Turkish Journal of Analysis and Number Theory, 2016, Vol. 4, No. 3, 54-59
Available online at http://pubs.sciepub.com/tjant/4/3/1

DOI:10.12691/tjant-4-3-1

Some new Formulas for the Kolakoski Sequence A000002

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Abstract We present here two formulas for the Kolakoski sequence $(K_n)_{n\geq 1}$. The first one is in concern with the frequency of the letters 1 and 2 in this sequence. Our investigation seems to support the hypothesis of the Keane's conjecture. In the second part of this paper, we give an expression of the n^{th} term of the sequence, K_n, according to $K_1, K_2, \ldots K_p$, with $p \approx \frac{4n}{9}$ improving so, the former known value $\frac{2n}{3}$ given by Bordellès [2].

Keywords: *Kolakoski sequence, recursion, recursive formula.*

Cite This Article: Abdallah Hammam, "Some new Formulas for the Kolakoski Sequence A000002." *Turkish Journal of Analysis and Number Theory*, vol. 4, no. 3 (2016): 54-59. doi: 10.12691/tjant-4-3-1.

1. Introduction

The Kolakoski sequence [13], $(K_n)_{n\geq 1}$ with $K_1 = 1$, named so in 1965 but used for the first time in 1935 by Oldenburger [16], is the unique sequence beginning with 1 and which is invariant by the run-length encoding operation Δ. It starts as follows: 12211212212211211221211…. This sequence is also known as the Sloane's sequence A000002 [17]. It has been investigated by many authors using different technics (see [1,3,5,4,7,20,21,22] for instance). In particular, Kimberling [12] asked many questions about this sequence. We try here to find the way to give answers to the two most known of them: The first, is asking whether the density ρ_n of the letter 2 in the word $K_1K_2K_3 \ldots K_n$, defined by

$$\rho_n = \frac{\left|\{1 \leq j \leq n : K_j = 2\}\right|}{n}$$

has an asymptotic limit when n goes to ∞ ? and if this limit exists, is it equal to 1/2 ? This question is also known as the Keane's conjecture [11]. Shallit [18] classified this question in the twelfth position in his list of some Open Problems in concern with the automatic sequences. No answer to this attractive question has been given until now. Chvàtal [6], used the structure of some special graphs and established that

$$\left|\lim_{n\to\infty} \rho_n - \frac{1}{2}\right| < 0.00058.$$

Kupin [14], used the Gouldon-Jackson's method and obtain

$$\left|\lim_{n\to\infty} \rho_n - \frac{1}{2}\right| < 0.0223097\ldots$$

Bordellès [2], applied the Dirichlet's pigeonhole principle and got a better bound:

$$\left|\lim_{n\to\infty} \rho_n - \frac{1}{2}\right| < 0.021703504\ldots$$

In this paper, we will present a recursive formula connecting the density of 2's with some differences between terms K_j of even and odd indices. This formula seems to support Keane's conjecture.

The second question we investigate here, looks for an explicit expression of the n^{th} term K_n. Steinsky [23] and Fédou [9] found two equivalent recursive formulas which give K_n according to $K_1, K_2, \ldots K_{n-1}$. Using an inductif argument, Bordellès [2] presented a relation between K_n and k_n defined by: $k_n = \min\left\{ j > 0 : \sum_{i=1}^{j} K_i \geq n \right\}$. We will try, in this paper, to improve these results by reducing the number of data we need to compute K_n. We shall use a certain k'_n smaller than k_n. Furthermore, we derive a simple expression of K_n using this time, the integer part of the truncated rational $0, K_1K_2K_3\ldots$.

2. Notation

We first introduce some definitions and notation:

Let Σ be the input alphabet set $\{1,2\}$ and let Σ^* be the set of all finite words over Σ. We let Δ denote the run-length encoding operator, such that if W is an element of Σ^*, then $\Delta(W)$ will be the new word containing lengths of blocs of similar digits, in the initial word W. As an example, $\Delta(112212112) = 221121$.

We now define the two useful primitives Δ_1^{-1} and $\Delta_{\neq}^{-1}$ in the same way than Oldenburger [16], Sing [20,21], Dekking [8], Fédou [9] and Huang [10] by:

If $W = \omega_1\omega_2...\omega_n$ is an element of Σ^*, then $\Delta_1^{-1}(W)$ (resp $\Delta_{\neq}^{-1}(W)$) will be the unique word W' starting from the left by 1 (resp $3-\omega_n$), consisting of, exactly $\sum_{j=1}^{n}\omega_j$ elements and satisfying the encoding condition: $\Delta(W') = W$. For instance, $\Delta_1^{-1}(2122) = 1121122$ and $\Delta_{\neq}^{-1}(221211) = 221121121$. As a consequence, if we let W_n denote, for every $n \geq 1$, the word $K_1K_2...K_n$, then $\Delta_1^{-1}(W_n) = W_{S_n}$. Here, S_n is the partial sum and also the Sloane's sequence A054353 defined by:

$S_n = \sum_{j=1}^{n} K_j$ and $S_0 = 0$, as it has been used by Bordellès [2]. Therefore, the infinite Oldenburger-Kolakoski word $K = K_1K_2K_3...$ can be seen as a fixed point of the function Δ_1^{-1}. Shallit [19] has defined K as a fixed point of some extendable homomorphism, such as $\Delta_{\neq}^{-1}$, on 12, since we can write:

$$K = 122\Delta_{\neq}^{-1}(2)\Delta_{\neq}^{-2}(2)\Delta_{\neq}^{-3}(2)...$$

More generally, for every $j \geq 2$, we can extend K as follows:

$$K = K_1...K_jK_{j+1}...K_{S_j}\Delta_{\neq}^{-1}\left(K_{j+1}...K_{S_j}\right)\Delta_{\neq}^{-2}\left(K_{j+1}...K_{S_j}\right)...$$

Table 1. How the 1's of W_n are transformed by Δ_1^{-1}

W_n	1^{111}	1^{112}	1^{122}	1^{121}	1^{211}	1^{212}	1^{222}	1^{221}
W_{S_n}	1^{11}	1^{12}	1^{22}	1^{21}	2^{11}	2^{12}	2^{22}	2^{21}
$W_{S_{2,n}}$	1^1	1^2	2^2	2^1	1^21^1	1^11^2	2^12^2	2^22^1
$W_{S_{3,n}}$	1	2	22	11	21	12	1122	2211

Table 2. How the 2's of W_n are transformed by Δ_1^{-1}

W_n	2^{111}	2^{112}	2^{122}	2^{121}	2^{211}	2^{212}	2^{222}	2^{221}
W_{S_n}	$1^{22}1^{11}$	$1^{21}1^{12}$	$1^{11}1^{22}$	$1^{12}1^{21}$	$2^{21}2^{11}$	$2^{22}2^{12}$	$2^{12}2^{22}$	$2^{11}2^{21}$
$W_{S_{2,n}}$	2^11^1	2^11^2	1^12^2	1^22^1	$2^22^11^21^1$	$2^12^21^11^2$	$1^11^22^12^2$	$1^21^12^22^1$
$W_{S_{3,n}}$	221	112	122	211	221121	112212	121122	212211

We also need to define the double sequence $\left(S_{i,n}\right)_{(i,n)\in N\times N^*}$ by :

$$(\forall n > 0): S_{0,n} = n, S_{1,n} = \sum_{j=1}^{n} K_j = S_n$$
$$and\ (\forall i > 0)\, S_{i+1,n} = S_{S_{i,n}}.$$

On the other hand, for each $j > 0$ and $i \geq 0$, we associate the set $\{a_0, a_1, ..., a_i\}$ of elements in $\{1,2\}$ such that:

$a_l \equiv S_{l,j} \pmod 2$ for $l = 0,1,...,i$. We will write in a symbolic way, what we call the index notation: $\left(K_j\right)^{a_0a_1a_2...a_i}$.

The existence of $a_2, a_3, ..., a_i$ is supported by Theorem 2 below. We also define the cardinals:

$$\left|K_j^{a_0a_1a_2...a_i}\right|_n = \left|\left\{l \in N^* : l \leq n, K_l = K_j \text{ and } a_m \equiv S_{m,l} \pmod 2 \text{ for } m = 0,1,...,i\right\}\right|.$$

As an example, we have $\left(K_1\right)^{1111...}$, $\left|1^{12}\right|_{10} = 1$ and $\left|2^{21}\right|_{10} = 2.$

In the end, let us remark that, when we integrate W_n by applying Δ_1^{-1}, every letter K_j of this word, will be transformed in a bloc whose the index of the last element is S_j as illustrated by Table 1 and Table 2. More generally, after each integration using Δ_1^{-1}, each $\left(K_j\right)^{a_0a_1a_2a_3...a_i}$ will give a bloc whose last element, in the extreme right, is $\left(\frac{3+(-1)^{a_0}}{2}\right)^{a_1a_2a_3...a_i}$.

3. A Recursive Formula for the Density of 2's in the Kolakoski Sequence

We now begin by presenting a useful connection between ρ_n, the density of 2's in the word W_n, and some linear combination of $\left|K_j\right|_n^{a_0a_1...a_i}$.

Lemma 1. *For every* $n > 0$:

$$\left|1^1\right|_n + \left|2^1\right|_n = \left|1^2\right|_n + \left|2^2\right|_n + \frac{1-(-1)^n}{2}$$

$$\left|1^{11}\right|_n + \left|1^{21}\right|_n = \left|1^{12}\right|_n + \left|1^{22}\right|_n + \frac{1-(-1)^{S_n}}{2}$$

$$\left|1^{111}\right|_n + \left|1^{121}\right|_n = \left|1^{112}\right|_n + \left|1^{122}\right|_n + \frac{1-(-1)^{S_{2,n}}}{2}$$

Proof. For the first, there are two cases: if n is even, then the number of odd indices in W_n is equal to the number of even ones. If n is odd, the number of odd indices is greater than the evens since the first term K_1 has an odd index. We will refer to this equation by “the index balance sheet in W_n ”.

We write now the same equation for W_{S_n} :

$$\left|1^1\right|_{S_n} + \left|2^1\right|_{S_n} = \left|1^2\right|_{S_n} + \left|2^2\right|_{S_n} + \frac{1-(-1)^{S_n}}{2}.$$

As the elements of W_{S_n} are created by the integration of W_n using the Δ_1^{-1} operator, this balance sheet equation, will not take into account the blocs of two similar elements in W_{S_n}, since they consist of an odd and an even index and do not have any impact on the balance sheet of indices. So, only the elements created by 1’s, as illustrated by Table 1 and Table 2, similar to those used by Nilsson [15] to perform his algorithm, will be useful in the final equation. This restriction allows us to get the desired equation. To prove the third relation, We just write the “index balance sheet” for $W_{S_{2,n}}$ just after eliminating all the blocs whose number of elements is even.

Theorem 2. *for all* $n \geq 1$, *we have*

$$t_n - o_n = 2\left(\left|2^1\right|_n - \left|1^2\right|_n\right) + \frac{\left((-1)^n - 1\right)}{2}$$

$$t_{S_n} - o_{S_n} = \left(\left|1^1\right|_n - \left|1^2\right|_n\right) + \left((-1)^n - 1\right)$$

$$t_{S_{2,n}} - o_{S_{2,n}} = \left(\left|1^{11}\right|_n - \left|1^{12}\right|_n\right) + \left((-1)^{S_n} - 1\right)$$

$$t_{S_{3,n}} - o_{S_{3,n}} = \left(\left|1^{111}\right|_n - \left|1^{112}\right|_n\right) + \left(\left|2^{111}\right|_n - \left|2^{112}\right|_n\right) + \left(\left|2^{122}\right|_n - \left|2^{121}\right|_n\right) + \left((-1)^{S_{2,n}} - 1\right).$$

Proof. Let n be an integer ≥ 1.

We begin by making useful connections between the four sequences:

$(t_n), (o_n), (\rho_n)$ and (S_n) which represent respectively, the number of 2s in the word W_n, the number of 1s, the density of 2s and the partial sum $K_1 + K_2 + \ldots + K_n$.

$$t_n = n\rho_n = n - o_n$$

$$t_n + n = S_n = (1+\rho_n)n$$

$$t_{S_n} - o_{S_n} = 2t_{S_n} - \left(t_{S_n} + o_{S_n}\right) = 2t_{S_n} - S_n = \left(2\rho_{S_n} - 1\right)S_n$$

$$\left(2\rho_{S_n} - 1\right)S_n = 2\left(\rho_{S_n} + 1\right)S_n - 3S_n = 2\left(t_{S_n} + S_n\right) - 3S_n = 2S_{2,n} - 3S_n.$$

Using the fact that $\Delta_1^{-1}(W_n) = W_{S_n}$, We find that

$$S_{2,n} = \sum_{j=1}^{S_n} K_j = \sum_{j=1}^{n} K_j \frac{3+(-1)^j}{2}$$

which becomes as follows:

$$2S_{2,n} - 3S_n = \sum_{j=1}^{n} (-1)^j K_j.$$

Bordellès et Al. [2], proved this equation using an induction on n. We now use the index notation and get:

$$2S_{2,n} - 3S_n = \left|1^2\right|_n + 2\left|2^2\right|_n - \left|1^1\right|_n - 2\left|2^1\right|_n.$$

The proof is finally achieved, by using the index balance sheet equation for W_n, given in Lemma 1. For the third equality, we replace n by S_n in the second equation to obtain:

$$2S_{3,n} - 3S_{2,n} = \left((-1)^{S_n} - 1\right) + \left(\left|1^1\right|_{S_n} - \left|1^2\right|_{S_n}\right).$$

If we take now a look at the second line of Table 1 and Table 2, we can see that, only blocs of single 1’s in W_{S_n} have to be taken into account. The conclusion is then, easily derived. To prove the fourth, We replace this time, n by $S_{2,n}$ and find:

$$2S_{4,n} - 3S_{3,n} = \left((-1)^{S_{2,n}} - 1\right) + \left(\left|1^1\right|_{S_{2,n}} - \left|1^2\right|_{S_{2,n}}\right).$$

Finally, a quick look at the third line of Table 1 and Table 2 gives the expected result.

As a conclusion, we proved that, for each $j \geq 1$, we can compute $S_{2,j}$, $S_{3,j}$ and so on, only from some elements between K_1 and K_j. This possibility is the answer to the existence of the sequence $(a_l)_{l \geq 2}$.

Corollary 3. *For each integer* $n \geq 1$ *fixed and* $i \geq 0$, *we define the following sequences by*:

$$U_i = \left(2\rho_{S_{i,n}} - 1\right), V_i = S_{i,n} \text{ and } W_i = U_i V_i.$$

Then, Theorem 2 allows us to write:

1. $U_0 V_0 = W_0 = \dfrac{(-1)^n - 1}{2} + 2\left(\left|2^1\right|_n - \left|1^2\right|_n\right)$
2. $U_1 V_1 = W_1 = \left((-1)^n - 1\right) + \left(\left|1^1\right|_n - \left|1^2\right|_n\right)$
3. $U_2 V_2 = W_2 = \left((-1)^{S_n} - 1\right) + \left(\left|1^{11}\right|_n - \left|1^{12}\right|_n\right)$
4. $U_3 V_3 = W_3 = \left((-1)^{S_{2,n}} - 1\right) + \left(\left|1^{111}\right|_n - \left|1^{112}\right|_n\right) + \left(\left|2^{111}\right|_n - \left|2^{112}\right|_n\right) + \left(\left|2^{122}\right|_n - \left|2^{121}\right|_n\right).$

More generally, we can show by induction that for each $i \geq 0$

$$U_{i+1}V_{i+1} = \left((-1)^{S_{i,n}} - 1\right) + \sum_{j=0\ldots i, a_j \in \{1,2\}} \lambda(a_0,\ldots,a_i)\left|1^{a_0 \ldots a_i}\right|_n + \sum_{j=0\ldots i, a_j \in \{1,2\}} \mu(a_0,\ldots,a_i)\left|2^{a_0 \ldots a_i}\right|_n.$$

The two coefficients λ and μ take quickly changing values and signum.

On the other hand, for enough great integer *i*, we have

$$\frac{4}{9} \le \rho_i \le \frac{5}{9}$$

and

$$V_{i+1} = \left(1+\rho_{S_{i,n}}\right)V_i \ge \frac{13}{9}V_i.$$

Hence, the sequence $(V_i)_{i\ge 0}$ increases at least exponentially. Finally, as a conclusion, we can say that the answer to Keane's conjecture, depends on the signum of the sequence $(W_i)_{i\ge 0}$. If this signum is always changing, it means that the sequence $(U_i)_{i\ge 0}$ has no limit or goes to zero, i.e., the frequency of 2's goes to $\frac{1}{2}$.

4. A New Formula for the n^{th} Term K_n

At first, we will present a simple recursive formula, which computes K_n, from the input values, $K_1, K_2, K_3 ...$ and K_{n-1}.

In the second part of this section, we will try to improve this result.

Lemma 4. *For every integer* $n \ge 3$,

$$K_n = \left[10^n\left(\sum_{j=1}^{n-1}\frac{3+(-1)^j}{2}10^{-S_j}\left(10\left(K_j-1\right)+1\right)-10^{-j}\right)\right].$$

Proof. We consider the decimal numbers $A_n \in \left]0,1\right[$ defined by :

$A_n = 0, K_1K_2K_3...K_{n-1} = \sum_{j=1}^{n-1} K_j 10^{-j}$ for every $n \ge 3$. Hence, by integration using the operator Δ_1^{-1}, we find a new rational :

$$B_n = \Delta_1^{-1}\left(A_n\right) = 0, K_1K_2K_3...K_n...K_{S_{n-1}}$$

also given by

$$B_n = \sum_{j=1}^{n-1}\frac{3+(-1)^j}{2}10^{-S_j}\left(10\left(K_j-1\right)+1\right)$$

and then, using the fact that the Kolakoski-Oldenburger constant 0, $K_1K_2K_3...$, is a fixed point of Δ_1^{-1}, we will have

$$B_n - A_n = 0,000...0K_n...K_{S_{n-1}}$$

from which we extract the n^{th} digit using the integer part: [ ].

In the end, let us remark that this recursive computation of K_n, depicted of its interesting mathematical side, takes too much time and obviously need to be improved.

Lemma 5. *For every* $n \ge 2$, *there exist two unique integers p and q, both* > 0 *such that*:

$$\left(n-S_p\right)\left(n+1-S_p\right) = 0 \quad \textit{and}$$

$$\left(n-S_{2,q}\right)\left(n+1-S_{2,q}\right)\left(n+2-S_{2,q}\right)\left(n+3-S_{2,q}\right) = 0$$

Proof. Let n be an integer ≥ 2, and assume that for every $p > 0\left(n-S_p\right)\left(n+1-S_p\right) \ne 0$. Then, in the best case, there will exist $r > 0 : S_r = n-1$ and $S_{r+1} = n+2$. This is not possible since $S_{r+1} - S_r = K_{r+1} \le 2$.

For the second, the basics of he proof come from the following two cases: for each $r > 0$, $K_{r+1} = 1$ and then $S_{2,r+1} - S_{2,r} = S_{1+S_r} - S_{S_r} = K_{1+S_r} \le 2$, or $K_{r+1} = 2$, which gives

$$\begin{aligned} S_{2,r+1} - S_{2,r} &= \left(S_{2+S_r} - S_{1+S_r}\right) + \left(S_{1+S_r} - S_{S_r}\right) \\ &\le 2+2 = 4. \end{aligned}$$

We will now derive a second recursive formula for K_n using less input parameters.

Theorem 6. *Let* $n \ge 2$ *and let* q *be the unique integer, such that* $S_{2,q-1} < n \le S_{2,q}$,

Then we have $K_n = \dfrac{\left(3+(-1)^{S_q}\right)}{2}$ *if* $S_{2,q} - n = 0$,

$$K_n = \frac{\left(3+(-1)^{q+S_q}\right)}{2}\left(K_q-1\right) + \frac{\left(1+(-1)^q\right)}{2}\frac{\left(3+(-1)^{S_q}\right)}{2}\left(2-K_q\right)$$

if $S_{2,q} - n = 1$ *and*

$$K_n = \frac{\left(3-(-1)^{S_q}\right)}{2}\left(K_q-1\right)\frac{\left(1+(-1)^q\right)}{2}$$

if

$$S_{2,q} - n \ge 2.$$

Proof. For each $N > 1$, let us remember that $A_N = \sum_{n=1}^{N-1} K_n 10^{-n}$.

Using the integration operator Δ_1^{-1}, we obtain

$$\begin{aligned} B_N &= \Delta_1^{-1}\left(A_N\right) \\ &= \sum_{p=1}^{N-1}\frac{3+(-1)^p}{2}\left(K_p-1\right)10^{1-S_p} + \frac{3+(-1)^p}{2}10^{-S_p} \end{aligned}$$

and

$$\begin{aligned} C_N &= \Delta_1^{-1}\left(B_N\right) \\ &= \sum_{q=1}^{N-1}\frac{1+(-1)^q}{2}\frac{3-(-1)^{S_q}}{2}\left(K_q-1\right)\left(10^{3-S_{2,q}} + 10^{2-S_{2,q}}\right) \\ &+ \left(\begin{aligned} &\frac{3+(-1)^{q+S_q}}{2}\left(K_q-1\right) \\ &+\frac{1+(-1)^q}{2}\frac{3+(-1)^{S_q}}{2}\left(2-K_q\right) \end{aligned}\right)10^{1-S_{2,q}} \\ &+ \frac{3+(-1)^{S_q}}{2}10^{-S_{2,q}}. \end{aligned}$$

When N goes to infinity, the three series A_N, B_N and C_N, converge absolutely to the same limit K, namely, the Oldenburger-Kolakoski constant 0, 1221121221..... . So, by identification of the coefficients in the $\left(10^{-i}\right)_{i\geq 1}$ base, we derive the expression of K_n, according to the value of the difference $S_{2,q} - n = 0,1,2 \text{ or } 3$.

We now try to simplify the expressions we got above.

Theorem 7. *Let $n \geq 2$ and let k_n' be the unique integer, such that: $S_{2,k_n'} \leq n < S_{2,k_n'+1}$, then,*

$$K_n = \frac{3-(-1)^{S_{k_n'}} + \left[\left|\cos\left(\frac{\left(n-S_{2,k_n'}\right)\pi}{3}\right)\right|\right]}{2}.$$

Proof. We separate the three cases: $n - S_{2,k_n'} = 1$ or 2 or 3 and use Table 1 and Table 2. We find almost easily that if $0 < n - S_{2,S_{k_n'}} \leq 2$ then $K_n = \frac{\left(3-(-1)^{S_{k_n'}}\right)}{2}$, and if $n - S_{2,k_n'} = 3$ then $K_n = \frac{\left(3+(-1)^{S_{k_n'}}\right)}{2}$. These two conditions can be replaced by

$$K_n = \frac{\left(3-(-1)^{S_{k_n'}} + \left[\frac{n-S_{2,k_n'}}{3}\right]\right)}{2}.$$

Finally, to include also the almost trivial case $n = S_{2,k_n'}$, we unify all the expressions in the following unique one:

$$K_n = \frac{\left(3-(-1)^{S_{k_n'}} + \left[\left|\cos\left(\frac{\left(n-S_{2,k_n'}\right)\pi}{3}\right)\right|\right]\right)}{2}.$$

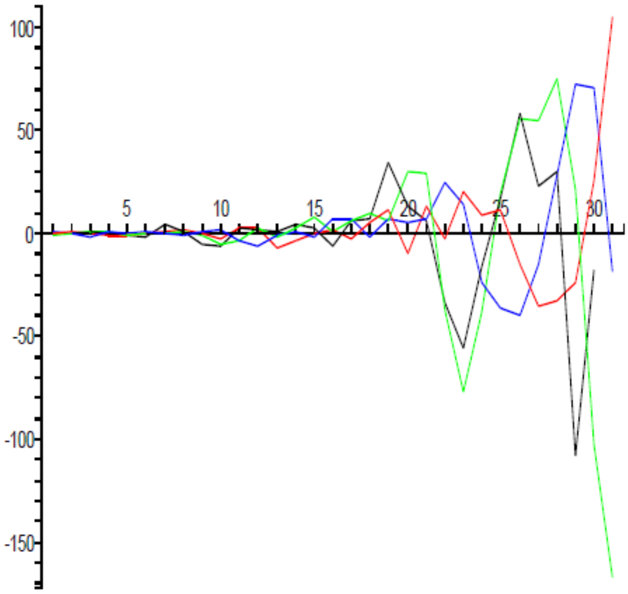

Figure 1. Signum variation of the product $U_i V_i$ for different values of n

Table 3. Comparison between n, k_n and k_n'

n	10	100	10^3	10^4	10^5	10^6
k_n	7	66	666	6667	66652	666673
k_n'	5	44	443	4446	44428	444461

5. Concluding Remarks

Our attempt to respond to the Keane's conjecture, does not allow us to give a final answer. However, the Corollary 3 seems clearly to support this hypothesis, since, in the left-hand side of its equations, the term V_i increases at least exponentially with i, while the right hand side does not and can not have a constant signum as illustrated by the graph in Figure 1. It contains a complicated linear combination of terms which are generated nearly randomly by the integration operator Δ_1^{-1} as seen in Corollary 3. In the future, we will focus our work, to investigate deeply this last assertion. On the other hand, to answer the question about the explicit expression of term K_n, let us begin by saying that the self-describing nature of the Oldenburger-Kolakoski sequence, makes it impossible to predict K_n from n only. What we can just do, is to give an expression of K_n, according to $K_1, K_2, ..., K_p$, where p is as smaller as we can. We have shown that, we can take $p = k_n'$ verifying $S_{2,p} < n < S_{2,p+1}$. This value is clearly smaller than, $p = n-1$ found by Steinsky [23] and $p = k_n$, presented by Bordellès [2]. We will try to improve this result by looking for simple expressions and much smaller values of p, like those satisfying $S_{i,p} < n < S_{i,p+1}$ with $i \geq 3$. In this case, p will be not far from $\left(\frac{2}{3}\right)^i n$, as illustrated in Table 3.

Acknowledgments

The author would like to thank the referees for their careful reading of the manuscript and the fruitful suggestions they provided, and Assim for helpful discussion.

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