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A Method to Construct Sets of Commuting Matrices

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Abstract

A method for constructing sets of matrices that pairwise commute is presented. The sets are defined such that each matrix is a combination of basic matrices. An iterative algorithm is given, where the construction approach aims to obtain appropriate basic matrices. A numerical example illustrate the proposed method.

Keywords: commuting matrices, iterative method

1. Introduction

Commuting matrices is an active topic both in pure and applied mathematics. They appear in a variety of applications in physical and general sciences (McCarthy & Shalit, 2013; Bourgeois, 2013; De Seguins, 2013; Ogata, 2013; Shastry, 2011; Yuzbashyan & Shastry, 2013), where several theoretical and numerical works on differential equations, matrix polynomials equations and general matrix equations get some properties of scalars (Brewer et al., 1986; Gohberg et al., 1982). In such contexts sets of matrices which pairwise commute are needed in numerical experiments. However, examples with commuting matrices often appear in works where commutativity is not the primary concern (Tisseur & Meerbergen, 2001; Higham & Kim, 2001; Guo et al., 2009; Han & Kim, 2010).

The classical way to obtain matrices that commute in pairs is to consider the solutions of equation $AX = XA$. In such case any two solutions of this equation commute if and only if the matrix A is nonderogatory (Gantmacher, 1960). Probably the most simple method for practical experiments is to consider the polynomials of a matrix B (Dennis & Weber, 1978), in the same way we have that two polynomials in B commute if and only if the matrix B is nonderogatory. Besides that, although there are works dealing with rings and other algebraic structures of commuting matrices, these are not of ease manipulation for numerical purposes (Suprenenko, 1968; Song, 1999; Britnell & Wildon, 2011).

Our objective here is to present a method for constructing sets of commuting matrices. Summarizing the remainder of this paper, in section 2 we develop the support theory, in section 3 we state the method and in section 4 we give a numerical example together with some practical considerations.

2. Support Theory

We consider the set of complex matrices of order n

$$\mathcal{V}_{n_k} = \left\{ \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{bmatrix}, \quad v_{ij} = \sum_{l=1}^{k} \alpha_{(ij)_l} y_l \right\},$$

where v_{ij} are multivariate linear polynomials in $y_1, y_2, \dots, y_k \in \mathbb{C}$, with coefficients $\alpha_{(ij)_l} \in \mathbb{C}$, $i, j = 1, \dots, n$ and $l = 1, \dots, k$ (Rosa et al., 2008).

Alternatively we can write this set as

$$\mathcal{V}_{n_k} = \{y_1A_1 + y_2A_2 + \dots + y_kA_k : \quad y_1, y_2, \dots, y_k \in \mathbb{C}\},$$

where A_i are $n \times n$ complex matrices, we call them the basic matrices of the set $\mathcal{V}_{n_k}$.

Example 1 Let

$$\mathcal{V}_{2_3} = \left\{ \begin{bmatrix} y_1 - 2y_2 & -y_1 + y_2 - 4y_3 \\ 2y_1 + 3y_2 + y_3 & y_1 + 4y_2 + y_3 \end{bmatrix} \right\}.$$

We can also write

$$\mathcal{V}_{2_3} = \{y_1A_1 + y_2A_2 + y_3A_3\},$$

in which

$$A_1 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 0 & -4 \\ 1 & 1 \end{bmatrix}.$$

Our concern is with the case when any two elements of the set $\mathcal{V}_{n_k}$ commute, that is when $\mathcal{V}_{n_k}$ is a commuting set. Conditions for this in terms of the basic matrices are stated next.

Proposition 1 $\mathcal{V}_{n_k} = \{y_1A_1 + y_2A_2 + \ldots + y_kA_k\}$ *is a commuting set if and only if*

$$A_iA_j = A_jA_i,$$

for $i, j = 1, \ldots, k$.

Proof. ($\Leftarrow$) Suppose that $A_iA_j = A_jA_i$, for $i, j = 1, 2, \ldots, k$.

Given $A, B \in \mathcal{V}_{n_k}$, then there are $\alpha_1, \alpha_2, \ldots, \alpha_k$ and $\beta_1, \beta_2, \ldots, \beta_k$, such that

$$A = \alpha_1A_1 + \alpha_2A_2 + \ldots + \alpha_kA_k \text{ and } B = \beta_1A_1 + \beta_2A_2 + \ldots + \beta_kA_k.$$

Hence,

$$AB = \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\beta_jA_iA_j \text{ and } BA = \sum_{j=1}^{k}\sum_{i=1}^{k} \beta_j\alpha_iA_jA_i.$$

From $A_iA_j = A_jA_i$, $i, j = 1, 2, \ldots, k$, it follows that $\alpha_i\beta_jA_iA_j = \beta_j\alpha_iA_jA_i$, then $AB = BA$.

($\Rightarrow$) For A_i and A_j, $i \neq j$, consider

$$A = 0A_1 + 0A_2 + \ldots + A_i + \ldots + 0A_j + \ldots + 0A_k \text{ and } B = 0A_1 + 0A_2 + \ldots + 0A_i + \ldots + A_j + \ldots + 0A_k.$$

Hence,

$$AB = A_iA_j \text{ and } BA = A_jA_i.$$

By hypothesis $\mathcal{V}_{n_k}$ is commuting, so $AB = BA$, then $A_iA_j = A_jA_i$, $i, j = 1, 2, \ldots, k$. □

Example 2 Let

$$\mathcal{V}_{3_2} = \left\{ \begin{bmatrix} y_1 + 21y_2 & 4y_1 - 24y_2 & y_1 \\ 3y_1 - 24y_2 & 2y_1 + 21y_2 & y_1 \\ 24y_2 - 5y_1 & 4y_1 - 24y_2 & 7y_1 - 3y_2 \end{bmatrix} \right\}.$$

We can also write

$$\mathcal{V}_{3_2} = \{y_1A_1 + y_2A_2\},$$

where

$$A_1 = \begin{bmatrix} 1 & 4 & 1 \\ 3 & 2 & 1 \\ -5 & 4 & 7 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 21 & -24 & 0 \\ -24 & 21 & 0 \\ 24 & -24 & -3 \end{bmatrix}$$

are commuting matrices, then $\mathcal{V}_{3_2}$ is a commuting set.

Next we inspect some basic facts related with a commuting $\mathcal{V}_{n_k}$.

If A is nonderogatory then the solution set of $AX = XA$ is a commuting set $\mathcal{V}_{n_n}$, which is closed under the product operation. This is illustrated in the following example.

Example 3 Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ -2 & -1 & -1 & -1 \end{bmatrix},$$

A is nonderogatory matrix. If we consider

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix},$$

and we choose x_{14}, x_{24}, x_{34} and x_{44} as arbitrary parameters, then the solution set of $AX = XA$ is

$$\mathcal{V}_{4_4} = \left\{ \begin{bmatrix} x_{24} + 2x_{34} + x_{44} & x_{14} + 2x_{24} + x_{34} & 2x_{14} + x_{24} & x_{14} \\ -2x_{24} - 2x_{34} & -2x_{14} - x_{24} + x_{34} + x_{44} & -2x_{14} + x_{24} + x_{34} & x_{24} \\ 2x_{24} + 2x_{34} & 2x_{14} + x_{24} & 2x_{14} + x_{34} + x_{44} & x_{34} \\ -2x_{24} - 2x_{34} & -2x_{14} - 2x_{24} - x_{34} & -2x_{14} - x_{24} - x_{34} & x_{44} \end{bmatrix} \right\},$$

which is a commuting set closed under the product.

On the other hand, it can be verified that the commuting set

$$\mathcal{V}_{3_3} = \left\{ \begin{bmatrix} x_1 & x_2 & x_3 \\ 0 & x_1 & 0 \\ 0 & 0 & x_1 \end{bmatrix} \right\}$$

is not a solution set of any equation $AX = XA$. Furthermore, considering

$$\mathcal{V}_{2_1} = \left\{ \begin{bmatrix} 0 & x_1 \\ x_1 & 0 \end{bmatrix} \right\},$$

we also can conclude that there are commuting sets $\mathcal{V}_{n_k}$ which are not closed under the product, that is, they are not rings. Although such cases can be always completed to a set closed under the product, this is an important issue to consider when dealing with commuting sets.

Now, we examine a crucial question: how many linearly independent matrices can a commuting set $\mathcal{V}_{n_k}$ have. The answer to this is not new. Schur gave it a century ago (Schur, 1905). The maximum number of linearly independent commutative $n \times n$ matrices is $N(n) = \frac{n^2}{4} + 1$, that is, the greater integer less than or equal to $\frac{n^2}{4} + 1$. Using our notation, given a commuting $\mathcal{V}_{n_k}$, for the basic matrices A_i, $i = 1, 2, \ldots, k$, commute it is necessary that $k \leq N(n)$. We use these in the development of our algorithm.

3. The Method

First we consider the set

$$E = \left\{ e_i e_j^T : i, j = 1, 2, \ldots, n \right\}$$

where e_i is $n \times 1$ with 1 in the i^{th} position and zeros elsewhere.

Conditions for the set E be commuting are stated next.

Lemma 1 *Let* $E = \left\{ e_i e_j^T,\ i, j = 1, 2, \ldots, n \right\}$, *if* $\Pi_a = e_g e_h^T$ *and* $\Pi_b = e_k e_l^T$ *are matrices of E, such that* $\Pi_a \neq \Pi_b$, *then* Π_a *commutes with* Π_b *if and only if* $g \neq l$ *e* $h \neq k$.

Proof. We have that

$$e_i^T e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus, $\Pi_a \Pi_b = e_g e_h^T e_k e_l^T$ and $\Pi_b \Pi_a = e_k e_l^T e_g e_h^T$. From $\Pi_a \neq \Pi_b$, we can conclude that $\Pi_a \Pi_b = \Pi_b \Pi_a$ if and only if $g \neq l$ and $h \neq k$. □

We observe that given two elements of E, $e_{i_1} e_{j_1}^T$ and $e_{i_2} e_{j_2}^T$, if each of them commutes with $e_{i_3} e_{j_3}^T$, then $e_{i_1} e_{j_1}^T + e_{i_2} e_{j_2}^T$ commutes with $e_{i_3} e_{j_3}^T$, even if $e_{i_1} e_{j_1}^T$ and $e_{i_2} e_{j_2}^T$ do not commute.

Let now

$$F = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \begin{bmatrix} e_1^T & e_2^T & \ldots, e_n^T \end{bmatrix} = \begin{bmatrix} e_1 e_1^T & e_1 e_2^T & \ldots & e_1 e_n^T \\ e_2 e_1^T & e_2 e_2^T & \ldots & e_2 e_n^T \\ \vdots & \vdots & \ddots & \vdots \\ e_n e_1^T & e_n e_2^T & \ldots & e_n e_n^T \end{bmatrix} \quad (1)$$

be an $n^2 \times n^2$ block matrix, where the (i, j) block is $e_i e_j^T$.

Using the matrix F we can determine all the matrices of the set E that commute with a given matrix $e_u e_v^T \in E$.

Proposition 2 *Let F be defined as above (1). Given a block $e_u e_v^T \in F$, let H be the set of blocks $e_i e_j^T$ of the submatrix resulting of F by deleting the block row v and the block column u, then $e_r e_s^T \in F$, such that $e_r e_s^T \neq e_u e_v^T$, commutes with $e_u e_v^T$ if and only if $e_r e_s^T \in H$.*

Proof. Given a block $e_u e_v^T \in F$, we have that

$$H = \left\{ e_r e_s^T : r, s = 1, 2, \ldots, n, r \neq v, s \neq u \right\}.$$

Supposing $r \neq v$ and $s \neq u$, it follows by Lemma 1 that $e_u e_v^T$ commutes with $e_r e_s^T$ if and only if $e_r e_s^T \in H$. □

The set H will be used in the method. It has $n^2 - 2n$ elements if $u \neq v$; otherwise, if $e_u e_u^T$ is a diagonal block of F, then H has $n^2 - 2n + 1$ elements. Besides that not all of its elements commute in pairs.

The next algorithm is a successive application of Proposition 2.

Algorithm 1

1) Given n, let

2) s: = 0

3) $G_s := \left\{ e_i e_j^T : i, j = 1, 2, \ldots, n \right\}$

4) $d := 0$

5) While $G_s \neq \emptyset$

5.1) Choose $e_u e_v^T \in G_s$ and let

5.2) $s := s + 1$

5.3) $A_s := e_u e_v^T$

5.4) If $u = v$

5.4.1) $d := d + 1$

5.5) $G_s := G_{s-1} - (\left\{ e_r e_s^T \in G_{s-1} : \ r = v \vee s = u \right\} \cup \left\{ e_u e_v^T \right\})$

If $d = n$

6.1) $k := s$

6.2) $\mathcal{U}_{n_k} := x_1 A_1 + x_2 A_2 + \ldots + x_s A_s$

If $d < n$

7.1) $k := s + 1$

7.2) $A_{s+1} := I_n$

7.3) $\mathcal{U}_{n_{k-1}} := x_1 A_1 + x_2 A_2 + \ldots + x_s A_s$

7.4) $\mathcal{U}_{n_k} := x_1 A_1 + x_2 A_2 + \ldots + x_s A_s + x_{s+1} A_{s+1}$

End.

If the matrices chosen from G_0 are $e_i e_i^T$, $i = 1, 2, \ldots, n$, then the algorithm gives only one commuting set, otherwise it gives two commuting sets, where in the second set A_{s+1} is the identity.

The fact that the sets $\mathcal{U}_{n_k}$ and $\mathcal{U}_{n_{k-1}}$ are commuting is a direct consequence of the matrices A_i pairwise commute. We also observe that the set of basic matrices $\{A_1, A_2, \ldots, A_s, A_{s+1}\}$ is linearly independent. Besides that we have the following.

Proposition 3 *The sets $\mathcal{U}_{n_k}$ and $\mathcal{U}_{n_{k-1}}$ are closed under the product operation.*

Proof. Let $A, B \in \mathcal{U}_{n_{k-1}}$, then there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ e $\beta_1, \beta_2, \ldots, \beta_{k-1}$ such that $A = \alpha_1 A_1 + \alpha_2 A_2 + \ldots + \alpha_{k-1} A_{k-1}$ and $B = \beta_1 A_1 + \beta_2 A_2 + \ldots + \beta_{k-1} A_{k-1}$, we have that $A_i A_j = 0 = A_j A_i$, and then $AB = 0 \in \mathcal{U}_{n_{k-1}}$.

Consider now $A', B' \in \mathcal{U}_{n_k}$, in a similar way, $A' = \alpha_1 A_1 + \alpha_2 A_2 + \ldots + \alpha_{k-1} A_{k-1} + \alpha_k I_n$ and $B' = \beta_1 A_1 + \beta_2 A_2 +$

$\ldots + \beta_{k-1}A_{k-1} + \beta_k I_n$, so we write $A' = A + \alpha_k I_n$ and $B' = B + \beta_k I_n$, where $A, B \in \mathcal{U}_{n_{k-1}}$, from $AB = 0$, it follows that

$$\begin{aligned} A'B' &= A\beta_k I_n + B\alpha_k I_n + \alpha_k\beta_k I_n \\ &= (\alpha_1 A_1 + \ldots + \alpha_{k-1}A_{k-1})\beta_k + (\beta_1 A_1 + \ldots + \beta_{k-1}A_{k-1})\alpha_k + \alpha_k\beta_k I_n \\ &= (\alpha_1\beta_k + \beta_1\alpha_k)A_1 + \ldots + (\alpha_{k-1}\beta_k + \beta_{k-1}\alpha_k)A_{k-1} + \alpha_k\beta_k I_n, \end{aligned}$$

then $A'B' \in \mathcal{U}_{n_k}$. □

4. Numerical Example

We implemented the algorithm in the *Matlab*. We use an auxiliary matrix to control the matrices of the set G_0 that make part of the commuting set $\mathcal{U}_{n_k}$. The following example is for matrices of order $n = 4$.

Consider

$$G_0 = \left\{ e_i e_j^T : \ i, j = 1, 2, 3, 4 \right\}$$

and let

$$M_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

be an $n \times n$ matrix, where the $1s$ in the positions (i, j) represent the elements $e_i e_j^T$ of the set G_0 that can be taken as the matrices A_l to construct the commuting set

$$\mathcal{U}_{n_k} = x_1 A_1 + x_2 A_2 + \ldots + x_s A_s.$$

We choose the element $(1, 2)$ of M_0, that is $A_1 = e_1 e_2^T$ as the first matrix, thus the row 2 and the column 1 of M_0 are set to zeros, to represent the elements in G_0 that were deleted to obtain G_1. Furthermore, setting $(M_0)_{12} = 2$ we indicate that the respective element was already chosen and therefore is neither in G_1. Hence we get

$$M_1 = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

In the same way the $1s$ in M_1 represent the elements of G_1, which are the available elements that commute with A_1, and therefore from those we have to pick the next one.

Choosing now the element $(1, 3)$ of M_1, that is $A_2 = e_1 e_3^T$. Thus deleting row 3 and column 1 from M_1 and setting $(M_1)_{13} = 2$, we obtain

$$M_2 = \begin{bmatrix} 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Again the $1s$ represent the elements that commute with the matrices already picked, that are represented by the $2s$. Continuing, we choose the element $(4, 2)$ of M_2, then $A_3 = e_4 e_2^T$, thus

$$M_3 = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix},$$

and finally we pick the only one left, $A_4 = e_4 e_3^T$. We can construct a commuting set with these elements

$$\mathcal{U}_{4_4} = \left\{ x_1 e_1 e_2^T + x_2 e_1 e_3^T + x_3 e_4 e_2^T + x_4 e_4 e_3^T : \ x_1, \ x_2, \ x_3, \ x_4 \in \mathbb{C} \right\},$$

or

$$\mathcal{U}_{4_4} = \left\{ \begin{bmatrix} 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x_3 & x_4 & 0 \end{bmatrix} : \ x_1, \ x_2, \ x_3, \ x_4 \in \mathbb{C} \right\}.$$

Adding the matrix I_4 we obtain the second commuting set

$$\mathcal{U}_{4_5} = \left\{x_1 e_1 e_2^T + x_2 e_1 e_3^T + x_3 e_4 e_2^T + x_4 e_4 e_3^T + x_5 I_4 : \; x_1,\; x_2,\; x_3,\; x_4,\; x_5 \in \mathbb{C}\right\},$$

or

$$\mathcal{U}_{4_5} = \left\{ \begin{bmatrix} x_5 & x_1 & x_2 & 0 \\ 0 & x_5 & 0 & 0 \\ 0 & 0 & x_5 & 0 \\ 0 & x_3 & x_4 & x_5 \end{bmatrix} : \; x_1,\; x_2,\; x_3,\; x_4,\; x_5 \in \mathbb{C} \right\}.$$

The maximum number of matrices linearly independent in $\mathcal{U}_{4_5}$ is the Schur number $\frac{n^2}{4} + 1 = 5 = k$. This evidently depends on the suitable choice we perform. We could get a lesser k, either with a different choice or stopping the iterations before the set G be empty, this can be achieved including an option in step 5 of the Algorithm 1 to terminate the iterations.

The sets $\mathcal{U}_{n_k}$ generated by the method have a very specific form. To obtain an aleatory form we can use a nonsingular matrix S and then $\mathcal{V}_{n_k} = S\,\mathcal{U}_{n_k} S^{-1}$ is also commuting. For example, from

$$\mathcal{U}_{3_3} = \left\{ \begin{bmatrix} x_1 & 0 & x_3 \\ 0 & x_1 & 0 \\ 0 & x_2 & x_1 \end{bmatrix} \right\},$$

obtained by Algorithm 1, if

$$S = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

then

$$\mathcal{V}_{3_3} = S\,\mathcal{U}_{3_3} S^{-1} = \left\{ \begin{bmatrix} x_1 + x_2 - x_3 & 4x_3 - x_2 & 2x_3 \\ x_2 - 2x_3 & x_1 - x_2 & x_3 \\ -4x_3 & 4x_3 & x_1 + 2x_3 \end{bmatrix} \right\}$$

is a commuting set with a different form from those generated by the algorithm.

As future prospects of the presented method, we cite the extensions to block versions, fact that will permit the application of it to generalized matrices partitioned into commuting blocks, like the block companion and the block Vandermonde, among others. Such matrices are linked to systems of higher order.

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