

# Using Poisson Integral Formula to Evaluate Four Types of Definite Integrals

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**Abstract** The present paper uses the mathematical software Maple for the auxiliary tool to study four types of definite integrals. The closed forms of these definite integrals can be obtained mainly using Poisson integral formula. On the other hand, we propose two examples to do calculation practically. The research method adopted in this study is to find solutions through manual calculations and verify the answers using Maple.

**Keywords:** definite integrals, closed forms, Poisson integral formula, Maple

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## 1. Introduction

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify our previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests.

In calculus and engineering mathematics, there are many methods to solve the integral problems including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. This article considers the following four types of definite integrals which are not easy to obtain their answers using the methods mentioned above.

$$\int_0^{2\pi} \frac{\sum_{k=0}^n \frac{(n)_k}{k!} r^k t^{n-k} \cos[(m-k)\theta - (n-k)\omega]}{[r^2 - 2rs \cos(\theta - \phi) + s^2][r^2 + 2rt \cos(\theta - \omega) + t^2]^n} d\theta \quad (1)$$

$$\int_0^{2\pi} \frac{\sum_{k=0}^n \frac{(n)_k}{k!} r^k t^{n-k} \sin[(m-k)\theta - (n-k)\omega]}{[r^2 - 2rs \cos(\theta - \phi) + s^2][r^2 + 2rt \cos(\theta - \omega) + t^2]^n} d\theta \quad (2)$$

$$\int_0^{2\pi} \frac{\exp[qr \cos(\theta + \psi)] \sum_{k=0}^n \frac{(n)_k}{k!} r^k t^{n-k} \cos[qr \sin(\theta + \psi) - k\theta - (n-k)\omega]}{[r^2 - 2rs \cos(\theta - \phi) + s^2][r^2 + 2rt \cos(\theta - \omega) + t^2]^n} d\theta \quad (3)$$

$$\int_0^{2\pi} \frac{\exp[qr \cos(\theta + \psi)] \sum_{k=0}^n \frac{(n)_k}{k!} r^k t^{n-k} \sin[qr \sin(\theta + \psi) - k\theta - (n-k)\omega]}{[r^2 - 2rs \cos(\theta - \phi) + s^2][r^2 + 2rt \cos(\theta - \omega) + t^2]^n} d\theta \quad (4)$$

where  $r, s, t, q, \phi, \omega, \psi$  are real numbers,  $|s| < |r| < |t|$ , and  $m, n$  are positive integers. The closed forms of these definite integrals can be obtained mainly using Poisson integral formula; these are the major results of this paper (i.e., Theorems 1 and 2). Adams et al. [1], Nyblom [2], and Oster [3] provided some techniques to solve the integral problems. Yu [4-29], Yu and B. -H. Chen [30], and T. -J. Chen and Yu [31,32,33] used complex power series method, integration term by term theorem, differentiation with respect to a parameter, Parseval's theorem, and generalized Cauchy integral formula to solve some types of integrals. In this paper, some examples are used to demonstrate the proposed calculations, and the manual calculations are verified using Maple.

## 2. Preliminaries and Main Results

Some notations and formulas used in this paper are introduced below.

### 2.1. Notations

Suppose that  $a$  is a real number, and  $p$  is a positive integer. Define  $(a)_p = a(a-1)\cdots(a-p+1)$ , and  $(a)_0 = 1$ .

## 2.2. Formulas

### 2.2.1. Euler's formula

$e^{ix} = \cos x + i \sin x$ , where  $i = \sqrt{-1}$ , and  $x$  is any real number.

### 2.2.2. DeMoivre's formula

$(\cos x + i \sin x)^m = \cos mx + i \sin mx$ , where  $m$  is an integer, and  $x$  is a real number.

The following is an important formula used in this study, which can be found in [[34], p 145].

### 2.2.3. Poisson integral formula

Suppose that  $r, s$  are real numbers, and  $|s| < |r|$ . If  $f$  is defined and continuous on the closed disc  $\{z \in C \mid |z| \leq |r|\}$  and is analytic on the open disc  $\{z \in C \mid |z| < |r|\}$ , then

$$f(se^{i\phi}) = \frac{r^2 - s^2}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})}{r^2 - 2rs \cos(\theta - \phi) + s^2} d\theta$$

### 2.2.4. Binomial theorem

$(u + v)^n = \sum_{k=0}^n \frac{(n)_k}{k!} u^k v^{n-k}$ , where  $u, v$  are complex numbers, and  $n$  is a positive integer.

In the following, we determine the closed forms of the definite integrals (1) and (2).

**Theorem 1.** *If  $r, s, t, \phi, \omega$  are real numbers,  $|s| < |r| < |t|$ , and  $m, n$  are positive integers, then the definite integrals*

$$\int_0^{2\pi} \frac{\sum_{k=0}^n \frac{(n)_k}{k!} r^k t^{n-k} \cos[(m-k)\theta - (n-k)\omega]}{[r^2 - 2rs \cos(\theta - \phi) + s^2][r^2 + 2rt \cos(\theta - \omega) + t^2]^n} d\theta$$

$$= \frac{2\pi}{r^2 - s^2} \left(\frac{s}{r}\right)^m \frac{\sum_{k=0}^n \frac{(n)_k}{k!} s^k t^{n-k} \cos[(m-k)\phi - (n-k)\omega]}{[s^2 + 2st \cos(\phi - \omega) + t^2]^n}$$

and

$$\int_0^{2\pi} \frac{\sum_{k=0}^n \frac{(n)_k}{k!} r^k t^{n-k} \sin[(m-k)\theta - (n-k)\omega]}{[r^2 - 2rs \cos(\theta - \phi) + s^2][r^2 + 2rt \cos(\theta - \omega) + t^2]^n} d\theta$$

$$= \frac{2\pi}{r^2 - s^2} \left(\frac{s}{r}\right)^m \frac{\sum_{k=0}^n \frac{(n)_k}{k!} s^k t^{n-k} \sin[(m-k)\phi - (n-k)\omega]}{[s^2 + 2st \cos(\phi - \omega) + t^2]^n}$$

**Proof** Let  $f(z) = \frac{z^m}{(z + te^{i\omega})^n}$ , then  $f(z)$  is defined and continuous on the closed disc  $\{z \in C \mid |z| \leq |r|\}$ , and it is analytic on the open disc  $\{z \in C \mid |z| < |r|\}$ . Let  $z = se^{i\phi}$ , then using Poisson integral formula for  $f(z)$  yields

$$\frac{(se^{i\phi})^m}{(se^{i\phi} + te^{i\omega})^n} = \frac{r^2 - s^2}{2\pi} \int_0^{2\pi} \frac{(re^{i\theta})^m}{[r^2 - 2rs \cos(\theta - \phi) + s^2]} d\theta$$

$$(re^{i\theta} + te^{i\omega})^n$$

By Euler's formula and DeMoivre's formula, we have

$$\int_0^{2\pi} \frac{e^{im\theta}}{[r^2 - 2rs \cos(\theta - \phi) + s^2](re^{i\theta} + te^{i\omega})^n} d\theta$$

$$= \frac{2\pi}{r^2 - s^2} \left(\frac{s}{r}\right)^m \frac{e^{im\phi}}{(se^{i\phi} + te^{i\omega})^n}$$

It follows that

$$\int_0^{2\pi} \frac{e^{im\theta} (re^{-i\theta} + te^{-i\omega})^n}{[r^2 - 2rs \cos(\theta - \phi) + s^2][r^2 + 2rt \cos(\theta - \omega) + t^2]^n} d\theta$$

$$= \frac{2\pi}{r^2 - s^2} \left(\frac{s}{r}\right)^m \frac{e^{im\phi} (se^{-i\phi} + te^{-i\omega})^n}{[s^2 + 2st \cos(\phi - \omega) + t^2]^n}$$

Using binomial theorem yields

$$\int_0^{2\pi} \frac{\sum_{k=0}^n \frac{(n)_k}{k!} r^k t^{n-k} e^{i[(m-k)\theta - (n-k)\omega]}}{[r^2 - 2rs \cos(\theta - \phi) + s^2][r^2 + 2rt \cos(\theta - \omega) + t^2]^n} d\theta$$

$$= \frac{2\pi}{r^2 - s^2} \left(\frac{s}{r}\right)^m \frac{\sum_{k=0}^n \frac{(n)_k}{k!} s^k t^{n-k} e^{i[(m-k)\phi - (n-k)\omega]}}{[s^2 + 2st \cos(\phi - \omega) + t^2]^n}$$

By the equality of real parts of both sides of Eq. (10), we obtain Eq. (5). Also, using the equality of imaginary parts of both sides of Eq. (10) yields Eq. (6) holds.

Next, the closed forms of the definite integrals (3) and (4) can be obtained below.

**Theorem 2.** *Let  $r, s, t, q, \phi, \omega, \psi$  be real numbers,  $|s| < |r| < |t|$ , and  $n$  be a positive integer, then the definite integrals*

$$\exp[qr \cos(\theta + \psi)]$$

$$\int_0^{2\pi} \frac{\sum_{k=0}^n \frac{(n)_k}{k!} r^k t^{n-k} \cos[qr \sin(\theta + \psi) - k\theta - (n-k)\omega]}{[r^2 - 2rs \cos(\theta - \phi) + s^2][r^2 + 2rt \cos(\theta - \omega) + t^2]^n} d\theta$$

$$\exp[qs \cos(\phi + \psi)]$$

$$= \frac{2\pi}{r^2 - s^2} \frac{\sum_{k=0}^n \frac{(n)_k}{k!} s^k t^{n-k} \cos[qs \sin(\phi + \psi) - k\phi - (n-k)\omega]}{[s^2 + 2st \cos(\phi - \omega) + t^2]^n}$$

and

$$\exp[qr \cos(\theta + \psi)]$$

$$\int_0^{2\pi} \frac{\sum_{k=0}^n \frac{(n)_k}{k!} r^k t^{n-k} \sin[qr \sin(\theta + \psi) - k\theta - (n-k)\omega]}{[r^2 - 2rs \cos(\theta - \phi) + s^2][r^2 + 2rt \cos(\theta - \omega) + t^2]^n} d\theta$$

$$\exp[qs \cos(\phi + \psi)]$$

$$= \frac{2\pi}{r^2 - s^2} \frac{\sum_{k=0}^n \frac{(n)_k}{k!} s^k t^{n-k} \sin[qs \sin(\phi + \psi) - k\phi - (n-k)\omega]}{[s^2 + 2st \cos(\phi - \omega) + t^2]^n}$$

**Proof** Let  $g(z) = \frac{\exp(qe^{i\psi} z)}{(z + te^{i\omega})^n}$ , then  $g(z)$  is defined and continuous on the closed disc  $\{z \in C \mid |z| \leq |r|\}$  and it is

analytic on the open disc  $\{z \in \mathbb{C} \mid |z| < |r|\}$ . If  $z = se^{i\phi}$ , then by Poisson integral formula for  $g(z)$  we obtain

$$\frac{\exp(qe^{i\psi} se^{i\phi})}{(se^{i\phi} + te^{i\omega})^n} = \frac{r^2 - s^2}{2\pi} \int_0^{2\pi} \frac{\exp(qe^{i\psi} re^{i\theta})}{[r^2 - 2rs \cos(\theta - \phi) + s^2] (re^{i\theta} + te^{i\omega})^n} d\theta \quad (13)$$

It follows that

$$\int_0^{2\pi} \frac{\exp[qre^{i(\theta+\psi)}] \cdot (re^{-i\theta} + te^{-i\omega})^n}{[r^2 - 2rs \cos(\theta - \phi) + s^2][r^2 + 2rt \cos(\theta - \omega) + t^2]^n} d\theta \quad (14)$$

$$= \frac{2\pi}{r^2 - s^2} \frac{\exp[qse^{i(\phi+\psi)}] \cdot (se^{-i\phi} + te^{-i\omega})^n}{[s^2 + 2st \cos(\phi - \omega) + t^2]^n}$$

By binomial theorem, we have

$$\exp[qr \cos(\theta + \psi)] = \sum_{k=0}^n \binom{n}{k} r^k t^{n-k} e^{i[qr \sin(\theta+\psi) - k\theta - (n-k)\omega]}$$

$$\int_0^{2\pi} \frac{\sum_{k=0}^n \binom{n}{k} r^k t^{n-k} e^{i[qr \sin(\theta+\psi) - k\theta - (n-k)\omega]}}{[r^2 - 2rs \cos(\theta - \phi) + s^2][r^2 + 2rt \cos(\theta - \omega) + t^2]^n} d\theta \quad (15)$$

$$\exp[qs \cos(\phi + \psi)] = \sum_{k=0}^n \binom{n}{k} s^k t^{n-k} e^{i[qs \sin(\phi+\psi) - k\phi - (n-k)\omega]}$$

$$= \frac{2\pi}{r^2 - s^2} \frac{\sum_{k=0}^n \binom{n}{k} s^k t^{n-k} e^{i[qs \sin(\phi+\psi) - k\phi - (n-k)\omega]}}{[s^2 + 2st \cos(\phi - \omega) + t^2]^n}$$

Using the equality of real parts of both sides of Eq. (15) yields Eq. (11) holds. Also, Eq. (12) can be obtained using the equality of imaginary parts of both sides of Eq. (15).

### 3. Examples

In the following, for the four types of definite integrals in this study, we provide two examples and use Theorems 1 and 2 to obtain their closed forms. In addition, Maple is used to calculate the approximations of these definite integrals and their solutions for verifying our answers.

#### 3.1. Example

In Eq. (5), if  $s = 2, r = 3, t = 4, \phi = \pi/6, \omega = \pi/3, m = 4$ , and  $n = 2$ , then the definite integral

$$\int_0^{2\pi} \frac{16 \cos\left(4\theta - \frac{2\pi}{3}\right) + 24 \cos\left(3\theta - \frac{\pi}{3}\right) + 9 \cos 2\theta}{\left[13 - 12 \cos\left(\theta - \frac{\pi}{6}\right)\right] \left[25 + 24 \cos\left(\theta - \frac{\pi}{3}\right)\right]^2} d\theta \quad (16)$$

$$= \frac{(36 + 16\sqrt{3})\pi}{14985 + 8100\sqrt{3}}$$

Next, we use Maple to verify the correctness of Eq. (16).

```
>evalf(int((16*cos(4*theta-2*Pi/3)+24*cos(3*theta-
Pi/3)+9*cos(2*theta))/((13-12*cos(theta-
Pi/6))*(25+24*cos(theta-Pi/3))^2),theta=0..2*Pi),18);
0.00689858296836092382
>evalf((36+16*sqrt(3))*Pi/(14985+8100*sqrt(3)),18);
0.00689858296836092381
```

On the other hand, if  $s = 3, r = -5, t = -6, \phi = -\pi/3, \omega = \pi/4, m = 5$ , and  $n = 3$  in Eq. (6), then

$$\int_0^{2\pi} \frac{-216 \sin\left(5\theta - \frac{3\pi}{4}\right) - 540 \sin\left(4\theta - \frac{\pi}{2}\right) - 450 \sin\left(3\theta - \frac{\pi}{4}\right) - 125 \sin 2\theta}{\left[34 + 30 \cos\left(\theta + \frac{\pi}{3}\right)\right] \left[61 + 60 \cos\left(\theta - \frac{\pi}{4}\right)\right]^3} d\theta \quad (17)$$

$$= \frac{-243\pi}{25000} \frac{216 \sin\left(\frac{5\pi}{12}\right) + 162 - 81\sqrt{2} - \frac{27\sqrt{3}}{2}}{\left[45 - 36 \cos\left(\frac{7\pi}{12}\right)\right]^3}$$

We also use Maple to verify the correctness of Eq. (17).

```
>evalf(int((-216*sin(5*theta-3*Pi/4)-540*sin(4*theta-
Pi/2)-450*sin(3*theta-Pi/4)-125*sin(2*theta))/
((34+30*cos(theta+Pi/3))*(61+60*cos(theta-i/4))^3),
theta=0..2*Pi),18);
-0.0000443409234263192035
>evalf(-243*Pi/25000*(216*sin(5*Pi/12)+162-
81*sqrt(2)-27/2*sqrt(3))/(45-36*cos(7*Pi/12))^3,18);
-0.0000443409234263192039
```

#### 3.2. Example

In Eq. (11), if  $s = -4, r = 6, t = -7, q = 2, \phi = 3\pi/4, \omega = -\pi/4, \psi = \pi/4$ , and  $n = 1$ , then

$$\int_0^{2\pi} \frac{\exp\left[12 \cos\left(\theta + \frac{\pi}{4}\right)\right] \left[ \begin{matrix} -7 \cos\left(12 \sin\left(\theta + \frac{\pi}{4}\right) + \frac{\pi}{4}\right) \\ + 6 \cos\left(12 \sin\left(\theta + \frac{\pi}{4}\right) - \theta\right) \end{matrix} \right]}{\left[52 + 48 \cos\left(\theta - \frac{3\pi}{4}\right)\right] \left[85 - 84 \cos\left(\theta + \frac{\pi}{4}\right)\right]} d\theta \quad (18)$$

$$= -\frac{\exp(8)\sqrt{2}\pi}{60}$$

Using Maple to verify the correctness of Eq. (18) as follows:

```
>evalf(int(exp(12*cos(theta+Pi/4))*(-7*cos(12*sin
(theta+Pi/4)+Pi/4)+6*cos(12*sin(theta+Pi/4)-theta))/
((52+48*cos(theta-3*Pi/4))*(85-4*cos(theta+Pi/4))),
theta=0..2*Pi),18);
-220.734122999910046
>evalf(-exp(8)*sqrt(2)*Pi/60,18);
-220.734122999910047
```

Also, let  $s = 3, r = -8, t = 9, q = 3, \phi = -\pi/4, \omega = 2\pi/3, \psi = -\pi/6$ , and  $n = 1$  in Eq. (12), then the definite integral

$$\int_0^{2\pi} \frac{\exp\left[-24 \cos\left(\theta - \frac{\pi}{6}\right)\right] \left[ \begin{matrix} 9 \sin\left(-24 \sin\left(\theta - \frac{\pi}{6}\right) - \frac{2\pi}{3}\right) \\ - 8 \sin\left(-24 \sin\left(\theta - \frac{\pi}{6}\right) - \theta\right) \end{matrix} \right]}{\left[73 + 48 \cos\left(\theta + \frac{\pi}{4}\right)\right] \left[145 - 144 \cos\left(\theta - \frac{2\pi}{3}\right)\right]} d\theta \quad (19)$$

$$= \frac{2\pi}{55} \frac{\exp\left(9 \cos \frac{5\pi}{12}\right) \left[ \begin{matrix} 9 \sin\left(-9 \sin \frac{5\pi}{12} - \frac{2\pi}{3}\right) \\ + 3 \sin\left(-9 \sin \frac{5\pi}{12} + \frac{\pi}{4}\right) \end{matrix} \right]}{90 + 54 \cos\left(\frac{11\pi}{12}\right)}$$

```

>evalf(int(exp(-24*cos(theta-Pi/6))*(9*sin(-
24*sin(theta-Pi/6)-2*Pi/3)-8*sin(-24*sin(theta-Pi/6)-
theta))/((73+48*cos(theta+Pi/4))*(145-144*cos(theta-
2*Pi/3))),theta=0..2*Pi),18);
0.1801875656110764
>evalf(2*Pi/55*(exp(9*cos(5*Pi/12))*(9*sin(-
9*sin(5*Pi/12)-2*Pi/3)+3*sin(-9*sin(5*Pi/12)+Pi/4)))/
(90+54*cos(11*Pi/12)),16);
0.1801875656110781

```

## 4. Conclusion

In this study, we mainly use Poisson integral formula to solve some definite integrals. In fact, the applications of this formula are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and use Maple to verify our answers. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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