Semigroups of Linear Operators

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Abstract

This paper will serve as a basic introduction to semigroups of linear operators. It will define a semigroup in the context of a physical problem which will serve to motivate further (elementary) theoretical development of linear semigroups including the *Hille-Yosida Theorem*. Applications and examples will also be discussed.

1 Introduction

Before defining what a semigroup is, one needs to recognize their global importance. Of course their importance cannot be fully realized until we have a clear definition and developed theory. However, in general, semigroups can be used to solve a large class of problems commonly known as evolution equations. These types of equations appear in many disciplines including physics, chemistry, biology, engineering, and economics. They are usually described by an initial value problem (IVP) for a differential equation which can be ordinary or partial. When we view the evolution of a system in the context of semigroups we break it down into transitional steps (i.e. the system evolves from state A to state B, and then from state B to state C). When we recognize that we have a semigroup, instead of studying the IVP directly, we can study it via the semigroup and its applicable theory. The theory of linear semigroups is very well developed [1]. For example, linear semigroup theory actually provides necessary and sufficient conditions to determine the well-posedness of a problem [3]. There is also theory for nonlinear semigroups which this paper will not address. This paper will focus on a special class of linear semigroups called C_0 *semigroups* which are semigroups of strongly continuous bounded linear operators. The theory of these semigroups will be presented along with some examples which tend to arise in many areas of application.

2 What is a Semigroup?

Let's begin with the most basic notion of a semigroup.

2.1 Definition (Semigroup) -

A *semigroup* is a set S coupled with a binary operation $*$ ($*$: $S \times S \to S$) which is associative. That is, $\forall\, x, y, z \in S$, $(x*y)*z = x*(y*z)$. Associativity can also be realized as $F(F(x, y), z) = F(x, F(y, z))$ where $F(x, y)$ serves as the mapping from $S \times S$ to S [4].

A semigroup, unlike a group, need not have an identity element e such that $x * e = x$, $\forall x \in S$. Further, a semigroup need not have an inverse. Therefore, many problems which can be solved with semigroups can only be solved in the forward direction (e.g. forward in time).

2.2 Simple Examples

Some of the simplest examples of semigroups are:

$$2S = \mathbb{R} \qquad * = \text{addition}$$
$$S = \mathbb{M}_{2\times 2}(\mathbb{R}) \qquad * = \text{matrix multiplication}$$

where $\mathbb{M}_{2\times 2}(\mathbb{R})$ = the set of 2×2 matrices with real entries [4].

While we have introduced the most general definition of a semigroup, this paper will focus on semigroups of linear operators. In particular, it will provide definitions, theory, examples, and applications of semigroups of linear operators (linear semigroups).

2.3 A More Concrete Example

To motivate the results about linear semigroups, consider the physical state of a system which is evolving with time (according to some physical law) as given by the following IVP (or abstract Cauchy problem):

$$\begin{aligned} \frac{d}{dt}u(t) &= A[u(t)] \qquad (t \geq 0) \\ u(0) &= f \end{aligned} \tag{1}$$

where $u(t)$ describes the state at time t which changes in time at a rate given by the function A. The solution of (1) is given by:

$$u(t) = e^{At} f. \tag{2}$$

A natural first question to ask is, "Is (1) well posed?" A well posed problem is one whose solution exists and is unique. Semigroup theory can determine when a problem is well posed and in order to use the theory, we need to know that we have a semigroup. So to continue with (2), let T operate on u as follows:

$$T(t) : u(s) \to u(t+s). \tag{3}$$

If we assume that A does not depend on time, then $T(t)$ is independent of s [3]. The solution, $u(t+s)$ at time $t+s$, can be computed as $T(t+s)$ acting on f. Likewise, if we painstakingly break down the process into two steps we have:

1^{st} Step: $T(s)(f) = u(s)$

2^{nd} Step: $T(t)(u(s)) = T(t)(T(s)(f)) = u(t+s) = T(t+s)(f)$.

2.4 The Semigroup Property

By transitionally breaking down the process of evolution, it is evident that we can reach the state of the system at time $t+s$ by either going directly from the initial condition to the state at time $t+s$ or by allowing the state to evolve over s time units (taking a snapshot), and then allowing it to evolve t more time units. Here the $T(\cdot)$ is acting like a transition operator [1]. The uniqueness of the solution gives reveals the *semigroup property* which is given by:

$$T(t+s) = T(t)T(s) \qquad (t, s > 0). \tag{4}$$

The semigroup property (4) of the family of functions, $\{T(t); t \geq 0\}$, is a composition (not a multiplication). Notice that $T(0)$ is the identity operator (I) (i.e. there is no transition at time zero and the initial data exists) [3].

2.5 More Properties

Now that we have seen the fundamental semigroup property, we want to understand how A (which governs the evolution of the system) and T relate to one another. We will first examine the scalar case. Two observations which may be preliminary indicators of the relationship are given as follows:

$$T(t)(f) = T(t)(u(0)) = u(t) = e^{At}f \tag{5}$$

$$\frac{d}{dt}T(t)(f) = A(T(t)(f)). \tag{6}$$

Notice that $u(t) = T(t)(f)$ solves (1) and suggests that:

$$T(t)(f) = e^{At}f \tag{7}$$

where A is the derivative of $T(t)$. In addition, each $T(t) : f \to e^{At}f$ is a continuous operator on $\mathbb{R}$, (or in an infinite dimensional setting, a Banach space X), which indicates the continuous dependence of $u(t)$ on f [3]. The initial data f should belong to the domain of A.

Upon inspection of (7), we have the following results:

i. $T(t)$ exhibits the semigroup property as in (4),

ii. $T(t)$ is a continuous function,

iii. $T(0)f = f$,

iv. $T(t) : \mathbb{R} \to \mathbb{R}$ is linear provided A is linear.

Again, since we are interested in linear semigroups, we will assume that A is linear. These observations bring forth the notion of C_0 semigroups.

3 C_0 Semigroups

Now that semigroups have been introduced in the framework of a physical problem, we should formally define a C_0 *semigroup*; a term which was introduced by Hille [3]. Generally we say that a C_0 semigroup is a strongly continuous one parameter semigroup of a bounded linear operator on a Banach Space X.

3.1 <u>Definition</u> (C_0 Semigroup) -

A C_0 *semigroup* (or strongly continuous semigroup) is a family, $T = \{T(t) \mid t \in \mathbb{R}^+\}$, of bounded linear operators from X to X satisfying:

i. $T(t+s) = T(t)T(s)\ \forall\ t, s \in \mathbb{R}^+$,

ii. $T(0) = I$, the identity operator on X, and

iii. $\lim_{t\to 0^+} T(t)f \to f$ for each $f \in X$ with respect to the norm on X [1].

The continuity condition given by (iii) arises naturally as we do not want our physical system in (1) to breakdown in time due to small measurement errors in the initial state (for example). From now on throughout this paper, the word *semigroup* will mean C_0 *semigroup.* A more careful inspection of the definition of semigroup may provoke the following question, "Can one replace (iii) by the condition:

$$\lim_{t\downarrow 0} \|T(t) - I\| = 0 \tag{8}$$

where $\|\cdot\|$ denotes the norm on X?" The answer is, NO! Semigroups that satisfy the property given in (8) are called *uniformly continuous* semigroups of bounded linear operators. The condition given in (8) is too strong for strongly continuous semigroups [1]. Uniformly continuous semigroups are thus a subset of strongly continuous semigroups. This paper will focus on the larger set of semigroups but will (as necessary) comment on the smaller set.

3.2 Some Questions

Now that we have an official definition of a semigroup, we want to answer the following three questions which attempt to unveil the relationship between $T(t)$ and A:

Q1. Given the semigroup $T(t)$, how can we find the operator A in $e^{At}f$?

Q2. Which operators A give rise to which semigroups?

Q3. Given A, how can we construct the corresponding semigroup $T(t)$?

The next section will address the answers to these questions.

4 Semigroup Generation

In this section we will explore the generation of semigroups. This will reveal the connection between A in (1) and $T(t)$ in (3). We will first examine the answer to $(Q1)$ as in the previous section. To motivate this, (7) suggests that "$T(t) = e^{tA}$." Notice that:

$$T'(t) = Ae^{tA} = AT(t)$$

and $T'(0) = A$. So perhaps we can obtain A by $A = \frac{d}{dt}T(t)|_{t=0}$. This leads to the following definition.

4.1 <u>Definition</u> (Generator) -

Let T be a semigroup. The *(infinitesimal) generator* of T, denoted by A, is given by the equation:

$$Af = \lim_{t \to 0^+} A_t f = \lim_{t \to 0^+} \frac{T(t)f - f}{t} \tag{9}$$

where the limit is evaluated in terms of the norm on X and f is in the domain of A iff this limit exists [1].

So, according to (9), the generator A is obtained by differentiating the semigroup T. From this we see that $u(\cdot) = T(\cdot)f$ solves (1). This answers ($Q1$).

4.2 On the Nature of A

Thus far, we have seen two types of linear semigroups, uniformly and strongly continuous semigroups. So in regard to ($Q2$), we pose the question, "Which operators A give rise to these two different types of semigroups?" Interestingly, we have yet to discuss what type of operator A is. In particular, is A a bounded (nice!) or unbounded (not so nice!) operator? Of course, as it almost always turns out, interesting problems are more difficult to work with. So in general, for most applications, A will be an unbounded operator. In fact, the difference between uniformly continuous and strongly continuous semigroups is just the nature of A. Precisely, A is the generator of a uniformly continuous semigroup T iff A is a bounded operator. So, if T is strongly continuous and fails to be uniformly continuous, then T will have an unbounded generator A [1], [2]. The fact that A can be unbounded in a setting such as that in (1) should not be startling since we may initially associate A with the operator $\frac{d}{dx}$ which we know is unbounded. This does not do ($Q2$) justice, and we will return to this question as we develop further.

4.3 On the Nature of $T(t)$

To examine ($Q3$), recall (7). It (i.e. (7)) should not be a clear fact at this time as it has not been rigorously proved. However, what should be clear is that the exponential in (7) may play a role in uncovering $T(t)$ from A. Therefore, we need to understand the exponential function in (perhaps) one of the following contexts:

i. $e^{At} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$

ii. $\lim_{n \to \infty} \left(1 - \frac{tA}{n}\right)^{-n}$

iii. $e^{At} = \mathcal{L}^{-1}\left(\frac{1}{\lambda - A}\right)$ for $\lambda > Re(A)$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform [3]. Furthermore, now that we know that the boundedness of A determines the type (continuity) of the semigroup generated, we should answer this question in the context of bounded/unbounded operators. Let's look at the easier (bounded) case first. This leads us to the following theorem where we view the exponential as a power series as in (i).

4.4 <u>Theorem</u> -

Let A be a bounded operator from X to X. Then,

$$T = \left\{ T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} : t \in \mathbb{R}^+ \right\} \tag{10}$$

is a uniformly continuous semigroup.

<u>Proof -</u>

Since A is bounded we know $\|A\| < \infty$ and thus

$$\sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$$

converges for each $t \geq 0$ to the bounded linear operator $T(t)$. We know the semigroup property (4) holds since

$$\left(\sum_{i=0}^{\infty} \frac{(t)^i}{i!} \right) \left(\sum_{j=0}^{\infty} \frac{(s)^j}{j!} \right) = \sum_{k=0}^{\infty} \frac{(t+s)^k}{k!}.$$

Clearly $T(0) = I$. Finally, to distinguish $T(t)$ as uniformly continuous semigroup:

$$\|T(t) - I\| = \left\| \sum_{n=1}^{\infty} \frac{(tA)^n}{n!} \right\| \leq \sum_{n=1}^{\infty} \frac{t^n \|A\|^n}{n!} = e^{t\|A\|} - 1$$

and $e^{t\|A\|} - 1 \to 0^+$ as $t \to 0$ and the proof is complete [2].

□

It should be noted, and it is not hard to show, that A is in fact the gererator of $T(t)$ [5]. So we have answered $(Q3)$ in the context of a bounded generator A in which case given A, we construct the semigroup $T(t)$ as $T(t) = e^{tA}$. Therefore, for a bounded generator A, the suggestion in (7) is true. Furthermore, the map $t \to T(t) = e^{tA}$ is differentiable. But how can we construct the semigroup when A is unbounded? Furthermore, getting back to $(Q2)$, what properties does A possess to make it a generator of strongly continuous semigroups? The answer to this question lies deeply within the "mother theorem" of linear semigroups called the *Hille-Yosida Theorem.* We will devote the entire next section to building up to and presenting this theorem.

5 Hille-Yosida Theorem

The previous section showed us the basic results for bounded generators and their uniformly continuous semigroups. In this section, we are forced to look at unbounded generators to investigate many "interesting" problems. It would be convenient to use the approach we took for bounded generators which was using the power series for e^{tA}. However, convergence of this series is not likely when A is unbounded. Recall that it remains to be shown what (exactly) makes A a generator of a semigroup and, once that is shown, how do we recover the semigroup $T(t)$ from the generator? Let's first address the question of what A is, exactly.

5.1 Notes on Resolvents

Ultimately we are looking for relationships between A and $T(t)$. In doing so, one may stumble upon a connection between $T(t)$ and the resolvent operator of A. Recall the resolvent set of A is given by $\rho(A)$ and is the set of complex numbers λ for which $\lambda I - A$ is invertible. The resolvent of A is a family of bounded linear operators which is denoted by $R(\lambda, A)$ and is given by $R(\lambda, A) = (\lambda I - A)^{-1}$ where $\lambda \in \rho(A)$. To see its connection to $T(t)$, consider the following:

$$\frac{1}{\lambda - A} = \int_0^\infty e^{-\lambda t} e^{tA} dt$$

where $A \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ with $Re(\lambda) > A$ [3]. This gives rise to the operator version:

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t) f dt$$

which is valid provided $\lambda > 0$ [3]. So the resolvent operator can be thought of as the Laplace transform of the semigroup. Furthermore, in light of viewing e^{tA} as $\lim_{n\to\infty} \left(1 - \frac{tA}{n}\right)^{-n}$, we can re-write this expression as:

$$e^{tA} = \lim_{n\to\infty} \left[\left(1 - \frac{tA}{n}\right)^{-1}\right]^n = \lim_{n\to\infty} \left[\frac{n}{t}\left(\frac{n}{t} I - A\right)^{-1}\right]^n. \tag{11}$$

So imbedded within this formula for e^{tA} is the resolvent operator $(\lambda I - A)^{-1}$ where $\lambda = \frac{n}{t}$. Before we proceed further with resolvent sets (which we will return to later), we will introduce corollaries and theorems that will lead up to the Hille-Yosida Theorem.

5.2 Theorem -

Let $T(t)$ be a semigroup. There exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that the following holds:

$$\|T(t)\| \leq M e^{\omega t} \quad \text{for } 0 \leq t < \infty. \tag{12}$$

Proof -

Choose a constant $M \geq 1$ such that $\|T(t)\| \leq M$ for all $0 \leq t \leq 1$. Let $\omega = log M$. Then for each $t > 0$ and if n is the least integer $\geq t$ then:

$$\|T(t)\| = \left\|T\left(\sum_{k=1}^{n} \frac{t}{n}\right)\right\| = \left\|\prod_{k=1}^{n}\left(T\left(\frac{t}{n}\right)\right)\right\| = \left\|\left(T\left(\frac{t}{n}\right)\right)^n\right\| \leq M^n \leq M^{t+1} = M e^{\omega t}$$

completing the proof.

5.3 Corollary -

If $T(t)$ is a semigroup then for each $f \in X$, $t \to T(t)f$ is a continuous function from $\mathbb{R}^+$ to X.

Proof -

Let $t, h \geq 0$ and $f \in X$. Then we have:

$$\|T(t+h)f - T(t)f\| = \|T(t)T(h)f - T(t)f\| \leq \|T(t)\|\|T(h)f - f\| \leq Me^{\omega t}\|T(h)f - f\|$$

and for $t \geq h \geq 0$ we have:

$$\begin{aligned}\|T(t-h)f - T(t)f\| = \|T(t-h)f - T(t-h+h)f\| &= \|T(t-h)f - T(t-h)T(h)f\| \\ &\leq \|T(t-h)\|\|f - T(h)f\| \leq Me^{\omega t}\|f - T(h)f\|\end{aligned}$$

by (12). Thus we have continuity [5].

□

5.4 Theorem -

Let $T(t)$ be a semigroup generated by A. Then the following hold:

i) For each $f \in \mathcal{D}(A)$,

$$T(t)f \in \mathcal{D}(A) \text{ (domain of } A) \quad \text{and} \quad AT(t)f = T(t)Af \quad \forall t \geq 0 \tag{13}$$

ii) For each $f \in \mathcal{D}(A)$ and $T(t)f \in \mathcal{D}(A)$,

$$\frac{d}{dt}T(t)f = AT(t)f = T(t)Af. \tag{14}$$

Proof -

i) Let $f \in \mathcal{D}(A)$ and fix $t \geq 0$. Then, for $s > 0$, A_s as in (Def. 4.1) and using $T(s)T(t) = T(t)T(s) = T(s+t)$:

$$A_sT(t)f = \frac{(T(s)T(t)f - T(t)f)}{s} = \frac{(T(t)T(s)f - T(t)f)}{s} = T(t)\frac{(T(s)f - f)}{s}. \tag{15}$$

As $s \to 0^+$, the right hand side of (15) converges to $T(t)(Af)$ (Def. 4.1) since $f \in \mathcal{D}(A)$ and $T(t)$ is continuous on X. Therefore,

$$\lim_{s \to 0^+} A_sT(t)f = T(t)Af$$

which gives $T(t)f \in \mathcal{D}(A)$ and $AT(t)f = T(t)Af$ as desired [1].

ii) Let $f \in \mathcal{D}(A)$ and $h > 0$. Consider the right-hand limit,

$$\begin{aligned}\lim_{h\to 0^+} \frac{(T(t+h)f - T(t)f)}{h} &= \lim_{h\to 0^+} \frac{T(t)T(h)f - T(t)f)}{h} \\ &= \lim_{h\to 0^+} \left(\frac{(T(h)-I)}{h}\right) T(t)f = AT(t)f = T(t)Af\end{aligned}$$

since $T(t)f \in \mathcal{D}(A)$ by (i) [1].
Consideration of the appropriate left-hand limit can be found in [5].

□

5.5 Theorem -

Let $T(t)$ be a semigroup generated by A. Then the following hold:

i) For each $f \in X$,

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} T(s)fds = T(t)f. \tag{16}$$

ii) For each $f \in X$,

$$\int_0^t T(s)fds \in \mathcal{D}(A) \quad \text{and} \quad A\left(\int_0^t T(s)fds\right) = T(t)f - f. \tag{17}$$

iii) For each $f \in \mathcal{D}(A)$,

$$T(t)f - T(s)f = \int_S^t T(\tau)Afd\tau = \int_s^t AT(\tau)fd\tau. \tag{18}$$

Proof -

i) The proof of (16) follows from the continuity of $t \to T(t)$ given in Corollary 5.3 [5].

ii) Let $f \in X$ and $h > 0$. Then,

$$A_h\left(\int_0^t T(s)fds\right) = \frac{T(h)-I}{h}\int_0^t T(s)fds = \frac{1}{h}\int_0^t (T(h)-I)T(s)fds.$$

Then, by the semigroup property,

$$\frac{1}{h}\int_0^t (T(h)-I)T(s)fds = \frac{1}{h}\int_0^t (T(s+h)f - T(s)f)\,ds$$

which gives:

$$\frac{1}{h}\int_{h}^{t+h} T(s')f ds' - \frac{1}{h}\int_{0}^{t} T(s)f ds = \frac{1}{h}\int_{t}^{t+h} T(s)f ds - \frac{1}{h}\int_{0}^{h} T(s)f ds. \tag{19}$$

Letting $h \to 0^+$ and applying the Fundamental Theorem of Calculus to (19) yields $T(t)f - T(0)f = T(t)f - f$ which proves (ii) [1].

iii) The proof of (18) follows from integrating (14) from s to t [5].

□

Recall that our goal is to describe the A's that generate the $T(t)$'s. The following theorem is a precursor to the Hille-Yosida Theorem and provides some additional information pertinent to our goal.

5.6 Theorem -

If A is the generator of a semigroup $T(t)$, then $\mathcal{D}(A)$ is dense in X and A is a closed operator.

Proof -

First, to show that $\mathcal{D}(A)$ is dense in X, we must show that $\overline{\mathcal{D}(A)} = X$. Take f an arbitrary element of X and let f_t be given by:

$$f_t = \frac{1}{t}\int_0^t T(s)f ds$$

By part (ii) of Theorem 5.5, $f_t \in \mathcal{D}(A)$ and furthermore, part (i) of the same theorem gives:

$$f_t = \frac{1}{t}\int_0^t T(s)f ds \to T(0)f = f \quad \text{as} \quad t \to 0^+$$

Thus, $f = \lim_{n\to\infty} f_{\frac{1}{n}} \subset \mathcal{D}(A)$ where f was chosen arbitrarily so that $\overline{\mathcal{D}(A)} = X$[1].

It remains to be shown that A is a closed operator. Recall that A closed means if $f_n \in \mathcal{D}(A)$, $f_n \to f$, and $Af_n \to g$, then $f \in \mathcal{D}(A)$ and $Af = g$. So let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{D}(A)$ with $f_n \to f$ and $Af_n \to g$ as $n \to \infty$. By part (iii) of Theorem 5.5, we have:

$$T(t)f_n - f_n = \int_0^t T(s)Af_n ds \tag{20}$$

for each $n \geq 1$ and $t > 0$. By Theorem 5.2, we have $\|T(s)\| \leq Ce^{\omega t}$ where C is a constant. Therefore,

$$\left\|\int_0^t T(s)Af_n ds - \int_0^t T(s)g ds\right\| \leq \int_0^t \|T(s)\|\|Af_n - g\|ds \leq C\int_0^t \|Af_n - g\|ds$$
$$\leq Ct\|Af_n - g\| \to 0 \quad \text{as} \quad n \to \infty$$

So, letting $n \to \infty$ in (20) yields

$$\begin{aligned} T(t)f - f &= \int_0^t T(s)g ds \\ \frac{T(t)f - f}{t} &= \frac{1}{t}\int_0^t T(s)g ds \\ \lim_{t\to 0^+}\frac{T(t)f - f}{t} &= \lim_{t\to 0^+}\frac{1}{t}\int_0^t T(s)g ds \\ A_t f &= g. \end{aligned}$$

Therefore, by Def. 4.1, we conclude that $f \in \mathcal{D}(A)$ and $Af = g$. Thus A is closed [3].

□

5.7 <u>Theorem</u> -

A semigroup is uniquely determined by its generator.

<u>**Proof**</u> -

Let T and S be two semigroups having the same generator A. Let $f \in \mathcal{D}(A)$ and let $t > 0$. Define $u : [0, t] \to X$ by $u(s) = T(s)S(t - s)f$. Then,

$$\frac{du(s)}{ds} = T(s)(-A)S(t - s)f + T(s)AS(t - s)f = 0$$

giving $u =$ *constant* on $[0, t]$. Therfore, since u is constant,

$$T(t)f = u(t) = u(0) = S(t)f.$$

Thus, $T = S$ [3].

□

Now that we have introduced some elementary theory, we will introduce a defintion which we will use in the Hille-Yosida Theorem since the details become much easier to work with.

5.8 <u>Definition</u> (Semigroup of Contractions) -

A semigroup $T(t)$ is a *semigroup of contractions* when $M = 1$ and $\omega = 0$ in (12) [5]. That is, $\|T(t)\| \leq 1$.

As stated in [3], "Roughly speaking, for most purposes it is enough to consider only contraction semigroups." For a full explanation see [3]. So, without further anticipation, we will present the Hille-Yosida Theorem.

5.9 Theorem (Hille-Yosida Theorem) -

A linear unbounded operator A is the generator of a (C_0) semigroup iff:

i. A is a closed operator,

ii. A has dense domain $(\mathcal{D}(A))$,

iii. for each $\lambda > 0$, $\lambda \in \rho(A)$, and

iv. $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$.

The Hille-Yosida Theorem is very powerful as it gives us both necessary and sufficient conditions. While the proof of this theorem is quite difficult and lengthy, the theorem itself provides a much more justified answer to $(Q2)$ as it describes A's character in further detail. We will prove the necessity to provide some insight into the theorem. The the proof of sufficiency will not be provided but can be found in [2], [3], and [5]. To achieve sufficiency, a more extensive background (in the form of lemmas) is required. In addition, the insight gained from proving sufficiency will not be put to full use as this paper will not rigorously discuss exactly how we view $T(t)$ as an exponential.

Proof (Hille-Yosida Theorem)

Necessity. The proofs of (i) and (ii) are given by Theorem 5.6. To prove (iii), notice that for each $\lambda > 0$, $\{e^{-\lambda t}T(t) : t \in \mathbb{R}\}$ is a semigroup of contractions whose generator is computed by,

$$\lim_{t \to 0^+} \frac{e^{-\lambda t}T(t)f - f}{t} = \lim_{t \to 0^+} \frac{-\lambda e^{-\lambda t}T(t)f + e^{-\lambda t}Af}{1} = -\lambda f + Af \tag{21}$$

using L'Hôpital's Rule which gives the generator as $A - \lambda I$ with domain $\mathcal{D}(A)$. Applying (17) to this semigroup gives:

$$\begin{aligned} -e^{-\lambda t}T(t)f + f &= (\lambda I - A)\int_0^t e^{-\lambda s}T(s)f ds \quad f \in X \\ -e^{-\lambda t}T(t)f + f &= \int_0^t e^{-\lambda s}T(s)(\lambda I - A)f ds \quad f \in \mathcal{D}(A) \end{aligned}$$

Letting $t \to \infty$ A we know $\int_0^\infty e^{-\lambda s}T(s)f ds \in \mathcal{D}(A)$ since A is closed. This result, together with the dominated convergence theorem give:

$$\begin{aligned} f &= (\lambda I - A)\int_0^\infty e^{-\lambda s}T(s)f ds \quad f \in X \\ f &= \int_0^\infty e^{-\lambda s}T(s)(\lambda I - A)f ds \quad f \in \mathcal{D}(A) \\ (\lambda I - A)^{-1}f &= \int_0^\infty e^{-\lambda s}T(s)f ds \quad f \in X, \lambda > 0. \end{aligned}$$

So we conclude that $(\lambda I - A) : \mathcal{D}(A) \to X$ is $1-1$ (i.e. bijective) and $(\lambda I - A)^{-1}$ is a bounded linear operator on X and we have $\lambda \in \rho(A)$. This proves (iii).

To prove (iv),

$$\|R(\lambda, A)\| = \|(\lambda I - A)^{-1} f\| \leq \int_0^\infty e^{-\lambda s} \|T(s)\| \|f\| ds \leq \frac{\|f\|}{\lambda} \quad f \in X, \lambda > 0.$$

Thus, the proof of necessity is complete [3].

□

One should notice that the resolvent of A is the Laplace Transform of the semigroup. ?Coincidently? we should expect to obtain the semigroup by inverting Laplace transform (which is one way to view e^{tA}). Probably not coincidence. So while the Hille-Yosida theorem fills in some of the gaps, one perplexing one remains.

5.10 The Missing Link

One missing link remains. We have seen to how to get from $T(t)$ to A via semigroup differentiation:

$$Af = \lim_{t \to 0^+} \frac{T(t)f - f}{t}.$$

We have also seen how to manuever from the A to its resolvent (which can be reversed) as:

$$\begin{aligned} R(\lambda, A) &= (\lambda I - A)^{-1} \\ A &= \lambda - R(\lambda, A)^{-1} \end{aligned}$$

and also from $T(t)$ to the resolvent of A as:

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t) dt. \tag{22}$$

More, importantly, we would like to touch on the answer to $(Q3)$ in which we are supposed to reconstruct $T(t)$ from A. In this section, we will provide results in the absence of rigor to give the reader an idea of how this can be accomplished. Recall that when A is bounded, we had $T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$. To handle the case of A unbounded, we can try to approximate A by a sequence $\{A_n\}_{n \in \mathbb{N}}$ of bounded operators and hope that:

$$e^{tA} = \lim_{n \to \infty} e^{tA_n}.$$

One such approximation of A, termed the *Yosida approximation*, is given by:

$$A_\lambda = \lambda A R(\lambda, A) = \lambda^2 R(\lambda, A) - \lambda I.$$

Interestingly, $T(t)f = \lim_{\lambda \to \infty} e^{tA_\lambda} f$ for $f \in X$ [5]. Therefore, in some sense, a strongly continuous semigroup with generator A is obtained as the limit of a sequence of uniformly continuous semigroups generated by the bounded linear operators given by the A_λ's.

As previously suggested in (11), one may also accomplish this task of going from A to $T(t)$ by:

$$T(t)f = \lim_{n \to \infty} \left(I - \frac{t}{n} A \right)^{-n} f = \lim_{n \to \infty} \left[\frac{n}{t} R\left(\frac{n}{t}, A \right) \right]^n f \quad \text{for } f \in X$$

as given in [5]. This method uses form (ii) of the exponential formula given in Sec. 4.3 coupled with the resolvent set. Therefore, despite the lack of rigor, this section gives the reader an idea of how we view $T(t)$ as an exponential.

6 Applications and Examples

As was stated at the beginning of the paper, recognizing problems to which semigroup theory can be applied to is important as the theory accompanying semigroups can be a powerful tool. In fact semigroup theory can determine if a problem is well-posed. This gives rise to the following Theorem.

6.1 Theorem (Well Posed Theorem) -

The IVP given by (1) (with A being linear) is well posed iff A is the generator of a semigroup T. In this case the unique solution of (1) is given by $u(t) = T(t)(f)$ for f in the domain of A [3].

This turns out to be quite important as it provides both necessary and sufficient conditions to determine if a problem is well-posed. We will now introduce some examples of semigroups. Many examples fall into the categories of: translations, fractional integration, harmonic functions, stochastic processes, difussion equations and ergodic theory [4]. We will look at three examples.

6.2 The Heat Equation

We are interested in using our knowledge of semigroups in a slightly more concrete example. In particular, we will look at the solution of the heat equation and show it is given by a semigroup. In this setting, let $X = L^p(\mathbb{R})$, $1 \leq p < \infty$. Recall that the heat equation as given by:

$$\begin{aligned} u_t &= u_{xx} \qquad -\infty \leq x \leq \infty \\ u(x,0) &= f. \end{aligned} \tag{23}$$

Using Fourier Transform methods, the solution to (23) can be written as

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} f(y)dy. \tag{24}$$

The *heat kernel* is given by $K_t(s) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(s)^2}{4t}}$ and we can write the solution to the heat equation as a convolution:

$$u(x,t) = K_t * f. \tag{25}$$

So the solution of (23) is a semigroup on X written as

$$T(t)f(s) = \frac{1}{4\pi t} \int_{-\infty}^{\infty} e^{\frac{-(s-r)^2}{4t}} f(r)dr \quad t > 0, x \in \mathbb{R}, \text{ and } f \in X \tag{26}$$

and we set $T(0) = I$. This ((26)) is called the *Gauss-Weierstrass semigroup.* To show it satisfies the semigroup property we must show:

$$T(a+b)f(s) = T(a)T(b)f(s). \tag{27}$$

Symbolically, $T(a+b)f(s)$ is given by:

$$T(a+b)f(s) = K_{a+b} * f(s). \tag{28}$$

Likewise, symbolically writing $T(a)T(b)f(s)$ we have:

$$T(a)T(b)f(s) = T(a)[K_b * f(s)] = K_a * [K_b * f(s)] = [K_a * K_b] * f(s) \tag{29}$$

since the operation of convolution is associative. Therefore, to show that (27) holds, it suffices to show that

$$K_{a+b}(x) = K_a * K_b(x) \tag{30}$$

which is equivalent to showing:

$$\frac{1}{\sqrt{4\pi(a+b)}} e^{\frac{-x^2}{4(a+b)}} = \frac{1}{\sqrt{4\pi a}} \frac{1}{\sqrt{4\pi b}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{4a}} e^{\frac{-(x-y)^2}{4b}} dy. \tag{31}$$

Although this requires many manipulations, we will go through the argument to prove that we do in fact have a semigroup.
Working with the right-hand side of (31) (without the constant) gives:

$$\int_{\mathbb{R}} e^{\frac{-[y^2(a+b)-2axy+x^2a]}{4ab}} dy = e^{\frac{-x^2a}{4ab}} \int_{\mathbb{R}} e^{\frac{-(a+b)}{4ab}[y^2 - \frac{2xa}{a+b}y]} dy = e^{\frac{-x^2}{4b}} \int_{\mathbb{R}} e^{\frac{-(a+b)}{4ab}[y-\frac{xa}{a+b}]^2 + \frac{x^2a}{4b(a+b)}} dy$$

$$= e^{\frac{-x^2}{4b}} e^{\frac{x^2a}{4b(a+b)}} \int_{\mathbb{R}} e^{\frac{-(a+b)}{4ab}[y-\frac{xa}{a+b}]^2} dy = e^{\frac{-x^2}{4(a+b)}} \int_{\mathbb{R}} e^{\frac{-(a+b)}{4ab}u^2} du = e^{\frac{-x^2}{4(a+b)}} \int_{\mathbb{R}} e^{-\left(\sqrt{\frac{a+b}{4ab}}\, u\right)^2} du$$

where $u = \left(y - \frac{xa}{a+b}\right)$. Now making the change of variables $t = \sqrt{\frac{a+b}{4ab}}\, u$ we have:

$$e^{\frac{-x^2}{4(a+b)}} 2\sqrt{\frac{ab}{a+b}} \int_{\mathbb{R}} e^{-t^2} dt = e^{\frac{-x^2}{4(a+b)}} 2\sqrt{\frac{ab\pi}{a+b}}$$

by evaluating the Gaussian integral. Now in order to show (31) we need to verify that:

$$\frac{1}{\sqrt{4\pi(a+b)}} = 2\sqrt{\frac{ab\pi}{a+b}} \frac{1}{\sqrt{4\pi a}\sqrt{4\pi b}}.$$

By simple calulation of the above, we see that (31) holds and the semigroup property is verified for the heat equation.

6.3 Poisson Semigroup

We introduce the *Poisson semigroup* within the space $X = L^p(\mathbb{R})$, $1 \le p < \infty$. For $t > 0$, define $T(t)$ on X by:

$$T(t)f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t}{t^2 + (x-y)^2} f(y) dy \quad x \in \mathbb{R} \text{ and } f \in X.$$

We have $T(t)f = P_t * f$ where the kernel is given as:

$$P_t(x) = \frac{1}{\pi}\frac{t}{t^2+x^2}.$$

We can evaluate the Fourier transform of P_t, denoted by $\mathcal{F}$, as:

$$(\mathcal{F}P_t)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t}{t^2+u^2}e^{ixu}du.$$

This can be evaluated using standard contour integration methods and the Residue Theorem. Using the Residue Theorem:

$$\frac{t}{\pi}\lim_{u\to it}\frac{e^{iux}}{u+it} = \frac{t}{\pi}\frac{e^{-tx}}{2it} = \frac{1}{2\pi i}e^{-xt}$$

which gives $(\mathcal{F}P_t)(x) = e^{-xt}$ for $x \geq 0$. To account for $x \in \mathbb{R}$, $(\mathcal{F}P_t)(x) = e^{-|x|t}$. So for $f \in X$, we have:

$$(\mathcal{F}(T(t)f))(x) = e^{-|x|t}(\mathcal{F}f)(x) \quad x \in \mathbb{R}.$$

Since $e^{-|x|s}e^{-|x|t} = e^{-|x|(s+t)}$, the semigroup property is satisfied. The Poisson semigroup arises in many instances since the kernel, $P_t x$, is a fundamental solution to Laplace's equation $\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0\right)$ in the region $\{(x,t)|x \in \mathbb{R}, t > 0\}$.

6.4 Translation Semigroups

Now we will introduce a class of semigroups called *translation semigroups*. In this setting, let $X = C_0([0,\infty))$ be the Banach space of functions f which are continuous on $[0,\infty)$ (continuous on the right at 0) and for which $f(x) \to M < \infty$ as $x \to \infty$ with respect to the sup norm. For $t \geq 0$, define $T(t)$ on X by:

$$(T(t)f)(x) = f(x+t).$$

Here the operator $T(t)$ translates the function $f \in C_0$ to the left by t units and forms a semigroup. The semigroup property is satisfied as translation by $t+s$ units is the same as a translation by t units followed by a translation by s units. Also, $T(0) = I$ is satisfied. Furthermore, $\lim_{t\to 0^+} T(t)f = f$ since:

$$\lim_{t\to 0^+}\sup_x \|f(x+t) - f(x)\| = 0.$$

In addition, our translation semigroup forms a contraction semigroup since $\|T(t)f\| = \|f\|$ which gives $\|T(t)\| = 1$.

6.5 What Can't Semigroups Do?

This section is intended to provide a short "list" of where (specifically) semigroups arise. This list should serve as evidence of how semigroups make their mark in many disciplines.

1. Feller Markov Processes
2. Conrol Theory
3. Population Growth Models
4. Linear Transport (Boltzmann) Equations
5. Delay Differential Equations
6. Integro-Differential Equations

These are just to name a few. To conclude, evolution equations arise in many disciplines of science. An abstract way to study and dissect these equations is through semigroups. Using semigroups is advantageous as the associated theory is quite rich. Studying semigroups, as I have for this paper, heighten your awareness of their prevalence throughout applied mathematics.

7 References

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