Barrier Option Pricing

Degree Project in Mathematics, First Level

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Abstract

This thesis examines the performance of five option pricing models with respect to the pricing of barrier options. The models include the Black-Scholes model and four stochastic volatility models ranging from the single-factor stochastic volatility model first proposed by Heston (1993) to a multi-factor stochastic volatility model with jumps in the spot price process. The stochastic volatility models are calibrated using four different loss functions to examine the loss functions effect on the resulting barrier option prices. Our results show that the Black-Scholes model yields significantly different prices than the stochastic volatility models for barriers far from the current spot price. The prices of the four stochastic volatility models are however very similar. We also show that the choice of loss function for parameter estimation has little effect on the obtained barrier option prices.

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1. Introduction

Ever since the presentation of the famous Black & Scholes model [7], academics and practitioners have made numerous attempts to relax the restrictive assumptions that make the model inconsistent with observed prices in the market. Particular interest has been directed towards the assumption of constant volatility, which makes the model unable to generate non-normal return distributions and the well-known volatility smile, consistent with empirical findings on asset returns [12].

Examples of extensions of the Black-Scholes model include models that allow for stochastic volatility and jumps in the underlying return process. Many different models have been proposed, ranging from single-factor stochastic volatility models, to multi-factor models with log-normally distributed jumps in the stock price process as well as stochastic interest rates (see e.g. [3], [4], [5], [11], [19] and [27]).

In a recognized paper, Schoutens, Simons & Tistaert [23] show that although there are many option pricing models available that very accurately can explain observed prices of plain vanilla options, the models may produce inconsistent prices when applied to more exotic derivatives. In this thesis, we extend the results of [23] by conducting a similar analysis with four stochastic volatility models, including two multi-factor stochastic volatility models not examined in [23]. All four models allow for non-normal return distributions and non-constant volatility and have proven to be effective in the pricing of plain vanilla call and put options (see e.g. [3], [5] and [23]). We also extend the estimation technique by calibrating the models using four different loss functions, and examine how the choice of loss function potentially affects the pricing performance of the models.

To estimate the model parameters, we use the same data set as in [23], i.e. the implied volatility surface of the Eurostoxx 50 index on the 7^{th} of October 2003. Using the estimated model parameters, we analyze the models' pricing performance with respect to the commonly traded exotic options called *barrier options*, and compare the obtained prices using the different models and loss functions.

Our results show that the prices of barrier options differ significantly between the Black-Scholes model and the stochastic volatility models. We also show that the choice of loss function for estimation of model parameters in the stochastic volatility models have little effect on the resulting barrier option prices.

1.1. Barrier options

A barrier option is a path-dependent option whose pay-off at maturity depends on whether or not the underlying spot price has touched some pre-defined barrier during the life of the option. In this thesis, we will limit our attention to four of the most common barrier options, namely up-and-in (UI), up-and-out (UO), down-and-in (DI) and down-and-out (DO) call options.

To describe the pay-off structures of the barrier options, define the maximum and the minimum of the spot price process $S = \{S_t, 0 < t < T\}$ as $m_S = \inf\{S_t; 0 < t < T\}$ and $M_S = \sup\{S_t; 0 < t < T\}$.

The UI call option is worthless unless the underlying spot price S hits a pre-determined barrier $H > S_0$ during the life of the option, in which case it becomes a standard call option. Hence, its pay-off at maturity is:

$$\Phi_{UI}(S_T, M_S) = (S_T - K)^+ \mathbf{1}(M_S \geq H) \tag{1.1}$$

where $\mathbf{1}(M_S \geq H)$ is an indicator function equal to one if $M_S \geq H$ and zero otherwise.

For the UO call option, the relationship is reversed: the UO call option is a standard call option unless the underlying spot price hits the barrier, in which case it becomes worthless:

$$\Phi_{UO}(S_T, m_S) = (S_T - K)^+ \mathbf{1}(m_S < H) \tag{1.2}$$

For DI and DO call options, the barrier H is set below the spot price at inception, $H < S_0$. The pay-off structures of the DI and DO options follow analogously: the DI call option is worthless unless the barrier H is reached some time during the life of the trade, in which case it becomes a plain vanilla call option. The DO option, on the other hand, is a standard call option unless the spot price reaches the lower barrier during the life of the option, in which case it becomes worthless. The pay-off structures of the DI and DO call options are:

$$\Phi_{DI}(S_T, m_S) = (S_T - K)^+ \mathbf{1}(m_S \leq H) \tag{1.3}$$

$$\Phi_{DO}(S_T, M_S) = (S_T - K)^+ \mathbf{1}(M_S > H) \tag{1.4}$$

2. Model presentation

We will assume that the reader has a basic knowledge of the basic Black-Scholes framework and will thus refrain from describing the Black-Scholes model in detail. The interested reader is referred to [21] for an economic outline and to [6] for a description of the mathematical framework.

For all models, we consider the risk-neutral dynamics of the stock price. We let $S = \{S_t, 0 \leq t \leq T\}$ denote the stock price process and $\varphi_T(\cdot)$ denote the characteristic function of the natural logarithm of the terminal stock price $s_T = \log(S_T)$, where $\log(\cdot)$ denotes the natural logarithm function. The constants r and δ will denote the, both constant and continuously compounded, interest rate and dividend yield, respectively. Further, we let $W_t^{\mathbb{Q}}$ denote a $\mathbb{Q}$-Wiener process.

2.1. Stochastic volatility model (SV)

Many different stochastic volatility models have been proposed, but we will limit our attention to the Heston [19] stochastic volatility model, henceforth denoted SV, in which the spot price is described by the following stochastic differential equations (SDEs) under $\mathbb{Q}$:

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sqrt{V_t}dW_t^{\mathbb{Q}\,(1)} \tag{2.1}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^{\mathbb{Q}\,(2)} \tag{2.2}$$

$$dW_t^{\mathbb{Q}\,(1)}dW_t^{\mathbb{Q}\,(2)} = \rho dt \tag{2.3}$$

where V_t is the stochastic variance and the parameters κ, θ, σ and ρ represent the speed of mean reversion, the long-run mean of the variance, the volatility of the variance process and the correlation between the variance and stock price processes, respectively. In addition to these parameters, the model requires the estimation of the instantaneous spot variance V_0.

Using the same representation of the parameters as in equations (2.1) – (2.3), the characteristic function of s_T takes the following form:

$$\varphi_T^{SV}(u) = S_0^{iu} f(V_0, u, T) \tag{2.4}$$

where

$$f(V_0, u, T) = \exp\{A(u,T) + B(u,T)V_0\} \tag{2.5}$$

$$A(u,T) = (r-\delta)iuT + \frac{\kappa\theta}{\sigma^2}\left[(\kappa - \rho\sigma iu - d)T - 2\log\left(\frac{1-ge^{-dT}}{1-g}\right)\right] \tag{2.6}$$

$$B(u,T) = \frac{\kappa - \rho\sigma iu - d}{\sigma^2}\left[\frac{1-e^{-dT}}{1-ge^{-dT}}\right] \tag{2.7}$$

$$d = \sqrt{(\rho\sigma iu - \kappa)^2 + \sigma^2(iu + u^2)} \tag{2.8}$$

$$g = (\kappa - \rho\sigma iu - d)/(\kappa - \rho\sigma iu + d) \tag{2.9}$$

2.2. Stochastic volatility model with jumps (SVJ)

We extend the SV model in the previous section along the lines of [4], by adding log-normally distributed jumps to the stock price process. In this model, here denoted SVJ, the return process of the spot price is described by the following set of SDEs under $\mathbb{Q}$:

$$\frac{dS_t}{S_t} = \left(r - \delta - \lambda\mu_J\right)dt + \sqrt{V_t}dW_t^{\mathbb{Q}\,(1)} + J_t dY_t \tag{2.10}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^{\mathbb{Q}\,(2)} \tag{2.11}$$

$$dW_t^{\mathbb{Q}\,(1)}dW_t^{\mathbb{Q}\,(2)} = \rho dt \tag{2.12}$$

where $Y = \{Y_t, 0 \le t \le T\}$ is a Poisson process with intensity $\lambda > 0$ and J_t is the jump size conditional on a jump occurring. All other parameters are defined as in (2.1) – (2.3). The subtraction of $\lambda\mu_J$ in the drift term compensates for the expected drift added by the jump component, so that the total drift of the process, as required for risk-neutral valuation, remains $(r-q)dt$.

As mentioned, the jump size is assumed to be log-normally distributed:

$$\log\ (1+J_t) \sim N\left(\log\ \left(1+\mu_J\right) - \frac{\sigma_J^2}{2}, \sigma_J^2\right) \tag{2.13}$$

Further, it is assumed that Y_t and J_t are independent of each other as well as of $W_t^{\mathbb{Q}\,(1)}$ and $W_t^{\mathbb{Q}\,(2)}$.

Using the independence of Y_t, J_t and the two Wiener processes, one can show that the characteristic function of the SVJ model is [15]:

$$\varphi_T^{SVJ}(u) = \varphi_T^{SV}(u) \cdot \varphi_T^{J}(u) \tag{2.14}$$

where:

$$\varphi_T^{J} = \exp\{-\lambda\mu_J\, iuT + \lambda T((1+\mu_J)^{\,iu} \exp(\sigma_J^2\,(iu/2)(iu-1)) - 1)\} \tag{2.15}$$

and $\varphi_T^{SV}(u)$ is defined as in (2.4).

2.3. Multi-factor stochastic volatility model (MFSV)

The multi-factor stochastic volatility model is an extension of the SV model and has been studied by e.g. [5] and [10]. We denote the multi-factor stochastic volatility model MFSV and let the following set of SDEs describe the return process under the risk-neutral measure:

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sqrt{V_t^{(1)}}dW_t^{\mathbb{Q}\,(1)} + \sqrt{V_t^{(2)}}dW_t^{\mathbb{Q}\,(2)} \tag{2.16}$$

$$dV_t^{(1)} = \kappa_1\left(\theta_1 - V_t^{(1)}\right)dt + \sigma_1\sqrt{V_t^{(1)}}dW_t^{\mathbb{Q}\,(3)} \tag{2.17}$$

$$dV_t^{(2)} = \kappa_2\left(\theta_2 - V_t^{(2)}\right)dt + \sigma_2\sqrt{V_t^{(2)}}dW_t^{\mathbb{Q}\,(4)} \tag{2.18}$$

where the parameters have the same meaning as in (2.1) – (2.3) and the subscripts 1 and 2 indicate to which variance process the parameter is related.

The dependence structure is assumed to be as follows:

$$dW_t^{\mathbb{Q}\,(1)}\, dW_t^{\mathbb{Q}\,(3)} = \rho_1 dt \tag{2.19}$$

$$dW_t^{\mathbb{Q}\,(2)} dW_t^{\mathbb{Q}\,(4)} = \rho_2 dt \tag{2.20}$$

$$dW_t^{\mathbb{Q}\,(i)} dW_t^{\mathbb{Q}\,(j)} = 0, \quad (i,j) = (1{,}2), (1{,}4), (2{,}3), (3{,}4) \tag{2.21}$$

In other words, each variance process is correlated with the corresponding Wiener process in the return process, i.e. the diffusion term of which the respective variance process determines the magnitude.

By the independence structure described above, the added diffusion term is independent of the nested SV model return SDE. Since the characteristic function of the sum of two independent variables is the product of their individual characteristic functions, the characteristic function of the MFSV model is determined as:

$$\varphi_T^{MFSV}(u) = \mathbb{E}_0^{\mathbb{Q}}\left[e^{ius_T}\right] = S_0^{iu} f\left(V_0^{(1)}, V_0^{(2)}, u, T\right) \tag{2.22}$$

where:

$$f\left(V_0^{(1)}, V_0^{(2)}, u, T\right) = \exp\left\{A(u,T) + B_1(u,T)V_0^{(1)} + B_2(u,T)V_0^{(2)}\right\} \tag{2.23}$$

$$A(u,T) = (r-\delta)iuT + \sum_{j=1}^{2} \sigma_j^{-2}\kappa_j\theta_j\left[\left(\kappa_j - \rho_j\sigma_j iu - d_j\right)T - 2\log\left(\frac{1-g_j e^{-d_jT}}{1-g_j}\right)\right] \tag{2.24}$$

$$B_j(u,T) = \sigma_j^{-2}(\kappa_j - \rho_j\sigma_j iu - d_j)\left[\frac{1-e^{-d_jT}}{1-g_j e^{-d_jT}}\right] \tag{2.25}$$

$$g_j = \frac{\kappa_j - \rho_j\sigma_j iu - d_j}{\kappa_j - \rho_j\sigma_j iu + d_j} \tag{2.26}$$

$$d_j = \sqrt{\left(\rho_j\sigma_j iu - \kappa_j\right)^2 + \sigma_j^2(iu + u^2)} \tag{2.27}$$

2.4. Multi-factor stochastic volatility model with jumps (MFSVJ)

The multi-factor stochastic volatility model with jumps that we use in this paper is a variation of the model presented in [5]. However, we use a different approach in the sense that we estimate the model to only one day's data whereas [5] uses a data set spanning over 7 years. We denote the multi-factor stochastic volatility model with jumps MFSVJ, and let the risk-neutral stock price dynamics be described by the following set of SDEs:

$$\frac{dS_t}{S_t} = \left(r - \delta - \lambda\mu_J\right)dt + \sqrt{V_t^{(1)}}dW_t^{\mathbb{Q}\,(1)} + \sqrt{V_t^{(2)}}dW_t^{\mathbb{Q}\,(2)} + J_t dY_t \tag{2.28}$$

$$dV_t^{(1)} = \kappa_1\left(\theta_1 - V_t^{(1)}\right)dt + \sigma_1\sqrt{V_t^{(1)}}dW_t^{\mathbb{Q}\,(3)} \tag{2.29}$$

$$dV_t^{(1)} = \kappa_2\left(\theta_2 - V_t^{(2)}\right)dt + \sigma_2\sqrt{V_t^{(2)}}dW_t^{\mathbb{Q}\,(4)} \tag{2.30}$$

where all parameters and variables are defined as in equations (2.1) – (2.3) and (2.10). The distributions of J_t and Y_t are log-normal and Poisson, respectively, according to equations (2.10) and (2.13), and the two variables are independent, both of each other and of the four Wiener processes. The dependence structure between the Wiener processes is the same as in the MFSV model according to equations (2.19) – (2.21).

Due to the independence between the added jump factor and the SDE of the MFSV model, the characteristic function of s_T is obtained in the same way as in the SVJ model, i.e. as the product of the jump-term characteristic function and the characteristic function of the MFSV model:

$$\varphi_T^{MFSVJ}(u) = \varphi_T^{MFSV}(u) \cdot \varphi_T^{J}(u) \tag{2.31}$$

where $\varphi_T^{MFSV}(u)$ and $\varphi_T^{J}(u)$ are defined in equations (2.22) and (2.15), respectively.

3. Vanilla option pricing

3.1. Option pricing in the Black-Scholes model

One of the most appealing features of the Black-Scholes model is the existence of an analytical formula for the pricing of European call and put options. Given that the model parameters (essentially σ) are known, the Black-Scholes price of a European call or put option is calculated as:

$$Price_t^{BS} = S_t e^{-\delta(T-t)} N(\omega d_1) - \omega K e^{-r(T-t)} N(\omega d_2) \tag{3.1}$$

where S_t denotes the spot price, K the strike price, r the interest rate, q the dividend yield, $T-t$ the time to maturity and ω is a binary operator equal to 1 for call options and -1 for put options. Further, we have that:

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{3.2}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{3.3}$$

in which σ is the volatility of the spot price.

3.2. Option pricing in stochastic volatility models

Assuming that the characteristic function of the log-stock price is known analytically, the price of plain vanilla options can be determined using the Fast Fourier Transform (FFT) method first presented in [8]. Using this approach, the call price is expressed in terms of an inverse Fourier transform of the characteristic function of the log-stock price under the assumed stochastic process. The resulting formula can then be re-formulated to enable computation using the FFT algorithm that significantly decreases computation time compared to standard numerical methods such as the discrete Fourier transform.

To derive the call price function in the FFT approach, we follow closely the method of [8], while also allowing for a continuous dividend yield (δ). Denote by s_T and k the natural logarithm of the terminal stock price and the strike price K, respectively. Further, let $C_T(k)$ denote the value of a European call option with pay-off function $f(S_T) = (S_T - K)^+ = (e^{s_T} - e^k)^+$ and maturity at time T. The discounted expected pay-off under $\mathbb{Q}$ is then:

$$C_T(k) = \mathbb{E}_t^{\mathbb{Q}}[e^{-rT}(S_T - K)^+] = \int_k^{\infty} e^{-rT}\left(e^{s_T} - e^k\right) q_T(s_T) ds_T \tag{3.4}$$

where $q_T(s)$ is the risk-neutral density of s_T. As k tends to $-\infty$, (3.4) translates to:

$$\lim_{k \to -\infty} C_T(k) = \int_{-\infty}^{\infty} e^{-rT} e^{s_T} q_T(s_T) ds_T = e^{-rT} \mathbb{E}_t^{\mathbb{Q}}[S_T] = S_0 \tag{3.5}$$

This is on the one hand reassuring, as the price of a call with a strike price of zero should equal S_0. On the other hand, in order to apply the Fourier transform to $C_T(k)$ it is required that the function is square integrable for all k, i.e. that $\int_{-\infty}^{\infty} |C_T(k)|^2 ds_T < \infty \ \forall\ k \in \mathbb{R}$. However, by (3.5), as k tends to $-\infty$:

$$\lim_{k \to -\infty} \int_{-\infty}^{\infty} |C_T(k)|^2\, ds_T = \int_{-\infty}^{\infty} |S_0|^2\, ds_T \to \infty \tag{3.6}$$

showing that $C_T(k)$ is not square integrable. This problem is solved by introducing the modified call price function:

$$c_T(k) = e^{\alpha k} C_T(k) \tag{3.7}$$

for some $\alpha > 0$. The modified call price function, $c_T(k)$, is then expected to be square integrable for all $k \in \mathbb{R}$, provided that α is chosen correctly. The Fourier transform of $c_T(k)$ takes the following form:

$$\mathfrak{F}\{c_T(k)\} = \int_{-\infty}^{\infty} c_T(k) e^{i\xi k}\, dk = \psi_T(\xi) \tag{3.8}$$

Combining (3.4), (3.7) and (3.8), we obtain:

$$\begin{aligned}
\psi_T(\xi) &= \int_{-\infty}^{\infty} e^{i\xi k} \int_k^{\infty} e^{\alpha k} e^{-(r-\delta)T}\left(e^{s_T} - e^k\right) q_T(s_T) ds_T\, dk \\
&= \int_{-\infty}^{\infty} e^{-(r-\delta)T} q_T(s_T) \int_{-\infty}^{s_T} \left(e^{s_T + \alpha k} - e^{(1+\alpha)k}\right) e^{i\xi k}\, dk ds_T \\
&= \int_{-\infty}^{\infty} e^{-(r-\delta)T} q_T(s_T) \left[\frac{e^{(\alpha+1+i\xi)s_T}}{\alpha + i\xi} - \frac{e^{(\alpha+1+i\xi)s_T}}{\alpha + 1 + i\xi}\right] ds_T \\
&= \frac{e^{-(r-\delta)T}}{\alpha^2 + \alpha - \xi^2 + i(2\alpha+1)\xi} \int_{-\infty}^{\infty} e^{(-\alpha i - i + \xi) i s_T} q_T(s_T) ds_T
\end{aligned} \tag{3.9}$$

where $\varphi_T(\cdot)$ denotes the characteristic function of s_T. The call price can then be obtained by Fourier inversion of $\psi_T(\xi)$ and multiplication by $e^{-\alpha k}$:

$$C_T(k) = e^{-\alpha k} \cdot \mathfrak{F}^{-1}\{\psi_T(\xi)\} = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi k} \psi_T(\xi)\, d\xi = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-i\xi k}\, \psi_T(\xi) d\xi$$
$$\approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-i\xi_j k} \psi_T(\xi_j)\eta, \qquad j = 1, \dots, N. \tag{3.10}$$

where $\xi_j = \eta(j-1)$ and η is the step size in the integration grid. Equation (3.10) can be re-written as:

$$C_T(k_u) = \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)}\, e^{ib\xi_j} \psi(\xi_j) \frac{\eta}{3} \left(3 + (-1)^j - \mathbf{1}(j-1)_{\{0\}}\right) \tag{3.11}$$

where $b = \pi/\eta$; $k_u = -b + (2b/N)(u-1)$, $u = 1, \dots, N+1$; and $\mathbf{1}(x)_{\mathcal{M}}$ is the indicator function equal to 1 if $x \in \mathcal{M}$ and 0 otherwise. The term $1/3 \cdot \left(3 + (-1)^j - \mathbf{1}(j-1)_{\{0\}}\right)$ is obtained using the Simpson rule for numerical integration. Note that evaluating (3.10) will give call prices for a range of strikes (k_u). The grid of strikes will be dependent on the choice of the parameters η and N, and call prices for specific strike prices can be obtained e.g. through interpolation.

Now, the idea of writing the call price on the form (3.11) is that it enables the use of the FFT. The FFT is an algorithm to efficiently evaluate summations on the form:

$$\mathrm{X}(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j), \qquad k = 1, \dots, N \tag{3.12}$$

With $x_j = e^{ib\xi_j} \psi(\xi_j) \frac{\eta}{3} \left(3 + (-1)^j - \mathbf{1}(j-1)_{\{0\}}\right)$, (3.11) is a special case of (3.12) and can thus be evaluated using the FFT.

4. Parameter estimation

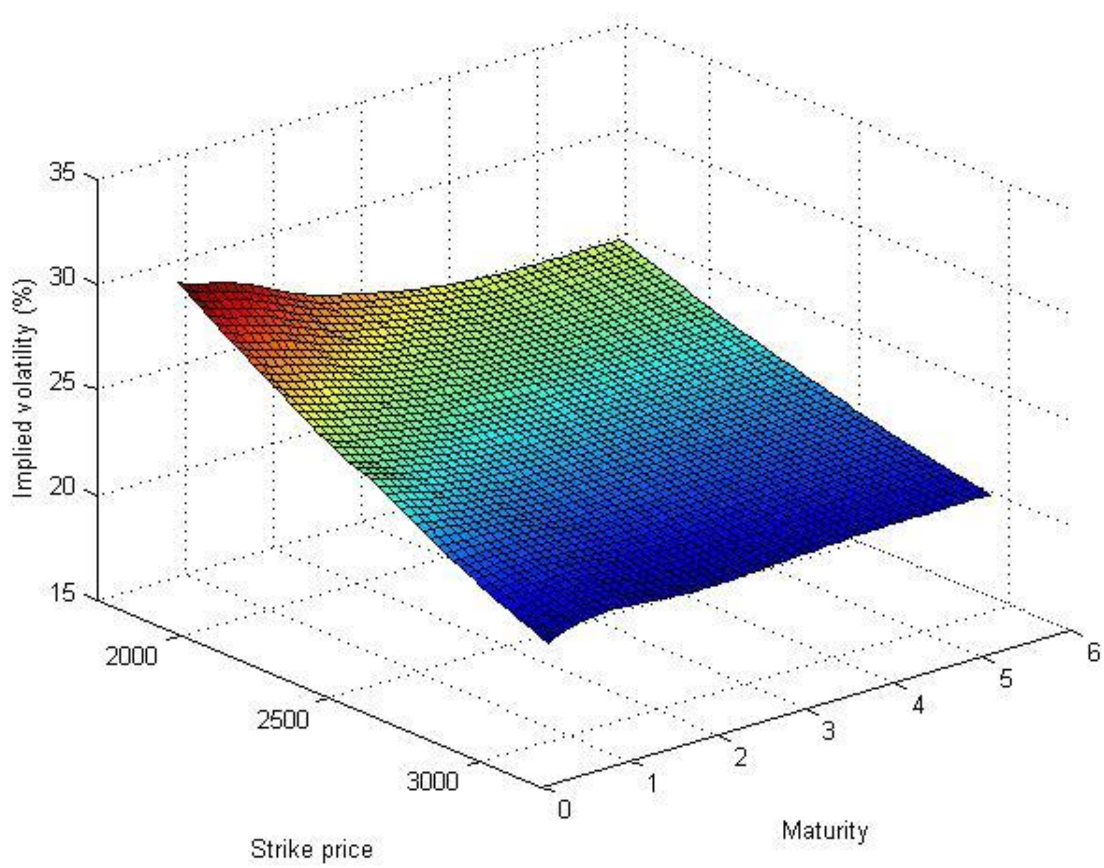
In order to use the defined models to price options, we must estimate model parameters under the equivalent martingale measure $\mathbb{Q}$. As all four stochastic volatility models describe incomplete markets, i.e. markets where we have more sources of risk than traded assets, it follows that the equivalent martingale measure is not unique [6]. In order to find parameter estimates under the equivalent martingale measure, we calibrate the models to fit observed market prices as closely as possible in some sense (see Section 4.2 below). As a consequence, the estimated model parameters will be according to the market's choice of $\mathbb{Q}$.

4.1. Data

The data set consists of 144 call options written on the Eurostoxx 50 index on the 7th of October 2003 and is the same data set used in [23]. The options have maturities between one month and five years and strike prices ranging from 1082 to 5440, with the spot price being 2461.44. As in [23], we also assume a constant interest rate of 3 % and a dividend yield of 0 %. The implied volatilities of all options in the data set are shown in Appendix B. Figure 1 shows the implied volatility surface spanned by the options in the data set. The surface has been obtained using the stochastic volatility inspired (SVI) method introduced in [17].

Figure 1

Implied volatility surface of the Eurostoxx 50 index on the 7th of October 2003 obtained using the stochastic volatility inspired (SVI) method described in [17].



4.2. Loss function

As discussed in the previous section, all models are defined under the risk-neutral measure. Hence, parameter estimates are obtained by calibrating the model to fit observed option prices (i.e. by making the model match observed option prices by altering the parameters). More formally, optimal parameter estimates under the risk-neutral measure are obtained by solving an optimization problem on the form:

$$\widehat{\Theta} = \arg\min_{\Theta} \mathfrak{L}\left(\{\hat{C}(\Theta)\}^n, \{C\}^n\right) \tag{4.1}$$

where Θ is the parameter vector. $\{\hat{C}(\Theta)\}^n$ is a set of n option prices obtained from the model, $\{C\}^n$ is the corresponding set of observed option prices in the market and $\mathfrak{L}(\cdot)$ is some loss function that quantifies the model's goodness of fit with respect to observed option prices. In this thesis, we consider four of the most common loss functions found in the literature, namely are the dollar mean squared error ($ MSE), the log-dollar mean squared error (L$ MSE) the percentage mean squared error (% MSE) and the implied volatility mean squared error (IV MSE):

$$\$\, MSE(\Theta) = \frac{1}{n}\sum_{i=1}^{n} w_i \left(C_i - \hat{C}_i(\Theta)\right)^2 \tag{4.2}$$

$$L\$\, MSE(\Theta) = \frac{1}{n}\sum_{i=1}^{n} w_i \left(\log(C_i) - \log\left(\hat{C}_i(\Theta)\right)\right)^2 \tag{4.3}$$

$$\%\, MSE(\Theta) = \frac{1}{n}\sum_{i=1}^{n} w_i \left(\frac{C_i - \hat{C}_i(\Theta)}{C_i}\right)^2 \tag{4.4}$$

$$IV\, MSE(\Theta) = \frac{1}{n}\sum_{i=1}^{n} w_i\left(\sigma_i - \hat{\sigma}_i(\Theta)\right)^2 \tag{4.5}$$

where σ_i is the Black-Scholes implied volatility of option i, and $\hat{\sigma}_i(\Theta)$ denotes the corresponding Black-Scholes implied volatility obtained using the model price as input, and w_i is an appropriately chosen weight.

The choice of loss function is important and has many implications. The $ MSE function minimizes the squared dollar error between model prices and observed prices and will thus favor parameters that correctly price expensive options, i.e. deep in-the-money (ITM) and long-dated options. The log-dollar MSE function mitigates this problem as the logarithm of the prices are more similar in magnitude than the actual prices. The % MSE function adjusts for price level by dividing the error by the market price, making it less biased towards correctly pricing expensive options. On the contrary, the % MSE function will put emphasis on options with prices close to zero, i.e. deep out-of-the-money (OTM) and short-dated options. The IV MSE function instead minimizes implied volatility errors, making options with higher implied volatility carry greater importance in the estimation. Due to the shape of the volatility smirk, this will in general put more weight on options with low strike prices, and less weight on options with high strike prices.

The existing literature has focused on the choice of loss function both for evaluation purposes [10], as well as for computational purposes. The reason for the latter is that most commonly proposed loss functions are non-convex and have several local (and perhaps global) minima, making standard optimization techniques unqualified [13]. Detlefsen & Härdle [14] study four different loss functions for estimation of stochastic volatility models and conclude that the most suitable choice once the models of interest have been specified is an implied volatility error metric, as this best reflects the characteristics of an option pricing model that is relevant for pricing out-of-sample. They also show that this choice leads to good calibrations in terms of relatively good fits and stable parameters. On a technical note, the IV MSE function is sometimes preferred to the other loss functions because it does not have the same problems with heteroskedasticity that can affect the estimation [10].

It has also been shown, e.g. by [22], that the choice of weighting (w_i) has a large influence on the behavior of the loss function for optimization purposes, and thus must be made with care. Two common methods are to either include the bid-ask spread of the options as a basis for weighting, giving less weight to options of which there is greater uncertainty of the true price (i.e. options wide a wide bid-ask spread) or to choose weights according to the number of options within different maturity categories.

In this thesis, we consider all loss functions (4.2) – (4.5). As the calculation of the Black-Scholes implied volatility has to be carried out numerically, calibration under the IV MSE loss function is very time consuming. To mitigate this problem, we instead consider an approximate IV MSE loss function:

$$IV\ MSE(\Theta) = \frac{1}{n}\sum_{i=1}^{n} w_i\big(\sigma_i - \hat{\sigma}_i(\Theta)\big)^2 \approx \frac{1}{n}\sum_{i=1}^{n} w_i\left(\frac{C_i - \hat{C}_i(\Theta)}{\mathcal{V}_i^{BS}}\right)^2 \tag{4.6}$$

where $\mathcal{V}_i^{BS}$ denotes the Black-Scholes Vega[1] of option i.

The modified IV MSE loss function (4.6), where the pricing error is divided by the Black-Scholes Vega, is obtained by considering the first order approximation:

$$\hat{C}_i(\Theta) \approx C_i + \mathcal{V}_i^{BS} \cdot (\hat{\sigma}_i(\Theta) - \sigma_i) \tag{4.7}$$

Re-arranging the terms yields:

$$\hat{\sigma}_i(\Theta) - \sigma_i \approx \frac{\hat{C}_i(\Theta) - C_i}{\mathcal{V}_i^{BS}} \tag{4.8}$$

Similar methods are used by [2], [9], [11], [24] and [25], among others, and significantly reduce computation time.

For all loss functions, we choose the weighting w_i such that all maturities are given the same weight in the calibration. Within each maturity category, all options are assigned equal weights. Our weighting is thus defined as:

$$w_i = \frac{1}{n_m \cdot n_{k_m}} \tag{4.9}$$

[1] Vega is the sensitivity of the option price with respect to volatility in the Black-Scholes model, i.e. $\mathcal{V}_i^{BS} = \partial C_i^{BS}/\partial\sigma_i$.

where n_m is the number of maturities and n_{k_m} is the number of options with the same maturity as option i. This choice of weighting is also used in [14].

4.3. Estimation procedure

The estimation was carried out using the `lsqnonlin` function in MATLAB. Since lsqnonlin is a local optimizer, we cannot know if the obtained solutions are the global minimums of the loss function. In order to maximize the chances of obtaining global solutions, each model was estimated ten times with different sets of starting values for the parameters. The starting values were randomly chosen on uniform intervals based on parameter estimates in previous studies such as [3], [11] and [23].

Apart from obvious bounds on the parameters, such as e.g. non-negative speed of mean-reversion and variances, we have implemented the so called Feller [15] condition, namely that $2\kappa\theta < \sigma^2$. The Feller condition ensures that the variance process is strictly positive and cannot reach zero. We implement the Feller condition by introducing the auxiliary variable $\Psi = 2\kappa\theta - \sigma^2$, and optimize using this variable rather than κ itself. The Feller condition then reduces to $\Psi > 0$, and κ can subsequently be calculated as $\kappa = (\Psi + \sigma^2)/2\theta$.

5. Barrier option pricing

5.1. The Black-Scholes model

One of the most appealing features of the Black-Scholes model is that it does not only provide analytical formulas for the pricing of vanilla options, but also for a range of exotic options. The price of a barrier option will depend on the regular Black-Scholes parameters $S_0, K, r, \delta, T, \sigma$ as well as on the barrier level, denoted H.

We obtain the pricing formulas from [26], where also derivations and intuition is provided. As mentioned in Section 1, we consider down-barrier call options with $H < K$ and up-barrier call options with $H > K$. All options considered are struck at the money (ATM), i.e. $K = S_0$.

Denote by $C_{BS}(S,K)$ and $P_{BS}(S,K)$ the Black-Scholes price of plain vanilla call and put options, respectively (the variables r, δ, T and σ are in all instances) and let $\nu = r - \delta - \frac{\sigma^2}{2}$ and $d_{BS}(S,K) = \frac{\log(S/K)+\nu T}{\sigma\sqrt{T}}$. Further, we denote by $\Phi(x)$ the standard normal cumulative distribution function.

Using this notation, the prices of the barrier options can be calculated as:

$$UI_{BS} = \left(\frac{H}{S}\right)^{\frac{2\nu}{\sigma^2}} \left\{ P_{BS}\left(\frac{H^2}{S}, K\right) - P_{BS}\left(\frac{H^2}{S}, H\right) + (H-K)e^{-rT}\Phi\big(-d_{BS}(H,S)\big) \right\} \\ + C_{BS}(S,H) + (H-K)e^{-rT}\Phi\big(d_{BS}(S,H)\big) \tag{5.1}$$

$$UO_{BS} = C_{BS}(S,K) - C_{BS}(S,H) - (H-K)e^{-rT}\Phi\big(d_{BS}(S,H)\big)$$
$$- \left(\frac{H}{S}\right)^{\frac{2\nu}{\sigma^2}} \left\{C_{BS}\left(\frac{H^2}{S},K\right) - C_{BS}\left(\frac{H^2}{S},H\right) - (H-K)e^{-rT}\Phi\big(d_{BS}(H,S)\big)\right\} \tag{5.2}$$

$$DI_{BS} = \left(\frac{H}{S}\right)^{\frac{2\nu}{\sigma^2}} C_{BS}\left(\frac{H^2}{S},K\right) \tag{5.3}$$

$$DO_{BS} = C_{BS}(S,K) - \left(\frac{H}{S}\right)^{\frac{2\nu}{\sigma^2}} C_{BS}\left(\frac{H^2}{S},K\right) \tag{5.4}$$

By definition, we have that $DI_{BS} + DO_{BS} = UI_{BS} + UO_{BS} = C_{BS}$, i.e. that the sum of a knock-in call option and a knock-out call option with the same strike price and barrier will equal the price of a vanilla call option.

To implement the Black-Scholes model, we need to estimate the volatility parameter σ. A common approach to finding the appropriate sigma is to observe the implied volatility surface (see Figure 1 in Section 4) and choose a volatility corresponding to the strike price and maturity in question. As neither the skew nor the term structure are incredibly steep around the ATM level for the considered maturities of one and three years, we have chosen to use the implied volatilities of the options with strike price and maturity closest to 2461.44 (ATM) and $T = 1$ and $T = 3$. This leads to $\sigma_{1y} = 24.46$ % and $\sigma_{3y} = 24.00$ %.[2]

5.2. Monte Carlo simulation

In order to price the path-dependant barrier options using the stochastic volatility models, we use Monte Carlo simulation. The first step towards pricing options using Monte Carlo simulation is to re-formulate the continuous processes of the various models to discrete time. For this purpose we use Euler-schemes.

Although we have implemented the Feller condition, there will be a risk that the variance process take negative values due to the discretization of the processes. For that reason, in each time step, we insert $V_t^+ = \max(V_t, 0)$ instead of V_t, i.e. we floor the variance at zero. Other methods include reflecting barriers, i.e. using $|V_t|$ rather than V_t^+. It has however been shown that the former method is less biased [15]. Note that when the variance is zero in a period, the variance process will have deterministic drift equal to $\kappa\theta dt$ in the next period.

For all models we use a time step of $dt = 1/252$, corresponding to one trading day, and 100 000 simulations.

[2] The actual strike price of the chosen options is 101.55 % of the spot price and the maturities are 1.19 and 2.19 years, respectively. See Appendix B for the implied volatilities of all options in the data set.

5.2.1. SV model

The Euler-scheme of the SV model takes the form:

$$S_t = S_{t-1} + (r - \delta)dt + \sqrt{V_t^+}\sqrt{dt}\, Z_t^{(1)} \tag{5.5}$$

$$V_t = V_{t-1}^+ + \kappa(\theta - V_{t-1}^+)dt + \sigma\sqrt{V_t^+}\sqrt{dt}\, Z_t^{(2)} \tag{5.6}$$

where $Z_t^{(1)}$ and $Z_t^{(2)}$are correlated $N(0,1)$ variables with correlation coefficient ρ.

5.2.2. SVJ model

The Euler-scheme of the SV model takes the form:

$$S_t = S_{t-1} + (r - \delta)dt + \sqrt{V_t^+}\sqrt{dt}\, Z_t^{(1)} + J_t X_t S_{t-1} \tag{5.7}$$

$$V_t = V_{t-1}^+ + \kappa(\theta - V_{t-1}^+)dt + \sigma\sqrt{V_t^+}\sqrt{dt}\, Z_t^{(2)} \tag{5.8}$$

where $Z_t^{(1)}$ and $Z_t^{(2)}$ are defined as in equation (5.5). dX_t is a Poisson counter with intensity λ and is simulated as $\Pr(X_t = 1) = \lambda dt$ and $\Pr(X_t = 0) = 1 - \lambda dt$. Recall from equation (2.13) that the jump size J_t is log-normally distributed. Regular standardization yields:

$$\frac{\log(1 + J_t) - \log(1 + \mu_J) + \sigma_J^2/2}{\sigma_J} \sim N(0,1) \tag{5.9}$$

If we let U_t be an $N(0,1)$ variable, we can simulate the jump size through:

$$J_t = \exp\left(\sigma_J U_t + \log(1 + \mu_J) - \frac{\sigma^2}{2}\right) - 1 \tag{5.10}$$

5.2.3. MFSV model

The discretization of the MFSV model is a natural extension of equations (5.5)–(5.6):

$$S_t = S_{t-1} + (r - \delta)dt + \sqrt{V_t^{(1)+}}\sqrt{dt}\, Z_t^{(1)} + \sqrt{V_t^{(2)+}}\sqrt{dt}\, Z_t^{(2)} \tag{5.11}$$

$$V_t^{(1)+} = V_{t-1}^{(1)+} + \kappa\left(\theta - V_{t-1}^{(1)+}\right)dt + \sigma\sqrt{V_t^{(1)+}}\sqrt{dt}\, Z_t^{(3)} \tag{5.12}$$

$$V_t^{(2)+} = V_{t-1}^{(2)+} + \kappa\left(\theta - V_{t-1}^{(2)+}\right)dt + \sigma\sqrt{V_t^{(2)+}}\sqrt{dt}\, Z_t^{(4)} \tag{5.13}$$

where $Z_t^{(1)}$ and $Z_t^{(3)}$ are correlated $N(0,1)$ variables with correlation ρ_1 and $Z_t^{(2)}$and $Z_t^{(4)}$ are correlated $N(0,1)$ variables with correlation ρ_2.

5.2.4. MFSVJ model

The discrete form of the MFSVJ model is simply obtained by adding the jump factor from equation (5.7) to equation (5.11):

$$S_t = S_{t-1} + (r-\delta)dt + \sqrt{V_t^{(1)+}}\sqrt{dt}\, Z_t^{(1)} + \sqrt{V_t^{(2)+}}\sqrt{dt}\, Z_t^{(2)} + J_t X_t S_{t-1} \tag{5.14}$$

$$V_t^{(1)+} = V_{t-1}^{(1)+} + \kappa\left(\theta - V_{t-1}^{(1)+}\right) dt + \sigma\sqrt{V_t^{(1)+}}\sqrt{dt}\, Z_t^{(3)} \tag{5.15}$$

$$V_t^{(2)+} = V_{t-1}^{(2)+} + \kappa\left(\theta - V_{t-1}^{(2)+}\right) dt + \sigma\sqrt{V_t^{(2)+}}\sqrt{dt}\, Z_t^{(4)} \tag{5.16}$$

where all variables are defined as in equations (5.10) – (5.13).

6. Results

6.1. Parameter estimates and in-sample fit

Table 1 summarizes the results of the parameter estimation and also display the root mean squared dollar error ($ RMSE) for the four models. Parameter estimates obtained using the IV RMSE, % MSE and Log $ MSE loss functions are shown in Tables A.1 – A.3 in Appendix A.

Table 1

Parameter estimates obtained by minimizing the squared dollar error to a sample of 144 call options on the Eurostoxx 50 index on the 7th of October 2003.

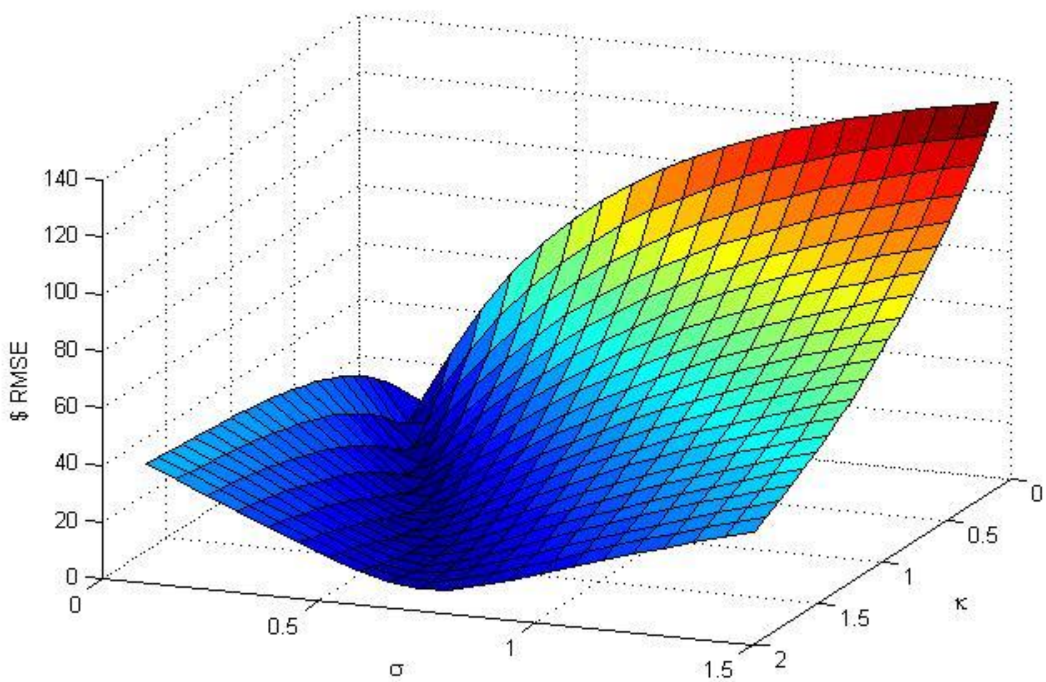
	κ	θ	σ	ρ	λ	μ_J	σ_J	V_0	$ $RMSE$
SV	0.5249	0.0705	0.2720	-0.7360				0.0649	2.4956
SVJ	0.5365	0.0630	0.2601	-0.9959	0.4978	0.1258	0.0534	0.0576	1.9178
MFSV	0.7057	0.0673	0.3082	-1.0000				0.0505	1.6832
	0.5545	0.0033	0.0602	0.8981				0.0154	
MFSVJ	0.6779	0.0670	0.3014	-0.9999	0.0706	0.0346	0.0536	0.0509	1.6808
	0.6068	0.0039	0.0690	0.8123				0.0150	

Unsurprisingly, the more advanced multi-factor model produces lower RMSEs than the single-factor models. Common for all models is a strong negative correlation between the return process and the variance process, which adds the empirically observed skew to the return distribution. Notable is however that for both multi-factor models, the second stochastic volatility factor, in both cases with significantly smaller magnitude than the first factor, is positively correlated with the return process.

The estimated parameters are similar across all loss functions. The most notable difference is that the IV MSE loss function result in higher estimates of the speed of mean-reversion (κ) and the volatility of the variance (σ) than the other loss functions. This however has rather small effect on the in-sample fit, regardless of loss function. The reason for this is that the parameters κ and σ have opposite effects on the behavior of the spot price process. A high value of κ will decrease volatility risk, as the volatility will be more quickly mean-reverting, making volatility shocks less persistent. On the contrary, a high value of σ will increase the magnitude of volatility shocks, increasing volatility risk. The trade-off is visualized in Figure 2 showing an error surface for different choices of κ and σ. We can see that the shape of the error surface more resembles a valley than a bowl, i.e. that the many combinations of κ and σ along the bottom of the valley yield errors of similar magnitude.

Figure 2

Error surface of the SV model for different choices of κ and σ.



As concluded in [23], the in-sample fit of the models is almost perfect. Figure A.1 in Appendix A shows the in-sample fits of the four stochastic volatility models. The figure shows the in-sample fit under the $ MSE loss function, but the corresponding plots for the other loss functions are almost identical.

6.2. Barrier option prices

Tables A.4 – A.11 in Appendix A show the obtained barrier option prices for the five models and the four loss functions used for estimation. We also show the probabilities that the studied barriers are breached for each model under each loss function in Tables A.12 – A.15.

The prices of the four stochastic volatility models are of similar magnitude in most instances. However, for down-and-in options with very low barriers and up-and-out options with barriers close to S_0, we observe large relative price differences simply because the prices are very close to zero making relative price differences very sensitive.

We do not observe any large differences in the prices between loss functions, pointing towards the conclusion that the choice of loss function is not essential for the purpose of barrier option pricing. It should however be noted that that may be a consequence of the data set at hand, in which all loss functions yield very similar parameter estimates. Rather than to neglect the importance of the choice of loss function, these results should be seen as a motivation to conducting larger scale studies on more extensive data sets to examine the importance of the choice of loss function for the purpose of exotic option pricing.

As for the Black-Scholes model, we observe large price differences in comparison to the stochastic volatility models. First, we note that the Black-Scholes price of down-and-in options is significantly lower than the corresponding prices of the stochastic volatility models for barriers below 85 % of the spot price. This observation is rather expected, as one of the main purposes of introducing stochastic volatility models is to model the empirical fact that volatility tends to increase in declining markets. Hence, the stochastic volatility models will introduce a higher probability of the stock price breaching the barrier far below the spot price. In other words, the probability distribution of the stock price at any future time point will be right skewed as compared to the normal distribution, implying higher probabilities of large declines in the stock price that are necessary for down-and-in options with low barriers to end up in the money. This is confirmed by the probabilities shown in Tables A.12 – A.15 in Appendix A. The probabilities of the spot price breaching the lower barriers is significantly higher in the stochastic volatility models for barriers of 80 % of the spot price and below, regardless of loss function used for parameter estimation. Given that there are rather small price differences between the vanilla call prices of the models, the relation $DO + DI = call\ option$ states that if there is a difference in price between down-and-in options, there must be a corresponding reverse price difference in the down-and-out options. However, as the prices of the down-and-out options for very low barriers obviously are much higher than the corresponding down-and-in options, the relative price difference is much smaller. Hence, the relative price differences of several hundred percent for down-and-in options only correspond to relative price differences of a few percent for the down-and-out options.

Second, we note that the Black-Scholes prices of up-and-in barrier options are significantly higher than the corresponding prices of the stochastic volatility models for high barriers, e.g. barriers above 25 % of the spot price. The pattern is most evident for the short maturity options (short in this case meaning a maturity of one year), whereas the difference for the 3-year options becomes evident at even higher levels of the barrier. The probabilities in Tables A.12 – A.15 in Appendix a reveal that this stems from an increased probability that the upper barrier is breached in the Black-Scholes model as compared to the stochastic volatility models. As we have not derived the actual distributions of the stock price under the stochastic volatility models, this pattern is more difficult to explain. A plausible explanation to the findings is however that the skewness of the distributions of the stock price under the stochastic volatility models decrease the amount of probability mass in the right tail of the distribution, making extreme events on the upside less likely than in the normal distribution assumed in the Black-Scholes model. Although the stochastic volatility models also add kurtosis to the stock price distribution, resulting in distributions with fatter tails than the normal distribution, it seems that in this case the effect of the skewness is more prominent. The pattern is confirmed when looking at the prices of the up-and-out barrier options. As expected given the recently discussed observations, the prices of the up-and-out options are lower in the Black-Scholes model than in the stochastic volatility models, with relative price differences being the largest for barriers close to the spot price.

7. Conclusion

This thesis examines the performance of the Black-Scholes model and four stochastic volatility models with respect to the pricing of barrier options. Our results show that the choice of loss function for estimation of the model parameters of the stochastic volatility models have little effect on the resulting prices of both vanilla options and barrier options. This result motivates further studies of the impact of the loss function on exotic option prices and parameter estimation.

Further, our results show that the Black-Scholes model yields barrier option prices that differ significantly from the corresponding prices obtained from the stochastic volatility models, although vanilla call prices are very similar. The reason for this is that the stochastic volatility models give rise to a skewed distribution of future spot prices, resulting in higher probabilities of breaching low barriers and lower probabilities of breaching high barriers as compared to the symmetrical normal distribution underlying the Black-Scholes model. For barrier levels close to the spot price, however, the prices are in many cases of similar magnitude.

As for the relationship between the stochastic volatility models, we find that all four models yield similar prices both with respect to vanilla call options and the path-dependent barrier options. This confirms the notion of [23], where it is concluded that large differences in exotic option prices are observed between different classes of models rather than between different models within the same category.

8. References

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Appendix A: Tables and graphs

Table A.1

Parameter estimates obtained by minimizing the squared implied volatility error (IV MSE) to a sample of 144 call options on the Eurostoxx 50 index on the 7th of October 2003.

	κ	θ	σ	ρ	λ	μ_J	σ_J	V_0	\$ $RMSE$
SV	0.8753	0.0691	0.3478	-0.7613				0.0673	0.0059
SVJ	0.7051	0.0609	0.2931	-0.9940	0.5286	0.0490	0.1202	0.0596	0.0039
MFSV	1.0637	0.0661	0.3749	-1.0000				0.0500	0.0051
	0.8527	0.0032	0.0734	0.5981				0.0169	
MFSVJ	0.9197	0.0644	0.3442	-1.0000	0.3402	-0.0703	0.0558	0.3402	0.0558
	0.8044	0.0024	0.0622	0.1411				-0.0703	

Table A.2

Parameter estimates obtained by minimizing the squared percentage error (% MSE) to a sample of 144 call options on the Eurostoxx 50 index on the 7th of October 2003.

	κ	θ	σ	ρ	λ	μ_J	σ_J	V_0	\$ $RMSE$
SV	0.4802	0.0676	0.2548	-0.6701				0.0631	0.0149
SVJ	0.4907	0.0631	0.2488	-0.9570	0.4588	0.1192	0.0669	0.0575	0.0121
MFSV	0.4040	0.0703	0.2383	-0.9241				0.0522	0.0122
	1.1390	0.0067	0.1231	0.7282				0.0113	
MFSVJ	0.4662	0.0686	0.2529	-0.9999	0.1105	0.1044	0.2099	0.1105	0.2099
	1.0426	0.0025	0.0716	-0.9755				0.1044	

Table A.3

Parameter estimates obtained by minimizing the squared log-dollar error (L$ MSE) to a sample of 144 call options on the Eurostoxx 50 index on the 7th of October 2003.

	$\boldsymbol{\kappa}$	$\boldsymbol{\theta}$	$\boldsymbol{\sigma}$	$\boldsymbol{\rho}$	$\boldsymbol{\lambda}$	$\boldsymbol{\mu_J}$	$\boldsymbol{\sigma_J}$	$\boldsymbol{V_0}$	$ $\boldsymbol{RMSE}$
SV	0.4803	0.0676	0.2548	-0.6692				0.0631	0.0149
SVJ	0.4947	0.0626	0.2489	-0.9432	0.5569	0.0899	0.0814	0.0576	0.0122
MFSV	0.3747	0.0714	0.2314	-0.9237				0.0527	0.0122
	1.2456	0.0067	0.1295	0.7891				0.0106	
MFSVJ	0.3860	0.0697	0.2319	-0.9999	0.1288	0.1040	0.1928	0.1288	0.1928
	1.1813	0.0038	0.0947	-0.9120				0.1040	

Figure A.1

In-sample fits of the stochastic volatility models with parameters estimated using the $ MSE loss function. The corresponding plots using the other loss functions are identical. In the plots, rings correspond to actual option prices and crosses to model prices.

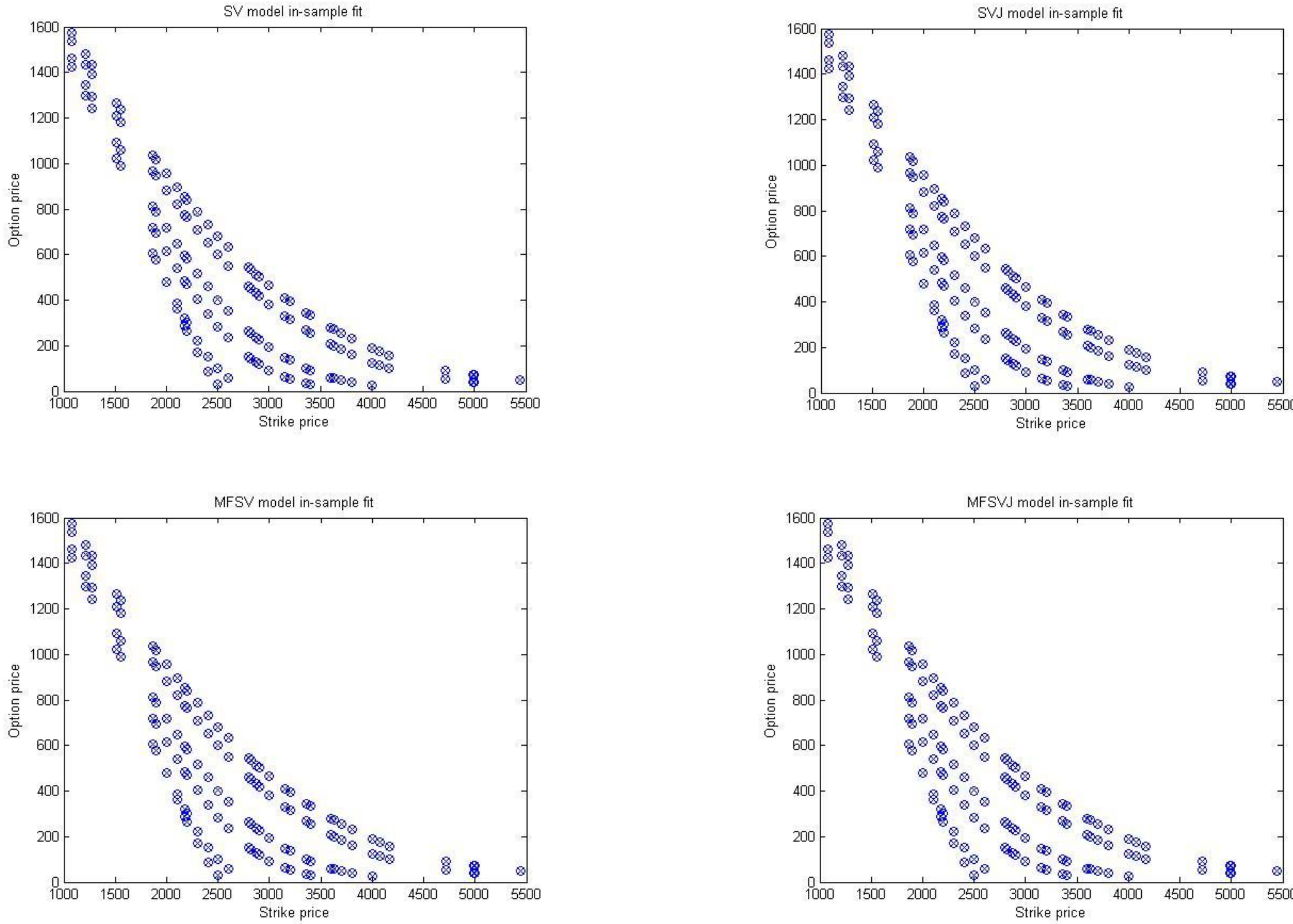

Table A.4

Prices for 1-year barrier options with parameters estimated using the $ MSE loss function.

	Down-and-in barrier option					Down-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$0.50S_0$	0.0000	0.0081	0.0060	0.0030	0.0081	274.1900	275.0006	277.3064	274.1703	274.5192
$0.55S_0$	0.0001	0.0263	0.0122	0.0098	0.0631	274.1900	274.9824	277.3002	274.1635	274.4642
$0.60S_0$	0.0020	0.1285	0.1884	0.0733	0.1399	274.1881	274.8803	277.1240	274.1000	274.3874
$0.65S_0$	0.0332	0.5236	0.5625	0.3668	0.4214	274.1568	274.4851	276.7499	273.8065	274.1059
$0.70S_0$	0.3203	1.6186	1.7434	1.4280	1.5160	273.8697	273.3901	275.5690	272.7453	273.0113
$0.75S_0$	1.9838	4.6841	5.0824	4.5179	4.7918	272.2062	270.3247	272.2299	269.6554	269.7355
$0.80S_0$	8.5831	12.9257	13.8146	12.1983	12.6703	265.6069	262.0830	263.4978	261.9750	261.8570
$0.85S_0$	27.7719	31.7671	33.3594	30.6729	30.4424	246.4182	243.2417	243.9530	243.5004	244.0849
$0.90S_0$	71.0483	69.0367	71.4286	68.5413	67.1866	203.1417	205.9721	205.8838	205.6320	207.3407
$0.95S_0$	150.4820	136.6473	139.7326	136.3818	135.4948	123.7081	138.3614	137.5798	137.7915	139.0325

	Up-and-in barrier option					Up-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$1.05S_0$	274.0687	274.7038	277.0395	273.9142	274.2368	0.1214	0.3050	0.2729	0.2591	0.2905
$1.10S_0$	272.5252	271.4839	273.9226	270.9683	271.2352	1.6648	3.5249	3.3898	3.2050	3.2921
$1.15S_0$	267.2057	260.8833	262.7319	260.8001	260.7290	6.9843	14.1254	14.5805	13.3732	13.7983
$1.20S_0$	256.3696	240.1548	238.9201	239.8235	239.5080	17.8205	34.8539	38.3922	34.3498	35.0193
$1.25S_0$	239.7287	209.6527	202.2006	209.3548	207.7613	34.4613	65.3561	75.1118	64.8185	66.7660
$1.30S_0$	218.3152	174.0841	157.7766	173.7639	168.6309	55.8748	100.9247	119.5358	100.4094	105.8964
$1.35S_0$	193.8843	136.4678	119.4497	137.0955	127.9099	80.3057	138.5409	157.8627	137.0778	146.6174
$1.40S_0$	168.3121	101.7691	91.8452	104.9172	91.7985	105.8780	173.2396	185.4672	169.2561	182.7288
$1.45S_0$	143.2018	73.7787	70.2569	78.0326	63.9862	130.9883	201.2300	207.0555	196.1407	210.5411
$1.50S_0$	119.7156	51.8687	52.1443	56.8645	44.5079	154.4744	223.1401	225.1681	217.3088	230.0194

Table A.5
Prices for 1-year barrier options with parameters estimated using the IV MSE loss function.

	Down-and-in barrier option					Down-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$0.50S_0$	0.0000	0.0086	0.0047	0.0023	0.0000	274.1900	277.9274	279.1101	277.0044	277.2151
$0.55S_0$	0.0001	0.0693	0.0204	0.0302	0.0142	274.1900	277.8667	279.0944	276.9765	277.2010
$0.60S_0$	0.0020	0.2525	0.1635	0.1180	0.0776	274.1881	277.6835	278.9514	276.8886	277.1376
$0.65S_0$	0.0332	0.7507	0.6064	0.4770	0.4064	274.1568	277.1853	278.5085	276.5296	276.8087
$0.70S_0$	0.3203	2.2140	2.0046	1.5776	1.5202	273.8697	275.7220	277.1103	275.4290	275.6949
$0.75S_0$	1.9838	6.0663	5.6497	4.7862	4.6622	272.2062	271.8696	273.4652	272.2205	272.5529
$0.80S_0$	8.5831	15.3985	14.2835	13.2063	12.8485	265.6069	262.5374	264.8314	263.8004	264.3667
$0.85S_0$	27.7719	33.9319	33.2627	31.8497	31.1045	246.4182	244.0040	245.8522	245.1569	246.1106
$0.90S_0$	71.0483	71.6079	70.5652	68.8335	67.9186	203.1417	206.3281	208.5497	208.1732	209.2966
$0.95S_0$	150.4820	139.8692	138.7984	136.3370	134.9312	123.7081	138.0667	140.3165	140.6697	142.2840

	Up-and-in barrier option					Up-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$1.05S_0$	274.0687	277.5996	278.8460	276.6731	276.9129	0.1214	0.3363	0.2689	0.3336	0.3023
$1.10S_0$	272.5252	274.0553	275.6274	273.1450	273.7037	1.6648	3.8807	3.4875	3.8616	3.5114
$1.15S_0$	267.2057	262.3678	264.5739	261.4563	263.0633	6.9843	15.5682	14.5410	15.5504	14.1518
$1.20S_0$	256.3696	239.2286	239.9937	238.4828	241.9611	17.8205	38.7074	39.1212	38.5238	35.2540
$1.25S_0$	239.7287	204.8924	198.7799	205.3381	210.1468	34.4613	73.0435	80.3350	71.6685	67.0683
$1.30S_0$	218.3152	163.4746	147.1112	167.3054	171.3472	55.8748	114.4614	132.0037	109.7012	105.8680
$1.35S_0$	193.8843	120.8413	100.4014	130.2004	131.6120	80.3057	157.0946	178.7135	146.8063	145.6031
$1.40S_0$	168.3121	84.2200	75.5922	96.7098	95.4904	105.8780	193.7159	203.5227	180.2968	181.7247
$1.45S_0$	143.2018	55.8416	60.2651	70.5457	67.2887	130.9883	222.0944	218.8497	206.4610	209.9265
$1.50S_0$	119.7156	34.8531	46.7368	48.9921	44.1533	154.4744	243.0829	232.3781	228.0145	233.0619

Table A.6

Prices for 1-year barrier options with parameters estimated using the % MSE loss function.

	Down-and-in barrier option					Down-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$0.50S_0$	0.50	0.0000	0.0081	0.0060	0.0030	274.1900	275.0006	277.3064	274.1703	274.5192
$0.55S_0$	0.55	0.0001	0.0263	0.0122	0.0098	274.1900	274.9824	277.3002	274.1635	274.4642
$0.60S_0$	0.60	0.0020	0.1285	0.1884	0.0733	274.1881	274.8803	277.1240	274.1000	274.3874
$0.65S_0$	0.65	0.0332	0.5236	0.5625	0.3668	274.1568	274.4851	276.7499	273.8065	274.1059
$0.70S_0$	0.70	0.3203	1.6186	1.7434	1.4280	273.8697	273.3901	275.5690	272.7453	273.0113
$0.75S_0$	0.75	1.9838	4.6841	5.0824	4.5179	272.2062	270.3247	272.2299	269.6554	269.7355
$0.80S_0$	0.80	8.5831	12.9257	13.8146	12.1983	265.6069	262.0830	263.4978	261.9750	261.8570
$0.85S_0$	0.85	27.7719	31.7671	33.3594	30.6729	246.4182	243.2417	243.9530	243.5004	244.0849
$0.90S_0$	0.90	71.0483	69.0367	71.4286	68.5413	203.1417	205.9721	205.8838	205.6320	207.3407
$0.95S_0$	0.95	150.4820	136.6473	139.7326	136.3818	123.7081	138.3614	137.5798	137.7915	139.0325

	Up-and-in barrier option					Up-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$1.05S_0$	274.0687	274.7038	277.0395	273.9142	274.2368	0.1214	0.3050	0.2729	0.2591	0.2905
$1.10S_0$	272.5252	271.4839	273.9226	270.9683	271.2352	1.6648	3.5249	3.3898	3.2050	3.2921
$1.15S_0$	267.2057	260.8833	262.7319	260.8001	260.7290	6.9843	14.1254	14.5805	13.3732	13.7983
$1.20S_0$	256.3696	240.1548	238.9201	239.8235	239.5080	17.8205	34.8539	38.3922	34.3498	35.0193
$1.25S_0$	239.7287	209.6527	202.2006	209.3548	207.7613	34.4613	65.3561	75.1118	64.8185	66.7660
$1.30S_0$	218.3152	174.0841	157.7766	173.7639	168.6309	55.8748	100.9247	119.5358	100.4094	105.8964
$1.35S_0$	193.8843	136.4678	119.4497	137.0955	127.9099	80.3057	138.5409	157.8627	137.0778	146.6174
$1.40S_0$	168.3121	101.7691	91.8452	104.9172	91.7985	105.8780	173.2396	185.4672	169.2561	182.7288
$1.45S_0$	143.2018	73.7787	70.2569	78.0326	63.9862	130.9883	201.2300	207.0555	196.1407	210.5411
$1.50S_0$	119.7156	51.8687	52.1443	56.8645	44.5079	154.4744	223.1401	225.1681	217.3088	230.0194

Table A.7

Prices for 1-year barrier options with parameters estimated using the L$ MSE loss function.

	Down-and-in barrier option					Down-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$0.50S_0$	0.0000	0.0009	0.0140	0.0000	0.0000	274.1900	274.8604	277.2472	274.8902	275.2893
$0.55S_0$	0.0001	0.0154	0.0437	0.0152	0.0225	274.1900	274.8459	277.2175	274.8750	275.2668
$0.60S_0$	0.0020	0.1650	0.1520	0.0968	0.1189	274.1881	274.6963	277.1093	274.7933	275.1704
$0.65S_0$	0.0332	0.5541	0.5494	0.4000	0.3789	274.1568	274.3072	276.7119	274.4901	274.9104
$0.70S_0$	0.3203	1.7318	1.8732	1.5002	1.3393	273.8697	273.1295	275.3880	273.3899	273.9500
$0.75S_0$	1.9838	5.0142	5.3108	4.5790	4.3979	272.2062	269.8471	271.9505	270.3111	270.8915
$0.80S_0$	8.5831	13.0531	13.7304	12.4231	12.3933	265.6069	261.8082	263.5309	262.4671	262.8960
$0.85S_0$	27.7719	30.6487	32.5291	30.3278	30.2435	246.4182	244.2126	244.7321	244.5624	245.0458
$0.90S_0$	71.0483	68.4832	70.6815	67.3655	67.3389	203.1417	206.3781	206.5797	207.5247	207.9504
$0.95S_0$	150.4820	136.1956	138.0895	136.6247	136.6379	123.7081	138.6657	139.1718	138.2655	138.6514

	Up-and-in barrier option					Up-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$1.05S_0$	274.0687	274.5579	276.9783	274.6182	275.0049	0.1214	0.3034	0.2829	0.2720	0.2845
$1.10S_0$	272.5252	271.3023	273.8376	271.8167	272.1384	1.6648	3.5589	3.4236	3.0735	3.1509
$1.15S_0$	267.2057	261.0447	262.9868	262.1867	262.4753	6.9843	13.8166	14.2744	12.7035	12.8141
$1.20S_0$	256.3696	240.6990	239.7447	241.8668	241.9726	17.8205	34.1623	37.5165	33.0234	33.3167
$1.25S_0$	239.7287	210.7009	203.6684	211.6293	210.4443	34.4613	64.1603	73.5928	63.2609	64.8450
$1.30S_0$	218.3152	174.7820	160.6701	174.9527	171.0036	55.8748	100.0793	116.5911	99.9375	104.2857
$1.35S_0$	193.8843	136.9364	122.0361	138.6404	127.7421	80.3057	137.9249	155.2251	136.2497	147.5473
$1.40S_0$	168.3121	102.4555	92.5817	105.3404	90.6287	105.8780	172.4058	184.6795	169.5497	184.6606
$1.45S_0$	143.2018	73.7968	69.9429	76.8820	62.3988	130.9883	201.0645	207.3184	198.0082	212.8906
$1.50S_0$	119.7156	52.2538	52.4665	56.4194	43.9130	154.4744	222.6075	224.7948	218.4708	231.3763

Table A.8
Prices for 3-year barrier options with parameters estimated using the $ MSE loss function.

	Down-and-in barrier option					Down-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$0.50S_0$	0.1166	1.9247	1.2530	1.0890	1.1407	502.5648	508.1323	509.9314	507.4075	507.4944
$0.55S_0$	0.6225	4.3752	3.4383	2.8419	2.9701	502.0589	505.6818	507.7460	505.6546	505.6650
$0.60S_0$	2.4522	8.9627	8.2221	6.9322	6.9653	500.2292	501.0943	502.9622	501.5643	501.6697
$0.65S_0$	7.6039	18.3658	17.0392	14.8029	14.8295	495.0775	491.6912	494.1451	493.6936	493.8056
$0.70S_0$	19.4850	32.9504	32.7477	28.8935	29.1749	483.1964	477.1066	478.4366	479.6030	479.4602
$0.75S_0$	42.8235	56.8495	56.5884	52.5148	52.7284	459.8578	453.2075	454.5960	455.9817	455.9067
$0.80S_0$	83.0905	93.2038	94.5855	90.0343	90.4168	419.5909	416.8532	416.5988	418.4622	418.2183
$0.85S_0$	145.6252	148.6478	150.1778	145.6288	145.6908	357.0562	361.4092	361.0065	362.8677	362.9443
$0.90S_0$	234.7899	227.1328	228.4069	224.5524	224.8729	267.8915	282.9242	282.7775	283.9441	283.7622
$0.95S_0$	353.4216	333.7625	335.6141	333.0173	332.7560	149.2598	176.2945	175.5702	175.4792	175.8791

	Up-and-in barrier option					Up-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$1.05S_0$	502.6577	509.9615	511.0834	508.4074	508.5364	0.0237	0.0955	0.1009	0.0892	0.0987
$1.10S_0$	502.3408	509.1370	510.1631	507.5046	507.6697	0.3406	0.9200	1.0213	0.9919	0.9654
$1.15S_0$	501.1482	506.3800	506.9940	504.4785	504.6730	1.5331	3.6769	4.1904	4.0180	3.9621
$1.20S_0$	498.4097	500.3307	500.1837	497.9418	498.1854	4.2716	9.7263	11.0007	10.5547	10.4497
$1.25S_0$	493.5500	490.1939	488.4480	486.9378	487.5194	9.1314	19.8631	22.7363	21.5587	21.1157
$1.30S_0$	486.1892	474.8442	471.0262	470.2683	471.0341	16.4922	35.2128	40.1581	38.2282	37.6010
$1.35S_0$	476.1741	454.9283	446.6772	448.2589	449.2228	26.5073	55.1287	64.5071	60.2377	59.4123
$1.40S_0$	463.5600	429.6798	416.8228	421.2235	421.8716	39.1213	80.3772	94.3616	87.2730	86.7635
$1.45S_0$	448.5679	399.7850	383.3252	390.7613	391.4846	54.1134	110.2720	127.8591	117.7352	117.1505
$1.50S_0$	431.5319	366.8493	348.2312	357.7234	357.9545	71.1494	143.2077	162.9531	150.7732	150.6806

Table A.9
Prices for 3-year barrier options with parameters estimated using the IV MSE loss function.

	Down-and-in barrier option					Down-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$0.50S_0$	0.1166	1.5235	1.3932	0.7854	0.8006	502.5648	512.7988	513.1338	510.7359	512.4673
$0.55S_0$	0.6225	3.8782	3.4164	2.3797	2.4523	502.0589	510.4441	511.1106	509.1416	510.8156
$0.60S_0$	2.4522	8.3593	7.7924	6.0164	6.1387	500.2292	505.9630	506.7346	505.5049	507.1292
$0.65S_0$	7.6039	17.0926	16.1282	13.5809	13.8102	495.0775	497.2296	498.3988	497.9404	499.4577
$0.70S_0$	19.4850	32.3303	31.1647	28.1240	28.3985	483.1964	481.9919	483.3623	483.3973	484.8694
$0.75S_0$	42.8235	56.2368	55.9693	52.0375	51.8495	459.8578	458.0854	458.5577	459.4838	461.4184
$0.80S_0$	83.0905	93.6866	93.4484	89.6516	89.6601	419.5909	420.6357	421.0786	421.8697	423.6078
$0.85S_0$	145.6252	150.0554	148.5529	145.3007	145.2336	357.0562	364.2669	365.9741	366.2206	368.0343
$0.90S_0$	234.7899	228.7264	227.7369	225.2156	224.8512	267.8915	285.5958	286.7901	286.3057	288.4167
$0.95S_0$	353.4216	335.4520	335.1454	333.4146	332.3035	149.2598	178.8703	179.3816	178.1067	180.9644

	Up-and-in barrier option					Up-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$1.05S_0$	502.6577	514.2255	514.4189	511.4200	513.1748	0.0237	0.0967	0.1081	0.1013	0.0931
$1.10S_0$	502.3408	513.2524	513.5219	510.4408	512.1824	0.3406	1.0699	1.0051	1.0805	1.0855
$1.15S_0$	501.1482	510.3158	510.3676	507.1494	509.3456	1.5331	4.0064	4.1594	4.3719	3.9223
$1.20S_0$	498.4097	504.3269	503.6635	500.5208	503.2223	4.2716	9.9954	10.8635	11.0005	10.0456
$1.25S_0$	493.5500	494.5497	492.1019	489.6629	492.9901	9.1314	19.7726	22.4251	21.8584	20.2778
$1.30S_0$	486.1892	479.7334	474.8366	473.6068	477.8161	16.4922	34.5888	39.6904	37.9145	35.4518
$1.35S_0$	476.1741	459.7841	451.6709	452.1582	457.8200	26.5073	54.5382	62.8561	59.3631	55.4479
$1.40S_0$	463.5600	434.7156	423.0536	425.5116	432.8808	39.1213	79.6066	91.4734	86.0097	80.3871
$1.45S_0$	448.5679	405.6125	390.3201	394.9992	404.7058	54.1134	108.7098	124.2069	116.5221	108.5621
$1.50S_0$	431.5319	371.9016	353.5464	362.1855	371.8962	71.1494	142.4207	160.9806	149.3358	141.3717

Table A.10

Prices for 3-year barrier options with parameters estimated using the % MSE loss function.

	Down-and-in barrier option					Down-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$0.50S_0$	0.1166	1.7404	1.5626	1.6054	1.3501	502.5648	503.6872	511.0489	506.3453	508.1788
$0.55S_0$	0.6225	4.1793	3.7445	3.7865	3.3151	502.0589	501.2484	508.8670	504.1643	506.2138
$0.60S_0$	2.4522	8.9276	8.1212	8.3693	7.5328	500.2292	496.5001	504.4903	499.5814	501.9962
$0.65S_0$	7.6039	17.4066	17.0913	17.5545	15.7692	495.0775	488.0211	495.5202	490.3962	493.7598
$0.70S_0$	19.4850	32.1160	32.7321	32.1154	29.8968	483.1964	473.3117	479.8794	475.8353	479.6322
$0.75S_0$	42.8235	55.7912	58.1686	56.7650	53.4617	459.8578	449.6365	454.4428	451.1858	456.0672
$0.80S_0$	83.0905	92.4371	95.9873	93.6330	90.6377	419.5909	412.9906	416.6241	414.3177	418.8913
$0.85S_0$	145.6252	144.7426	150.3621	147.5742	145.3039	357.0562	360.6851	362.2494	360.3766	364.2250
$0.90S_0$	234.7899	223.3466	227.8481	226.2646	223.6534	267.8915	282.0811	284.7634	281.6862	285.8755
$0.95S_0$	353.4216	330.7993	337.0078	332.8813	331.7967	149.2598	174.6284	175.6037	175.0695	177.7323

	Up-and-in barrier option					Up-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$1.05S_0$	502.6577	505.3378	512.5204	507.8871	509.4388	0.0237	0.0898	0.0911	0.0636	0.0901
$1.10S_0$	502.3408	504.4254	511.6130	507.1414	508.5102	0.3406	1.0023	0.9985	0.8094	1.0188
$1.15S_0$	501.1482	501.5254	508.6607	504.6485	505.5199	1.5331	3.9022	3.9508	3.3023	4.0090
$1.20S_0$	498.4097	495.4492	502.0150	498.8955	499.1183	4.2716	9.9785	10.5965	9.0552	10.4107
$1.25S_0$	493.5500	485.2651	490.7827	488.9692	488.0400	9.1314	20.1626	21.8287	18.9816	21.4889
$1.30S_0$	486.1892	469.7046	473.1524	473.3601	472.2255	16.4922	35.7231	39.4591	34.5907	37.3034
$1.35S_0$	476.1741	449.3348	449.8940	452.7337	450.1750	26.5073	56.0929	62.7174	55.2171	59.3539
$1.40S_0$	463.5600	424.4657	420.8375	426.6148	423.1582	39.1213	80.9619	91.7740	81.3359	86.3707
$1.45S_0$	448.5679	394.8396	387.6430	396.2492	391.1637	54.1134	110.5881	124.9685	111.7016	118.3652
$1.50S_0$	431.5319	362.7651	351.5801	363.5622	355.6086	71.1494	142.6626	161.0314	144.3885	153.9204

Table A.11

Prices for 3-year barrier options with parameters estimated using the L$ MSE loss function.

	Down-and-in barrier option					Down-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$0.50S_0$	0.1166	1.8213	1.4078	1.5752	1.2631	502.5648	506.4842	509.9652	509.6144	508.0357
$0.55S_0$	0.6225	4.1917	3.4542	3.7666	3.3675	502.0589	504.1137	507.9188	507.4230	505.9312
$0.60S_0$	2.4522	8.7155	8.0933	8.6851	7.5311	500.2292	499.5899	503.2797	502.5045	501.7676
$0.65S_0$	7.6039	17.4098	16.8459	17.7770	16.0722	495.0775	490.8957	494.5271	493.4125	493.2266
$0.70S_0$	19.4850	31.7345	32.0136	32.8229	29.9369	483.1964	476.5709	479.3594	478.3666	479.3619
$0.75S_0$	42.8235	55.0120	55.9222	57.3852	54.0816	459.8578	453.2935	455.4508	453.8044	455.2172
$0.80S_0$	83.0905	91.8509	94.3488	95.7342	91.5664	419.5909	416.4545	417.0241	415.4554	417.7323
$0.85S_0$	145.6252	147.1173	149.3706	151.1974	147.6444	357.0562	361.1881	362.0024	359.9922	361.6544
$0.90S_0$	234.7899	223.9602	227.4672	229.3696	224.7491	267.8915	284.3452	283.9058	281.8200	284.5497
$0.95S_0$	353.4216	330.1831	333.2652	335.8637	332.4321	149.2598	178.1223	178.1078	175.3259	176.8666

	Up-and-in barrier option					Up-and-out barrier option				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$1.05S_0$	502.6577	508.2358	511.2855	511.1219	509.2215	0.0237	0.0697	0.0875	0.0677	0.0773
$1.10S_0$	502.3408	507.3842	510.3989	510.4288	508.4099	0.3406	0.9212	0.9740	0.7607	0.8889
$1.15S_0$	501.1482	504.3765	507.4626	508.1051	505.7000	1.5331	3.9289	3.9104	3.0845	3.5988
$1.20S_0$	498.4097	498.1795	500.8697	502.6892	499.7790	4.2716	10.1260	10.5033	8.5003	9.5197
$1.25S_0$	493.5500	487.6471	489.1923	492.8148	489.3262	9.1314	20.6584	22.1806	18.3747	19.9725
$1.30S_0$	486.1892	472.2957	471.6178	477.7649	473.4243	16.4922	36.0098	39.7552	33.4247	35.8744
$1.35S_0$	476.1741	451.6957	448.6112	457.0704	452.1559	26.5073	56.6098	62.7618	54.1192	57.1428
$1.40S_0$	463.5600	426.7686	419.8896	431.0010	424.8611	39.1213	81.5368	91.4834	80.1886	84.4377
$1.45S_0$	448.5679	397.3830	388.3640	400.2333	392.4640	54.1134	110.9225	123.0090	110.9563	116.8348
$1.50S_0$	431.5319	365.1018	353.7794	367.0294	357.3844	71.1494	143.2037	157.5935	144.1602	151.9144

Table A.12

Probabilities that the spot price breaches the respective barriers with model parameters estimated using the $ MSE loss function.

	1 year to maturity					3 years to maturity				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$0.50S_0$	0.45%	3.68%	3.92%	4.15%	4.12%	9.83%	17.86%	18.38%	18.68%	18.61%
$0.55S_0$	1.33%	5.90%	6.28%	6.61%	6.56%	15.18%	22.37%	22.96%	23.18%	23.17%
$0.60S_0$	3.40%	9.14%	9.65%	9.87%	9.82%	21.88%	27.53%	28.05%	28.94%	28.91%
$0.65S_0$	7.33%	13.59%	14.12%	14.38%	14.37%	29.97%	33.50%	34.02%	34.74%	34.77%
$0.70S_0$	13.67%	19.50%	19.94%	20.06%	20.03%	38.81%	40.17%	40.56%	41.30%	41.20%
$0.75S_0$	22.58%	26.95%	27.28%	27.28%	27.29%	48.12%	47.54%	47.91%	48.62%	48.54%
$0.80S_0$	34.29%	36.24%	36.51%	36.45%	36.47%	58.50%	55.65%	55.97%	56.55%	56.48%
$0.85S_0$	48.31%	47.69%	47.75%	47.77%	47.82%	68.66%	64.89%	64.95%	65.56%	65.44%
$0.90S_0$	64.08%	61.34%	61.23%	61.37%	61.44%	78.68%	74.90%	75.20%	75.32%	75.31%
$0.95S_0$	80.54%	77.46%	77.35%	77.49%	77.48%	88.65%	85.71%	85.80%	86.43%	86.36%
$1.05S_0$	81.24%	82.90%	82.32%	83.28%	83.21%	89.14%	89.74%	89.37%	89.80%	89.71%
$1.10S_0$	67.14%	68.74%	68.12%	69.24%	69.24%	80.65%	81.85%	80.94%	81.58%	81.60%
$1.15S_0$	54.46%	55.22%	53.89%	55.45%	55.45%	72.55%	74.07%	72.79%	73.73%	73.83%
$1.20S_0$	43.47%	42.49%	40.35%	42.51%	42.51%	65.17%	66.32%	64.74%	66.17%	66.21%
$1.25S_0$	34.19%	31.48%	28.11%	31.39%	31.33%	58.40%	58.85%	57.11%	58.89%	58.76%
$1.30S_0$	26.62%	22.18%	17.89%	22.27%	22.21%	52.21%	52.04%	49.74%	51.66%	51.49%
$1.35S_0$	20.55%	14.96%	12.21%	15.39%	15.35%	46.82%	45.64%	42.89%	44.78%	44.91%
$1.40S_0$	15.92%	9.62%	8.70%	10.39%	10.36%	41.70%	39.66%	36.46%	38.64%	38.78%
$1.45S_0$	12.11%	5.87%	5.85%	6.73%	6.71%	37.16%	33.90%	30.68%	32.83%	32.92%
$1.50S_0$	9.09%	3.55%	3.82%	4.35%	4.35%	33.14%	28.80%	25.54%	28.03%	27.97%

Table A.13
Probabilities that the spot price breaches the respective barriers with model parameters estimated using the IV MSE loss function.

	1 year to maturity					3 years to maturity				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$0.50S_0$	0.40%	4.26%	4.22%	4.14%	4.20%	9.71%	18.23%	18.28%	18.06%	18.60%
$0.55S_0$	1.29%	6.73%	6.69%	6.57%	6.57%	15.06%	22.58%	22.90%	23.01%	23.14%
$0.60S_0$	3.33%	10.04%	10.03%	9.82%	9.89%	21.86%	27.75%	28.19%	28.58%	28.36%
$0.65S_0$	7.27%	14.48%	14.55%	14.47%	14.25%	29.51%	33.95%	33.99%	34.54%	34.26%
$0.70S_0$	13.52%	20.24%	20.34%	20.19%	20.09%	38.42%	40.66%	40.92%	41.20%	40.95%
$0.75S_0$	22.50%	27.54%	27.77%	27.56%	27.32%	48.14%	47.97%	48.02%	48.69%	48.51%
$0.80S_0$	34.25%	36.58%	36.84%	36.85%	36.49%	58.36%	56.34%	56.19%	56.79%	56.86%
$0.85S_0$	48.19%	47.67%	47.84%	48.08%	47.74%	68.70%	65.26%	65.37%	65.92%	66.30%
$0.90S_0$	63.93%	60.91%	61.02%	61.46%	61.23%	79.06%	75.35%	75.40%	75.75%	76.11%
$0.95S_0$	80.54%	77.04%	77.02%	77.41%	77.36%	88.96%	86.16%	85.88%	86.46%	86.50%
$1.05S_0$	81.31%	83.06%	82.83%	83.66%	83.19%	89.21%	89.89%	89.24%	90.46%	89.98%
$1.10S_0$	67.06%	68.95%	68.46%	69.93%	69.10%	80.92%	81.86%	80.86%	82.21%	81.92%
$1.15S_0$	54.41%	55.20%	54.32%	56.76%	55.48%	72.72%	74.20%	72.68%	74.46%	73.94%
$1.20S_0$	43.50%	42.28%	40.57%	44.21%	42.85%	65.00%	66.35%	64.80%	66.82%	66.04%
$1.25S_0$	34.23%	30.62%	28.07%	32.72%	31.41%	58.44%	59.34%	57.41%	59.62%	58.92%
$1.30S_0$	26.58%	20.94%	17.36%	22.87%	22.14%	52.12%	52.47%	50.00%	52.97%	52.02%
$1.35S_0$	20.51%	13.51%	9.99%	15.06%	14.87%	46.73%	45.86%	42.94%	46.63%	45.60%
$1.40S_0$	15.53%	8.38%	6.72%	9.20%	9.59%	41.33%	39.48%	36.60%	40.58%	39.50%
$1.45S_0$	11.76%	4.88%	4.85%	5.14%	5.98%	36.59%	33.88%	30.75%	35.02%	33.94%
$1.50S_0$	8.87%	2.74%	3.46%	2.66%	3.69%	32.27%	28.79%	25.48%	29.95%	28.91%

Table A.14

Probabilities that the spot price breaches the respective barriers with model parameters estimated using the % MSE loss function.

	1 year to maturity					3 years to maturity				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$0.50S_0$	0.43%	3.00%	3.57%	3.16%	3.21%	9.85%	16.74%	18.24%	18.15%	17.90%
$0.55S_0$	1.32%	5.06%	5.80%	5.40%	5.39%	15.34%	21.42%	22.69%	22.56%	22.41%
$0.60S_0$	3.33%	8.29%	9.05%	8.52%	8.48%	22.26%	26.71%	27.85%	27.68%	27.50%
$0.65S_0$	7.11%	12.51%	13.55%	12.84%	12.94%	29.95%	32.55%	33.90%	33.56%	33.67%
$0.70S_0$	13.37%	18.22%	19.50%	18.62%	18.63%	38.91%	39.24%	40.46%	40.19%	40.42%
$0.75S_0$	22.53%	26.06%	27.12%	26.30%	26.31%	48.30%	46.99%	47.74%	47.74%	47.79%
$0.80S_0$	34.39%	35.43%	36.30%	35.86%	35.84%	58.99%	55.58%	56.19%	55.78%	56.29%
$0.85S_0$	48.18%	47.00%	47.49%	47.48%	47.59%	69.20%	64.91%	65.23%	65.02%	65.25%
$0.90S_0$	63.89%	60.98%	61.15%	61.23%	61.19%	79.21%	74.78%	74.73%	74.90%	75.32%
$0.95S_0$	80.37%	77.40%	77.37%	77.56%	77.50%	88.80%	85.58%	85.78%	85.62%	85.80%
$1.05S_0$	81.36%	82.58%	82.19%	83.03%	82.28%	89.23%	89.69%	89.34%	90.13%	89.48%
$1.10S_0$	67.05%	68.16%	67.57%	68.78%	67.51%	80.55%	81.39%	80.77%	82.29%	80.94%
$1.15S_0$	54.56%	54.62%	53.53%	55.00%	53.73%	72.49%	73.43%	72.64%	74.28%	72.76%
$1.20S_0$	43.55%	42.03%	40.22%	42.28%	40.88%	64.97%	65.52%	64.71%	66.49%	64.70%
$1.25S_0$	34.44%	31.14%	28.44%	31.26%	29.86%	58.00%	57.90%	56.63%	58.90%	57.28%
$1.30S_0$	26.76%	22.16%	18.83%	22.20%	20.58%	51.72%	50.63%	49.35%	51.96%	49.99%
$1.35S_0$	20.60%	15.29%	12.45%	15.35%	13.55%	46.11%	44.24%	42.42%	45.15%	43.13%
$1.40S_0$	15.66%	10.05%	8.56%	10.31%	8.43%	40.98%	38.00%	36.10%	38.82%	36.87%
$1.45S_0$	11.85%	6.51%	5.92%	6.89%	5.12%	36.40%	32.40%	30.56%	33.12%	31.24%
$1.50S_0$	8.93%	4.08%	4.06%	4.52%	3.21%	32.43%	27.60%	25.66%	28.00%	26.10%

Table A.15

Probabilities that the spot price breaches the respective barriers with model parameters estimated using the L$ MSE loss function.

	1 year to maturity					3 years to maturity				
Barrier (H)	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**	**BS**	**SV**	**SVJ**	**MFSV**	**MFSVJ**
$0.50S_0$	0.39%	3.06%	3.57%	3.21%	3.08%	9.80%	16.69%	17.70%	17.83%	17.68%
$0.55S_0$	1.27%	5.22%	5.83%	5.43%	5.24%	15.07%	21.20%	22.36%	22.21%	22.36%
$0.60S_0$	3.37%	8.33%	9.10%	8.60%	8.46%	21.90%	26.40%	27.73%	27.75%	27.70%
$0.65S_0$	7.29%	12.67%	13.60%	12.97%	12.82%	29.80%	32.32%	33.77%	33.54%	33.65%
$0.70S_0$	13.58%	18.45%	19.60%	18.83%	18.72%	38.89%	39.18%	40.29%	40.37%	40.40%
$0.75S_0$	22.60%	26.10%	26.98%	26.43%	26.12%	48.48%	46.97%	47.68%	47.95%	47.64%
$0.80S_0$	34.38%	35.60%	36.31%	35.77%	35.55%	58.60%	55.30%	55.96%	56.01%	56.23%
$0.85S_0$	48.22%	47.13%	47.47%	47.34%	47.18%	68.46%	64.20%	64.92%	65.05%	65.22%
$0.90S_0$	64.00%	60.98%	61.17%	61.46%	61.08%	78.73%	74.44%	74.70%	74.92%	75.23%
$0.95S_0$	80.59%	77.47%	77.52%	77.66%	77.44%	88.66%	85.51%	85.58%	85.71%	86.02%
$1.05S_0$	81.27%	82.53%	82.23%	83.09%	82.42%	89.09%	89.54%	89.38%	90.04%	89.42%
$1.10S_0$	67.11%	68.11%	67.75%	69.01%	67.84%	80.51%	81.34%	80.90%	82.07%	80.94%
$1.15S_0$	54.45%	54.59%	53.67%	55.29%	53.92%	72.61%	73.16%	72.55%	74.28%	72.77%
$1.20S_0$	43.52%	41.98%	40.35%	42.68%	41.33%	65.22%	65.70%	64.58%	66.54%	65.04%
$1.25S_0$	34.39%	31.11%	28.72%	31.52%	30.10%	58.25%	58.48%	56.82%	59.20%	57.44%
$1.30S_0$	26.81%	22.26%	19.34%	22.46%	20.80%	52.10%	51.30%	49.61%	52.33%	50.11%
$1.35S_0$	20.74%	15.24%	12.68%	15.46%	13.64%	46.63%	44.54%	42.79%	45.89%	43.42%
$1.40S_0$	15.83%	10.14%	8.62%	10.36%	8.46%	41.50%	38.67%	36.53%	39.43%	37.50%
$1.45S_0$	11.99%	6.60%	5.99%	6.85%	5.09%	36.66%	33.03%	30.94%	33.81%	31.90%
$1.50S_0$	9.00%	4.22%	4.08%	4.49%	3.28%	32.55%	28.17%	26.18%	28.75%	26.54%

Appendix B: Data

Table B.1

Implied volatilities of options written on the Eurostoxx 50 index on the 7th of October 2003. The data set was obtained from [23].

	Maturity (years)					
Strike price	**0.0361**	**0.2000**	**1.1944**	**2.1916**	**4.2056**	**5.1639**
1081.82			0.3804	0.3451	0.3150	0.3137
1212.12			0.3667	0.3350	0.3082	0.3073
1272.73			0.3603	0.3303	0.3050	0.3043
1514.24			0.3348	0.3116	0.2920	0.2921
1555.15			0.3305	0.3084	0.2899	0.2901
1870.3		0.3105	0.2973	0.2840	0.2730	0.2742
1900.00		0.3076	0.2946	0.2817	0.2714	0.2727
2000.00		0.2976	0.2858	0.2739	0.2660	0.2676
2100.00	0.3175	0.2877	0.2775	0.2672	0.2615	0.2634
2178.18	0.3030	0.2800	0.2709	0.2619	0.2580	0.2600
2200.00	0.2990	0.2778	0.2691	0.2604	0.2570	0.2591
2300.00	0.2800	0.2678	0.2608	0.2536	0.2525	0.2548
2400.00	0.2650	0.2580	0.2524	0.2468	0.2480	0.2505
2499.76	0.2472	0.2493	0.2446	0.2400	0.2435	0.2463
2500.00	0.2471	0.2493	0.2446	0.2400	0.2435	0.2463
2600.00		0.2405	0.2381	0.2358	0.2397	0.2426
2800.00			0.2251	0.2273	0.2322	0.2354
2822.73			0.2240	0.2263	0.2313	0.2346
2870.83			0.2213	0.2242	0.2295	0.2328
2900.00			0.2198	0.2230	0.2288	0.2321
3000.00			0.2148	0.2195	0.2263	0.2296
3153.64			0.2113	0.2141	0.2224	0.2258
3200.00			0.2103	0.2125	0.2212	0.2246
3360.00			0.2069	0.2065	0.2172	0.2206
3400.00			0.2060	0.2050	0.2162	0.2196
3600.00				0.1975	0.2112	0.2148
3626.79				0.1972	0.2105	0.2142
3700.00				0.1964	0.2086	0.2124
3800.00				0.1953	0.2059	0.2099
4000.00				0.1931	0.2006	0.2050
4070.00					0.1988	0.2032
4170.81					0.1961	0.2008
4714.83					0.1910	0.1957
4990.91					0.1904	0.1949
5000.00					0.1903	0.1949
5440.18						0.1938