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Some Fixed Point Theorems in Complete Dislocated Quasi-b-metric Space

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Abstract

In this paper, we main introduced some concepts and Ciric cyclic fixed point theorem in the complete dislocated quasi-b-metric space. We also can improve some fixed point theorems by Ciric cyclic fixed point theorem such as Kannan cyclic fixed point theorem. It is consist with [Klin-Eam. C, 2016]. Our results for such space consist with the metric space. And our theorems generalization and extend some results in the literature.

Keywords: fundamental mathematics, fixed point, dislocated quasi-b-metric space, Ciric cyclic contraction, Kannan cyclic contraction, Hardy-Roges cyclic contraction, Zamfirescu cyclic contraction

1. Introduction

It is well-known that fixed point theory is an important mathematical displine. It has been study extensively. The metric space is very clearly and some contraction have only one fixed point in metric space, such as Ciric contraction [Ciric, 1974; Senapati. T, 2016]. And Ciric contraction includes Kannan contraction [Choudhury. B. S, 2014], Hardy-Roges contraction [Kumari. P. S, 2016] and Zamfirescu contraction [Zamfirescu. T, 1972]. So the Ciric fixed point theorem can deduces the Kannan fixed point theorem, Hardy-Roges fixed point theorem and Zamfirescu fixed point theorem. They all the classical fixed point theorems in metric space. For the past years, some new type of metric spaces was introduced. Such as quasi-metric space was introduced in [Alegre. C, 2015; Gaba. Y. U, 2016; Liu Z, 1997; Noorwali. M, 2016], dislocated metric space was introduced in [Pasicki. L, 2015] and b-metric space was introduced in [Zhu. C, 2014]. In [Klin-Eam. C, 2015] introduce a new metric space called dislocated quasi-b-metric space. It is more generalization than others. In this paper, we main introduced Ciric cyclic fixed point theorem in complete dislocated quasi-b-metric space. And deduces some fixed point theorems such as Kannan cyclic fixed point theorem, Hardy-Roges cyclic fixed point theorem and Zamfirescu cyclic fixed point theorem. Our result generalization and consists the fixed point theorems with metric space.

2. Preliminaries

Definition 1 *L*et X is non-empty set, $w : X^2 \to [0,\infty)$, such that there exist a constant $\alpha > 1$, satisfied as follow:
(i)$w(x,y) = w(y,x) = 0$ implies that $x = y$, for all $x, y \in X$;
(ii)$w(x,y) \leq \alpha[w(x,z) + w(z,y)]$, for all $x, y, z \in X$.
Then, we called (X,w) is a dislocated quasi-b-metric space.

Definition 2 *L*et (X,w) is a dislocated quasi-b-metric space, $\{x_n\}$ is a sequence in it.Then x is called the limit of $\{x_n\}$, if

$$\lim_{n\to\infty} w(x_n, x) = lim_{n\to\infty} w(x, x_n) = 0.$$

Definition 3 *L*et (X,w) is a dislocated quasi-b-metric space, $\{x_n\}$ is a sequence in it, if

$$\lim_{m,n\to\infty} w(x_n, x_m) = lim_{n,m\to\infty} w(x_m, x_n) = 0.$$

Then $\{x_n\}$ is called a cauchy sequence.

Definition 4 *L*et (X,w) is a dislocated quasi-b-metric space, if all of the cauchy sequences are convergents in X, then

(X, w) is called the complete.

Definition 5 *(*[Klin-Eam, 2016])Let (X, w) is a dislocated quasi-b-metric space, A and B are nonempty closed subsets in it. A map $T : A \cup B \to A \cup B$ is called cyclic map, if $T(A) \subseteq B$ and $T(B) \subseteq A$.

Definition 6 *L*et (X, w) is a dislocated quasi-b-metric space, A and B are nonempty closed subsets in it. A map T is called Ciric cyclic contraction, if T is a cyclic mapping on $A \cup B$ and exist $h \in (0, 1)$, such that for $\alpha \geq 1, \alpha h < 1$ and all $x \in A$, $y \in B$,

$$w(Tx, Ty) \leqslant hmax\{w(x, y), w(x, Tx), w(y, Ty), w(x, Ty), w(y, Tx)\}.$$

Definition 7 *L*et (X, w) is a dislocated quasi-b-metric space, A and B are nonempty closed subsets in it. A map T is called Kannan cyclic contraction, if T is a cyclic mapping on $A \cup B$ and exist $h \in [0, \frac{1}{2})$, such that for $\alpha \geq 1, \alpha h < \frac{1}{2}$ and all $x \in A$, $y \in B$,

$$w(Tx, Ty) \leqslant h[w(x, Tx) + w(y, Ty)].$$

Definition 8 *L*et (X, w) is a dislocated quasi-b-metric space, A and B are nonempty closed subsets in it. T is called Hardy-Roges cyclic contraction, if T is a cyclic mapping on $A \cup B$ and exist $\sum_{i=1}^{5} a_i < 1$ such that for $\alpha \geq 1, \sum_{i=1}^{5} \alpha a_i < 1$ and all $x \in A$, $y \in B$,

$$w(Tx, Ty) \leqslant a_1 w(x, y) + a_2 w(x, Tx) + a_3 w(y, Ty) + a_4 w(x, Ty) + a_5 w(y, Tx).$$

Definition 9 *L*et (X, w) is a dislocated quasi-b-metric space, A and B are nonempty closed subsets in it. T is called Zamfirescu cyclic contraction, if T is a cyclic mapping on $A \cup B$ and exist $h \in (0, 1)$ such that for $\alpha \geq 1, \alpha h < 1$ and all $x \in A$, $y \in B$,

$$w(Tx, Ty) \leqslant hmax\{w(x, y), \tfrac{1}{2}[w(x, Tx) + w(y, Ty)], \tfrac{1}{2}[w(x, Ty) + w(y, Tx)]\}.$$

Lemma 1 *(X, w) is a dislocated quasi-b-metric space, T is Ciric cyclic contraction in it, for $x \in X$ and all $1 \leq i, j \leq n$, $i, j \in Z^+$. Then*

$$w(T^i x, T^j x) \leqslant h\delta(O^T(x; 0, n)).$$

Which $O^T(x; 0, n) = \{x, Tx, T^2x, ..., T^n x\}$ and $\delta(H) = sup\{w(x, y), x, y \in H\}$.

Proof.

$$\begin{aligned} w(T(T^{i-1}x), T(T^{j-1}x)) &\leq hmax\{w(T^{i-1}x, T^{j-1}x), w(T^{i-1}x, T^i x), w(T^{j-1}x, T^j x), w(T^{i-1}x, T^j x), w(T^{j-1}x, T^i x)\} \\ &\leq h\delta(O^T(x; 0, n)). \end{aligned}$$

Then, there exists some $1 \leq k_i \leq n$, such that

$$M_1 = max\{w(x, T^{k_1}x), w(T^{k_2}x, x), w(x, x)\} = \delta(O^T(x; 0, n)). \quad (1)$$

□

Lemma 2 $\delta(O^T(x; 0, \infty)) \leq M_2 \frac{\alpha}{1-\alpha h}$. *Which*

$$M_2 = max\{w(x, Tx), w(Tx, x), w(x, x)\}.$$

Proof. From lemma 1 , we obtain that

$$\delta(O^T(x; 0, 1)) \leq \delta(O^T(x; 0, 2)) \leq ... \leq \delta(O^T(x; 0, n)).$$

$$\delta(O^T(x; 0, \infty)) = sup\{\delta(O^T(x; 0, n)), n = 1, 2, ...\}.$$

When $\delta(O^T(x;0,n)) = w(x,x)$,

we obtain that

$$\delta(O^T(x;0,\infty)) \leq M_2 \frac{\alpha}{1-\alpha h}. \tag{2}$$

When $\delta(O^T(x;0,n)) = w(x,T^{k_1}x)$,

we obtain that

$$\begin{aligned} w(x,T^{k_1}x) &\leq \alpha[w(x,Tx) + w(Tx,T^{k_1}x)] \\ &\leq \alpha w(x,Tx) + \alpha h\delta(O^T(x;0,n)) \\ &= \alpha w(x,Tx) + \alpha h w(x,T^{k_1}x). \end{aligned}$$

Then

$$w(x,T^{k_1}x) \leq \frac{\alpha}{1-\alpha h} w(x,Tx). \tag{3}$$

Similarly, when $\delta(O^T(x;0,n)) = w(T^{k_2}x,x)$,

$$w(T^{k_2}x,x) \leq \frac{\alpha}{1-\alpha h} w(Tx,x). \tag{4}$$

So by (2), (3), (4),

$$\delta(O^T(x;0,\infty)) \leq M_2 \tfrac{\alpha}{1-\alpha h}.$$

□

3. Main Results

Theorem 1 *(X,w) is a complete dislocated quasi-b-metric space, if T is a Ciric cyclic contraction in it. Then, in $A \cap B$, T has only one fixed point.*

Proof. Let $x_0 \in A(fixed)$, suppose $m \geq n$.

First, we will proof $\{T^n x_0\}$ are Cauchy sequence.

$w(T^m x_0, T^n x_0) = w(T^{m-n+1}(T^{n-1}x_0), T(T^{n-1}x_0)) \leq h\delta(O^T(T^{n-1}x_0); 0, m-n+1))$.

So from (1), for some k_i, k_j, $1 \leq i, j \leq m-n+1$,

we have

$\delta(O^T(T^{n-1}x_0; 0, m-n+1)) =$

$$max\{w(T^{n-1}x_0, T^{k_i}(T^{n-1}x_0)), w(T^{k_j}(T^{n-1}x_0), T^{n-1}x_0), w(T^{n-1}x_0, T^{n-1}x_0)\}.$$

When $\delta(O^T(T^{n-1}x_0; 0, m-n+1)) = w(T^{n-1}x_0, T^{k_i}(T^{n-1}x_0))$,

as to

$$\begin{aligned} w(T^{n-1}x_0, T^{k_i}(T^{n-1}x_0)) &= w(T(T^{n-2}x_0), T^{k_i+1}(T^{n-2}x_0)) \\ &\leq h\delta(O^T(T^{n-2}x_0; 0, k_i+1)) \\ &\cdots \end{aligned}$$

$$\leq h^n \delta(O^T(x_0; 0, m)). \tag{5}$$

Similarly, when $\delta(O^T(T^{n-1}x_0; 0, m-n+1)) = w(T^{k_j}(T^{n-1}x_0), T^{n-1}x_0)$,

$$w(T^{k_j}(T^{n-1}x_0), T^{n-1}x_0) \leq h^n \delta(O^T(x_0; 0, m)). \tag{6}$$

When $\delta(O^T(T^{n-1}x_0; 0, m-n+1)) = w(T^{n-1}x_0, T^{n-1}x_0)$,

$$\begin{aligned} w(T^m x_0, T^n x_0) &\leq hw(T^{n-1}x_0, T^{n-1}x_0) \\ &= hw(T(T^{n-2}x_0), T(T^{n-2}x_0)) \\ &\leq h^2\delta(O^T(T^{n-2}x_0; 0, 1)) \\ &... \\ &\leq h^n\delta(O^T(x_0; 0, n)) \end{aligned}$$

$$\leq h^n\delta(O^T(x_0; 0, m)) \tag{7}$$

Thus, by (5), (6), (7), we have

$$w(T^m x_0, T^n x_0) \leq h^n\delta(O^T(x_0; 0, m)).$$

Similarly,

$$w(T^n x_0, T^m x_0) \leq h^n\delta(O^T(x_0; 0, m)).$$

Take $n \to \infty$, we get $w(T^m x_0, T^n x_0) \to 0$ and $w(T^n x_0, T^m x_0) \to 0$.

Since (X, w) is complete, we obtain that $\{T^n x_0\}$ is Cauchy sequence.

Second, we note $\{T^n x_0\}$ converges to some $z \in X$. And $\{T^{2n}x_0\}$ is a sequence in A, $\{T^{2n-1}x_0\}$ is a sequence in B. It is all tends to the same limit z.

Consider

$$w(T^n x_0, Tz) \leq hmax\{w(T^{n-1}x_0, z), w(T^{n-1}x_0, T^n x_0), w(z, Tz), w(T^{n-1}x_0, Tz), w(z, T^n x_0)\}.$$

Take limit as $n \to \infty$. we have

$$w(z, Tz) \leq hw(z, Tz).$$

as to $0 \leq h < 1$, we have $w(z, Tz) = 0$.

Similarly,

$w(Tz, T^n x_0) \leq hmax\{w(z, T^{n-1}x_0), w(z, Tz), w(T^{n-1}x_0, T^n x_0), w(z, T^n x_0), w(T^{n-1}x_0, Tz)\}$.

Take limit as $n \to \infty$. we have

$$w(Tz, z) \leq hw(z, Tz).$$

As to $w(z, Tz) = 0$, we have $w(Tz, z) = 0$.

Third, we prove z is the only one fixed point. Suppose there have another fixed point $z^* \in X$ on T, such that $Tz^* = z^*$.

$$w(z, z^*) = w(Tz, Tz^*) \leq hmax\{w(z, z^*), w(z^*, z)\}$$

and

$$w(z^*, z) = w(Tz^*, Tz) \leq hmax\{w(z, z^*), w(z^*, z)\}.$$

Then

$$w(z, z^*) \leq h[w(z, z^*) + w(z^*, z)];$$
$$w(z^*, z) \leq h[w(z, z^*) + w(z^*, z)].$$

We have

$$w(z, z^*) \leq \frac{h}{1-h} w(z^*, z);$$
$$w(z^*, z) \leq \frac{h}{1-h} w(z, z^*).$$

We obtain that $w(z, z^*) = w(z^*, z) = 0$.

Hence, z is the only one fixed point of T. The proof are completes.

□

Corollary 1 *(X, w) is a complete dislocated quasi-b-metric space, if T is Kannan cyclic contraction in it. Then, T has only one fixed point in $A \bigcap B$.*

Corollary 2 *(X, w) is a complete dislocated quasi-b-metric space, if T is Hardy-Roges cyclic contraction in it. Then, T has only one fixed point in $A \bigcap B$.*

Corollary 3 *(X, w) is a complete dislocated quasi-b-metric space, if T is Zamfirescu cyclic contraction in it. Then, T has only one fixed point in $A \bigcap B$.*

4. Discussion

Example 1 Let $X = [-1, 1]$, $T : X \longrightarrow X$ defined by $Tx = -\frac{x}{5}$ and $x \in A$, $y \in B$, if $A = [-1, 0]$, $B = [0, 1]$, then $Tx \in B$ and $Ty \in A$, defined $w : X^2 \rightarrow [0, \infty)$ that

$$w(x, y) = |x - y|^2 + 4|x| + 3|y|.$$

Proof. It is clearly that (X, w) is a complete dislocated quasi-b-metric space and T is a cyclic map on X.

Thus,

$$\begin{aligned}
w(Tx, Ty) &= |Tx - Ty|^2 + 4|Tx| + 3|Ty| \\
&= \frac{1}{25}|x - y|^2 + \frac{4}{5}|x| + \frac{3}{5}|y| \\
&\leq \frac{1}{25}(|x| + |y|)^2 + \frac{4}{5}|x| + \frac{3}{5}|y| \\
&\leq \frac{2}{25}|x|^2 + \frac{2}{25}|y|^2 + \frac{4}{5}|x| + \frac{3}{5}|y| \\
&\leq \frac{2}{25}[\frac{36}{25}|y|^2 + 4|y| + \frac{3}{5}|y|] + \frac{2}{25}[\frac{36}{25}|x|^2 + 4|x| + \frac{3}{5}|x|] \\
&\leq \frac{2}{25}[w(y, Ty) + w(x, Tx)] \\
&\leq \frac{2}{25}[w(x, y) + w(x, Tx) + w(y, Ty) + w(x, Ty) + w(y, Tx)]
\end{aligned}$$

As to $\frac{2}{25} \times 5 < 1$,

then

$$\begin{aligned}
w(Tx, Ty) &= |Tx - Ty|^2 + 4|Tx| + 3|Ty| \\
&\leq \frac{2}{25}[w(x, y) + w(x, Tx) + w(y, Ty) + w(x, Ty) + w(y, Tx)] \\
&\leq \frac{11}{25} max\{w(x, y), w(x, Tx), w(y, Ty), w(x, Ty), w(y, Tx)\}
\end{aligned}$$

$0 \leq h = \frac{11}{25} < 1$. It is satisfied theorem 1. Thus, 0 is the only one fixed point of T. □

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