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Algebraic Design Techniques for Reliable Stabilization

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Abstract—**In this paper we study two problems in feedback stabilization. The first is the simultaneous stabilization problem, which can be stated as follows. Given plants $G_0, G_1, \cdots, G_l$, does there exist a single compensator C that stabilizes all of them? The second is that of stabilization by a stable compensator, or more generally, a "least unstable" compensator. Given a plant G, we would like to know whether or not there exists a stable compensator C that stabilizes G; if not, what is the smallest number of right half-place poles (counted according to their McMillan degree) that any stabilizing compensator must have? We show that the two problems are equivalent in the following sense. The problem of simultaneously stabilizing $l+1$ plants can be reduced to the problem of simultaneously stabilizing l plants using a stable compensator, which in turn can be stated as the following purely algebraic problem. Given $2l$ matrices $A_1, \cdots, A_l$, $B_1, \cdots, B_l$, where A_i, B_i are right-coprime for all i, does there exist a matrix M such that $A_i + MB_i$ is unimodular for all i? Conversely, the problem of simultaneously stabilizing l plants using a stable compensator can be formulated as one of simultaneously stabilizing $l+1$ plants.**

The problem of determining whether or not there exists an M such that $A + BM$ is unimodular, given a right-coprime pair (A, B), turns out to be a special case of a question concerning a matrix division algorithm in a proper Euclidean domain. We give an answer to this question, and we believe this result might be of some independent interest. We show that, given two $n \times m$ plants G_0 and G_1, we can generically stabilize them simultaneously provided *either* n or m is greater than one. In contrast, simultaneous stabilizability of two single-input–single-output plants, g_0 and g_1, is *not* generic.

I. Introduction

In this paper we study two problems in feedback stabilization. The first is the simultaneous stabilization problem, which can be stated as follows. Given plants $G_0, G_1, \cdots, G_l$, does there exist a single compensator C that stabilizes all of them? This can be viewed as a problem of reliable stabilization, where G_0 is the nominal description of a particular plant, which changes to $G_1, \cdots, G_l$ in the case of some failures (e.g., failure of sensors, severance of loops or software breakdown). The second problem tackled in this paper is that of stabilization by a stable compensator or, more generally, a "least unstable" compensator. Given a plant G, we would like to know whether or not there exists a stable compensator C that stabilizes G; if not, what is the smallest number of right half-plane poles (counted according to their McMillan degree) that any stabilizing compensator must have?

We show that the two problems are equivalent in the following sense. The problem of simultaneously stabilizing $l+1$ plants can be reduced to the problem of simultaneously stabilizing l plants using a stable compensator, which in turn can be stated as the following purely algebraic problem. Given $2l$ matrices $A_1, \cdots, A_l$, $B_1, \cdots, B_l$, where A_i, B_i are right coprime for all i, does there exist a matrix M such that $A_i + MB_i$ is unimodular for all i? Conversely, the problem of simultaneously stabilizing l plants using a stable compensator can be formulated as one of simultaneously stabilizing $l+1$ plants. All this is done in Section III.

The problem of determining whether or not there exists an M such that $A + BM$ is unimodular, given a right-coprime pair (A, B), turns out to be a special case of a question concerning a matrix division algorithm in a proper Euclidean domain. We give an answer to this question in Section IV, and we believe this result might be of some independent interest. Using this result, in Section V we study the problem of stabilization using a "least unstable" compensator.

One of the surprising aspects of the problems studied here is the generic nature of the solutions. We show that, given two $n \times m$ plants G_0 and G_1, we can generically stabilize them simultaneously provided *either* n or m is greater than one. In other words, even if the given G_0 and G_1 cannot be simultaneously stabilized, there exist plants $\bar{G}_1$ arbitrarily close to G_1 such that $\bar{G}_1, G_0$ can be simultaneously stabilized. In contrast, simultaneous stabilizability

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of two single-input–single-output plants g_0 and g_1 is *not* generic. This is shown in Section VI.

It turns out that the results given here for simultaneous stabilizability can be readily extended to the case of an arbitrary (i.e., not necessarily finite) family of plants. However, the resulting necessary and sufficient conditions are not computationally verifiable, except in special cases (e.g., the case where $l=1$, i.e., there are only two plants, or, where all plants are "close" to a nominal plant). As far as we are aware, the problem of simultaneous stabilization has been studied only in [1], [18]. Our results were derived independently of [1], [18] and generalize those of [1] to the case of multiinput–multioutput systems.

The problem of stabilizing a single plant using a stable compensator is studied in [2], where necessary and sufficient conditions are given for the existence of a stable stabilizing compensator for a given plant. These conditions are very elegant and easily verifiable, involving some interlacing of plant poles and zeros, and they figure in an important way in our proofs. The question of a "least unstable" stabilizing compensator has not previously been addressed in the literature, but is resolved here.

To keep the exposition simple, we deal for the most part with plants whose transfer matrices contain only rational functions of s, and define a "stable" transfer function to be a proper rational function whose poles are in the open left half-plane. However, the generalizations to the case of distributed systems and/or systems whose transfer functions have poles in a prescribed region of the complex plane (not necessarily the open left half-plane) are straightforward and are indicated in Section VII.

II. Background and Notation

Throughout the paper we let $R(s)$ denote the set of rational functions in s with real coefficients, and we let $\mathcal{H}$ denote the subset of $R(s)$ consisting of proper rational functions whose poles lie in the open left half-plane. In the case of lumped linear time-invariant systems, $\mathcal{H}$ consists of precisely the transfer functions of BIBO stable systems. The set $\mathcal{H}$ is a ring; thus, if two functions f_1 and f_2 belong to $\mathcal{H}$, so do their difference and product.[1] The ring $\mathcal{H}$ is clearly commutative ($f_1 f_2 = f_2 f_1$) and is an integral domain ($f_1 f_2 = 0$ implies $f_1 = 0$ or $f_2 = 0$). The set $R(s)$ is the quotient field generated by $\mathcal{H}$; i.e., every $g \in R(s)$ can be written as $g = f_1/f_2$, $f_1, f_2 \in \mathcal{H}$, $f_2 \neq 0$, and conversely, every ratio f_1/f_2 where $f_1, f_2 \in \mathcal{H}, f_2 \neq 0$, belongs to $R(s)$.

A function f in $\mathcal{H}$ is called a *unit* if its reciprocal belongs to $\mathcal{H}$. Clearly the units in $\mathcal{H}$ are the properly invertible minimum phase transfer functions.

Given any rational function h, we can find two functions f and g in $\mathcal{H}$ such that $h = f/g$, and such that f and g are relatively prime (i.e., 1 is a greatest common divisor of f and g). Such a pair (f, g) is called a *coprime factorization* of h. It is essential to recognize that we are doing factorizations in the ring $\mathcal{H}$, and not in the ring of polynomials. In other words, we are expressing a given rational function h as a ratio of proper stable transfer functions with no common factors, rather than as a ratio of polynomials with no common zeros.

[1]See [3], [4] for the requisite background in abstract algebra.

We let $\mathcal{H}^{n\times m}$ denote the set of $n \times m$ matrices whose elements all belong to $\mathcal{H}$. Thus $\mathcal{H}^{n\times m}$ is the set of transfer functions of BIBO stable lumped linear time-invariant systems with m inputs and n outputs. A matrix $F \in \mathcal{H}^{n\times n}$ is *unimodular* if its inverse belongs to $\mathcal{H}^{n\times n}$. Clearly F is unimodular if and only if $\det F$ is a unit.

Given any $H \in R^{n\times m}(s)$ (which means H is an $n\times m$ matrix whose elements are rational functions of s), we can find matrices $N \in \mathcal{H}^{n\times m}$, $D \in \mathcal{H}^{m\times m}$ such that $H(s) = N(s)[D(s)]^{-1}$ and the matrices N, D are *right coprime*, i.e., there exist $P \in \mathcal{H}^{m\times n}$, $Q \in \mathcal{H}^{m\times m}$ such that

$$P(s)N(s)+Q(s)D(s)=I_m, \qquad \forall s. \tag{2.1}$$

Similarly, we can find $\tilde{N} \in \mathcal{H}^{n\times m}$, $\tilde{D} \in \mathcal{H}^{n\times n}$, $\tilde{P} \in \mathcal{H}^{m\times n}$, $\tilde{Q} \in \mathcal{H}^{n\times n}$ such that $H(s) = [\tilde{D}(s)]^{-1}\tilde{N}(s)$, and

$$\tilde{N}(s)\tilde{P}(s)+\tilde{D}(s)\tilde{Q}(s)=I_n, \qquad \forall s. \tag{2.2}$$

We refer to (N, D) as a *right-coprime factorization* (*r.c.f.*) of H and to $(\tilde{D}, \tilde{N})$ as a *left-coprime factorization* (*l.c.f.*) of H.

If (N, D) is an r.c.f. of $H \in R^{n\times m}(s)$, so is (NU, DU) whenever U is an $m \times m$ unimodular matrix. The converse is also true, i.e., if $(N_1, D_1), (N_2, D_2)$ are two r.c.f.'s of H, then there exists a unimodular matrix U such that $N_1 = N_2U$, $D_1 = D_2U$. Similar statements apply to l.c.f.'s.

Given an $n\times m$ rational matrix H, it is possible to select an r.c.f. (N, D), an l.c.f. $(\tilde{D}, \tilde{N})$, and matrices $P, Q, \tilde{P}, \tilde{Q}$ such that

$$\begin{bmatrix} Q & P \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D & -\tilde{P} \\ N & \tilde{Q} \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_n \end{bmatrix}. \tag{2.3}$$

For further discussion of these topics, see [5]–[8].

Next, we state and prove a result concerning coprime factorizations of a strictly proper matrix.

Lemma 2.1: Let H be strictly proper, let (N, D) be any r.c.f. of H, and let P, Q be any matrices over $\mathcal{H}$ such that $PN + QD = I_m$. Then, i) N is strictly proper, and ii) $\det D(\infty) \neq 0$, $\det Q(\infty) \neq 0$ (i.e., both $\det D$ and $\det Q$ have relative degree zero).

Proof: Since H is strictly proper, we can write every element of H as $h_{ij} = \bar{n}_{ij}/\bar{d}_{ij}$, where $\bar{n}_{ij}, \bar{d}_{ij} \in \mathcal{H}$, $\bar{n}_{ij}$ is strictly proper, and $\bar{d}_{ij}(\infty) \neq 0$. One way of doing this is as follows. If

$$h_{ij}(s) = \frac{\alpha_{ij}(s)}{\beta_{ij}(s)}, \tag{2.4}$$

let

$$\bar{n}_{ij}(s) = \frac{\alpha_{ij}(s)}{(s+1)^{\gamma_{ij}}},$$

$$\bar{d}_{ij}(s) = \frac{\beta_{ij}(s)}{(s+1)^{\gamma_{ij}}},$$

$$\gamma_{ij} = \deg \beta_{ij}. \tag{2.5}$$

Now let $\bar{d} = \Pi_i \Pi_j \bar{d}_{ij}$; then $\bar{d} \in \mathcal{H}$ and $\bar{d}(\infty) \neq 0$. Moreover, we can write $H(s) = N_1(s)(\bar{d}(s))^{-1}$, where $N_1 \in \mathcal{H}^{n \times m}$ is strictly proper. Of course, the matrices N_1 and $\bar{d}I_m$ need not be right coprime. However, we can extract a greatest common right divisor $R \in \mathcal{H}^{m \times m}$ using standard methods [8, Theorem 2.1], [11, pp. 30–35]. Let $N_1 = \bar{N}R$, $\bar{d}I_m = \bar{D}R$; then $(\bar{N}, \bar{D})$ is an r.c.f. of H. Since $\det \bar{D}(\infty) \cdot \det R(\infty) = (\bar{d}(\infty))^m \neq 0$, it follows that $\det R(\infty) \neq 0$. Hence $\bar{N}(\infty) = N_1(\infty)[R(\infty)]^{-1} = 0$, since N_1 is strictly proper; this shows that $\bar{N}$ is also strictly proper. In this way we have constructed *one* r.c.f. $(\bar{N}, \bar{D})$ of H such that $\bar{N}$ is strictly proper. Now let (N, D) be *any* r.c.f. of H. Then there exists a unimodular matrix U such that $N = \bar{N}U$, $D = \bar{D}U$. Hence $N(\infty) = \bar{N}(\infty)U(\infty) = 0$, which shows that N is strictly proper. This completes the proof of i).

To prove ii), let P, Q be any matrices over $\mathcal{H}$ such that

$$P(s)N(s) + Q(s)D(s) = I_m, \qquad \forall s. \tag{2.6}$$

Letting $s \to \infty$ in (2.6) gives $Q(\infty)D(\infty) = I_m$, which shows that $\det D(\infty) \neq 0$, $\det Q(\infty) \neq 0$. □

A similar result holds for l.c.f.'s.

Now we briefly summarize some results on feedback stability, taken from [8]. Consider the feedback system shown in Fig. 1, where G and C are rational matrices of order $n \times m$ and $m \times n$, respectively, and assume that $\det(I_n + GC) \not\equiv 0$ (otherwise the system is not well-defined).

Then it is easy to verify that

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} (I_m + CG)^{-1} & -C(I_n + GC)^{-1} \\ G(I_m + CG)^{-1} & (I_n + GC)^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2.7}$$

or, more concisely,

$$e = Hu. \tag{2.8}$$

We will say that the pair (G, C) *is stable* if $H \in \mathcal{H}^{(n+m)\times(n+m)}$. (It is necessary to consider all four transfer functions in (2.4) because any three of these can be BIBO stable while the fourth is not; see [9] for some examples.) We say that C *stabilizes* G if (G, C) is stable. Note that (2.7) is essentially symmetric in G and C, so that C stabilizes G if and only if G stabilizes C.

Next, we state without proof the necessary and sufficient conditions for a pair (G, C) to be stable, which are taken from [8], [10].

Lemma 2.2: Let $(N, D), (\tilde{D}, \tilde{N})$ be any r.c.f. and l.c.f. of $G \in R^{n \times m}(s)$, and let $(N_C, D_C), (\tilde{D}_C, \tilde{N}_C)$ be any r.c.f. and l.c.f. of $C \in R^{n \times m}(s)$. Suppose $\det(I_n + GC) = \det(I_m + CG) \neq 0$. Then the following conditions are equivalent.

i) The pair (G, C) is stable.

ii) The matrix $\tilde{D}_C D + \tilde{N}_C N$ is unimodular.

iii) The matrix $\tilde{D} D_C + \tilde{N} N_C$ is unimodular.

Corollary 2.2.1: Let $(N, D), (\tilde{D}, \tilde{N})$ be any r.c.f. and l.c.f. of $G \in R^{n \times m}(s)$, and suppose $C \in \mathcal{H}^{m \times n}$. Then the following conditions are equivalent.

i) The pair (G, C) is stable.

ii) $D + CN$ is unimodular.

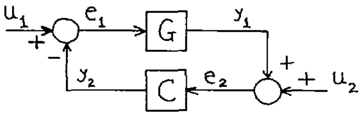

Fig. 1.

iii) $\tilde{D} + \tilde{N}C$ is unimodular.

The next result characterizes all compensators that stabilize a given strictly proper plant. The proof is given in [8, Theorem 3.1].

Lemma 2.3: Let $G \in R^{n \times m}(s)$ be strictly proper, and let $(N, D), (\tilde{D}, \tilde{N})$ be any r.c.f. and l.c.f. of G. Select matrices $P, Q, \tilde{P}, \tilde{Q}$ such that

$$PN + QD = I_m, \qquad \tilde{N}\tilde{P} + \tilde{D}\tilde{Q} = I_n. \tag{2.9}$$

Then

i) every C such that (G, C) is stable is proper;

ii) the set of C such that (G, C) is stable is given by

$$\begin{aligned} \mathcal{C}(G) &= \{(Q - R\tilde{N})^{-1}(P + R\tilde{D}), R \in \mathcal{H}^{m \times n}\} \\ &= \{(\tilde{P} + DS)(\tilde{Q} - NS)^{-1}, S \in \mathcal{H}^{m \times n}\}. \end{aligned} \tag{2.10}$$

Corollary 2.3.1: Suppose $G \in \mathcal{H}^{n \times m}$ is strictly proper. Then

$$\begin{aligned} \mathcal{C}(G) &= \{(I - RG)^{-1}R, R \in \mathcal{H}^{m \times n}\}. \\ &= \{S(I - GS)^{-1}, S \in \mathcal{H}^{m \times n}\}. \end{aligned} \tag{2.11}$$

III. Simultaneous Stabilization

In this section we study the problem of simultaneously stabilizing $l+1$ plants $G_0, G_1, \cdots, G_l$ using the *same* compensator C. We begin by studying the case of two plants G_0 and G_1, and show that G_0, G_1 can be simultaneously stabilized if and only if an associated system can be stabilized using a stable compensator. Since necessary and sufficient conditions for this are known [2], our test for simultaneous stabilizability of two plants is computationally verifiable. We then show that the problem of simultaneously stabilizing $l+1$ plants can be reduced to one of simultaneously stabilizing l plants using a stable compensator. At present, no computable tests are available for the latter, so that this result can only be viewed as a starting point for further work.

We begin with the problem of simultaneously stabilizing two given $n \times m$ strictly proper plants G_0 and G_1.[2] Without loss of generality we assume that for $i = 0, 1$, we have available an r.c.f. (N_i, D_i) and an l.c.f. $(\tilde{D}_i, \tilde{N}_i)$ of G_i, together with matrices $P_i, Q_i, \tilde{P}_i, \tilde{Q}_i$, such that

$$\begin{bmatrix} Q_i & P_i \\ -\tilde{N}_i & \tilde{D}_i \end{bmatrix} \begin{bmatrix} D_i & -\tilde{P}_i \\ N_i & \tilde{Q}_i \end{bmatrix} = I_{n+m}. \tag{3.1}$$

[2] The assumption that G_0 and G_1 are strictly proper is not essential to the theory and is only made to simplify a few expressions. The general case is discussed in Section VII.

Theorem 3.1: Define

$$A_1 = Q_0 D_1 + P_0 N_1, \qquad B_1 = -\tilde{N}_0 D_1 + \tilde{D}_0 N_1. \quad (3.2)$$

Then $\det A_1 \not\equiv 0$, so that A_1^{-1} is well-defined, and A_1, B_1 are right coprime. Moreover, there exists a C that stabilizes both G_0 and G_1 if and only if there exists an $M \in \mathcal{H}^{m\times n}$ that stabilizes $B_1 A_1^{-1}$.

Proof: First, from Lemma 2.1 it follows that $A_1(\infty) = Q_0(\infty)D_1(\infty)$ (since $N_1(\infty)=0$), and that $\det A_1(\infty) \neq 0$ (since $\det Q_0(\infty) \neq 0$ and $\det D_1(\infty) \neq 0$). Hence, A_1^{-1} is well-defined. Next, since

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} Q_0 & P_0 \\ -\tilde{N}_0 & \tilde{D}_0 \end{bmatrix} \begin{bmatrix} D_1 \\ N_1 \end{bmatrix}, \quad (3.3)$$

it follows from (3.1) that A_1, B_1 are right coprime.

From Lemma 2.3, there exists a C that stabilizes both G_0 and G_1 if and only if there exist R_0, R_1 in $\mathcal{H}^{m\times n}$ such that

$$(Q_0 - R_0\tilde{N}_0)^{-1}(P_0 + R_0\tilde{D}_0) = (Q_1 - R_1\tilde{N}_1)^{-1}(P_1 + R_1\tilde{D}_1). \quad (3.4)$$

Observe now that $Q_0 - R_0\tilde{N}_0, P_0 + R_0\tilde{D}_0$ are right coprime, and that $Q_1 - R_1\tilde{N}_1, P_1 + R_1\tilde{D}_1$ are right coprime. Hence, (3.4) holds if and only if there exists a unimodular matrix U such that

$$Q_0 - R_0\tilde{N}_0 = U(Q_1 - R_1\tilde{N}_1);$$
$$P_0 + R_0\tilde{D}_0 = U(P_1 + R_1\tilde{D}_1). \quad (3.5)$$

Thus, we have shown that G_0 and G_1 can be simultaneously stabilized if and only if there exist *stable* R_0 *and* R_1 and a *unimodular* U such that (3.5) holds. We now show that this is the case if and only if there exists a stable M such that $A_1 + MB_1$ is unimodular.

To do this, rewrite the two equations in (3.5) as

$$[I \quad R_0]\begin{bmatrix} Q_0 & P_0 \\ -\tilde{N}_0 & \tilde{D}_0 \end{bmatrix} = U[I \quad R_1]\begin{bmatrix} Q_1 & P_1 \\ -\tilde{N}_1 & \tilde{D}_1 \end{bmatrix} \quad (3.6)$$

and recall from (3.1) that

$$\begin{bmatrix} Q_1 & P_1 \\ -\tilde{N}_1 & \tilde{D}_1 \end{bmatrix}^{-1} = \begin{bmatrix} D_1 & -\tilde{P}_1 \\ N_1 & \tilde{Q}_1 \end{bmatrix}. \quad (3.7)$$

Multiplying both sides of (3.6) by the matrix in (3.7) gives

$$[I \quad R_0]\begin{bmatrix} Q_0 & P_0 \\ -\tilde{N}_0 & \tilde{D}_0 \end{bmatrix}\begin{bmatrix} D_1 & -\tilde{P}_1 \\ N_1 & \tilde{Q}_1 \end{bmatrix} = U[I \quad R_1] \quad (3.8)$$

$$[I \quad R_0]\begin{bmatrix} A_1 & X_1 \\ B_1 & Y_1 \end{bmatrix} = U[I \quad R_1] \quad (3.9)$$

as the equations to be satisfied, where the definitions of X_1 and Y_1 are self-evident. We will now show that there exist stable R_0, R_1 and a unimodular U satisfying (3.9) if and only if there exists a stable M such that $A_1 + MB_1$ is unimodular. To prove the "if" part, select M such that $A_1 + MB_1$ is unimodular and let $R_0 = M$, $U = A_1 + MB_1$, and $R_1 = U^{-1}(X_1 + MY_1)$. To prove the "only if" part, select R_0, R_1, U such that (3.9) holds, and let $M = R_0$; then $A_1 + MB_1 = U$ is unimodular.

To complete the proof observe that $M \in \mathcal{H}^{m\times n}$ stabilizes $B_1A_1^{-1}$ if and only if $A_1 + MB_1$ is unimodular. □

During the course of the above proof we have actually characterized *all* compensators that simultaneously stabilize G_0 and G_1. Let $\mathfrak{M}$ denote the set of all $M \in \mathcal{H}^{m\times n}$ such that $A_1 + MB_1$ is unimodular; thus $\mathfrak{M}$ is the set of all stable compensators that stabilize $B_1A_1^{-1}$. Then the set of all compensators that simultaneously stabilize G_0 and G_1 is given by

$$\{(Q_0 - R\tilde{N}_0)^{-1}(P_0 + R\tilde{D}_0), R \in \mathfrak{M}\}. \quad (3.10)$$

In the multivariable case, an explicit expression for the set $\mathfrak{M}$ is not available, but $\mathfrak{M}$ can be explicitly described in the case $m = n = 1$ [17].

To clarify the result contained in Theorem 3.1 we now study the case where one of the systems (say G_0) is stable. Actually, there is no loss of generality in making this assumption. Suppose that we are given two plants $\overline{G}_0$ and $\overline{G}_1$, and we would like to know whether or not they can be simultaneously stabilized. First, we select a compensator $\overline{C}$ that stabilizes $\overline{G}_0$, and define $G_0 = \overline{G}_0(I + \overline{C}\overline{G}_0)^{-1}$, $G_1 = \overline{G}_1(I + \overline{C}\overline{G}_1)^{-1}$. A little reflection will show that i) $\overline{G}_0, \overline{G}_1$ can be simultaneously stabilized if and only if G_0, G_1 can be simultaneously stabilized, and ii) if C stabilizes both G_0 and G_1, then $C + \overline{C}$ stabilizes both $\overline{G}_0$ and $\overline{G}_1$.

Corollary 3.1.1: Suppose G_0 is strictly proper and stable, and G_1 is strictly proper. Then G_0 and G_1 can be simultaneously stabilized if and only if $G_1 - G_0$ can be stabilized by a stable compensator.

Proof: Since G_0 is stable, we can apply Theorem 3.1 with $N_0 = \tilde{N}_0 = G_0$, $\tilde{D}_0 = I$, $D_0 = I$, $P_0 = 0$, $\tilde{P}_0 = 0$, $Q_0 = I$, $\tilde{Q}_0 = I$. This gives $B_1A_1^{-1} = G_1 - G_0$. However, we give a proof independent of Theorem 3.1.

From Corollary 2.3.1 the set of compensators that stabilize G_0 is given by

$$\mathcal{C}(G_0) = \{(I - RG_0)^{-1}R, R \in \mathcal{H}^{m\times n}\}. \quad (3.11)$$

Now, from Fig. 2 we see that a compensator of the form $R(I - G_0R)^{-1}$ stabilizes G_1 if and only if R stabilizes $G_1 - G_0$. □

To aid in the application of Theorem 3.1 we quote below a result from [2] on stabilization using a stable compensator, which we shall encounter again in Section V.

Lemma 3.1 [2]: Let $\sigma_1, \cdots, \sigma_{l-1}, \sigma_l = \infty$ denote the extended real nonnegative zeros of the largest invariant factor of B_1.[3] Then there exists an $M \in \mathcal{H}^{m\times n}$ that stabilizes

[3] The *blocking zeros* of the rational matrix $B_1A_1^{-1}$ are the values of s such that $B_1(s)[A_1(s)]^{-1} = 0$. Since B_1 and A_1 are right coprime, the blocking zeros of $B_1A_1^{-1}$ in the right half-plane are the same as the zeros of the largest invariant factor of B_1 (in the right half-plane). Note that $B_1(\infty) = 0$.

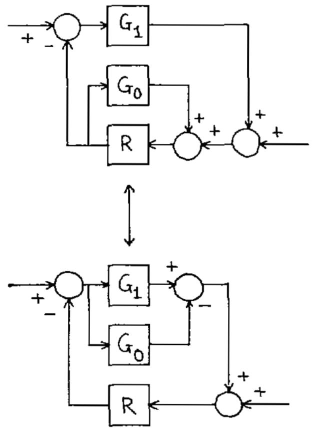

Fig. 2.

$B_1 A_1^{-1}$ if and only if the number of (real) zeros of $\det A_1$ in the interval (σ_i, σ_j), counted according to multiplicity, is even for every i, j.

In the case where G_0 is strictly proper and stable, we have seen that $B_1 A_1^{-1}$ equals $G_1 - G_0$. Thus $\sigma_1, \cdots, \sigma_{l-1}$ are precisely the real nonnegative blocking zeros of $G_1 - G_0$, i.e., the real nonnegative values of s such that $G_1(s) = G_0(s)$. The zeros of $\det A_1$ in the right half-plane are precisely the poles of $G_1 - G_0$, which in turn are the poles of G_1 (since G_0 is stable). In the general case, where neither G_0 nor G_1 is assumed to be stable, $\sigma_1, \cdots, \sigma_{l-1}$ are still the real nonnegative blocking zeros of $G_1 - G_0$. This can be seen by observing that

$$B_1(s) = 0 \Leftrightarrow [-\tilde{N}_0 D_1 + \tilde{D}_0 N_1](s) = 0$$
$$\Leftrightarrow [\tilde{D}_0^{-1} \tilde{N}_0](s) = [N_1 D_1^{-1}](s). \quad (3.12)$$

However, the interpretation of the zeros of $\det A_0$ is no longer simple.

Next, we consider the problem of simultaneously stabilizing several strictly proper plants $G_0, G_1, \cdots, G_l$. Without loss of generality we assume that we have available matrices satisfying (3.1) for $i = 0, \cdots, l$. By proceeding as in the proof of Theorem 3.1, we can derive the following result.

Theorem 3.2: Define

$$A_i = Q_0 D_i + P_0 N_i, \quad B_i = -\tilde{N}_0 D_i + \tilde{D}_0 N_i, \qquad i = 1, \cdots, l. \quad (3.13)$$

Then $\det A_i \neq 0$ for all i, and B_i, A_i are right coprime for all i. Moreover, there exists a C that stabilizes G_i for $i = 0, \cdots, l$ if and only if there exists an $M \in \mathcal{H}^{m \times n}$ that stabilizes $B_i A_i^{-1}$ for $i = 1, \cdots, l$.

Outline of Proof: We leave it to the reader to verify that $\det A_i \neq 0$ and that B_i, A_i are right coprime for all i. Now, there exists a C that stabilizes G_i for all i if and only if there exist $R_0, \cdots, R_l$ in $\mathcal{H}^{m \times n}$ such that

$$(Q_0 - R_0 \tilde{N}_0)^{-1}(P_0 + R_0 \tilde{D}_0)$$
$$= (Q_i - R_i \tilde{N}_i)^{-1}(P_i + R_i \tilde{D}_i) \qquad \text{for } i = 1, \cdots, l. \quad (3.14)$$

Next, (3.14) is true if and only if there exist unimodular matrices $U_1, \cdots, U_l$ such that

$$Q_0 - R_0 \tilde{N}_0 = U_i (Q_i - R_i \tilde{N}_i),$$
$$P_0 + R_0 \tilde{D}_0 = U_i (P_i + R_i \tilde{D}_i) \qquad \text{for } i = 1, \cdots, l. \quad (3.15)$$

The rest of the steps are as in the proof of Theorem 3.1. □

Theorem 3.2 shows that the simultaneous stabilization of $l+1$ plants can be reduced to the simultaneous stabilization of l plants using a stable compensator. The converse is also true. Given l plants $H_1, \cdots, H_l$, there exists a stable compensator stabilizing all the plants $H_1, \cdots, H_l$ if and only if there exists a compensator that simultaneously stabilizes $H_0 = 0, H_1, \cdots, H_l$ [the fact that the compensator stabilizes the zero plant implies that the compensator must be stable; see (2.7)]. However, at present the criterion of Theorem 3.2 is not computationally verifiable except when $l = 1$.

If we wish to study the simultaneous stabilization of an arbitrary (i.e., not necessarily finite) family of plants $\{G_0, G_\alpha, \alpha \in \mathcal{Q}\}$, the required generalization of Theorem 3.2 is readily apparent. For an interpretation of this problem in a differential-geometric setting, see [18]. Single-input–single-output versions of Theorems 3.1, 3.2 can be found in [1].

IV. Euclidean Division in the Ring $\mathcal{H}$

In this section we derive two results concerning Euclidean division in the ring $\mathcal{H}$. These results prove to be useful when we study the problem of stabilization using a "least unstable" compensator in Section V. The simple statements of these results (namely, Lemmas 4.1 and 4.2) are in sharp contrast with the tediousness and technical nature of their proofs. Thus, for clarity of exposition, we state the two lemmas in succession, and then give the proof of each.

Given a nonzero function $f \in \mathcal{H}$, define its *gauge* $\gamma(f)$ as

$$\gamma(f) = \text{relative degree of } f + \# \text{ zeros of } f \text{ in closed right half-plane}$$
$$= \# \text{zeros of } f \text{ in closed RHP, including } \infty. \quad (4.1)$$

Thus $\gamma(f)$ is a well-defined nonnegative integer for all nonzero $f \in \mathcal{H}$. Moreover, it can be shown [13] that $\mathcal{H}$ is a Euclidean domain; that is, given any f in $\mathcal{H}$ and any $g \neq 0$ in $\mathcal{H}$, there exists an $h \in \mathcal{H}$ such that either $f + gh = 0$, or else $\gamma(f + gh) < \gamma(g)$. In other words, a division algorithm can be performed in $\mathcal{H}$, and a greatest common divisor of a given pair of functions f and g can be found in a finite number of steps using the familiar Euclidean algorithm.

With the above definition of gauge, $\mathcal{H}$ is actually a *proper* Euclidean domain, i.e., we have $\gamma(fg) = \gamma(f) + \gamma(g)$ for all nonzero f, g. However, it is possible to have $\gamma(f+g) > \max\{\gamma(f), \gamma(g)\}$.[4] For example, consider

$$f = \frac{s+4}{s+1}, \quad g = -\frac{2s+3}{s+1}, \quad f + g = \frac{-s+1}{s+1}. \quad (4.2)$$

[4] Contrast this with the case of polynomials, where $\deg(f+g) \geq \max\{\deg f, \deg g\}$.

Then $\gamma(f)=\gamma(g)=0$, but $\gamma(f+g)=1$. This has an important consequence; namely, given f and $g \neq 0$, there may exist more than one h such that $\gamma(f+gh)<\gamma(g)$. [Contrast this with the case of polynomials. Given polynomials f, g with $g \neq 0$, there exists a *unique* polynomial h such that $\deg(f-gh)<\deg(g)$]. With this in mind, we define

$$I(f,g)=\min_{h \in \mathcal{H}} \gamma(f+gh). \tag{4.3}$$

If f is a multiple of g, so that $f=gh$ for some h, we set $I(f,g)=-\infty$.

Observe now that f is a unit in $\mathcal{H}$ if and only if $\gamma(f)=0$. Also, given f and g in $\mathcal{H}$, their greatest common divisor is well-defined to within a unit. Thus, if we let $\langle f, g\rangle$ denote a g.c.d. of f and g, then $\gamma(\langle f,g\rangle)$ is a well-defined integer[5] even though $\langle f,g\rangle$ is only defined to within a unit factor. Now suppose f and g are not relatively prime, and let w be a g.c.d. of f and g. If $w \neq 0$, it is easy to see that

$$I(f,g)=\gamma(w)+I(f/w, g/w). \tag{4.4}$$

Thus, for computational purposes, it is enough if we can calculate $I(f,g)$ when f, g are relatively prime.

Lemma 4.1: Let f, g be two elements of $\mathcal{H}$ with a greatest common divisor of 1. Let $\sigma_1,\cdots,\sigma_k$ denote the distinct nonnegative real zeros of the function g, including ∞ as appropriate, arranged in ascending order. Then

$$\begin{aligned} I(f,g) &= \#\text{ sign changes in the sequence} \\ &\qquad \{f(\sigma_i), i=1,\cdots,k\} \\ &\triangleq \nu. \end{aligned} \tag{4.5}$$

Remarks: Since f, g are relatively prime, $f(\sigma_i)\neq 0$, so that $f(\sigma_i)$ has a definite sign for all i.

The next lemma presents a result on matrix Euclidean division, which might be of some independent interest.

Lemma 4.2: Let $\mathfrak{R}$ be a proper Euclidean domain, with gauge γ. Given f, g in $\mathfrak{R}$, define

$$I(f,g)=\min_{h \in \mathfrak{R}} \gamma(f+gh) \tag{4.6}$$

where we take $\gamma(0)=-\infty$. Let $A \in \mathfrak{R}^{m\times m}$, $B \in \mathfrak{R}^{n \times m}$ be right coprime, with $\det A \neq 0$. Then

$$\min_{M \in \mathfrak{R}^{m\times n}} \gamma(\det(A+MB)) = I(a, b_1) \tag{4.7}$$

where $a=\det A$ and b_1 is the greatest common divisor of all elements of B.

Remarks: Suppose that A and B are *not* right coprime; then we can write $A=A_1F$, $B=B_1F$, where A_1, B_1 are right coprime and F is a greatest common right divisor of A and B. Since $\gamma(\det(A+MB))=\gamma(\det(A_1+MB_1)+\gamma(\det F)$, Lemma 4.2 can be used to compute the minimum value of $\gamma(\det(A+MB))$ even when A and B are not right coprime.

We now give the proofs of Lemmas 4.1 and 4.2; these borrow heavily from [2].

[5]We can accommodate the case $\langle f,g\rangle=0$ by defining $\gamma(0)=-\infty$.

Proof of Lemma 4.1: We will first show that

$$\gamma(f+gh)\geq \nu \qquad \text{for all } h \in \mathcal{H}. \tag{4.8}$$

Then we will show how to construct an $h \in \mathcal{H}$ such that $\gamma(f+gh)=\nu$.

First, define

$$r=f/(f+gh). \tag{4.9}$$

Then

$$1-r=hg/(f+gh). \tag{4.10}$$

We now observe that

i) $r(s)=1$ at all RHP zeros of g; moreover, the multiplicity of s as a zero of $1-r$ is at least equal to its multiplicity as a zero of g.

ii) Every zero of r in the closed right half-plane is also a zero of f; moreover, the multiplicity of any such zero of r is less than or equal to its multiplicity as a zero of f.

iii) Conversely, let r be any function satisfying i) and ii) above, and define

$$h=f(1-r)/gr. \tag{4.11}$$

Then $h \in \mathcal{H}$.

To show that $\gamma(f+gh)\geq\nu$ for all h, let h be selected arbitrarily; then the resulting function r satisfies i) and ii). Now, write f and $f+gh$ as

$$f(s)=u_1(s)\phi_1(s)/[s^{l_1}\psi_1(s)] \tag{4.12}$$

$$f(s)+g(s)h(s)=u_2(s)\phi_2(s)/[s^{l_2}\psi_2(s)] \tag{4.13}$$

where u_1, u_2 are units of $\mathcal{H}$, ϕ_1, ϕ_2 are monic polynomials whose zeros are all in the closed right half-plane, and ψ_1, ψ_2 are monic strictly Hurwitz polynomials. Since i) holds, we have

$$\frac{\phi_1(\sigma_i)}{\phi_2(\sigma_i)}\cdot\frac{u_1(\sigma_i)s^{l_2}\psi_2(\sigma_i)}{u_2(\sigma_i)s^{l_1}\psi_1(\sigma_i)}=1, \qquad \text{for all } i. \tag{4.14}$$

Since the second term in (4.14) does not change sign, $\phi_1(\sigma_i)/\phi_2(\sigma_i)$ must always be of the same sign; i.e., the sequence $\{\phi_2(\sigma_i)\}$ must contain exactly as many sign changes as the sequence $\{\phi_1(\sigma_i)\}$. It is easy to see that the latter number of sign changes is ν, so that the sequence $\{\phi_2(\sigma_i)\}$ must have exactly ν sign changes. From this, it follows that $f+gh$ has at least ν zeros in the closed RHP, i.e., $\gamma(f+gh)\geq\nu$.[6]

We now give an iterative procedure for constructing an $h \in \mathcal{H}$ such that $\gamma(f+gh)=\nu$. Given f and g, find a polynomial $\xi(s)$ which has only ν nonnegative real zeros, such that the sequence $\{f(\sigma_i)/\xi(\sigma_i)\}$ does not change sign. This is clearly possible; indeed, if $f(\sigma_i)$ has a different sign from $f(\sigma_{i+1})$, we select any real number in the interval (σ_i, σ_{i+1}) to be a zero of $\xi(\cdot)$. Define

[6]Some care is needed in applying this argument when ∞ is a zero of $1-r$. But the required modifications are minor and are left to the reader.

$$r_0(s)=\frac{f(s)(s+1)^{\nu}}{\xi(s)}. \tag{4.15}$$

Then r_0 satisfies ii) above, but not necessarily i). We show, following [2], that given any r_i, we can construct an r_{i+1} such that r_{i+1} satisfies ii).

iv) Given any s_0 in the closed RHP, $1-r_{i+1}$ can be made to have a zero at s_0; moreover, the multiplicity of the zero of $1-r_{i+1}$ at s_0 exceeds that of the zero of $1-r_i$ at s_0, if any. Finally, the RHP poles of r_{i+1} are exactly the same as the RHP poles of r_i, multiplicities included.

If we can indeed accomplish the construction of such an r_{i+1}, we can eventually find an r_j with exactly ν RHP poles satisfying i) and ii). For such a choice of $r=r_j$, we see that h given by (4.11) belongs to $\mathcal{H}$, and $\gamma(f+gh)=\#$ RHP poles of $r=\nu$.

The iterative procedure is

$$r_{i+1}=u_{i+1}r_i \tag{4.16}$$

where u_{i+1} is a unit. The details of the procedure are exactly as in [2], with very minor differences. Reference [2, eq. (25)] is replaced by

$$\epsilon d_p^+(\sigma_i)/\xi(\sigma_i)>0 \qquad \forall i \tag{4.17}$$

while [2, eq. (26)] is replaced by

$$S(s)=\frac{\epsilon d_p^-(s)h(s)}{\xi(s)g(s)}. \tag{4.18}$$

This completes the proof of the lemma. □

Proof of Lemma 4.2:[7] In the first part of the proof we show that

$$\gamma(\det(A+MB))\geq I(a,b_1) \qquad \text{for all } M\in R^{m\times m}. \tag{4.19}$$

Since $a\neq 0$, BA^{-1} is well-defined. Put BA^{-1} in Smith–McMillan form, and suppose U,V are unimodular matrices such that

$$U^{-1}(BA^{-1})V=\operatorname{diag}[s_1/t_1,\cdots,s_k/t_k,0,\cdots,0]\triangleq D \tag{4.20}$$

where k is the rank of B, s_i divides s_{i+1}, t_{i+1} divides t_i, and s_i, t_i are relatively prime. Then clearly the ordered pair

$$(U\operatorname{diag}[s_1,\cdots,s_k,0,\cdots,0],\ V\operatorname{diag}[t_1,\cdots,t_k,1,\cdots,1]) \tag{4.21}$$

is also an r.c.f. of BA^{-1}. Thus there exists a unimodular matrix W such that

$$B=U\operatorname{diag}[s_1,\cdots,s_k,0,\cdots,0]W \tag{4.22}$$

$$A=V\operatorname{diag}[t_1,\cdots,t_k,1,\cdots,1]W. \tag{4.23}$$

In particular, we see that

$$s_1=b_1=\text{g.c.d. of all elements of } B \tag{4.24}$$

[7] For notational simplicity we assume that $m=n$. If $m\neq n$, the only change is that all Smith–McMillan forms are bordered by zero matrices.

$$a\sim\prod_{i=1}^{k}t_i \tag{4.25}$$

where "$\sim$" denotes "is equivalent to."

Next, let A^{adj} denote the adjoint matrix of A. Then $A^{-1}=A^{\mathrm{adj}}a^{-1}$, and we have

$$BA^{\mathrm{adj}}=BA^{-1}a\sim Da. \tag{4.26}$$

Thus, if

$$C=\operatorname{diag}[c_1,\cdots,c_k,0,\cdots,0] \tag{4.27}$$

is a Smith form for BA^{adj}, we have

$$c_i\sim\frac{s_i}{t_i}a. \tag{4.28}$$

Hereafter, we suppose, without loss of generality, that

$$a=\prod_{i=1}^{k}t_i, \qquad c_i=\frac{s_i}{t_i}a=s_i\prod_{j\neq i}t_j. \tag{4.29}$$

Next, observe that

$$\begin{aligned}\gamma(\det(A+MB))&+(n-1)\gamma(a)\\ &=\gamma(\det(A+MB))+\gamma(\det A^{\mathrm{adj}})\\ &=\gamma\big(\det(AA^{\mathrm{adj}}+MBA^{\mathrm{adj}})\big)\\ &=\gamma\big(\det(aI_m+MBA^{\mathrm{adj}})\big)\end{aligned} \tag{4.30}$$

so that

$$\begin{aligned}\min_M&\,\gamma(\det(A+MB))\\ &=\min_M\gamma\big(\det(aI_m+MBA^{\mathrm{adj}})\big)-(n-1)\gamma(a).\end{aligned} \tag{4.31}$$

Suppose that

$$UBA^{\mathrm{adj}}V=C; \tag{4.32}$$

then

$$\begin{aligned}\gamma&\big(\det(aI_m+MBA^{\mathrm{adj}})\big)\\ &=\gamma\big(\det(AA^{\mathrm{adj}}+MBA^{\mathrm{adj}})\big)\\ &=\gamma\big(\det(V^{-1}AA^{\mathrm{adj}}V+V^{-1}MBA^{\mathrm{adj}}V)\big)\\ &\qquad\text{since } \gamma(\det V)=\gamma(\det V^{-1})=0\\ &=\gamma\big(\det(aI_m+M_1UBA^{\mathrm{adj}}V)\big)\\ &\qquad\text{where } M_1=V^{-1}MU^{-1}\\ &=\gamma\big(\det(aI_m+M_1C)\big).\end{aligned} \tag{4.33}$$

Hence minimizing $\gamma(\det(aI_m+MBA^{\mathrm{adj}}))$ is equivalent to minimizing $\gamma(\det(aI_m+M_1C))$.

By a well-known expansion formula [11, p. 9], we have

$$\begin{aligned}\det(aI_m+M_1C)&=a^m+\text{a multiple of } a^{m-1}c_1\\ &\quad+\text{a multiple of } a^{m-2}c_1c_2+\cdots\\ &\quad+\text{a multiple of } a^{m-k}c_1c_2\cdots c_k\end{aligned} \tag{4.34}$$

where we use the fact that c_i divides c_{i+1}. Now, (4.34) implies that

$$\det(aI_m + M_1C) = a^{m-k}\big[a^k + \text{a multiple of g.c.d. of } \{a^{k-1}c_1, a^{k-2}c_1c_2, \cdots, ac_1c_2\cdots c_{k-1}, c_1c_2\cdots c_k\}\big]. \quad (4.35)$$

Now, note that, from (4.29), we have

$$\begin{aligned} a^{k-1}c_1 &= a^k s_1/t_1 = a^{k-1}s_1t_2t_3\cdots t_k \\ a^{k-2}c_1c_2 &= a^k\frac{s_1s_2}{t_1t_2} = a^{k-1}s_1s_2t_3\cdots t_k \\ c_1\cdots c_k &= a^{k-1}s_1s_2s_3\cdots s_k. \end{aligned} \quad (4.36)$$

Since g.c.d. $\{t_2t_3\cdots t_k, \cdots, s_2s_3\cdots s_k\} = 1$,[8] we have that g.c.d. $\{a^{k-1}c_1, \cdots, c_1\cdots c_k\} = a^{k-1}s_1$. Hence

$$\begin{aligned} \det(aI_m + M_1C) &= a^{m-k}\big(a^k + \text{a multiple of } a^{k-1}s_1\big) \\ &= a^{m-1}(a + \text{a multiple of } s_1). \end{aligned} \quad (4.37)$$

Hence

$$\gamma(\det(aI_m + M_1C)) \geqslant \gamma(a^{m-1}) + I(a, s_1). \quad (4.38)$$

From (4.31), (4.33), and (4.38), we get

$$\gamma(\det(A + MB)) \geqslant I(a, s_1) = I(a, b_1) \quad \text{since } s_1 = b_1. \quad (4.39)$$

This proves (4.18).

To prove that the bound (4.18) is exact, let U, V be as in (4.32), and select $\theta_1, \cdots, \theta_k$ such that

$$\theta_1(t_2\cdots t_k) + \cdots + \theta_k(s_2\cdots s_k) = 1 \quad (4.40)$$

(this is possible because g.c.d. $(t_2\cdots t_k, \cdots, s_2\cdots s_k) = 1$); let r be chosen so that $\gamma(a + rb_1) = I(a, b_1)$; let

$$E = \begin{bmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{bmatrix} \in \mathfrak{R}^{m\times m} \quad (4.41)$$

and define

$$M = EV\begin{bmatrix} r\theta_m & 0 & 0 & \cdots & 0 \\ r\theta_{m-1} & 0 & 0 & & 1 \\ \vdots & & & & \vdots \\ r\theta_2 & 0 & 1 & & 0 \\ r\theta_1 & 1 & 0 & & 0 \end{bmatrix} U \triangleq EVRU, \quad \text{say}. \quad (4.42)$$

where $\theta_i = 0$ if $i > k$. Then (noting that E is unimodular), we have

$$\begin{aligned} &\gamma(\det(A + MB))\cdot\gamma(\det A^{\text{adj}}) \\ &= \gamma(\det E^{-1}V^{-1}A + E^{-1}V^{-1}MB)\cdot\gamma(\det(A^{\text{adj}}VE)) \\ &= \gamma(\det(aI_m + E^{-1}V^{-1}MBA^{\text{adj}}VE)) \\ &= \gamma(\det(aI_m + RUBVE)) = \gamma(\det(aI_m + RCE)). \end{aligned} \quad (4.43)$$

However,

$$\begin{aligned} \det(aI_m + RCE) &= a^m\det(I_m + RCa^{-1}E) \\ &= a^m\left(1 + r\frac{b_1}{a}\right) = a^{m-1}(a + rb_1) \end{aligned} \quad (4.44)$$

where the last calculation is tedious, but straightforward (see [2, eqs. (79), (80)]. Hence, for this choice of M,

$$\begin{aligned} &\gamma(\det(A + MB)) + \gamma(\det A^{\text{adj}}) \\ &= \gamma(\det(A + MB)) + \gamma(a^{m-1}) \\ &= \gamma\big[a^{m-1}(a + rb_1)\big] = \gamma(a^{m-1}) + \gamma(a + rb_1) \end{aligned} \quad (4.45)$$

which shows that

$$\gamma(\det(A + MB)) = \gamma(a + rb_1) = I(a, b_1). \quad (4.46)$$

□

V. Stabilization Using a "Least Unstable" Compensator

In this section we study the following problems. Given a strictly proper plant G, what is the smallest number of right half-plane poles (counted according to their McMillan degree) that any stabilizing compensator for G can have? This question is of interest for two reasons: i) it generalizes the question of stabilization using a stable compensator, and ii) if the answer to this question is known, it is possible to obtain a lower bound on the dynamic orders of all stabilizing compensators for G. The main result of this section is given next.

Theorem 5.1: Let $G \in R^{n\times m}(s)$ be strictly proper, and let $\sigma_1, \cdots, \sigma_{l-1}$, $\sigma_l = \infty$ denote the nonnegative real blocking zeros of G (i.e., the nonnegative real values of s such that $G(s) = 0$), arranged in ascending order. Define

$$n_i = \begin{cases} 1 & \text{if the number of poles of } G \text{ in } (\sigma_i, \sigma_{i+1}) \text{ is even} \\ -1 & \text{if the number of poles of } G \text{ in } (\sigma_1, \sigma_{i+1}) \text{ is odd} \end{cases} \quad (5.1)$$

where the poles of G are counted according to their McMillan degree. Let ν denote the number of -1's in the

[8] This follows from the up and down divisibility properties of the s_i's, t_i's, and the primeness of (s_i, t_i).

sequence $n_1,\cdots,n_{l-1}$. Then every compensator C that stabilizes G has at least ν poles (counted according to McMillan degree) in the closed right half-plane. Moreover, there exists a compensator C_0 with exactly ν poles in the closed right half-plane that stabilizes G.

Proof: Let $(N, D), (\tilde{D}, \tilde{N})$ be any r.c.f. and l.c.f. of G, and select P, Q such that

$$PN + QD = I. \tag{5.2}$$

Then, by Lemma 2.3, every C that stabilizes G must be of the form

$$C = (Q - R\tilde{N})^{-1}(P + R\tilde{D}) \qquad \text{for some } R \in \mathcal{H}^{m\times n}. \tag{5.3}$$

By a result in [14], the number of RHP poles of C, counted according to their McMillan degree, is equal to the number of RHP zeros of $\det(Q - R\tilde{N})$. Since $\det[Q(\infty) - R(\infty)\tilde{N}(\infty)] = \det Q(\infty) \neq 0$, the number of RHP zeros of $\det(Q - R\tilde{N})$ equals $\gamma(\det(Q - R\tilde{N}))$. By Lemma 4.2, the minimum value of $\gamma(\det(Q - R\tilde{N}))$ as a function of R is $I(q, n_1)$, where $q = \det Q$ and n_1 is the smallest invariant factor of $\tilde{N}$. Now observe that the RHP zeros of n_1 are precisely the RHP blocking zeros of G. Thus, from Lemma 4.1, $I(q, n_1)$ equals the number of sign changes in the sequence $\{q(\sigma_i)\}_{i=1}^{l}$. Next, from (5.2) we get

$$Q(\sigma_i)D(\sigma_i) = I, \qquad \forall i \tag{5.4}$$

since $N(\sigma_i) = 0$. This shows that the signs of $\det Q(\sigma_i)$ and $\det D(\sigma_i)$ are the same. To complete the proof, note that, from [14],

$$\det D(s) \sim \prod_{i=1}^{k} \left[\frac{s - p_i}{s+1}\right]^{m_i} \tag{5.5}$$

where $p_1,\cdots,p_k$ are the RHP poles of G, with McMillan degrees $m_1,\cdots,m_k$, respectively. It is easy to verify that the number of sign changes in the sequence $\{\det D(\sigma_i)\}_{i=1}^{l}$ equals the number of -1's in the sequence $\{n_i\}_{i=1}^{l-1}$. □

Note that the previous result on stabilization using a stable compensator, which is proved in [2] and stated here as Lemma 3.1, is a corollary of Theorem 5.1.

VI. Genericity of Simultaneous Stabilizability of Two Plants

In this section we study the genericity of simultaneous stabilizability of two plants, and of stabilizability of a single plant using a stable compensator. We show that both properties are generic in the case of multivariable systems, but not in the case of single-input–single-output systems.

We begin by reviewing the conditions for two functions f and g in $\mathcal{H}$ to be relatively prime (i.e., the common divisors of f and g are units).

Lemma 6.1: Two functions f and g in $\mathcal{H}$ are relatively prime if and only if i) at least one of them is nonzero at infinity, and ii) they have no common zeros in the closed right half-plane.

The proof is easily deduced from [16].

Next, we define a norm on $\mathcal{H}$, so that we then have a natural notion of neighborhoods. For any $f \in \mathcal{H}$, we define

$$\|f\| = \sup_{\omega \in R} |f(j\omega)|. \tag{6.1}$$

Thus, a *ball* $\mathcal{B}(f;\epsilon)$ is defined by

$$\mathcal{B}(f;\epsilon) = \{g: \|f - g\| < \epsilon\}. \tag{6.2}$$

A set $\mathcal{F}$ in $\mathcal{H}$ is *open* if, for every $f \in \mathcal{H}$, there is an $\epsilon > 0$ such that $\mathcal{B}(f;\epsilon) \subset \mathcal{F}$. A *neighborhood* of f is any open set containing f. A sequence $\{f_i\}$ in $\mathcal{H}$ *converges* to f if every neighborhood of f contains all but a finite number of terms in the sequence $\{f_i\}$. Finally, a set $\mathcal{F}$ in $\mathcal{H}$ is *dense* if every $f \in \mathcal{H}$ is the limit of a sequence in $\mathcal{F}$.

Recall that a *binary relation* on $\mathcal{H}$ is a subset of $\mathcal{H} \times \mathcal{H}$. If $\mathcal{R} \subset \mathcal{H} \times \mathcal{H}$ is a binary relation (or a *relation* for short), we write $a\mathcal{R}b$ to denote $(a, b) \in \mathcal{R}$. In other words, we say that a is related to b via $\mathcal{R}$ if $(a, b) \in R$. Given a relation $\mathcal{R}$ on $\mathcal{H}$, we define, for every $a \in \mathcal{H}$,

$$\mathcal{S}(a;\mathcal{R}) = \{b \in \mathcal{H}: (a, b) \in \mathcal{R}\}. \tag{6.3}$$

Finally, we say that the relation $\mathcal{R}$ is *generic* if $\mathcal{S}(a;\mathcal{R})$ is open and dense in $\mathcal{H}$ for every a in $\mathcal{H}$. Thus, if $\mathcal{R}$ is a generic relation on $\mathcal{H}$, then for any a, b in $\mathcal{H}$ one of two things is true: i) $a\mathcal{R}b$, and moreover, there is a ball $\mathcal{B}(b;\epsilon)$ such that $a\mathcal{R}\bar{b}$ for every $\bar{b}$ in $\mathcal{B}(b;\epsilon)$; or ii) a is not related to b via $\mathcal{R}$, in which case there is a sequence $\{b_i\}$ converging to b such that $a\mathcal{R}b_i$. Roughly speaking, if $\mathcal{R}$ is a generic relation and if $a\mathcal{R}b$, then small perturbations in b will not destroy the relationship, while if a is not related to b, then arbitrarily small perturbations in b will cause the relationship to hold true.

Lemma 6.2: Define a relation $\mathcal{C}$ on $\mathcal{H}$ by

$$\mathcal{C} = \{(a, b): a, b \in \mathcal{H} \text{ and are relatively prime}\}. \tag{6.4}$$

Then $\mathcal{C}$ is generic.

The proof using Lemma 6.1 is easy and therefore omitted.

In order to consider multivariable systems, we define a norm on $\mathcal{H}^{n\times m}$. Given $F \in \mathcal{H}^{n\times m}$, we define

$$\|F\| = \max_{1 \leq i \leq n} \sum_{j=1}^{m} \|f_{ij}\|. \tag{6.5}$$

This norm defines a topology on $\mathcal{H}^{n\times m}$, as before. It is easy to verify that a sequence $\{F^k\}$ in $\mathcal{H}^{n\times m}$ converges to $F \in \mathcal{H}^{n\times m}$ if and only if each of the component sequences $\{f_{ij}^k\}$ converges to f_{ij}.

Lemma 6.3: let $F \in \mathcal{H}^{n\times m}$ and suppose either n or m is greater than one. Then either 1 is the smallest invariant factor of F, or else there is a sequence $\{F_i\}$ converging to F such that 1 is the smallest invariant factor of F_i for all i.

Proof: Recall that the smallest invariant factor of a matrix is the greatest common divisor of all of its elements. The result now follows readily from Lemma 6.2. □

Up to now, we have only defined notions of neighborhood and convergence on $\mathcal{H}^{n\times m}$, which is the set of *stable* $n\times m$ transfer functions. We now extend these notions to $R_s^{n\times m}(s)$, which is the set of $n\times m$ matrices whose elements are strictly proper rational functions of s. Given a $G\in R^{n\times m}(s)$, a *neighborhood* of G consists of all ratios $N_1(s)[D_1(s)]^{-1}$, where (N, D) is any r.c.f. of G, and N_1, D_1 belong to some neighborhoods of N, D in $\mathcal{H}_s^{n\times m}$, $\mathcal{H}^{m\times m}$, respectively. A sequence $\{G_i\}$ in $R_s^{n\times m}(s)$ *converges* to $G\in R_s^{n\times m}(s)$ if there are r.c.f.'s (N_i, D_i) of G_i and (N, D) of G such that $N_i\to N$ in $\mathcal{H}_s^{n\times m}$ and $D_i\to D$ in $\mathcal{H}^{m\times m}$, respectively. The reader is referred to [8, sect. 4] for further details of the above topology. Once we have a topology on $R_s^{n\times m}(s)$, it is clear what is meant by a relation on $R_s^{n\times m}(s)$ being generic.

Theorem 6.1: Define a relation $\mathcal{SS}$ on $R_s^{n\times m}(s)$ as follows:[9]

$$\mathcal{SS}=\{(G_0,G_1): G_0,G_1\in R_s^{n\times m}(s), \text{ and } G_0,G_1 \text{ can be simultaneously stabilized}\}. \quad (6.6)$$

If either n or m is greater than one, then $\mathcal{SS}$ is generic.

Proof: Suppose that either n or m is greater than one. In order to show that $\mathcal{SS}$ is generic, we must establish two things: 1) whenever $(G_0,G_1)\in\mathcal{SS}$, there is a neighborhood $\mathfrak{N}_1$ of G_1 such that $(G_0,\overline{G}_1)\in\mathcal{SS}$ for all $\overline{G}_1\in\mathfrak{N}_1$, and 2) for any G_1, there is a sequence $\{G_1^{(i)}\}$ converging to G_1 such that $(G_0,G_1^{(i)})\in\mathcal{SS}$.

To prove the first statement, we prove first that the set of units in $\mathcal{H}$ is open. Let f be a unit in $\mathcal{H}$; then

$$\inf_{\operatorname{re} s\geqslant 0}|f(s)|\triangleq\epsilon>0 \quad (6.7)$$

i.e., f has no zeros in the closed RHP, including infinity. Now suppose $g\in\mathfrak{B}(f;\epsilon/2)$. Then

$$\inf_\omega|g(j\omega)|\geqslant\inf_\omega|f(j\omega)|-\sup_\omega|f(j\omega)-g(j\omega)| >\epsilon/2>0. \quad (6.8)$$

Hence, by the Nyquist criterion, $g(\cdot)$ has no zeros in the closed RHP and is thus a unit. Since the mapping $F\to\det F$ from $\mathcal{H}^{m\times m}$ to $\mathcal{H}$ is continuous, and since F is unimodular if and only if $\det F$ is a unit, it follows that the set of unimodular matrices in $\mathcal{H}^{m\times m}$ is open.

Now suppose $(G_0,G_1)\in\mathcal{SS}$, and let $(N_0,D_0),(N_1,D_1)$ be any r.c.f.'s of G_0 and G_1, respectively. Define $A_1=Q_0D_1+P_0N_1$, $B_1=-\tilde N_0D_1+\tilde D_0N_1$, as in (3.2). Then, by Theorem 3.1, there is a matrix $M\in\mathcal{H}^{m\times n}$ such that A_1+MB_1 is unimodular. Since the set of unimodular matrices is open, there is a neighborhood $\mathfrak{N}$ of B_1 such that A_1+MB is unimodular for every $B\in\mathfrak{N}$. It is easy to see also that there exist neighborhoods $\mathfrak{N}_1$ of N_1 and $\mathfrak{N}_2$ of D_1 such that $-\tilde N_0D+\tilde D_0N\in\mathfrak{N}$ whenever $N\in\mathfrak{N}_1$, $D\in\mathfrak{N}_2$. To summarize, whenever $N\in\mathfrak{N}_1$, $D\in\mathfrak{N}_2$, the matrix $A_1+MB=A_1+M(-\tilde N_0D+\tilde D_0N)$ is unimodular, so that $(G_0,G)\in\mathcal{SS}$. Since the set of ratios $\{ND^{-1}, N\in\mathfrak{N}_1, D\in\mathfrak{N}_2\}$ is a neighborhood of $G_1=N_1D_1^{-1}$, we have shown that $\mathcal{S}(G_0;\mathcal{SS})$ is open for every G_0.[10]

To complete the proof that $\mathcal{SS}$ is generic, let $G_0\in R_s^{n\times m}(s)$ be given, let $G_1\in R_s^{n\times m}(s)$ be arbitrary, and define $A_1=Q_0D_1+P_0N_1$, $B_1=-\tilde N_0D_1+\tilde D_0N_1$ as before. Clearly B_1 is strictly proper, so let $B_1=F/(s+1)^\alpha$, where $F\in\mathcal{H}^{n\times m}$ and $F(\infty)\neq 0$. By Lemma 6.3, either 1 is the smallest invariant factor of F, or else there is a sequence $\{F_i\}$ converging to F such that 1 is the smallest invariant factor of G_i for all i. In the first case, $b_1=1/(s+1)^\alpha$ is the smallest invariant factor of B_1 and b_1 vanishes only at infinity. Hence, by Lemma 3.1, $(G_0,G_1)\in\mathcal{SS}$. In the second case, select a sequence $\{F_i\}$ converging to F such that 1 is the smallest invariant factor of F_i for all i. Now let

$$N_i=N_1+\tilde Q_0(F_i-F)/(s+1)^\alpha,$$
$$D_i=D_1-\tilde P_0(F_i-F)/(s+1)^\alpha \quad (6.9)$$
$$G_i=N_iD_i^{-1} \quad (6.10)$$

where $\tilde P_0$, $\tilde Q_0$ are selected such that

$$\tilde N_0\tilde P_0+\tilde D_0\tilde Q_0=I_n. \quad (6.11)$$

Clearly, $N_i\in\mathcal{H}_s^{n\times m}$ for all i and $N_i\to N_1$ in $\mathcal{H}_s^{n\times m}$ as $i\to\infty$.[11] Similarly, $D_i\to D_1$ in $\mathcal{H}^{m\times m}$ as $i\to\infty$. Hence $G_i\in R_s^{n\times m}(s)$ and $G_i\to G$ in $R_s^{n\times m}(s)$ as $i\to\infty$ (recall our definition of convergence in $R_s^{n\times m}(s)$). Moreover,

$$B_i\triangleq-\tilde N_0D_i+\tilde D_0N_i =B_1+(F_i-F)/(s+1)^\alpha=F_i/(s+1)^\alpha \quad (6.12)$$

has $1/(s+1)^\alpha$ as its smallest invariant factor. Hence, by Lemma 3.1, $(G_0,G_i)\in\mathcal{SS}$ for all i. □

Using exactly the same reasoning, one can show that the set of G in $R^{n\times m}(s)$ that can be stabilized by a stable compensator is dense in $R^{n\times m}(s)$, provided either n or m is greater than one. This is a formalization of an observation made in [2].

VII. Conclusions

In this paper, we have studied the problems of simultaneous stabilization and stabilization using a stable compensator. We have shown that the simultaneous stabilization of $l+1$ plants is equivalent to the stabilization of l plants using a stable compensator. We have given computationally verifiable tests for the simultaneous stabilizability of two plants, and have shown that this property is generic in the case of multivariable systems. Finally, we have derived an expression for the least unstable compensator that stabilizes a given plant.

In order to simplify the presentation, we have only studied the case of strictly proper plants. An examination

[9] $\mathcal{SS}$ is the set of pairs that can be simultaneously stabilized. See (3.1) for the definition of $g(G_0)$.

[10] This statement is true even if $n=m=1$; i.e., G_0 is a single-input–single-output system.

[11] Here we use $\mathcal{H}_s$ to denote the subset of $\mathcal{H}$ consisting of strictly proper functions.

of our proofs reveals that this assumption is quite unnecessary; it is only made so that various inverses are guaranteed to exist.

In some applications, we may wish to place the poles of the closed-loop system not just in the open left half-plane, but in some subset thereof. This would be the case, for example, if we wish the closed-loop system to have a certain maximum settling time and minimum damping factor. The generalization of our results to this case is extremely straightforward. Suppose $\mathbb{S}$ is a region in the complex plane which is symmetric about the real axis, and let $\mathcal{H}_{\mathbb{S}}$ denote the set of proper rational functions with real coefficients whose poles are all in $\mathbb{S}$. Then [13] $\mathcal{H}_{\mathbb{S}}$ is also a proper Euclidean domain, and the gauge $\gamma_{\mathbb{S}}(f)$ of a function in $\mathcal{H}_{\mathbb{S}}$ is defined by

$$\gamma_{\mathbb{S}}(f) = \text{relative degree of } f + \#\text{ zeros of } f \text{ outside } \mathbb{S}.$$

We can also study distributed systems by letting $\mathcal{H}$ be a ring and $\mathcal{H}_s$ a prime ideal in $\mathcal{H}$; see [6], [8] for details.

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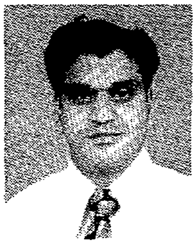

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