

Theorem 2.1] such that  $G := F^{-1}D_c$  is a gcd of  $(\bar{D}_s\bar{D}_wN_p, D_c)$ , implying

$$\bar{D}_s\bar{D}_wN_p = F_1G, \quad D_c = FG \tag{34}$$

with  $F, F_1, G$  matrices with elements in  $\hat{\mathcal{Q}}_-$  and

$$\det F, \det G \in \hat{\mathcal{Q}}_-^\infty. \tag{35}$$

Then by the equations above

$$\begin{bmatrix} A_2 \\ E_2 \end{bmatrix} = \begin{bmatrix} F & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ F_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G & 0 & 0 & 0 \\ U_2\bar{D}_wD_p & G_s & 0 & 0 \\ U_1N_p & -U_1D_p & G_w & 0 \\ 0 & 0 & 0 & I \end{bmatrix} =: \begin{bmatrix} \bar{A}_2 \\ \bar{E}_2 \end{bmatrix} G_2 \tag{36}$$

where

$$(\bar{E}_2, \bar{A}_2) \text{ is r.c.} \tag{37}$$

Hence,  $G_2$  is a gcd of  $(E_2, A_2)$  with

$$\det G_2 = \det G \det G_s \det G_w. \tag{38}$$

Now, by Lemma 5 with (28), (11), (32), (29), and (30), by (18), and by Lemma 1, any gcd of  $(E_2, A_2)$  must be such that its determinant is equivalent to  $\det D_s \det D_w$ . Hence, from (38), (21), and (25),  $\det G \sim \det \bar{D}_s \det \bar{D}_w$ . So, from the arguments above

$$\left. \begin{matrix} \epsilon \in \mathcal{Q}^n & \forall u_0 \in \mathbb{C}^n, \forall v_0 \in \mathbb{C}^m \\ \text{loop } \hat{\mathcal{Q}}\text{-stability} \end{matrix} \right\} = (\bar{D}_s\bar{D}_wN_p, D_c)$$

admits a gcd  $G \in \hat{\mathcal{Q}}^{m \times m}$  such that  $\det G \sim \det \bar{D}_s \det \bar{D}_w$ . (39)

Hence, the hypothesis of (39) implies also that the conditions of Theorem 1 are necessary.

Notice also from (14) and (34) that  $(N_p, G)$  has to be r.c., a sideline result.

*Sufficiency:* Assume a compensator  $\hat{C} \in \hat{\mathcal{Q}}^{m \times n}$  exists with l.r.  $(D_c, N_c)$  as in the theorem statement. That  $F^{-1}N_c$  is an  $\hat{\mathcal{Q}}$ -stabilizer of the loop with  $D_c = FG$  means that loop  $\hat{\mathcal{Q}}$ -stability is realized, i.e., the characteristic function of (14) is invertible in  $\hat{\mathcal{Q}}_-$ .

On the other hand, if  $G \in \hat{\mathcal{Q}}^{m \times m}$  is a gcd of  $(\bar{D}_s\bar{D}_wN_p, D_c)$  with  $\det G \sim \det \bar{D}_s \det \bar{D}_w$ , then in view of Lemma 5, (36), (37), (38), (21), (25), (28), (29), and (30) exists  $G_1$ , a gcd of  $(E, A)$  such that  $\det G_1 \sim \det D_s \det D_w$ . Since the loop is  $\hat{\mathcal{Q}}$ -stable, we have in view of (10) and (14) and applying Lemma 4

$$EA^{-1} \in \hat{\mathcal{Q}}^{n \times (2n+2m)}.$$

In light of (8), the definition of  $B$  in (3), (7), and (2) and in view of the fact that  $\hat{f} \in \hat{\mathcal{Q}}_- \Rightarrow \hat{f} \in \hat{\mathcal{Q}}$ , we conclude  $\hat{\epsilon} \in \mathcal{Q}^n \forall u_0 \in \mathbb{C}^n, \forall v_0 \in \mathbb{C}^m$ , completing the proof.

*Remark 2:* A careful look at the sufficiency proof of Theorem 1 reveals that condition b) of Theorem 1 may be relaxed to

b')  $G$  is a gcd of  $(\bar{D}_s\bar{D}_wN_p, D_c)$ .

Call this *Theorem 1'*. Theorem 1' has great value in that it requires only b') for constructing the compensator  $\hat{C}$ . ■

The conditions established by Theorem 1 are perfectly analogous to the lumped parameter problem [4], [5]. It should be remembered, however, that  $\epsilon \in \mathcal{Q}^n$  does not imply in the general case that  $\lim_{t \rightarrow \infty} \epsilon(t) = 0$ , a fact which has been overlooked in [3]. The next theorem establishes a sufficient condition for the last. Before stating it, let us define

$\check{N}_s =$  inverse Laplace transform of  $N_s$

$\check{N}_w =$  inverse Laplace transform of  $N_w$ .

*Theorem 2:* Assume that the compensator satisfies the conditions of Theorem 1. Assume also that the elements of  $\check{N}_s$  and  $\check{N}_w$  belong to  $L_1$ , while their derivatives belong to  $\mathcal{Q}$ . Then  $\hat{\mathcal{Q}}$ -stability is achieved and  $\epsilon(t) \rightarrow 0$  as  $t \rightarrow \infty \forall u_0 \in \mathbb{C}^n, \forall v_0 \in \mathbb{C}^m$ .

*Proof:* Follows straightforwardly from [2, Lemma 5.1].

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New Relationships Between Input-Output and Lyapunov Stability

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**Abstract**—In this note we define a concept called "small signal  $L_p$ -stability," and show its relationship to Lyapunov stability.

In this note we present several new results concerning the relationship between the input-output and Lyapunov stability of autonomous nonlinear systems described by

$$\dot{x}(t) = f(x(t), u(t)) \tag{1a}$$

$$y(t) = g(x(t), u(t)) \tag{1b}$$

where  $u \in R^m, x \in R^n, y \in R^k$  denote the input, state, and output of the system, respectively. Such relationships are valuable for the following reason. Both Lyapunov theory and input-output theory are well-developed in their own right for single-loop systems [1]-[3], as well as for large-scale systems [4]-[6]; however, the interrelationships between the two theories are not completely understood. An examination of [1]-[6] reveals that many stability criteria are easier to state and prove in the input-output setting than in the state-space (Lyapunov) setting; yet, many problems are more naturally stated in the state-space setting than in the input-output setting. Thus, if one could conclude input-output stability from Lyapunov stability, or vice versa, it would be possible to use the techniques and results from one theory to solve the problems of the other.

In what follows we assume that  $f(0,0) = 0$  and that  $g(0,0) = 0$ ; further we assume that, corresponding to each initial condition  $x(t_0) = x_0$  and each input  $u(\cdot)$ , (1a) has a unique solution for all  $t \geq t_0$ , whose value at  $t$

Manuscript received June 10, 1981; revised September 28, 1981. This work was supported by the Natural Sciences and Engineering Council of Canada under Grant A-1240 and by the U.S. Department of Energy under Contract DE-AC01-78ET-29127.

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is denoted by  $\phi(t, t_0, x_0, u)$ . It is clear that  $\phi(t, t_0, x_0, u)$  is independent of  $u(\tau)$  for  $\tau \notin [t_0, t]$ .

In order to state our results precisely, as well as to compare our results to existing ones, it is desirable to introduce a few definitions and state two known results.

**Definition 1:** System (1) is *reachable* if there exists a time  $t^* > 0$  such that for every  $x \in R^n$  there exists a  $u(\cdot)$  satisfying  $\phi(t^*, 0, 0, u) = x$ .

**Definition 2:** System (1) is *uniformly observable* if there exists a function  $\alpha$  of class KR [7, p. 114] such that

$$\|g(\phi(\cdot, 0, x, 0), 0)\|_2 \geq \alpha(|x|) \quad \forall x \quad (2)$$

where  $|\cdot|$  denotes the Euclidean norm on  $R^n$  and  $\|\cdot\|_2$  denotes the  $L_2$ -norm.

**Fact 1 [8]:** Suppose system (1) is reachable and uniformly observable, and that it is also  $L_2$ -stable in the following sense: whenever  $x(0) = 0$  and  $u(\cdot) \in L_2^m$ , the resulting  $y(\cdot) \in L_2^k$ . Under these conditions the equilibrium point  $x = 0$  of

$$\dot{x}(t) = f(x(t), 0) \quad (3)$$

is globally attractive, i.e.,  $\phi(t, 0, x, 0) \rightarrow 0$  as  $t \rightarrow \infty$  for all  $x \in R^n$ .

**Fact 2 [9]:** Suppose  $f$  can be written as

$$f(x, u) = f_1(x) + f_2(u) \quad (4)$$

and suppose that  $f_1, f_2$ , and  $g$  are all globally Lipschitz continuous. Under these conditions, if  $x = 0$  is a globally exponentially stable equilibrium point of (3), then system (1) is finite-gain  $L_2$ -stable, i.e., there exists a finite constant  $\gamma$  such that whenever  $x(0) = 0$  and  $u(\cdot) \in L_2^m$  the resulting  $y(\cdot) \in L_2^k$  and satisfies

$$\|y\|_2 \leq \gamma \|u\|_2. \quad (5)$$

In this note we give *local* versions of Facts 1 and 2 by introducing the new concept of *small-signal*  $L_p$ -stability, which is defined next.

**Definition 3:** System (1) is said to be *small-signal*  $L_p$ -stable for a given  $p \in [1, \infty]$  if there exist finite constants  $\gamma_p$  and  $c_p$  such that the following is true: whenever  $x(0) = 0$ ,  $u(\cdot) \in L_p^m$  and  $\|u(t)\|_p \leq c_p$  for all  $t$ , we have that  $y(\cdot) \in L_p^k$  and

$$\|y\|_p \leq \gamma_p \|u\|_p. \quad (6)$$

In Definition 3 it is important to note that even though  $u(\cdot)$  is pointwise-bounded (since  $\|u(t)\|_p \leq c_p$  for all  $t$ ), the quantity  $\|u\|_p$  can be arbitrarily large if  $p < \infty$ .

**Theorem 1:** Suppose  $f, g$  are continuously differentiable at  $x = 0, u = 0$ , and that  $x = 0$  is an exponentially stable equilibrium point of (3). Under these conditions

- i) system (1) is small-signal  $L_p$ -stable for all  $p \in [1, \infty]$  and  $c_p = c_\infty \forall p \in [1, \infty)$  and
- ii) if  $\|u\|_\infty \leq c_\infty$  and  $u(t) \rightarrow 0$  as  $t \rightarrow \infty$ , then both  $x(t) \rightarrow 0$  and  $y(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

Theorem 1 is thus a local version of Fact 2 in that the hypothesis is the (local) exponential stability of the equilibrium  $x = 0$  of system (3), and the conclusions about system (1) are also local.

**Proof:** The hypotheses imply that  $f$  and  $g$  satisfy local Lipschitz conditions at  $x = 0, u = 0$ , i.e., there exist constants  $k_f, k_g$ , and  $\epsilon$  such that

$$\|f(x, u) - f(x_1, u_1)\| \leq k_f(|x - x_1| + |u - u_1|) \quad (7a)$$

$$\|g(x, u) - g(x_1, u_1)\| \leq k_g(|x - x_1| + |u - u_1|) \quad (7b)$$

$$\forall x \in B_\epsilon, u \in B_\epsilon$$

where  $B_\epsilon$  denotes the closed ball of radius  $\epsilon$  centered at 0.

The hypotheses also imply [7, p. 273] that there exists a Lyapunov function  $V: R^n \rightarrow R_+$  that is continuously differentiable at  $x = 0$ , and positive constants  $\alpha, \beta, \gamma, \delta$  such that

$$\alpha^2|x|^2 \leq V(x) \leq \beta^2|x|^2 \quad (8a)$$

$$\dot{V}_{(3)}(x) \leq -|x|^2 \quad (8b)$$

$$|\nabla V(x)| \leq \gamma|x| \quad (8c)$$

$$\forall x \in B_\delta$$

where  $\dot{V}_{(3)}$  denotes the derivative of  $V$  along the trajectories of (3).

Suppose  $u(\cdot)$  is a fixed input such that  $\|u(t)\| \leq \epsilon \forall t \geq 0$ . If we evaluate the derivative of  $V$  along the trajectory of (1a) starting at  $x(0) = 0$ , we get

$$\begin{aligned} \left. \frac{d}{dt} V(x(t)) \right|_{(1a)} &= \nabla V(x)' f(x, u) \\ &= \nabla V(x)' f(x, 0) + \nabla V(x)' [f(x, u) - f(x, 0)] \\ &\leq -|x|^2 + \gamma k_f |x| \|u\| \quad \text{if } |x(t)| \leq \min\{\delta, \epsilon\} \end{aligned} \quad (9)$$

where in the last step we use (7a), (8b), and (8c). Since  $|x(0)| \leq \delta$ , there exists a  $T > 0$  such that  $|x(t)| \leq \delta \forall t \in [0, T]$ . Hence, (9) holds  $\forall t \in [0, T]$ . Observe now that the right side of (9) is negative whenever  $|x| > \gamma k_f \|u\|$ . This implies that

$$V(x(t)) \leq \max_{|x| \leq \gamma k_f \|u\|} V(x) \leq \beta^2 \gamma^2 k_f^2 \|u\|^2, \quad \forall t \in [0, T] \quad (10)$$

$$|x(t)| \leq \frac{\beta \gamma k_f}{\alpha} \|u\|, \quad \forall t \in [0, T]. \quad (11)$$

If we now select the constant  $c_\infty$  such that

$$c_\infty < \min \left\{ \epsilon, \frac{\epsilon}{\beta \gamma k_f} \cdot \min\{\epsilon, \delta\} \right\} \quad (12)$$

we can repeat the above argument to show that (11) holds for all  $t \geq 0$ .<sup>1</sup> Hence, whenever  $\|u(t)\| \leq c_\infty, \forall t \geq 0$ , we have

$$|x(t)| \leq \frac{\beta \gamma k_f}{\alpha} \|u(t)\|, \quad \forall t \geq 0. \quad (13)$$

It now readily follows from (1b) and (7b) that

$$\|y(t)\| \leq \left( \frac{\beta \gamma k_f}{\alpha} + 1 \right) \cdot k_g \|u(t)\|. \quad \forall t \geq 0 \quad (14)$$

This shows that system (1) is small-signal  $L_\infty$ -stable.

Now let  $p \in [1, \infty)$ , and let  $u(\cdot) \in L_p$  satisfy  $\|u(t)\| \leq c_\infty \forall t \geq 0$ . Then, as before, we have

$$\left. \frac{d}{dt} V(x(t)) \right|_{(1a)} \leq -|x|^2 + \gamma k_f |x| \|u\|, \quad \forall t \geq 0. \quad (15)$$

Let  $w = V^{1/2}$ . Then  $w$  is continuously differentiable except at  $x = 0$ , and there it is directionally differentiable. Hence, if  $x \neq 0$ , we have  $2w\dot{w} = \dot{V}$  so that (15) becomes

$$\begin{aligned} 2w(x) \dot{w}_{(1a)}(x) &\leq -|x|^2 + \gamma k_f |x| \|u\| \\ &\leq -\frac{1}{\beta^2} w^2(x) + \gamma k_f \frac{w(x)}{\alpha} \|u\| \end{aligned} \quad (16)$$

where we use (8a). Dividing both sides of (16) by  $2w$  gives

$$\dot{w}_{(1a)}(x) \leq \frac{-1}{2\beta^2} w(x) + c \|u\| \quad (17)$$

where  $c$  is an appropriate constant. Now the comparison principle [10] gives

$$w(x(t)) \leq \int_0^t c e^{-\tau/2\beta^2} \|u(t-\tau)\| d\tau. \quad (18)$$

As shown in [11], if  $u(\cdot) \in L_p$ , then so is the convolution on the right side of (18). Hence,  $w(x(\cdot)) \in L_p$ , and since

$$|x| \leq w(x)/\alpha \quad (19)$$

from (8a), it follows that  $x(\cdot) \in L_p$ . Moreover, from [11],

$$\|w(x(\cdot))\|_p \leq 2\beta^2 c \|u(\cdot)\|_p \quad (20a)$$

<sup>1</sup>Suppose (12) holds, and let  $T^*$  be the first time such that  $|x(T^*)| = \delta$ . Then  $|x(t)| < \delta \forall t \in [0, T^*]$ ; hence (11) holds with  $T$  replaced by  $T^*$ , and we get  $|x(T^*)| < \delta$ . This contradiction shows that no such  $T^*$  exists, and (11) holds  $\forall t \geq 0$ .

$$\|x(\cdot)\|_p \leq \frac{\sqrt{2c\beta}}{\alpha} \|u(\cdot)\|_p. \quad (20b)$$

It is now easy to see that system (1) is small-signal  $L_p$ -stable.

Finally, it is shown in [1] that if  $u(\cdot) \in L_\infty$  and  $u(t) \rightarrow 0$  as  $t \rightarrow \infty$ , then so does the convolution on the right side of (18). From this it readily follows that  $x(t) \rightarrow 0, y(t) \rightarrow 0$  as  $t \rightarrow \infty$ .  $\square$

*Corollary 1.1:* Suppose  $x = 0$  is a globally exponentially stable equilibrium of (3), and the functions of  $f, g$  are globally Lipschitz continuous. Then system (1) is  $L_p$ -stable for all  $p \in [1, \infty]$ .

To give a local version of Fact 1 we need another pair of definitions.

*Definition 4:* System (1) is *locally reachable* if there is a function  $\beta$  of class  $K$  such that the following is true: there exists a  $t^* > 0$  and a  $c > 0$  such that whenever  $x \in R^n$  and  $|x| \leq c$ , there exists a  $u(\cdot)$  satisfying

$$i) |u(t)| \leq \beta(|x|) \quad \text{for all } t \geq 0 \quad (21a)$$

$$ii) \phi(t^*, 0, 0, u) = x. \quad (21b)$$

*Definition 5:* System (1) is *locally uniformly observable* if there exists a function  $\alpha$  of class  $K$  and a  $c > 0$  such that, whenever  $|x| < c$ , we have

$$\|g(\phi(\cdot, 0, x, 0), 0)\|_2 \geq \alpha(|x|). \quad (22)$$

*Theorem 2:* Suppose system (1) is locally reachable, locally uniformly observable, and small-signal  $L_2$ -stable. Then the equilibrium  $x = 0$  of (3) is attractive.

*Proof:* Suppose  $|x| \leq c$ , and select an input  $\bar{u}(\cdot)$  with  $|\bar{u}(t)| \leq \beta(|x|)$  such that  $\phi(t^*, 0, 0, \bar{u}) = x$ . Let  $u(\cdot)$  be the input in  $L_2$  defined by

$$u(t) = \begin{cases} \bar{u}(t), & 0 \leq t \leq t^* \\ 0, & t > t^*. \end{cases} \quad (23)$$

Clearly,  $\phi(t^*, 0, 0, u) = x$  and  $|u(t)| \leq \beta(|x|) \forall t \geq 0$ . Now consider the solution trajectory  $\phi(t, 0, x, 0)$ . It is easy to see, because system (1a) is autonomous, that  $\phi(t, 0, x, 0) = \phi(t + t^*, 0, 0, u)$ . Since system (1) is small-signal  $L_2$ -stable, it follows that for sufficiently small  $|x|$  we have

$$\|g(\phi(\cdot, 0, 0, u), u(\cdot))\|_2 \leq c_2 \|u(\cdot)\|_2 < \infty. \quad (24)$$

However, for  $t \geq t^*$ , we have, by (22) that

$$\begin{aligned} \alpha(|\phi(t, 0, 0, u)|) &= \alpha(|\phi(t - t^*, 0, 0, u)|) \\ &\leq \left\{ \int_{t^*}^t y^2(\tau) d\tau \right\}^{1/2} \end{aligned} \quad (25)$$

where  $y(\cdot) = g(\phi(\cdot, 0, 0, u), u(\cdot))$ . Since the right side of (25) approaches zero as  $t \rightarrow \infty$ , it follows that  $\phi(t, 0, x, 0) \rightarrow 0$  as  $t \rightarrow \infty$ , provided, of course, that  $|x|$  is sufficiently small.

As shown in [12] the property of uniform observability is not in general preserved under feedback. In [12] two new forms of detectability (rather than observability) are introduced that are preserved under arbitrary interconnection, and these are used to prove alternate forms of Fact 1 that are especially suited to large-scale systems. We can prove local versions of these results as well. They are not listed here for want of space, but are left to the reader.

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A Note on the Routh-Hurwitz Test

M. M. FAHMY AND J. O'REILLY

**Abstract**—This note demonstrates how the  $\epsilon$ -method in the original Routh-Hurwitz test can be applied when the Routh array contains a row with zero leftmost element together with an all-zero row. The complete root distribution is determined by applying the criterion only once to the given polynomial  $D(s)$  without either factoring out a common divisor or shifting the imaginary axis.

Since the test may become computationally tedious due to the inclusion of cumbersome  $\epsilon$ -terms, the recently reported method of Shamash [1] is used to simplify the computational scheme. This method, being restricted to the case where  $D(s)$  has purely imaginary roots constituting the whole set of roots of the greatest common even divisor, is here refined and generalized to accommodate all cases in which all-zero rows appear.

I. INTRODUCTION

In recent years there has been a revival of interest in the classical Routh-Hurwitz criterion [1]-[13]. A singular case arises when the leftmost element of a certain row in the Routh array is zero. One efficient remedy for this case is to replace the resulting zero by an infinitesimal quantity  $\epsilon$  and continue to fill in the array. However, there is a history of statements, unsupported by example, e.g., [7], [8], [14], to the effect that problems invalidating the original test may be encountered when the array contains  $\epsilon$ -terms and the given polynomial  $D(s)$  possesses imaginary-axis roots (which produce all-zero rows).

Accordingly, some modifications have been introduced to determine the complete root distribution. Many authors [3], [4], [7] suggest that the common divisor be factored out of  $D(s)$  or some subsidiary polynomials of smaller degree. Others [13] apply the criterion twice, to  $D(s)$  and  $D(-s)$ . In [10], alternatively, the imaginary axis is shifted to the right and to the left by an infinitesimal amount.

All the above methods, though legitimate, obscure the fact that the Routh-Hurwitz criterion, without any modification, gives the complete root distribution *even* in the presence of imaginary-axis roots and  $\epsilon$ -terms. It is our contention that no problems are encountered for any case and that the correct answer is always arrived at.

In Section II we demonstrate how the root distribution can be determined when the Routh array contains  $\epsilon$ -terms and all-zero rows. The criterion is applied only once to  $D(s)$  without either factoring out a common divisor or shifting the imaginary axis. This fact must be emphasized as an intrinsic merit of the original Routh-Hurwitz test.

Nonetheless, the computations may become tedious to carry out due to the inclusion of cumbersome  $\epsilon$ -terms and, consequently, methods for simplifying the computational scheme are valuable. Shamash [1] has recently proposed a novel method to form a modified, computationally less involved Routh's array. His proposition is based on an implicit assumption that the polynomial  $D(s)$  possesses imaginary-axis roots that constitute the *whole* set of roots of the greatest common even divisor. Since, at the outset, one is not always aware whether such a property is satisfied and, moreover, the appearance of an all-zero row is not solely attributable to the presence of imaginary-axis roots, Shamash's method seems to be of restricted use. In Section III we generalize this method such that it works irrespective of the location of roots and in all cases in which all-zero rows appear. Three examples are given in Section IV to illustrate the feasibility and generality of the procedure.

Manuscript received June 2, 1981; revised August 18, 1981.

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