L(p)$, and q can be as close to p as desired. (Of course, we could have applied Theorem 3.5 here.)

The above arguments can also be used for some other examples involving real line bundles over certain manifolds. We state the next step in somewhat general form in order to contrast the situation over $\mathbb{R}$ with the situations that we will meet shortly over $\mathbb{C}$. The manifold in the statement of the following proposition will often be a manifold covering the one that we are dealing with, just as $\mathbb{R}/2\mathbb{Z}$ covers $\mathbb{R}/\mathbb{Z}$ in the discussion above.

Proposition 8.4. *Let M be a compact connected Riemannian manifold, with its usual ordinary metric, and let u be a smooth function from M to $\mathbb{R}^n$ such that $u \cdot u = 1$. Define the projection-valued function p on M by $p(m) = \langle u(m), u(m)\rangle_0$. Then*

$$L(p) = L(u).$$

Proof. Let $m \in M$ and let X be a tangent vector at m. Let D denote "total derivative", so that D_X denotes differentiation at m in the direction of X. Then

$$D_X p = \langle D_X u, u(m)\rangle_0 + \langle u(m), D_X u\rangle_0.$$

Since $u \cdot u = 1$ we have $u(m) \cdot (D_X u) = 0$. We state the next step as a lemma for later reference. It is easy to prove by arguments similar to those used in the proof of Proposition 8.2.

Lemma 8.5. *Let $\mathcal{H}$ be a Hilbert space, over $\mathbb{R}$ or $\mathbb{C}$ and let $v, w \in \mathcal{H}$ with $\|v\| = 1$ and $\langle v, w\rangle = 0$. Let*

$$T = \langle v, w\rangle_0 + \langle w, v\rangle_0.$$

Then $\|T\| = \|w\|$.

On applying this lemma we see that $\|D_X p\| = \|D_X u\|$. Since

$$\|(Dp)(m)\| = \sup\{\|D_X p\| : \|X\| \le 1\}$$

and similarly for $\|(Du)(m)\|$, we see that they are equal. Consequently $\|Dp\|_\infty = \|Du\|_\infty$. But standard arguments (see the discussion early in Section 11 after Corollary 14.2) show that $L(p) = \|Dp\|_\infty$ and similarly for $L(u)$. $\square$

We remark that Proposition 8.4 is false for functions from M to $\mathbb{C}^n$ because the phase of u can vary while leaving p fixed.

Actually, Proposition 8.4 is true for path-length metric spaces. We show this in Appendix A.

We now return to the Möbius-strip bundle, and apply to it the observations made above. The consequence of the observations is that, for our present purposes, it suffices to work with smooth u's and projections. So let u be, as earlier, a smooth function from $\mathbb{R}$ to $\mathbb{R}^n$ with $u \cdot u = 1$ and $u(r+1) = -u(r)$. As r goes from 0 to 1 the vector $u(r)$ traces a curve on the unit sphere of $\mathbb{R}^n$ from $u(0)$ to its antipodal point $-u(0)$. The length of this curve is $\int_0^1 \|u'(r)\|dr$. But the shortest path from $u(0)$ to $-u(0)$ will be along one of the great-circle geodesics, and it will have length π. Since $\|p'(r)\| = \|u'(r)\|$ by Lemma 8.5, it follows that $\int_0^1 \|p'(r)\|dr \geq \pi$. By the mean-value theorem there must be at least one point, r_0, where $\|p'(r_0)\| \geq \pi$. Thus $L(p) \geq \pi$. By the approximations discussed above, it follows that for any projection $p \in M_n(A)$ such that $\Xi \cong pA^n$ we have $L(p) \geq \pi$. Also, we can achieve $L(p) = \pi$ by choosing u such that $u(r)$ moves along a great circle at speed π.

When we want to use Theorem 6.2, we see that it is best to keep the size of our matrices as small as possible. The simplest choice is then $u(r) = (\cos(\pi r), \sin(\pi r))$. The components of this u form a standard module frame for Ξ. We summarize what we have found by:

Proposition 8.6. *For any $p \in M_n(A)$ which represents the Möbius-strip bundle we have $L(p) \geq \pi$. For any $n \geq 2$ we can find such a p with $L(p) = \pi$.*

We remark that for any positive integer k the space of $\mathbb{R}$-valued continuous functions ξ which satisfy

$$\xi(r-k) = -\xi(r)$$

is an A-module for pointwise multiplication, and it is an entertaining and instructive exercise to show that these modules are projective, to determine which are free, to find standard module frames, etc. In fact, the same is true for the modules consisting of functions satisfying

$$\xi(r-k) = +\xi(r).$$

We now want to illustrate another aspect of our theory by examining briefly what happens when one changes the metric on the circle. Our discussion will be at the qualitative level, but with more effort it could be made quantitative.

Consider a smooth embedding of $M = \mathbb{T}$ into $\mathbb{R}^2$ with its Euclidean metric, and assume that the image is approximately two far-away disjoint round circles connected by a very narrow "tube". Give M the metric coming from restricting the ordinary Euclidean metric from $\mathbb{R}^2$ to this embedding. Let $p_0 \in M_2(C(M))$ be the projection for a Möbius-strip bundle such that p_0 is a constant function on one of the almost-circles and the tube, so that the twist takes place over the other almost-circle. Let $p_1 \in M_2(C(M))$ be the projection for a Möbius-strip bundle such that p_1 is a constant function on the tube and on the almost-circle where p_0 has its twist. Then one can show that for a suitable constant, c, depending on the specific choice of embedding (especially the narrowness of the tube), p_0 and p_1 can have been chosen to have $L(p_j) < c$ for $j = 0, 1$ but there is no homotopy $\{p_t\}$ of projections from p_0 to p_1 such that $L(p_t) < c$ for all t. Thus the set of projections $p \in M_2(C(M))$ which represent the Möbius-strip bundle and have $L(p) < c$ has more than one path-component, and from our metric point of view the different path components can be viewed as representing genuinely different vector bundles over M. Indeed, M with its given metric can be made very close, for Hausdorff distance in $\mathbb{R}^2$, to the disjoint union of two circles (as seen by cutting the very narrow tube), and p_0 and p_1 then correspond to vector bundles on the disjoint union which are a Möbius-strip bundle on one circle and a trivial bundle on the other, but in different ways. One can make many variations of the above example, involving embedding M as a greater number of almost-circles connected by narrow tubes.

One might object that the metrics involved in these examples are not path-length metrics. But one can use the same idea to smoothly embed a 2-sphere into $\mathbb{R}^3$ as a collection of far away disjoint round almost-spheres connected by narrow tubes. Then instead of putting on M the restriction of the ordinary Euclidean metric on $\mathbb{R}^3$, one equips M with the Riemannian metric from the embedding, and then the ordinary metric from the Riemannian metric. This is a path-length metric. Finally, one can consider projections corresponding to putting on the various almost-spheres line bundles which on corresponding actual spheres would have various Chern classes (as discussed in Section 13).

9. Approximate Möbius-strip bundles

We now use the Möbius-strip bundle to further illustrate our earlier considerations, in a quantitative way. The circle $\mathbb{T}$ will now play the role of the larger metric space Z of our earlier discussion, and so we denote the circle $\mathbb{T}$ by Z for the rest of this section. Let m be a large positive integer, and let $X = \{j/m : 0 \leq j \leq m-1\} \sim \mathbb{Z}(1/m)/\mathbb{Z}$ where the j's are integers. We view X as a subset of Z, and equip X with the metric from Z. We now let q_1 denote the specific projection p determined as in the previous section in terms of the standard frame $(\cos(\pi r), \sin(\pi r))$. Then we let p_1 denote the restriction of q_1 to X. Let us determine $L(p_1)$. If v and w are two unit-length vectors such that the angle between them is θ, then it follows from Proposition 8.2

that the norm-distance between the projections along these vectors is $|\sin\theta|$. Thus for $0 \le j,k < m$ we have

$$\|p_1(j/m) - p_1(k/m)\| = \sin(\pi|j-k|/m),$$

and from this it is not hard to see that

$$L(p_1) = \frac{\sin(\pi/m)}{(1/m)} = \pi\frac{\sin(\pi/m)}{(\pi/m)}.$$

Notice that this approaches π as m goes to $+\infty$, consistent with the fact that $L(q_1) = \pi$ as seen in the previous section.

Since X is finite, every vector bundle over X is specified (up to isomorphism) just by giving the dimension of the fiber vector-space over each point. Our projection p_1 represents the real vector bundle whose fiber at each point has dimension 1. But this vector bundle is equally well represented by the projection p_0 defined by $p_0(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for each $t \in X$. Note that $L(p_0) = 0$. Let q_0 denote the projection for Z defined by $q_0(r) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for all $r \in Z$, so that p_0 is the restriction of q_0 to X. Then q_0 represents the rank-1 trivial vector bundle over Z, and this bundle is not isomorphic to the Möbius-strip bundle determined by q_1.

The reason that this situation is possible, from our metric point-of-view, is that there is no path, p, of projections from p_0 to p_1 such that $L(p_t)$ is sufficiently small for all t. This can be seen directly, by examining what such a path must do at various neighboring points of X. But let us instead apply the general considerations given in Theorem 6.4. (We will not expect this to give as sharp an estimate as a direct argument would give.) Set $\varepsilon = (2m)^{-1}$ and note that X is ε-dense in Z for this ε. Let p be a path of projections from p_0 to p_1, and let N be a constant such that $L(p_t) \le N$ for all t. Note that $\varepsilon L(q_0) = 0$, while $\varepsilon L(q_1) = \pi/2m < 1/2$ as soon as $m \ge 4$. Now $M_2^s(\mathbb{R})$ is a 3-dimensional vector space, and from theorem 1.1a of [29] we find that the projection constant of any 3-dimensional real Banach space is no greater than 3. (The techniques of section 7 of [49] can be used to obtain the precise value of $\mathcal{PC}(M_2^s(\mathbb{R}))$.) We will apply Theorem 6.4, but by the observation just made we can replace λ_3 there with 6. Suppose now that $m > 42N$ so that $\varepsilon 6N < 1/14$. We conclude from Theorem 6.4 that there exists a continuous path of projections from q_0 to q_1. But we know that this is not possible since q_0 and q_1 determine non-isomorphic A-modules. Consequently we must have $N \ge m/42$. Now $L(p_1) < \pi$ while $L(p_0) = 0$. Thus if $m \ge 4 \cdot 42$ so that $N > 4 > \pi$, we see that the collection of projections in $M_2(A)$ which can be connected to p_1 by paths p of projections for which $L(p_t) \leq 4$ for all t, does not include p_0. Consequently the projections in this collection can be viewed from our metric point of view as giving approximate Möbius-strip bundles on our finite set X, which are not equivalent in our metric sense to the trivial bundle of rank 1 on X.

Of course, similar considerations apply to other closed subsets of Z which are ε-dense, and to other compact metric spaces Y whose Gromov–Hausdorff distance from Z is less than ε and for which a corresponding metric on $Z \dot{\cup} Y$ has been chosen.

10. Lower bounds for $L(p)$ from Chern classes

In this section we will indicate how Chern classes can sometimes be used to obtain a lower bound on $L(p)$ for projections representing a given vector bundle. In Section 12 we will illustrate this approach by considering vector bundles on a two-torus. Our discussion here is brief, and there is much more to be explored in this direction.

For our purposes, and in particular for the two-torus, it is simplest to work in the framework of Connes' 1980 paper [10], which initiated the subject of non-commutative differential geometry, and which uses the Chern–Weil approach to Chern classes. We briefly sketch the setting. We have a unital C^*-algebra A, together with an action α of a connected Lie group G by automorphisms of A. (For our present purposes it will be quite sufficient for the reader to have in mind just the case in which $A = C(G/H)$ where H is some cocompact closed subgroup of G, with α the evident action [50].) We let A^∞ denote the dense $*$-subalgebra of smooth elements of A with respect to α. Then α lifts to a homomorphism of the Lie algebra, $\mathfrak{g}$, of G (and its complexification) into the Lie algebra $\mathrm{Der}(A^\infty)$ of derivations of A^∞ into itself. We denote this homomorphism again by α. We must also have a tracial state, τ, on A which is invariant for the action α. For the case $A = C(G/H)$ this will just be a G-invariant probability measure on G/H (unique if it exists).

Let Ξ be the smooth version of a projective A-module, that is, Ξ is a projective A^∞-module. On Ξ there always exists a connection (i.e., covariant derivative), that is, a linear map $\nabla : \mathfrak{g} \to \mathrm{Lin}(\Xi)$ which satisfies the Leibniz property

$$\nabla_X(\xi a) = (\nabla_X(\xi))a + \xi(\alpha_X(a))$$

for $X \in \mathfrak{g}$. The curvature of ∇ is the alternating 2-form Θ on $\mathfrak{g}$ defined by

$$\Theta(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

One finds that it has values in $\mathrm{End}_{A^\infty}(\Xi)$. Denote $\mathrm{End}_{A^\infty}(\Xi)$ by E.

Let $\bigwedge(\mathfrak{g}')$ denote the complexified exterior algebra over the dual vector space, $\mathfrak{g}'$, of $\mathfrak{g}$. Then $\Omega^* = E \otimes \bigwedge(\mathfrak{g}')$ has in a natural way the structure of a differential graded algebra, and we can view Θ as an element of Ω^2. Thus $\Theta \wedge \cdots \wedge \Theta$ (k-times) can be viewed as an element of Ω^{2k}, by using the product in E. The tracial state τ on A induces in a natural way an unnormalized trace on E, which we denote by τ_E. It is characterized [41] by the property that

$$\tau_E(\langle \xi, \eta \rangle_E) = \tau(\langle \eta, \xi \rangle_A)$$

for $\xi, \eta \in \Xi$, where $\langle \xi, \eta \rangle_E$ is the element (a "rank-one" operator) of E defined by $\langle \xi, \eta \rangle_E \zeta = \xi \langle \eta, \zeta \rangle_A$. Then $\tau_E(\Theta \wedge \cdots \wedge \Theta)$, defined in the evident way, is an element of $\bigwedge^{2k}(\mathfrak{g}')$. The main theorem [10] is that this element is a closed form, and that its cohomology class, ch_k, depends only on Ξ. (There are various choices of normalizing constants that are used here. We choose to use the constant 1.) If we pair ch_k with a $2k$-homology class, we obtain a number. This number may be related to $L(p)$ when p represents Ξ, but it is independent of the choice of such (smooth) p.

Suppose now that we have a specific projection $p \in M_n(A^\infty)$ such that $\Xi = p(A^\infty)^n$. On $(A^\infty)^n$ we have the evident flat connection ∇^0 given by $\nabla^0_X((a_j)) = (\alpha_X(a_j))$. Then there is a canonically associated connection, ∇, on Ξ, defined by $\nabla_X(\xi) = p\nabla^0_X(\xi)$. It is often called the Grassmann (or Levi–Civita) connection for p. It is natural to use the Grassmann connection in the setting sketched above. The curvature of the Grassmann connection is given [10] by

$$\Theta(X, Y) = p(\alpha_X(p)\alpha_Y(p) - a_Y(p)\alpha_X(p))p,$$

where here α denotes the evident action of G on $M_n(A)$. Clearly $E = \mathrm{End}_{A^\infty}(\Xi) = pM_n(A^\infty)p$. Then τ_E is the restriction to E of the canonical unnormalized trace τ on $M_n(A)$ coming from τ on A. In particular $\|\tau_E\| = \tau(p)$. (We remark that $\tau(p)$ is the 0-th Chern class of Ξ.) If we define ω by

$$\omega(X, Y) = \tau_E(p(\alpha_X(p)\alpha_Y(p) - \alpha_Y(p)\alpha_X(p))),$$

then ω is a cocycle whose cohomology class, ch_1, is independent of p representing Ξ. If we then pair ω with a cycle in $\bigwedge^2 \mathfrak{g}$, then we obtain a number which is independent of p. Now for any $X \wedge Y \in \bigwedge^2 \mathfrak{g}$ we have $d(X \wedge Y) = [X, Y]$. (See equation 3.1 of [30].) Thus $X \wedge Y$ is a cycle exactly if $[X, Y] = 0$. Consequently, if $[X, Y] = 0$ then

$$c_{XY}(\Xi) = \tau_E(p(\alpha_X(p)\alpha_Y(p) - \alpha_Y(p)\alpha_X(p)))$$

is a number independent of p representing Ξ.

Suppose now that we have a norm, ν, on $\mathfrak{g}$, and that we define a seminorm, L, on each $M_n(A)$ by

$$L(a) = \sup\{\|\alpha_X(a)\| : \nu(X) \leq 1\},$$

for α extended to $M_n(A)$. (See the next section.) Then we find that

$$|c_{XY}(\Xi)| \leq \|\tau_E\| 2L(p)^2 \nu(X)\nu(Y).$$

Since $\|\tau_E\| = \tau(p)$, we obtain a lower bound for $L(p)$. But by Theorem 3.5, due to Hanfeng Li, this same lower bound applies to any projection representing Ξ. Thus we obtain:

Theorem 10.1. *For every p representing Ξ we have*

$$(L(p))^2 \geq (2\tau(p))^{-1} \sup\{|c_{XY}(\Xi)| : [X,Y] = 0, \nu(X) \leq 1, \nu(Y) \leq 1\}.$$

To the extent that we have in hand cycles in $\bigwedge^4 \mathfrak{g}$, we can also pair them with $\tau(\Theta \wedge \Theta)$ to obtain other lower-bounds for $L(p)$, and similarly for higher dimensions. More generally, for any ordinary compact Riemannian manifold, to the extent that one has in hand specific even homology classes, one can pair them with corresponding Chern classes of a given vector bundle Ξ to try to obtain lower bounds on $L(p)$ for smooth p's representing Ξ. But consideration of flat bundles which are not trivial shows that lower bounds from Chern classes may well not be optimal bounds.

A possible related way of measuring the twisting of Ξ would be just by the size of the curvature Θ of various of its connections, where by the size of Θ we mean $\sup\{\|\Theta(X,Y)\| : \nu(X), \nu(Y) \leq 1\}$ for a norm ν on $\mathfrak{g}$ as above. This measure of size is what is used in working with "almost flat bundles". See section 3 of [58] and the references it contains. It would be interesting to investigate what could be done in this direction. But again, there are non-trivial flat bundles, that have connections whose size measured this way is 0.

11. Vector bundles on the two-torus

We illustrate the considerations of the previous section by examining the two torus. One of our aims is to show that it can be feasible to find projections p with close-to-minimal $L(p)$. We will let $G = \mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and $A = C(\mathbb{T}^2)$, with the evident action α of G on A by translation. On $\mathfrak{g} = \mathbb{R}^2$ we place the standard inner product and corresponding norm. This gives the standard flat Riemannian metric on $\mathbb{T}^2$, which in turn gives the usual ordinary metric on $\mathbb{T}^2$ given by

$$d((r,s),(t,u)) = \min\{((r-t-m)^2 + (s-u-n)^2)^{1/2} : m,n \in \mathbb{Z}\},$$

where our notation on the left omits $\mathbb{Z}^2$. For $a \in A^\infty$ the Lipschitz constant, $L(a)$, of a can be conveniently calculated as

$$L(a) = \sup\{\|\alpha_X(a)\|_\infty : X \in \mathfrak{g},\ \|X\| \le 1\},$$

and similarly for $a \in M_n(A^\infty)$. See the proof of theorem 3.1 of [44] and lemma 3.1 of [46].

The projective modules corresponding to the complex line bundles over $\mathbb{T}^2$ can all be realized [42, 37] in the form

$$\Xi_k = \{\xi \in C(\mathbb{T} \times \mathbb{R}, \mathbb{C}) : \xi(r, s+1) = e(kr)\xi(r, s)\},$$

where $k \in \mathbb{Z}$ and $e(t) = e^{2\pi i t}$. The action of elements of A is by pointwise multiplication.

We need to find projections for these modules. For this purpose it is clearer to work at first in somewhat greater generality. Let B be a unital C^*-algebra, and let

$$A = TB = \{a \in C(\mathbb{R}, B) : a(s+1) = a(s)\}\ .$$

Our application will be to the case in which $B = C(\mathbb{T})$ so that $A = C(\mathbb{T}^2)$. We use a variation of the familiar clutching construction to construct projective A-modules, along the lines used in the proof of theorem 8.4 of [43]. Let w be a unitary element of B. (In our application w will be the function $e(kr)$.) Set

$$\Xi_w = \{\xi \in C(\mathbb{R}, B) : \xi(s+1) = w\xi(s)\}\ .$$

Each Ξ_w is a right A-module for pointwise multiplication, and has a natural Hermitian metric given by $\langle \xi, \eta\rangle_A(s) = \xi(s)^*\eta(s)$. Furthermore, Ξ_w is a projective A-module, for reasons that we now need to review. (See lemma 8.8 of [43].) We will apply Proposition 7.2, which will also give us a projection determining Ξ_w. So we must find a standard module frame for Ξ_w. Two elements of Ξ_w suffice, and we will denote them by η_1 and η_2. We assume that w is not connected to the identity through invertible elements, since otherwise Ξ_w is easily seen to be a free module (lemma 8.5 of [43]). It is reasonable to assume that $\eta_1(0) = 1_A$. Since $\eta_1(s+1) = w\eta_1(s)$, we then see that η_1 must fail to be invertible at some point, since otherwise w would be connected to the identity. In the absence of any further information about w we will assume that η_1 actually takes value 0 at some point t_0. Since $\|\eta_1(1)\| = \|\eta_1(0)\| = 1$, we see that to have a small Lipschitz norm it is best if $t_0 = 1/2$. By the reconstruction property, for any $\xi \in \Xi_w$ we must have

$$\xi = \sum_{j=1}^{2} \eta_j \langle \eta_j, \xi\rangle_A = (\eta_1\eta_1^* + \eta_2\eta_2^*)\xi,$$

so that we must have $\eta_1\eta_1^* + \eta_2\eta_2^* = 1_A$. It follows that $\eta_2(0) = 0$ and $\|\eta_2(1/2)\| = 1$. Much as in [37] we will take η_1 and η_2 of the form

$$\eta_1(s) = J_1(s)\cos(\pi s), \quad \eta_2(s) = J_2(s)\sin(\pi s)$$

where

$$\begin{aligned} J_1(s) &= (-w)^n \quad \text{for } n - 1/2 \leq s < n + 1/2, \\ J_2(s) &= (-w)^n \quad \text{for } n \leq s < n + 1. \end{aligned}$$

Of course, J_1 and J_2 are discontinuous, but their discontinuities are at the points where $\cos(\pi s)$ or $\sin(\pi s)$ takes value 0, and η_1 and η_2 are not only continuous, but are actually Lipschitz as functions on $\mathbb{R}$. For example, η_2 near 0 is given by $\eta_2(s) = \int_0^s h_2(t)dt$ where

$$h_2(t) = \begin{cases} \pi(-w^*)\cos(\pi t) & \text{for } -1 < t < 0 \\ \pi\cos(\pi t) & \text{for } 0 \leq t < 1 \end{cases},$$

which is the derivative of η_2 where the derivative exists. If we let η_2' denote this not-everywhere-defined derivative, we see that $L(\eta_2) = \|\eta_2'\|_\infty = \pi$. Similarly $L(\eta_1) = \|\eta_1'\|_\infty = \pi$. Since $J_j(s+1) = -wJ_j(s)$ while $\cos(\pi s)$ and $\sin(\pi s)$ satisfy $g(s+1) = -g(s)$, we see that $\eta_j(s+1) = w\eta_j(s)$, so that $\eta_j \in \Xi_w$ for $j = 1, 2$. Furthermore, we clearly have $\eta_1\eta_1^* + \eta_2\eta_2^* = 1_A$, so that $\{\eta_1, \eta_2\}$ is a standard module frame for Ξ_w. To express the corresponding projection in $M_2(A)$ it is convenient to set $H = J_2^*J_1$, so that H is the discontinuous periodic function of period 1 taking value 1_A for $0 \leq s < 1/2$ and value $-w$ for $1/2 \leq s < 1$. Then, for example, $\langle \eta_2, \eta_1\rangle_A(s) = H(s)\cos(\pi s)\sin(\pi s)$. Thus

$$p(s) = \begin{pmatrix} \cos^2(\pi s)1_A & \overline{H}(s)\cos(\pi s)\sin(\pi s) \\ H(s)\cos(\pi s)\sin(\pi s) & \sin^2(\pi s)1_A \end{pmatrix}.$$

We now apply all of this to the case in which $B = C(\mathbb{T})$ and $w(r) = e(kr)$ for a fixed k. Then $A = C(\mathbb{T}^2)$, and p is a continuous projection-valued function on $\mathbb{T}^2$ which is differentiable in r and piecewise differentiable in s. Since $w(r) = e(kr)$, we see that H, as defined above but now for our special case, is given by

$$H(s, r) = 1 \quad \text{for} \quad 0 \leq s < 1/2 \quad \text{and} \quad -e(kr) \quad \text{for} \quad 1/2 \leq s < 1\ ,$$

extended with period 1. Thus H is continuously differentiable on $\mathbb{R}^2$ whenever $s \notin \mathbb{Z}/2$. From this we see that p is continuously differentiable on $\mathbb{R}^2$ whenever $s \notin \mathbb{Z}/2$. We now make the following observation. Let f be a continuous and continuously piecewise-differentiable function, defined on an interval I in $\mathbb{R}$, with values in a Banach space, such that at the points of discontinuity of f' the left-hand and right-hand limits exist. Then for any $s, t \in I$ we have $f(t) - f(s) = \int_s^t f'(r)dr$

in the evident sense, given that f' is not defined at a finite number of points. Thus if there is a constant K such that $\|f'(r)\| \leq K$ when it is defined, then $L(f) \leq K$. It is easily seen that for any two points $m, n \in \mathbb{R}^2$ the restriction of p to the straight line-segment joining those two points satisfies the conditions on f just stated. On applying the above observation and using the periodicity of p, we see quickly that $L(p) \leq \|Dp\|_\infty$ where Dp denotes the total derivative of p, defined when $s \notin \mathbb{Z}/2$.

We now express p in terms of a unit vector field u. Most natural would be to take $u = (\zeta_1, \zeta_2)'$, where $'$ denotes transpose. But calculations are a bit simpler if we set $u(r,s) = (H(r,s)\cos(\pi s),\ \sin(\pi s))'$ and check that we still have $p = \langle u, u\rangle_0$, where, as in our discussion of the Möbius bundle, $\langle u, u\rangle_0(r,s)$ denotes the rank-1 projection along the vector $u(r,s)$. Notice that u is not even continuous where $s \in \mathbb{Z}$. For any $X \in \mathbb{R}^2$ let D_X denote the directional derivative along X if it is defined. Then for $s \notin \mathbb{Z}/2$ we have

$$D_X p = \langle u, D_X u\rangle_0 + \langle D_X u, u\rangle_0.$$

Since $u \cdot u = 1$, we have $\mathrm{Re}(\langle u, D_X u\rangle_A) = 0$. Thus we need, here and later, the following small generalization of Lemma 8.5, whose proof is obtained by using the techniques of the proof of Proposition 8.2.

Lemma 11.1. *Let v and w be vectors in a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, with $\|v\| = 1$ and $\mathrm{Re}(\langle v, w\rangle_{\mathcal{H}}) = 0$. Let*

$$T = \langle v, w\rangle_0 + \langle w, v\rangle_0.$$

Then $\|T\| = \|w - v\langle v, w\rangle_{\mathcal{H}}\| \leq \|w\|$.

We thus see that

$$\|D_X p\|_\infty = \|D_X u - u\langle u, D_X u\rangle_{\mathcal{H}}\|_\infty \leq \|D_X u\|_\infty.$$

Now the total derivative, Du, of u, for $s \notin Z/2$, is

$$Du(r,s) = \begin{pmatrix} \frac{\partial H}{\partial r}(r,s)\cos(\pi s) & -\pi H(r,s)\sin(\pi s) \\ 0 & \pi\cos(\pi s) \end{pmatrix}.$$

This is complex, whereas we need the supremum of $\|D_X u\|$ for X real with $\|X\| = 1$. By splitting Du into its real and imaginary parts it is not difficult to see that this supremum is $2\pi|k|$ for $k \neq 0$, obtained for $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and for s approaching 1 from the left. Thus $L(p) \leq 2\pi|k|$ by Lemma 11.1.

Let us now obtain a lower bound for $L(p)$. Straightforward calculations show that, for the evident abbreviations,

$$D_X u - u\langle u, D_X u\rangle_{\mathcal{H}} = \begin{pmatrix} 2\pi ikH\cos\sin^2 & -\pi H\sin \\ -2\pi ik\sin\cos^2 & \pi\cos \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

for $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$. At $s = 1/4$ and $r = 0$ this becomes

$$\frac{\pi}{\sqrt{2}}\begin{pmatrix} ik & -1 \\ -ik & 1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \frac{\pi}{\sqrt{2}}(ikX_1 - X_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

whose norm is $\pi(k^2X_1^2 + X_2^2)^{1/2}$. For $\|X\| = 1$ this has maximum $\pi|k|$ for $k \neq 0$. Altogether we thus obtain

$$\pi|k| \leq L(p) \leq \pi 2|k|.$$

It is no surprise that $L(p)$ increases with $|k|$, since intuitively the larger $|k|$ is the more rapidly Ξ_k twists. But further investigation would be needed if one wanted to know whether p is a projection for which $L(p)$ is minimal among all projections representing Ξ_k. In the next section we will see what information Chern classes can give about this.

Let us now apply Theorem 4.7, using Notation 4.6, to obtain the uniqueness of corresponding bundles on nearby spaces. Let $X = \mathbb{T}^2$ and let ρ_X be the metric defined at the beginning of this section. For each $k \in \mathbb{Z}$ let p^k be the projection for Ξ_k defined above. Suppose we have a metric space (Y, ρ_Y) and a metric ρ on $X\dot{\cup}Y$ which restricts to ρ_X and ρ_Y, such that $\mathrm{dist}_H^\rho(X,Y) < \varepsilon$. Let r be given with $r\varepsilon < 1/2$. For each k such that $2\pi|k| < r$, so that $\varepsilon L(p^k) < 1/2$, let Π_k denote the path component of p^k in $\mathcal{P}_2^r(X)$. Let p' be another projection in Π_k, and let $s \geq r$ be such that $\varepsilon s < 1/2$ and there is a path p in Π_k from p^k to p' such that for each t there is a $\bar{q}_t \in \mathcal{P}_2(Y)$ with $p_t \oplus \bar{q}_t \in \mathcal{P}_2^s(X\dot{\cup}Y)$. Then according to Theorem 4.7 for any q_0 and $q_1 \in \mathcal{P}_2^s(Y)$ such that $L(p^k \oplus q_0) < s$ and $L(p' \oplus q_1) < s$, and for any δ with $2\varepsilon s < \delta < 1$, there is a path $\tilde{p}$ from p^k to p' and a path q from q_0 to q_1 such that $L(\tilde{p}_t \oplus q_t) < (1-\delta)^{-1}s$ for each t. In this controlled sense q_0 and q_1 are equivalent projections corresponding to the equivalent projections p^k and p'.

In the same way we can apply Theorem 6.6 to obtain the existence of vector bundles on nearby spaces, and in fact to lift homotopies. Let X and Y be as in the previous paragraph. But now assume that $\varepsilon\lambda_2 r < 1/12$. Since $\lambda_2 \leq 10/3$ by formula (5.3) and the comments after Notation 5.4, it suffices to have $\varepsilon r < 1/40$. As in the previous paragraph let Π_k be the path component of p^k in $\mathcal{P}_2^r(X)$ for each k such that $2\pi|k| < r$. Set $s = \lambda_2 r(1 - 12\varepsilon\lambda_2 r)^{-1}$. Then for any path p

in Π_k there exists a path q in $\mathcal{P}^s(Y)$ such that $L^\rho(p_t \oplus q_t) < s$ for each t.

12. Chern class estimates for the two-torus

We now apply Theorem 10.1 to obtain lower bounds for $L(p)$ for p's representing the line bundles over $\mathbb{T}^2$ discussed in the previous section. Thus, in the notation of Section 10, we need to calculate $c_{XY}(\Xi_k)$, where now X and Y are in the commutative Lie algebra $\mathbb{R}^2$ of $\mathbb{T}^2$. To calculate this Chern class we can choose any convenient smooth projection p representing Ξ_k. We modify slightly our construction above for p so as to obtain a p which is smooth. We let φ_1 and φ_2 be real-valued functions in $C^\infty(\mathbb{R})$ which are close to $\cos(\pi s)$ and $\sin(\pi s)$ respectively, and satisfy both $\varphi_j(s+1) = -\varphi_j(s)$ and $\varphi_1^2 + \varphi_2^2 = 1$, but are such that φ_1 takes value 0 in all of a small neighborhood of $1/2$, while φ_2 takes value 0 in all of a small neighborhood of 0. For J_1 and J_2 as above for our given k, we set $\eta_1(r,s) = J_1(r,s)\varphi_1(s)$ and $\eta_2(r,s) = J_2(r,s)\varphi_2(s)$. Then η_1 and η_2 are infinitely differentiable, and they form a standard module frame for Ξ_k. We let p denote the corresponding projection, and much as before we define u by $u(r,s) = \begin{pmatrix} \eta_1(r,s) \\ \eta_2(r,s) \end{pmatrix}$. Then, as before, we have $p = \langle u, u\rangle_0$.

We want to express $c_{XY}(\Xi_k)$ in terms of u, and later, of η_1 and η_2. For $X \in \mathbb{R}^2$ denote derivation in the X direction by ∂_X. Then

$$\partial_X p = \langle \partial_X u, u\rangle_0 + \langle u, \partial_X u\rangle_0.$$

But the r-component of $\partial_X u$ need not be a bounded function on $\mathbb{R}^2$, since u satisfies $u(r, s+1) = e(kr)u(r,s)$. We introduce some notation to handle this. Set $B = C^\infty(\mathbb{R}^2)$, the algebra of smooth complex-valued, possibly unbounded, functions on $\mathbb{R}$. Thus we can view (the smooth part of) Ξ_k as a subspace of B. Clearly $u \in B^2$, and $p \in M_2(B)$. Actually much of our calculation below works for any standard module frame for Ξ_k whose elements are smooth. Thus we can assume just that $u \in B^n$ for some n (with entries in Ξ_k) so that $p \in M_n(B)$. The main simplification which we will be using is that we are dealing with a line bundle, so that the B-valued (equivalently A-valued) inner product on Ξ_k is given just by pointwise multiplication. If instead we were dealing with the higher-rank (smooth) projective modules

$$\Xi_{q,k} = \{\xi \in B : \xi(r, s+q) = e(kr)\xi(r,s),\ \xi(r+1,s) = \xi(r,s)\}$$

for a $q \geq 2$, then the inner product would involve a sum over $\mathbb{Z}/q\mathbb{Z}$ in order to make its values periodic of period 1 in the s variable. This would complicate matters. Because we are in the line-bundle situation, we can identify $E = \mathrm{End}_A(\Xi_k)$ with A itself, and we have $\langle \xi, \eta \rangle_E = \langle \eta, \xi \rangle_A = \langle \eta, \xi \rangle_B$ as functions. (Note the reversal of order.) Thus the key property of u can be rewritten as $\langle u, u \rangle_B = 1$. For $v, w \in B^n$ we let $\langle v, w \rangle_0$ denote the corresponding "rank-1" operator defined by $\langle v, w \rangle_0 x = v \langle w, x \rangle_B$ for $x \in B^n$. Then $\langle v, w \rangle_0 \in M_n(B)$, with $(\langle v, w \rangle_0)_{jk} = \langle w_k, v_j \rangle_B$.

For any $X \in \mathbb{R}^2$ we will have the Leibniz rule

$$\partial_X(\langle v, w \rangle_0) = \langle \partial_X v, w \rangle_0 + \langle v, \partial_X w \rangle_0,$$

and similarly for $\langle v, w \rangle_B$. We now begin calculating, using strongly the line-bundle aspect for u. Note first that

$$p \langle \partial_X u, u \rangle_0 = \langle \langle u, u \rangle_0 \partial_X u, u \rangle_0 = \langle u, \partial_X u \rangle_B \, p.$$

It follows that

$$p(\partial_X p) = p(\langle \partial_X u, u \rangle_0 + \langle u, \partial_X u \rangle_0) = \langle u, \partial_X u \rangle_B \, p + \langle u, \partial_X u \rangle_0.$$

Now from $p^2 = p$ and the Leibniz rule we have $p(\partial_Y p)p = 0$. Then, since $(\partial_Y p)p$ is the adjoint of $p(\partial_Y p)$, we have for $X, Y \in \mathbb{R}^2$

$$\begin{aligned} p(\partial_X p)(\partial_Y p)p &= (\langle u, \partial_X u \rangle_B \, p + \langle u, \partial_X u \rangle_0)(\partial_Y p)p \\ &= \langle u, \partial_X u \rangle_0 (\partial_Y p)p = \langle u, \partial_X u \rangle_0 (p \langle \partial_Y u, u \rangle_B + \langle \partial_Y u, u \rangle_0) \\ &= \langle \langle u, \partial_X u \rangle_0 u, u \rangle_0 \langle \partial_Y u, u \rangle_B + \langle \langle u, \partial_X u \rangle_0 (\partial_Y u), u \rangle_0 \\ &= (\langle \partial_X u, u \rangle_B \langle \partial_Y u, u \rangle_B + \langle \partial_X u, \partial_Y u \rangle_B)p. \end{aligned}$$

When we subtract from this the corresponding expression with X and Y interchanged, we find that

$$p(\partial_X(p)\partial_Y(p) - \partial_Y(p)\partial_X(p))p = 2i \, \mathrm{Im}(\langle \partial_X u, \ \partial_Y u \rangle_B)p.$$

The right-hand side is in $M_n(A)$ since the left-hand side is. The translation-invariant trace τ on $M_n(A)$ is given by taking the ordinary trace of matrices followed by integration over the fundamental domain $[0,1)^2 \sim \mathbb{T}^2$. Since p is a rank-1 projection at each point of $\mathbb{T}^2$, we have $\tau(p) = 1$. Thus when we apply the formula for $c_{X,Y}$ given somewhat before Theorem 10.1 we find that

$$c_{X,Y}(\Xi_k) = 2i \int_{\mathbb{T}^2} \mathrm{Im}(\langle \partial_X u, \partial_Y u \rangle_B).$$

(We are using the fact that our Lie algebra is commutative.)

For our specific u defined in terms of η_1 and η_2 we then find that

$$c_{X,Y}(\Xi_k) = 2i \int_{\mathbb{T}^2} \mathrm{Im}((\partial_X \eta_1)^-(\partial_Y \eta_1) + (\partial_X \eta_2)^-(\partial_Y \eta_2)).$$

Let us take the particular case in which $X = \partial/\partial_r$ and $Y = \partial/\partial_s$, and denote these by ∂_1 and ∂_2 respectively. Then for $j = 1, 2$ we have $\partial_1\eta_j = (\partial_1 J_j)\varphi_j$ and $\partial_2\eta_j = J_j(\partial_2\varphi_j)$, since J_j is locally constant in s and φ_j vanishes in a neighborhood of the discontinuities of J_j. Examination of the definitions of J_1 and J_2 shows that

$$\partial_1 J_1(r,s) = \begin{cases} 0 & \text{for } 0 < s < 1/2 \\ -2\pi ike(kr) & \text{for } 1/2 < s < 1, \end{cases}$$

while $\partial_1 J_2(r,s) = 0$ for $0 < s < 1$. Consequently $\partial_1\eta_2 = 0$, and since $\partial_1\eta_1 = (\partial_1 J_1)\varphi_1$ while $\partial_2\eta_1 = J_1(\partial_2\varphi_1)$, we find that

$$\begin{aligned} c_{\partial_1,\partial_2}(\Xi_k) &= 2i\int_{1/2}^{1} \mathrm{Im}((\partial_1 J_1)^{-}\varphi_1 J_1(\partial_2\varphi_1))ds \\ &= 4\pi ik\int_{1/2}^{1}(1/2)\partial_1(\varphi_1^2)ds = 2\pi ik(\varphi_1^2(1) - \varphi_1^2(1/2)) = 2\pi ik. \end{aligned}$$

We assume that L is defined in terms of the standard inner product on our Lie algebra $\mathbb{R}^2$. Since ∂_1 and ∂_2 come from elements of our Lie algebra that commute and have norm 1, we see from Theorem 10.1 that $L(p) \geq (\pi|k|)^{1/2}$. From Theorem 3.5, due to Hanfeng Li, we then obtain:

Proposition 12.1. *For every projection p representing Ξ_k (of any size) we have*

$$L(p) \geq (\pi|k|)^{1/2}.$$

This estimate perhaps is not optimal, but it does show that as $|k|$ goes to $+\infty$ the lower bound for the $L(p)$'s goes to $+\infty$, which is interesting, and consistent with what we found in the previous section.

13. Projections for monopole and induced bundles

In this section we begin to consider the complex line-bundles over the 2-sphere S^2, though most of our discussion will take place in more general contexts that can be useful in treating other examples, notably the coadjoint orbits of compact Lie groups considered in [48], as well as instantons [33]. See also section 2 of [50]. The term "monopole bundles" traditionally refers to the non-trivial complex line-bundles on S^2. Our eventual aim is to understand what should be meant by "monopole bundles" on spaces close to S^2, in analogy with the non-commutative case considered by physicists (for which see the references at the beginning of the introduction). From the action of $SO(3)$ on $\mathbb{R}^3$ we can view S^2 as $SO(3)/SO(2)$. By means of the adjoint representation of $SU(2)$ one obtains a 2-sheeted covering of $SO(3)$ by $SU(2)$, and through this covering we can also view S^2 as $SU(2)/U(1)$, where $U(1)$ is the diagonal subgroup of $SU(2)$. Set $G = SU(2)$ and $H = U(1)$. We can view

H as $\mathbb{R}/\mathbb{Z}$, and then for each $n \in \mathbb{Z}$ we can view the function $s \mapsto e(ns)$ as a character of H. With this understanding, we set

$$\Xi_n = \{\xi \in C(G) : \xi(xs) = \bar{e}(ns)\xi(x) \text{ for } x \in G,\ s \in H\}.$$

With pointwise operations Ξ_n is clearly a $C(G/H)$-module. We will see the well-known fact that it is projective, so corresponds to a complex vector bundle, and in fact to a line-bundle. For $n \neq 0$ these are the monopole bundles. By definition their "charge" is $|n|$. Clearly Ξ_n is carried into itself by the action of G on functions on G by left translation, reflecting the fact that the corresponding vector bundle is G-equivariant.

We seek projections for the Ξ_n's. The feature that we will use to obtain projections is the well-known fact that the one-dimensional representations of H occur as subrepresentations of the restrictions to H of finite-dimensional unitary representations of G. Since $H = U(1)$ is a maximal torus in $SU(2)$, the one-dimensional representations which occur on restricting a representation of G are the weights of that representation. We will treat the more general situation in which we have some compact group G, a closed subgroup H, a unitary representation V of H, and a finite-dimensional unitary representation $(U, \mathcal{K})$ of G whose restriction to H contains V as a subrepresentation. Our approach generalizes that given by Landi for the case of 2-spheres and 4-spheres in [31, 32]. (See also appendix A of [33].)

The above set-up means that there is a subspace, $\mathcal{H}$, of $\mathcal{K}$ that is carried into itself by the restriction of U to H, and such that this restricted representation of H is equivalent to V. From now on we simply let V denote this restricted representation. In the next few paragraphs we work with continuous functions, but if G is a Lie group then everything has a version involving smooth functions. Set

$$\Xi_V = \{\xi \in C(G, \mathcal{H}) : \xi(xs) = V_s^*(\xi(x)) \text{ for } x \in G,\ s \in H\}.$$

Clearly Ξ_V is a module over $A = C(G/H)$ for pointwise operations. Also, Ξ_V is clearly carried into itself by the action of G by left translation, i.e. Ξ_V is G-equivariant. (The completion of Ξ_V for the inner product determined by a G-invariant measure on G/H is the Hilbert space for the unitary representation of G induced from the representation V of H.) On Ξ_V we define an A-valued inner product by

$$\langle \xi, \eta \rangle_A(x) = \langle \xi(x), \eta(x) \rangle_{\mathcal{K}}.$$

(We take the inner product on $\mathcal{K}$, and so on $\mathcal{H}$, to be linear in the second variable.)

We want to show that Ξ_V is a projective A-module, and find a projection representing it. Let $\Omega_{\mathcal{K}} = C(G/H, \mathcal{K})$. Then $\Omega_{\mathcal{K}}$ is a free A-module in the evident way. For $\xi \in \Xi_V$ set $(\Phi\xi)(x) = U_x\xi(x)$ for $x \in G$, and notice that $(\Phi\xi)(xs) = (\Phi\xi)(x)$ for $s \in H$ and $x \in G$, so that $\Phi\xi \in \Omega_{\mathcal{K}}$. It is clear that Φ is an injective A-module homomorphism from Ξ_V into $\Omega_{\mathcal{K}}$. We show that the range of Φ is projective by exhibiting the projection onto it from $\Omega_{\mathcal{K}}$. When applied to our earlier Ξ_n it will give a projection that we can then use in later sections. Let P be the projection from $\mathcal{K}$ onto $\mathcal{H}$. Note that $U_sPU_s^* = P$ for $s \in H$ by the invariance of $\mathcal{H}$. Let $\mathcal{E}$ denote the C^*-algebra $C(G/H, \mathcal{B}(\mathcal{K}))$, where $\mathcal{B}(\mathcal{K})$ denotes the algebra of operators on $\mathcal{K}$. In the evident way $\mathcal{E} = \mathrm{End}_A(\Omega_{\mathcal{K}})$. Define p on G by

$$p(x) = U_xPU_x^*,$$

and notice that $p(xs) = p(x)$ for $s \in H$ and $x \in G$, so that $p \in \mathcal{E}$. Clearly p is a projection in $\mathcal{E}$. If $\xi \in \Xi_V$, then $p(x)(\Phi\xi)(x) = U_xPU_xU_x^*\xi(x) = (\Phi\xi)(x)$, so that $\Phi\xi$ is in the range of p. Suppose, conversely, that $F \in \Omega_{\mathcal{K}}$ and that F is in the range of p. Set $\eta_F(x) = U_x^*F(x) = U_x^*p(x)F(x) = PU_x^*F(x)$. Then the range of η_F is in $\mathcal{H}$, and we see easily that $\eta_F(xs) = U_s^*\eta_F(x)$. Thus $\eta_F \in \Xi_V$. Furthermore, $(\Phi\eta_F)(x) = F(x)$. Thus F is in the range of Φ. This shows that the range of p as a projection on $\Omega_{\mathcal{K}}$ is exactly the range of Φ. Thus this range, and so Ξ_V, are projective A-modules. Furthermore, p is a projection for Ξ_V, so we have attained our goal of finding a projection for Ξ_V.

To express p as an element of $M_n(A)$ for some n we need only choose an orthonormal basis, $\{e_j\}_{j=1}^n$, for $\mathcal{K}$, and view it as a basis (so standard module frame) for $\Omega_{\mathcal{K}}$, and express p in terms of this basis. Furthermore, if we define η_j on G by $\eta_j(x) = PU_x^*e_j$, then it is easily seen that each η_j is in Ξ_V, and that $\{\eta_j\}$ is a standard module frame for Ξ_V. The basis also gives us an isomorphism of $\mathcal{E}$ with $M_n(A)$.

We remark that there will usually be many representations of G whose restriction to H contains V as a subrepresentation, and each of these representations will give a projection for Ξ_V. From the statements of our earlier uniqueness and existence theorems we see that it is probably best to take n as small as possible, that is, to take $\mathcal{K}$ of as small dimension as possible. Also, as we will see in Section 16, there may well be projections for Ξ_V of even smaller size that do not come from our construction above.

Finally, we remark that when $\mathcal{H}$ is one-dimensional, p is very close to being a coherent state. See the discussion in section 3 of [48].

14. Metrics on homogeneous spaces

Homogeneous spaces are quotient spaces, and the metrics that we will use on them are quotients of metrics on the big space. For technical reasons we will need a strong understanding of the relation between a quotient metric and the metric on the big space from which it comes, both for ordinary metrics and for Riemannian metrics. The definition of (ordinary) quotient metrics is somewhat complicated. A nice exposition is given in [65], where three equivalent definitions of quotient metrics are considered. One of these definitions fits especially well with our methods, and it goes as follows. Let (Z, ρ) be a metric space, and let $\sim$ be an equivalence relation on Z. Let $A = C(Z)$, and define a closed subalgebra of A by

$$A/\sim \; = \; \{f \in C(Z) : f(z) = f(w) \text{ if } z \sim w\}.$$

(The functions in $C(Z)$ can be either real-valued or complex-valued.) Let L^ρ be the Lipschitz seminorm on $C(Z)$. Define a pseudometric, $\tilde{\rho}$, on Z by

$$\tilde{\rho}(z, w) = \sup\{|f(z) - f(w)| : f \in A/\sim \text{ and } L^\rho(f) \leq 1\}.$$

Note that if $z \sim w$ then $\tilde{\rho}(z, w) = 0$. Thus $\tilde{\rho}$ drops to a pseudometric on the quotient space $Z/\sim$. But even if Z is compact and all the equivalence classes are closed (so that $Z/\sim$ with the quotient topology is Hausdorff compact) $\tilde{\rho}$ may still fail to be a metric, and the quotient metric space is obtained by identifying points which are at distance 0 for $\tilde{\rho}$. However (see 1.16_+ of [20]),

Proposition 14.1. *Let (Z, ρ) be a compact metric space, and let G be a compact group with an action α on Z, so that the quotient space Z/α is compact and Hausdorff for the quotient topology. If the action α is by isometries, then the corresponding pseudometric, $\tilde{\rho}$, on Z/α is in fact a metric, giving the quotient topology. Let $\pi : Z \to Z/\alpha$ be the quotient map. Then for any $f \in C(Z/\alpha)$ we have*

$$L^{\tilde{\rho}}(f) = L^\rho(f \circ \pi).$$

Proof. Let $\mathcal{O}_z$ be the α-orbit of a point $z \in Z$, and define f_z by $f_z(w) = \rho(w, \mathcal{O}_z)$, for the evident meaning. Then from the fact that α is an action by isometries it is easily seen that f_z is α-invariant and so constant on orbits. Furthermore it separates $\mathcal{O}_z$ from any other orbit, and $L^\rho(f_z) \leq 1$. From this it follows easily that $\tilde{\rho}$ is a metric which gives the quotient topology. By composing with π it is natural to view $C(Z/\alpha)$ as a (closed $*$-) subalgebra of $C(Z)$, and then it consists of all of the functions in $C(Z)$ which are constant on all orbits. For any

$z \in Z$ and any $f \in C(Z/\alpha)$ we have $f(\pi(z)) = f(\mathcal{O}_z)$, and from this it follows readily that $L^{\tilde{\rho}}(f) = L^{\rho}(f \circ \pi)$. $\square$

By applying linear functionals we quickly obtain:

Corollary 14.2. *Let (Z, ρ), α, and $(X/\alpha, \tilde{\rho})$ be as above. For any Banach space $\mathcal{V}$ and any function F from X/α to $\mathcal{V}$ we have*

$$L^{\tilde{\rho}}(F) = L^{\rho}(F \circ \pi).$$

We now consider homogeneous spaces and Riemannian metrics. Let G be a compact connected semisimple Lie group, with Lie algebra $\mathfrak{g}$. On $\mathfrak{g}$ we choose an Ad-invariant inner product, for example the negative of the Killing form. We denote this inner product by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. By translation this inner product defines a bi-invariant Riemannian metric on G, with corresponding ordinary metric ρ.

Let $\mathcal{V}$ be a Banach space, over $\mathbb{R}$ or $\mathbb{C}$, and let F be a smooth function from G into $\mathcal{V}$. Let DF denote the total derivative of F for left translation, so that at each point $x \in G$ we can view D_xF as a real-linear operator from $\mathfrak{g}$ into $\mathcal{V}$, defined by

$$(D_xF)(X) \ = \ (d/dt)|_{t=0}F(\exp(-tX)x).$$

Simple arguments similar to those given in the proofs of theorem 3.1 of [44] and lemma 3.1 of [46] show that

$$L^{\rho}(F) = \|DF\|_{\infty} = \sup\{\|D_xF\| : x \in G\},$$

where on $\mathfrak{g}$ we use the norm from its inner product. (Our Proposition 2.5 is also helpful here.) Similar considerations appeared already early in Section 11.

Suppose now that H is a closed subgroup of G, with Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$. Let G/H be the corresponding homogeneous space. We can view functions in $C(G/H)$ as H-invariant functions on G. Now G/H inherits a structure of differentiable manifold [64] and we need to relate the tangent space at points of G/H to those at points of G. For this it is convenient to use an explicit description of the projective module of smooth cross-sections of the tangent bundle of G/H, along the lines given in [59, 16, 4, 50]. This fits very well into the approach we have given earlier concerning projective modules for vector bundles. Considerable guidance in treating the tangent bundle of G/H can also be found, for example, in the proof of proposition 3.16 of [9] and its surrounding discussion. (A substantial part of our discussion can easily be generalized to the more general setting of chapter 3 of [9], but that would take us beyond compact groups and spaces.)

We begin by letting $\mathfrak{m}$ denote the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$. Then $\mathfrak{m}$ is invariant under Ad_s for $s \in H$, and $\mathfrak{m}$ can be identified with the tangent space at $eH \in G/H$. At xH each element of $\mathrm{Ad}_x(\mathfrak{h})$ acts as the zero derivation, and the tangent space at xH can be identified with $\mathrm{Ad}_x(\mathfrak{m})$. But one must be careful with this, since the function $\mathrm{Ad}_{xs}(Y)$ for $Y \in \mathfrak{m}$ will not in general be constant in s, hence the specific identification of $\mathrm{Ad}_x(\mathfrak{m})$ with the tangent space at xH depends on the choice of the coset representative x. This can be handled by the following description of the module $\mathcal{T}(G/H)$ of smooth cross-sections of the tangent bundle of G/H:

Proposition 14.3. *The cross-section module $\mathcal{T}(G/H)$ has a natural realization as*

$$\{Z \in C^\infty(G, \mathfrak{g}) : Z(x) \in \mathrm{Ad}_x(\mathfrak{m}) \text{ and } Z(xs) = Z(x) \text{ for } x \in G, s \in H\}.$$

Given $F \in C^\infty(G/H, \mathcal{V})$ for some Banach space $\mathcal{V}$, the action of $Z \in \mathcal{T}(G/H)$ on F is given by

$$(ZF)(x) = (d/dt)|_{t=0} F(\exp(-tZ(x))x).$$

Equivalently, the cross-section module could be given by

$$\{W \in C^\infty(G, \mathfrak{m}) : W(xs) = \mathrm{Ad}_s^{-1}(W(x)) \text{ for all } x \in G, s \in H\},$$

with the action of W on F given by

$$(WF)(x) = (d/dt)|_{t=0} F(x \exp(-tZ(x))).$$

Proof. It is clear that $\mathcal{T}(G/H)$ is a $C^\infty_{\mathbb{R}}(G/H)$-module for pointwise operations. Let Q denote the orthogonal projection of $\mathfrak{g}$ onto $\mathfrak{m}$. Then $Q\mathrm{Ad}_s = \mathrm{Ad}_s Q$ for each $s \in H$. Each $X \in \mathfrak{g}$ determines an element $\tilde{X}$ of $\mathcal{T}(G/H)$, defined by

$$\tilde{X}(x) = \mathrm{Ad}_x(Q(\mathrm{Ad}_{x^{-1}} X))$$

for $x \in G$. Note that there may well be points x where $\mathrm{Ad}_{x^{-1}} X \in \mathfrak{h}$, so that $\tilde{X}(x) = 0$. (We are feeling the effect here of the fact that G/H may well not be parallelizable.) However, we see that for any $X \in \mathrm{Ad}_x(\mathfrak{m})$ we have $\tilde{X}(x) = X$, so that the evaluations at x of elements of $\mathcal{T}(G/H)$ fill the whole tangent space at xH. From this and the fact that $\mathcal{T}(G/H)$ is a $C^\infty_{\mathbb{R}}(G/H)$-module it follows that $\mathcal{T}(G/H)$ does represent the full space of smooth cross-sections of the tangent bundle for G/H.

When we use the fact that

$$\exp((Z(y))x = x \exp(\mathrm{Ad}_x^{-1}(Z(y))$$

we obtain the second description of the cross-section module. ∎

We will use the first description of $\mathcal{T}(G/H)$ given by the above proposition. We define a $C^\infty_{\mathbb{R}}(G/H)$-valued inner product on $\mathcal{T}(G/H)$ by

$$\langle Z, W \rangle_{G/H}(x) = \langle Z(x), W(x) \rangle_{\mathfrak{g}}.$$

Calculations very similar to those in the previous section show that if $\{X_j\}$ is an orthonormal basis for $\mathfrak{g}$, then $\{\tilde{X}_j\}$ is a standard module frame for $\mathcal{T}(G/H)$. (Here, of course, we are working over $\mathbb{R}$.) Furthermore, the $C^\infty_{\mathbb{R}}(G/H)$-valued inner product gives the Riemannian metric on G/H which is induced from that on G as discussed in chapter 3 of [9].

The Riemannian metric on G/H will define an ordinary metric $\hat{\rho}$ on G/H. Given a smooth function F on G/H with values in some Banach space, its Lipschitz constant, $L^{\hat{\rho}}(F)$, for the ordinary metric $\hat{\rho}$ on G/H is defined. But, for the kinds of reasons discussed above for ρ, we also have

$$L^{\hat{\rho}}(F) = \sup\{\|ZF\|_\infty : Z \in \mathcal{T}(G/H) \text{ and } \langle Z, Z\rangle_{G/H} \leq 1\}.$$

However, we can also view F as a function on G, and calculate its Lipschitz constant $L^\rho(F)$ there. As seen above, we have

$$L^\rho(F) = \sup\{\|XF\|_\infty : X \in \mathfrak{g},\ \|X\| \leq 1\},$$

where the elements of $\mathfrak{g}$ are viewed as right-invariant vector fields. The relation that we need for later calculations is:

Proposition 14.4. *Let $F \in C^\infty(G/H, \mathcal{V})$ for some Banach space $\mathcal{V}$, and view F also as a function on G. Then*

$$L^{\hat{\rho}}(F) = \sup\{\|XF\|_\infty : X \in \mathfrak{g},\ \|X\| \leq 1\}.$$

Consequently $L^{\hat{\rho}}(F) = L^\rho(F)$, and $\hat{\rho}$ is the quotient metric from ρ.

Proof. Let $Z \in \mathcal{T}(G/H)$ with $\langle Z, Z\rangle_{G/H} \leq 1$, so that $\|Z_x\| \leq 1$ for all $x \in G$. From Proposition 14.3 we see that $\|(ZF)(x)\| = \|Z_x F\|$, and from this it is evident that $L^{\hat{\rho}}(F) \leq L^\rho(F)$. On the other hand, let $x \in G$ be given, and let $X \in \mathrm{Ad}_x(\mathfrak{m})$ with $\|X\| \leq 1$. Define $\tilde{X} \in \mathcal{T}(G/H)$ as done earlier, and recall that $\tilde{X}(x) = X$. It follows that $\|\tilde{X}F\|_\infty \geq \|(XF)(x)\|$. Notice from the definition of $\tilde{X}$ that $\langle \tilde{X}, \tilde{X}\rangle_{G/H} \leq 1$. It now follows that $L^\rho(F) \leq L^{\hat{\rho}}(F)$, so that they are equal. If we apply this for $F \in C^\infty(G/H)$, we see that $\hat{\rho}$ is the quotient of ρ. $\square$

15. Lipschitz constants of induced bundles

We now combine the results of Sections 13 and 14 to obtain a formula for the Lipschitz constants of the projections obtained in Section 13 when our setting is compact semisimple Lie groups as in Section 14. Thus we let G, $\mathfrak{g}$, H, etc., be as in Section 14, and we now assume that we have a unitary representation $(U, \mathcal{K})$ of G and an H-invariant subspace $\mathcal{H}$ of $\mathcal{K}$, with the restriction of U to H and $\mathcal{H}$ denoted by V.

As before we let $p(x) = U_xPU_x^*$, where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. Now finite-dimensional representations of a Lie group are smooth, and so p is a smooth function. We let Dp denote its total derivative. From the previous section we see that $L(p) = \|Dp\|_\infty$. Now a simple calculation shows that for any $x \in G$ and $X \in \mathfrak{g}$ we have

$$(Dp)_x(X) = [U_xPU_x^*,\ U_X]$$

where U_X refers to the infinitesimal version of U as a representation of $\mathfrak{g}$ on $\mathcal{K}$. Then

$$\|(Dp)_x(X)\| = \|U_x[P, U_x^*U_XU_x]U_x^*\| = \|[P, U_{\mathrm{Ad}_{x^{-1}}(X)}]\|.$$

Since we have chosen the inner product on $\mathfrak{g}$ so that it is preserved by Ad, it follows that

$$\|(Dp)_x\| = \sup\{\|[P, U_X]\| : \|X\|_{\mathfrak{g}} \le 1\}.$$

Since the right-hand side is independent of x, we see that it also equals $\|Dp\|_\infty$. But each U_X is skew adjoint, and so

$$[P, U_X] = PU_X - U_XP = PU_X + (PU_X)^* = 2\mathrm{Re}(PU_X).$$

When we combine this with Proposition 14.4 we obtain:

Proposition 15.1. *With notation as above,*

$$L_{G/H}(p) = \|Dp\|_\infty = 2\sup\{\|\mathrm{Re}(PU_X)\| : X \in \mathfrak{g},\ \|X\| \le 1\},$$

where $L_{G/H}$ denotes the Lipschitz seminorm for the metric $\hat{\rho}$ on G/H.

Suppose now that our representation V of H is one-dimensional, so that Ξ_V is the space of cross-sections for a line bundle. Let v_0 be a unit vector in the Hilbert subspace $\mathcal{H} \subset \mathcal{K}$ for V, so that $P = \langle v_0, v_0\rangle_0$, the rank-1 operator determined by v_0. Notice that since U_X is a skew-adjoint operator, $\mathrm{Re}(\langle v_0, U_Xv_0\rangle_{\mathcal{H}}) = 0$. Then on combining Proposition 15.1 with Lemma 11.1 we obtain:

Proposition 15.2. *With notation as at the beginning of this section, but with $\mathcal{H}$ of dimension 1, spanned by the unit-vector v_0, we have*

$$L_{G/H}(p) = \sup\{\|(I - P)U_Xv_0\| : X \in \mathfrak{g},\ \|X\| \le 1\}.$$

For use in the next section we also need to consider tensor powers of one-dimensional representations. For $(U, \mathcal{K})$, v_0, $(V, \mathcal{H})$, P and p as above and for a given n let $\mathcal{K}^{\otimes n}$ be the n-fold full tensor power of $\mathcal{K}$, with $U^{\otimes n}$ the corresponding representation of G. Let v_0^n be the n-fold tensor power of v_0, and let $\mathcal{K}_n$ be the G-invariant subspace of $\mathcal{K}^{\otimes n}$ generated by v_0^n. Let P_n be the projection of $\mathcal{K}_n$ onto the one-dimensional subspace, $\mathcal{H}_n$, spanned by v_0^n, which is H-invariant. Let $U^{(n)}$ be the restriction of $U^{\otimes n}$ to $\mathcal{K}_n$, and let V^n be the restriction of

$U^{\otimes n}$ to a representation of H on $\mathcal{H}_n$. Finally, let $p_n \in C(G/H, B(\mathcal{K}_n))$ be defined by $p_n(x) = U_x^{(n)} P_n (U_x^{(n)})^*$, so that we are in a version of the setting discussed above. Then

$$L(p_n) = \sup\{\|(I - P_n)U_X^{(n)} v_0^n\| : X \in \mathfrak{g},\ \|X\| \leq 1\}.$$

But

$$U_X^{(n)} v_0^n = (U_X v_0) \otimes v_0 \cdots \otimes v_0 \quad + \quad v_0 \otimes (U_X v_0) \otimes v_0 \dots v_0 \quad + \quad \dots$$

so that

$$\langle v_0^n, U_X^{(n)} v_0^n \rangle \ = \ n\langle v_0, U_X v_0\rangle.$$

Let $w_X \ = \ U_X v_0 - v_0 \langle v_0, U_X v_0\rangle_{\mathcal{H}}$. Then

$$\begin{aligned} U_X^{(n)} v_0^n \quad - \quad v_0^n \langle v_0^n, U_X^{(n)} v_0^n \rangle \ &= \ U_X^{(n)} v_0^n - n v_0^n \langle v_0, U_X v_0\rangle_{\mathcal{H}} \\ &= \ w_X \otimes v_0 \otimes \cdots \otimes v_0 \ + \ v_0 \otimes w_X \otimes v_0 \otimes \cdots \otimes v_0 \ + \ \dots \end{aligned}$$

Thus, since $w_X \perp v_0$, we find that

$$\|U_X^{(n)} v_0^n \ - \ v_0^n \langle v_0^n, U_X^{(n)} v_0^n\rangle\|^2 \ = \ n\langle w_X, w_X\rangle.$$

Combining this with Lemma 11.1, we obtain:

Proposition 15.3. *With notation as above,*

$$L(p_n) = n^{1/2} L(p).$$

16. The sphere, $SU(2)$, and monopole and induced bundles

To treat the two-sphere S^2 we view it as G/H where $G = SU(2)$ and H is the diagonal subgroup $\begin{pmatrix} e(t) & 0 \\ 0 & \bar{e}(t) \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$, as discussed at the beginning of Section 13. For each $n \in \mathbb{Z}$ we consider the representation $t \mapsto e(nt)$ of $H \cong \mathbb{R}/\mathbb{Z}$, and the corresponding induced A-module Ξ_n, where $A = C(S^2)$. We begin by considering the case $n = 1$. The corresponding representation occurs in the restriction to H of the standard representation of $SU(2)$ on $\mathbb{C}^2$. As v_0 we can take the lowest-weight vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then the projection onto its span is $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. For the corresponding projection p_1 in $M_2(A)$ we have, by Proposition 15.2,

$$L(p_1) = \sup\{\|(I - P)U_X v_0\| : X \in \mathfrak{g},\ \|X\| \leq 1\}.$$

A general element of $\mathfrak{g} = su(2)$ will be of the form $X = \begin{pmatrix} ir & -\bar{w} \\ w & -ir \end{pmatrix}$ where $r \in \mathbb{R}$ and $w \in \mathbb{C}$. Then $\|(I - P)U_X v_0\| = \left\| \begin{pmatrix} -\bar{w} \\ 0 \end{pmatrix} \right\| = |w|$. We choose our normalization of the Ad-invariant inner product on $su(2)$ to be $\langle X, Y\rangle_{\mathfrak{g}} = 2^{-1}\mathrm{tr}(XY^*)$, so that for X as above we have $\|X\| = (r^2 + |w|^2)^{1/2}$. For this choice we see that $L(p_1) \leq 1$. But one choice of X is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and from this choice we see that $L(p_1) = 1$. If instead we had chosen as v_0 the highest-weight vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we would be dealing with the case $n = -1$, and so Ξ_{-1}. All of the above discussion applies with almost no change to this case too. We have thus obtained:

Proposition 16.1. *Let p be the projection in $M_2(C^\infty(S^2))$ constructed above for either of the monopole bundles Ξ_1 and Ξ_{-1}. For the Riemannian metric on S^2 coming from the choice of normalization of the inner product on $su(2)$ made above, we have $L(p) = 1$.*

Let us now apply the results of Sections 4 and 6. We could do this in the same way as we did near the end of Section 11 for the two torus. But for the sake of variety, let us take a slightly different approach, involving specific choices of numbers, and more focused on our monopole projection above while less focused on homotopies. So let p_* denote our projection above for Ξ_1 or Ξ_{-1}. For the moment let us denote S^2 by X and let ρ_X be its round metric corresponding to our choice above of an Ad-invariant inner product on $su(2)$.

Choose $\varepsilon < 1/60$, so that $12\varepsilon\lambda_2 L(p_*) < 2/3$, since $\lambda_2 \leq 10/3$ by equation 5.3. Let (Y, ρ_Y) be another compact metric space, and suppose that we have a metric ρ on $X \dot{\cup} Y$ that restricts to ρ_X and ρ_Y, and for which $\mathrm{dist}^\rho_H(X, Y) < \varepsilon$. Then according to Corollary 6.3 there is a projection $q_1 \in M_2(C(Y))$ such that

$$L(p_* \oplus q_1) < \lambda_2 L(p_*)(1 - (2/3))^{-1} \leq (10/3)(3/1) = 10.$$

Then $\varepsilon L(p_* \oplus q_1) < 1/6 < 1/2$, and so from Theorem 4.5 we find that if q_0 is any other projection in $M_2(C(Y))$ such that $L(p_* \oplus q_0) < 10$ then there is a path q of projections in $M_2(C(Y))$ going from q_0 to q_1 such that

$$L(p_* \oplus q_t) < (1 - 20/60)^{-1} 10 = 15.$$

In particular q_0 and q_1 (and all the q_t's) will determine isomorphic vector bundles on Y, and so it is reasonable to consider the corresponding isomorphism class of bundles on Y to be the "monopole" bundle on Y for the given proximity of Y to X, although the statement in terms of projections is more precise.

We now consider Ξ_n for $|n| \geq 2$. We carry out the discussion for $n \geq 2$, but a very parallel discussion works for $n \leq -2$. We seek to apply

Proposition 15.3. Let $v_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2 = \mathcal{H}$, and let U be the standard representation of $SU(2)$ on $\mathcal{K} = \mathbb{C}^2$ as above. Form $U^{\otimes n}$ and $\mathcal{K}^{\otimes n}$ as done before Proposition 15.3, and then form v_0^n, $\mathcal{H}_n$ and $U^{(n)}$. Because v_0 is a lowest weight vector, it is not difficult to see that $U^{(n)}$ on $\mathcal{K}_n$ is the $(n+1)$-dimensional irreducible representation of $SU(2)$ and that v_0^n is a lowest-weight vector for it. (See for example proposition VII.2 of [57] for a more general context.) We clearly have $U_s^{(n)} v_0^n = \bar{e}(ns) v_0^n$ for $s \in H$, so the restriction of $U^{(n)}$ to H and to the one-dimensional subspace, $\mathcal{H}_n$, spanned by v_0^n is the representation of H which defines Ξ_n. Thus if, as before, we let P_n denote the projection from $\mathcal{K}_n$ onto $\mathcal{H}_n$, and if we set $p_n(x) = U_x^{(n)} P_n (U_x^{(n)})^*$, then p_n is a projection in $C(G/H, \mathcal{B}(\mathcal{K}_n))$ which represents Ξ_n. From Proposition 15.3, applied also for $n \leq -2$ (and also from Proposition 16.1 for $n = \pm 1$, and for p_0 a constant projection for the free rank-1 projective module Ξ_0 over G/H), we obtain:

Theorem 16.2. *For any $n \in Z$ and for the projection p_n defined above for Ξ_n we have*
$$L(p_n) = |n|^{1/2}.$$

We notice however that since $\mathcal{K}_n$ has dimension $|n|+1$, the standard module frame for Ξ_n corresponding to p_n and a basis for $\mathcal{K}_n$ will have $|n|+1$ elements. This corresponds to the fact that in the present setting Ξ_n is embedded in a free module of rank $|n|+1$. But any complex line-bundle on a compact space of dimension 2 can be obtained as a summand of a rank-2 trivial bundle. (See, for example, proposition 1.1 and theorem 1.5 of chapter 8 of [24].) So we should be able to obtain Ξ_n as a summand of a free module of rank 2, with a frame consisting of just two elements and a corresponding projection in $M_2(A)$. We can indeed do this, as follows.

Fix n, and let ζ_1 and ζ_2 be the functions on $SU(2)$ defined by $\zeta_1(x) = \bar{z}_1^n$ and $\zeta_2(x) = \bar{z}_2^n$ respectively when $n > 0$, and by their complex conjugates when $n < 0$, where much as before $x = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$ with $|z_1|^2 + |z_2|^2 = 1$. It is clear that ζ_1 and ζ_2 are in Ξ_n. From now on we assume that $n > 0$, but entirely parallel considerations apply for $n < 0$. Let $h \in A$ be defined by $h(x) = (|z_1|^{2n} + |z_2|^{2n})^{-1/2}$, and set $\eta_j = \zeta_j h$ for $j = 1, 2$, so that $\langle \eta_1, \eta_1 \rangle_A + \langle \eta_2, \eta_2 \rangle_A = 1$. Since Ξ_n corresponds to a line bundle, $\mathrm{End}_A(\Xi_n) = A$, and so this last relation is basically the reconstruction formula of Definition 7.1. Thus $\{\eta_1, \eta_2\}$ is a standard module frame for Ξ_n, and this frame provides a projection, p, in $M_2(A)$

for Ξ_n. Specifically, $p_{jk} = \langle \eta_j, \eta_k \rangle_A = \bar{\eta}_j \eta_k$, so that

$$p_{jk}(x) = z_j^n \bar{z}_k^n / (|z_1|^{2n} + |z_2|^{2n})$$

for $j, k = 1, 2$. If we define u on $SU(2)$ by $u(x) = h(x) \begin{pmatrix} z_1^n \\ z_2^n \end{pmatrix}$, then $p(x) = \langle u(x), u(x) \rangle_0$, the rank-1 operator on $\mathcal{K} = \mathbb{C}^2$ determined by the unit-vector $u(x)$. (Recall that we take the inner product on $\mathcal{K}$ to be linear in the second variable.) Note that u is not constant on cosets of H, but that p is.

We can now try to calculate the Lipschitz constant of p, first as a function on G and then on G/H by using Proposition 14.4 much as before. It is clear that p and u are smooth. Thus we can use derivatives to calculate $L(p)$. The advantage of working on G is that while u is not a function on G/H, we can make use of its derivatives on G to calculate those of p.

Suppose now that u is any smooth function from G into a Hilbert space $\mathcal{K}$ such that $\langle u(x), u(x) \rangle_{\mathcal{K}} = 1$ for all x, and define p by $p(x) = \langle u(x), u(x) \rangle_0$. Then for any $X \in \mathfrak{g}$ and $x \in G$ we have, much as earlier,

$$(Xp)(x) = \langle (Xu)(x), u(x) \rangle_0 + \langle u(x), (Xu)(x) \rangle_0.$$

Because each $u(x)$ is a unit-vector we see that $\mathrm{Re}(\langle (Xu)(x), u(x) \rangle_{\mathcal{K}}) = 0$. For any fixed $x \in G$ we can then apply Lemma 11.1 to conclude that

$$\|(Xp)(x)\| = \|(Xu)(x) - u(x)\langle u(x), (Xu)(x) \rangle\| = \|(I_{\mathcal{K}} - p(x))((Xu)(x))\|.$$

Consequently, much as just after Lemma 11.1, we obtain:

Proposition 16.3. *Let u be a smooth function from G to a Hilbert space $\mathcal{K}$ such that $\langle u(x), u(x) \rangle_{\mathcal{K}} = 1$ for all $x \in G$. Let p be defined by $p(x) = \langle u(x), u(x) \rangle_0$. Then*

$$L(p) = \sup\{\|(I_{\mathcal{K}} - p(x))((Xu)(x))\| : x \in G, \ \|X\| \le 1\}.$$

When the dimension of $\mathcal{K}$ is 2, the projection $I_{\mathcal{K}} - p(x)$ will be of rank 1, and so is given by a unit vector, $v(x)$, orthogonal to $u(x)$. Then

$$\begin{aligned} \|(I_{\mathcal{K}} - p(x))((Xu)(x))\| &= \|\langle v(x), v(x) \rangle_0((Xu)(x))\| \\ &= \|v(x)\langle v(x), (Xu)(x) \rangle_{\mathcal{K}}\| = |\langle v(x), (Xu)(x) \rangle_{\mathcal{K}}|. \end{aligned}$$

For $u(x) = h(x)(\bar{\zeta}_1(x), \bar{\zeta}_2(x))'$ as before, where the prime denotes transpose, we can choose $v(x)$ to be defined by $v(x) = h(x)(\zeta_2(x), -\zeta_1(x))'$. Now

$$(Xu)(x) = (Xh)(x)u(x)(h(x))^{-1} \ + \ h(x)((X\bar{\zeta}_1)(x), (X\bar{\zeta}_2)(x))',$$

so that, since $v(x) \perp u(x)$,

$$\begin{aligned} \langle v(x), (Xu)(x) \rangle_{\mathcal{K}} &= \langle v(x), h(x)((X\bar{\zeta}_1)(x), (X\bar{\zeta}_2)(x))' \rangle_{\mathcal{K}} \\ &= h^2(x)(\zeta_2(x)(X\zeta_1)(x) - \zeta_1(x)(X\zeta_2)(x)). \end{aligned}$$

Since ζ_1 and ζ_2 are defined for all $(z_1, z_2) \in \mathbb{C}^2$, and as functions on G they are determined by the first column of x, we can calculate $(X\zeta_j)(x)$ by means of the straight-line path $I - tX$ instead of the path $\exp(-tX)$, for $t \in \mathbb{R}$. For $X = \begin{pmatrix} ir & -\bar{w} \\ w & -ir \end{pmatrix}$ as before,

$$\zeta_1((I - tX)x) = (\bar{z}_1 + t(ir\bar{z}_1 + w\bar{z}_2))^n.$$

Taking the derivative at $t = 0$, we obtain

$$(X\zeta_1)(x) = n\bar{z}_1^{n-1}(ir\bar{z}_1 + w\bar{z}_2).$$

In the same way

$$(X\zeta_2)(x) = -n\bar{z}_2^{n-1}(\bar{w}\bar{z}_1 + ir\bar{z}_2).$$

Thus

$$\langle v(x), (Xu)(x) \rangle_{\mathcal{K}} \;=\; h^2(x)n(ir2\bar{z}_1^n\bar{z}_2^n \;+\; w\bar{z}_2^{n+1}\bar{z}_1^{n-1} + \bar{w}\bar{z}_1^{n+1}\bar{z}_2^{n-1}).$$

To obtain a lower bound for $L(p)$ we evaluate the absolute value of this expression at $x_* = (2^{-1/2}, 2^{-1/2})$ and with $r = 1$, $w = 0$. We obtain n. Thus $L(p) \geq n$. On the other hand, just taking absolute values, and using the fact that $|z_1|^2 + |z_2|^2 = 1$, we find that for all $x \in G$

$$|\langle v(x), (Xu)(x) \rangle_{\mathcal{K}}| \leq nh^2(x)(r2|z_1z_2|^n + |w||z_1z_2|^{n-1}).$$

The maximum of the right-hand side for $r^2 + |w|^2 = 1$ is

$$nh^2(x)(4|z_1z_2|^{2n} + |z_1z_2|^{2(n-1)})^{1/2}.$$

Again using $|z_1|^2 + |z_2|^2 = 1$ we see easily that the maximum value of $h^2(x)$ is 2^{n-1} while the maximum value of $|z_1z_2|$ is $1/2$, and from this we obtain $L(p) \leq n\sqrt{2}$. We summarize our findings with:

Proposition 16.4. *For u defined by $u(x) = h(x)(z_1^n, z_2^n)'$ for $n > 0$, or $u(x) = h(x)(\bar{z}_1^n, \bar{z}_2^n)'$ for $n < 0$, and for p the corresponding projection-valued function, we have*

$$|n| \leq L(p) \leq |n|\sqrt{2}.$$

It is interesting to notice that as soon as $|n| \geq 2$ the Lipschitz constant for these projections, coming from standard module frames with only two elements, is strictly larger, and more rapidly increasing with n, then the projections considered in Theorem 16.2, which come from standard module frames with $n + 1$ elements. It is reasonable to guess that these projections have minimal Lipschitz constant for their two situations, but I have not examined this matter.

The results of Sections 4 and 6 can now be applied in the same way as discussed early in this section for the particular case of $|n| = 1$.

17. Appendix A: Path-length spaces

The purpose of this appendix is to prove:

Proposition 17.1. *Let (X, ρ) be a compact metric space which is a path-length space, and let $\mathcal{H}$ be a real Hilbert space. Let $u : X \to \mathcal{H}$ be a continuous function such that $\|u(x)\| = 1$ for all $x \in X$, and define the projection-valued function p by $p(x) = \langle u(x),\ u(x)\rangle_0$. Then $L(p) = L(u)$.*

Proof. Let v and $w \in \mathcal{H}$ with $\|v\| = 1 = \|w\|$ and $\langle v,\ w\rangle_{\mathcal{H}} > 0$. Since $\mathcal{H}$ is over $\mathbb{R}$,

$$\langle v,\ w\rangle_{\mathcal{H}} = 1 - (1/2)\|v - w\|^2,$$

so that from Proposition 8.2

$$\|\langle v,\ v\rangle_0 - \langle w,\ w\rangle_0\| = \|v - w\|(1 - (1/4)\|v - w\|^2)^{1/2}.$$

Thus, since u is uniformly continuous, given any $\varepsilon > 0$ we can find $\delta > 0$ small enough that whenever $\rho(x, y) < \delta$ then $\|u(x) - u(y)\| \le (1 + \varepsilon)\|p(x) - p(y)\| \le (1 + \varepsilon)L(p)\rho(x, y)$. Thus we need the following lemma, which is essentially known — see 1.8bis of [20].

Lemma 17.2. *Let (X, ρ) be a path-length metric space, let B be a Banach space, and let f be a B-valued function on X. Suppose that there are constants K and $d > 0$ such that for any $x, y \in X$ with $\rho(x, y) < d$ we have $\|f(x) - f(y)\| \le K\rho(x, y)$. Then $L^\rho(f) \le K$.*

Proof. Let $x, y \in X$ and $\varepsilon > 0$ be given, with $\varepsilon < d$. Let γ be a path in X from x to y, with domain $I = [0,\ t_*]$, whose length is $\le (1 + \varepsilon/K)\rho(x, y)$. Then γ is uniformly continuous, so there is a $\delta > 0$ such that if $s, t \in I$ with $|s - t| < \delta$ then $\rho(\gamma(s),\ \gamma(t)) < \varepsilon < d$. Choose $\{t_j\}_{j=0}^m$ in I such that $t_0 = 0,\ t_m = t_*,\ t_{j+1} > t_j$, and $|t_{j+1} - t_j| < \delta$ for all j. Then

$$\begin{aligned}
\|f(y) - f(x)\| &\le \sum_{j=0}^{m-1} \|f(\gamma(t_{j+1})) - f(\gamma(t_j))\| \\
&\le K \sum \rho(\gamma(t_{j+1}),\ \gamma(t_j)) \\
&\le K(1 + \varepsilon/K)\rho(x,\ y) = (K + \varepsilon)\rho(y,\ x).
\end{aligned}$$

Since ε is arbitrary, we obtain $\|f(x) - f(y)\| \le K\rho(x, y)$, as desired. (Actually, we only need that B be a metric space for the above proof to work.) $\square$

$\square$

The above proposition and lemma are false if the path-length assumption is omitted. An example pertinent to our earlier constructions of vector bundles can be produced as follows. In $\mathbb{R}^2$ consider an equilateral triangle with base the unit interval $I = [0,\ 1]$ on the x-axis of $\mathbb{R}^2$. Let X be obtained from the triangle by removing a very small open ball about the vertex opposite to I. Thus X is homeomorphic to a closed interval. Let ρ be the restriction to X of the Euclidean metric on $\mathbb{R}^2$, rather than the evident path-length metric on X. Let $u : X \to \mathbb{R}^2$ be defined by $u(t, 0) = (\cos(\pi t),\ \sin(\pi t))$ on I, and by taking u to be continuous and constant on each of the two "legs" of X. Thus on one leg u has value $(1, 0)$ while on the other it has value $(-1, 0)$. Since the two end-points of X are very close together, $L^\rho(u)$ is very large. But one can find a very small d for which the hypotheses of the lemma are satisfied with a smallish K. On the other hand, when we set $p(x) = \langle u(x), u(x)\rangle_0$, then p has the same value on the two legs of X, and $L(p) < L(u)$.

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